LPV Controller Interpolation for Improved Gain-Scheduling Control Performance

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In this paper, a new gain-scheduling control design approach is proposed by combining LPV control theory with interpolation techniques. The improvement of gain-scheduled controllers can be achieved from local synthesis of Lyapunov functions and continuous construction of a global Lyapunov function by interpolation. It has been shown that this combined LPV control design scheme is capable of improving closed-loop performance derived from local performance improvement. The gain of the LPV controller will also change continuously across parameter space. The advantages of the newly proposed LPV control is demonstrated through a detailed AMB controller design example.

Introduction

The gain-scheduling approach is perhaps one of the most popular nonlinear control design techniques that has been widely used in fields ranging from aerospace to process control. Although it seems to work well in practice, this heuristic design procedure does not take the parameter variations into account.1,2 In its early practice, the control design came with virtually no guarantee on performance, robustness, or even nominal stability. Recently, a systematic gain-scheduling design technique was developed in the form of linear parameter-varying (LPV) control theory.3-8 This class of systems is different from the standard linear time-varying counterpart due to the causal dependence of its controller gains on the variations of the plant dynamics. The implications of parameter-dependent systems theory for gain scheduling is obvious because control scheduling conceptually involves a linear, parameter-dependent plant. The LPV design technique provides guaranteed stability and performance properties, and simplifies the interpolation and realization problems associated with conventional gain-scheduling methods. However, one potential problem associated with parameter-dependent Lyapunov function approach is the lack of guidance for choosing "right" basis functions to parameterize infinite dimensional functional space.

Interpolation is an important step toward synthesis of gain-scheduled controllers and has not been adequately addressed in a systematic way. Some ad-hoc interpolation techniques have been proposed in the past: (1) linear interpolation of poles, zeros, and gains of local controllers; (2) linear interpolation of solutions of Riccati equations; and (3) linear interpolation of state-space matrices of balanced controller realizations. These approaches are intuitively appealing but could generate destabilizing controllers. Stillwell and Rugh9 proposed a gain-scheduled control design based on interpolation techniques. The interpolated parameter-varying controller preserves point-wise stability of local LTI controllers for all frozen parameter values. However, the global control does not provide a priori stability and performance guarantee from locally designed controllers for fast time-varying parameters.

In this research, we propose an interpolating LPV control approach by combining LPV control theory with interpolation techniques. The interpolated LPV control design is capable of improving controlled performance by finding the most appropriate Lyapunov functions in a local sense. Moreover, the local stability property will be extended to the entire parameter range by a globally constructed Lyapunov function through interpolation. The proposed interpolating LPV control technique has great potential to many industrial applications including active magnetic bearings (AMBs), for which the controller adjusts its gain based on changing rotor speed and provides accurate control of flexible mode variation in terms of natural...
functions of one can analyze the performance of
the achievable performance relies on the choice of
the sets that have compatible dimensions. It is as-

monotonic decrease of its value over any allowable tra-

where \( x, \dot{x} \in \mathbb{R}^n \), \( d \in \mathbb{R}^n \), \( e \in \mathbb{R}^n \). All the
matrices have compatible dimensions. It is as-

For the parameter
set \( P \subset \mathbb{R}^n \). Its time derivative is bounded and satisfies the constraint \( -v_i \leq \dot{v}_i \leq v_i, i = 1, 2, \ldots, s \). For notational purposes, \( \mathcal{V} := \{ v: -v_i \leq v_i \leq v_i, i = 1, 2, \ldots, s \} \), where \( \mathcal{V} \) is a given convex polytope in \( \mathbb{R}^n \) that contains the origin. Given the sets \( P \) and \( \mathcal{V} \), we define the parameter \( \nu \)-variation set as

\[
\mathcal{F}_P = \{ \rho \in C^1(\mathbb{R}_+, \mathbb{R}^s): \rho(t) \in P, \rho(t) \in \mathcal{V}, \forall t \geq 0 \}
\]  

Therefore the dynamics of the LPV system are char-
characterized by the parameter value \( \rho \) and its variation along time. Previous research on LPV control the-
theory mainly focused on a single Lyapunov function (quadratic or parameter-dependent) over the entire
parameter set. For the given LPV system, it is clear that the achievable performance relies on the choice of
the Lyapunov function. However, it would be benefi-
cial if one can analyze the performance of the LPV system over different parameter ranges using different
Lyapunov functions with a stability guarantee.

To simplify the presentation, let us assume that the parameter set has dimension one. That is, \( P \subset \mathbb{R}^1 \). Suppose \( P_1, P_2 \) is an overlapped partition of the para-
ter set \( P \), and define the intersection of \( P_1 \) and \( P_2 \) as

\[
P^{12} = P_1 \cap P_2
\]  

Then the parameter space \( P \) is the union of two subsets \( P_1, P_2 \). Through linear interpolation of the Lyapunov
functions \( X_1(\rho), X_1(\rho) \) over subsets \( P_1 \) and \( P_2 \), one would obtain a continuous Lyapunov function over
the entire parameter space. The continuity property of the Lyapunov function is important to guarantee
monotonic decrease of its value over any allowable tra-
jectories dictated by the LPV system dynamics. A new LPV stability and performance analysis condition based on the partitioned parameter subsets is then proposed.

**Theorem 1** For the parameter set \( P = [p_1, p_2] \) with
its overlapped partition \( P^1 = [p, b] \) and \( P^2 = [a, p] \)
\((a < b)\), the LPV system (1), and a given symmetric
matrix function \( Q_X(\rho) \geq 0 \), if there exists a group of
continuously differentiable positive-definite matrix
functions \( X_i(\rho), i = 1, 2 \) such that for any \( \rho \in P^i \),

\[
\begin{bmatrix}
A^T(\rho)X_i(\rho) + X_i(\rho)A(\rho)
\pm \nu \left( Q_X(\rho) + \frac{\partial Q_X}{\partial \rho} \right)
B^T(\rho)X_i(\rho)
C(\rho)
\end{bmatrix}
\begin{bmatrix}
X_i(\rho)B(\rho)
D^T(\rho)
\end{bmatrix}
< 0
\]  

and for all \( \rho \in P^{12} \)

\[
X_2(\rho) - X_1(\rho) + (b - a)Q_X(\rho) \geq 0
\]  

then the LPV system (1) is exponentially stable and
its induced \( \ell_2 \) norm with \( x(0) = 0 \) is bounded by \( \gamma = \max \{ \gamma_1, \gamma_2 \} \).

Given the above conditions over each parameter subset, a continuous Lyapunov function in the form of

\[
V(x, \rho) = x^T \left[ \begin{array}{cc}
\frac{b - a}{b - a} & X_1(\rho) + \left( \frac{b - a}{b - a} \right) X_2(\rho)
\end{array} \right] x
\]  

can be constructed and used to verify the desired sta-
ability and performance properties of the system for any
parameter trajectories within \( \mathcal{F}_P \). Then a non-smooth dissipative system theory is applied to address possible discontinuity of the Lyapunov function derivative at the boundary of the parameter subsets.\(^{10}\) The perfor-
ance bound derived in Theorem 1 states the "worst-case"
performance, which is not often achievable. For example, if the parameter trajectory is constrained in
the subset \( P^i \), then the system’s performance is over-
bounded by \( \gamma_i \), which could be less than \( \max \{ \gamma_1, \gamma_2 \} \).

Theoretically, this theorem simply states the fact that the performance level of the LPV system relates to
the existence of a continuous Lyapunov function. However, in practical situations, it is usually hard
to identify a suitable parameter-dependent Lyapunov
function without resorting to a global search. In this
sense, the improved analysis condition will be help-
ful to find a sharper performance bound through local
studies. Then the global Lyapunov function will be
constructed by interpolation.

**Interpolating LPV Control Synthesis**

Given a generalized open-loop LPV system

\[
\begin{bmatrix}
\dot{x}(t)
\dot{e}(t)
y(t)
\end{bmatrix} =
\begin{bmatrix}
A(\rho(t)) & B_1(\rho(t)) & B_2(\rho(t))
C_1(\rho(t)) & D_{11}(\rho(t)) & D_{12}(\rho(t))
C_2(\rho(t)) & D_{21}(\rho(t)) & D_{22}(\rho(t))
\end{bmatrix}
\begin{bmatrix}
x(t)
d(t)
u(t)
\end{bmatrix}
\]  

where all of the state-space data are continuous functions of the scheduling parameter \( \rho \), it is assumed that
the parameter trajectory resides in the set \( \mathcal{F}_P \). For
simplicity, we assume that
(A1) \((A(p), B_2(p), C_2(p))\) is parameter-dependently stabilizable and detectable for all \(p\),

(A2) \(D_{21}(p)\) and \(D_{22}(p)\) have full column and row rank respectively for all \(p\),

(A3) \(D_{11}(p) = 0\) and \(D_{22}(p) = 0\).

We consider the class of interpolating LPV controller in the form of

\[
\begin{bmatrix}
\dot{x}_k(t) \\
y(t)
\end{bmatrix} = \begin{bmatrix} A_k(p(t), \dot{p}(t)) & B_k(p(t), \dot{p}(t)) \\
C_k(p(t), \dot{p}(t)) & D_k(p(t), \dot{p}(t))
\end{bmatrix} \begin{bmatrix} x_k(t) \\
y(t)
\end{bmatrix}
\]

In general, the controller gain is a continuous function of parameter \(p\) and its derivative. The control design objective is to minimize the energy (\(L_2\) norm) of the output signal \(y(t)\) of the closed-loop system in the presence of bounded energy disturbance \(d(t)\). Specifically for the magnetic bearing control problem, we would like to synthesize a gain-scheduled control with combined disturbance rejection, gyroscopic compensation, and automatic balancing capability. This can be formulated as an optimization problem with minimization of displacement of the rotor from its centerline at selected points subject to unknown torque disturbances.

Define \(x_T^T := [x^T \ x_k^T]\), then the closed-loop system can be written as

\[
\begin{bmatrix}
\dot{x}_c \\
d
\end{bmatrix} = \begin{bmatrix} A_c(p, \dot{p}) & B_c(p, \dot{p}) \\
C_c(p, \dot{p}) & D_c(p, \dot{p})
\end{bmatrix} \begin{bmatrix} x_c \\
d
\end{bmatrix}
\]

where

\[
\begin{bmatrix} A_c & B_c \\
C_c & D_c
\end{bmatrix} = \begin{bmatrix} A & B_1 \\
0 & 0 \\
C_1 & 0 \\
0 & D_{11}
\end{bmatrix} + \begin{bmatrix} 0 & B_2 \\
0 & 0 \\
0 & D_{12}
\end{bmatrix} \begin{bmatrix} A_k & B_k \\
C_k & D_k
\end{bmatrix} \begin{bmatrix} 0 & I \\
C_2 & 0 \\
0 & D_{21}
\end{bmatrix}
\]

Next, we propose a synthesis condition for an improved LPV controller using multiple parameter-dependent Lyapunov functions and an interpolation scheme. For clarity, only one parameter is considered and the parameter space \(P\) is covered by two overlapped subsets \(P^1\) and \(P^2\). For each parameter subset, we seek to design one LPV controller as stated in the form of eqn. (9). The overall gain-scheduling controller is then constructed by interpolating local LPV controllers. It is clear that the global controller is capable of achieving tighter performance due to smaller parameter range. However, a critical issue associated with the proposed controller interpolation scheme is the stability of the global LPV controller. This will be guaranteed by constructing a globally continuous Lyapunov function over the entire parameter set.

**Theorem 2** For a partition of parameter space \(P = [a, b]\) as subsets \(P^1 = [a, b]\) and \(P^2 = [a, b]\) \((a < b)\), and given \(Q_R, Q_S \geq 0\) matrix functions, if one of the following equivalent conditions are satisfied:

1. there exist continuously differentiable matrix functions \(R_i(p), S_i(p), i = 1, 2\) such that for \(p \in P^i\)

\[
N_R^T \begin{bmatrix} \mp \nu \left(Q_R + \frac{\partial R_1}{\partial p}\right) & R_i C_i^T \\ C_i R_i & -\gamma_i I \end{bmatrix} N_R < 0
\]

2. there exist continuously differentiable matrix functions \(R_i(p), S_i(p), i = 1, 2\) and continuous matrix functions \(R_i, S_i, i = 1, 2\) such that

\[
N_S^T \begin{bmatrix} \mp \nu \left(Q_S + \frac{\partial S_1}{\partial p}\right) & S_i B_1 \\ B_1^T S_i & -\gamma_i I \end{bmatrix} N_S < 0
\]

with \(N_R(p) = \text{Ker} \left[ B_1^T(p) \ D_{12}(p) \ 0 \right], N_S(p) = \text{Ker} \left[ C_2(p) \ D_{21}(p) \ 0 \right],\) and for \(p \in P^1\)

\[
B_2(p) - R_1(p) + (b - a)Q_R \geq 0 \quad (15)
\]

\[
B_2(p) - R_1(p) - (b - a)Q_R \leq 0 \quad (16)
\]

\[
S_2(p) - S_1(p) + (b - a)Q_S \geq 0 \quad (17)
\]

\[
S_2(p) - S_1(p) - (b - a)Q_S \leq 0. \quad (18)
\]
then the closed-loop LPV system can be stabilized by a continuous LPV controller with induced $\mathcal{L}_2$ performance less than $\gamma = \max\{\gamma_1, \gamma_2\}$.

Furthermore, let

$$\gamma(\rho) = \begin{cases} \gamma_1 \left( \frac{b-a}{b-a} \right) + \left( \frac{a-\rho}{a-\rho} \right) \gamma_2 & \rho \leq a \\ \gamma_1 & a < \rho < b \\ \gamma_2 & \rho \geq b \end{cases}$$

$$R(\rho), S(\rho) = \begin{cases} R_1(\rho), S_1(\rho) & \rho \leq a \\ \left( \frac{a-\rho}{a-\rho} \right) \{R_1(\rho), S_1(\rho)\} + \left( \frac{b-a}{b-a} \right) \{R_2(\rho), S_2(\rho)\} & a < \rho < b \\ R_2(\rho), S_2(\rho) & \rho \geq b \end{cases}$$

and $M(\rho)N^T(\rho) := I - R(\rho)S(\rho)$, calculate

$$\dot{A}_k(\rho, \hat{\rho}) = S(\rho)\frac{dR}{dt} + N(\rho)\frac{dM^T}{dt} - A^T(\rho)$$

$$- \frac{1}{\gamma(\rho)} \left[ S(\rho)B_1(\rho) + \hat{B}_k(\rho)D_{21}(\rho) \right] C_1^T(\rho)]$$

$$\times \left[ C_1(\rho)R(\rho) + D_{21}(\rho)\hat{C}_k(\rho) \right]$$

$$\dot{B}_k(\rho) = - \left[ \gamma(\rho)C_2^T(\rho) + S(\rho)B_1(\rho)D_{21}(\rho) \right]$$

$$\times \left[ D_{21}(\rho)D_{12}(\rho) \right]^{-1}$$

$$\dot{\hat{C}}_k(\rho) = - \left[ D_{12}^T(\rho)D_{12}(\rho) \right]^{-1}$$

$$\times \left[ \gamma(\rho)B_2^T(\rho) + D_{12}(\rho)C_1(\rho)R(\rho) \right]$$

$$\dot{\hat{D}}_k(\rho) = 0$$

Then the interpolated LPV controller $K_\rho$ will be constructed as

$$A_k(\rho, \hat{\rho}) = N^{-1}(\rho) \left\{ \dot{A}_k(\rho, \hat{\rho}) - S(\rho)B_2(\rho)\hat{C}_k(\rho) \right\} M^{-T}(\rho)$$

$$- \dot{B}_k(\rho)C_2(\rho)R(\rho) - S(\rho)A(\rho)R(\rho) \}$$

$$B_k(\rho) = N^{-1}(\rho)\dot{\hat{B}}_k(\rho)$$

$$C_k(\rho) = \hat{C}_k(\rho)M^{-T}(\rho)$$

$$D_k(\rho) = 0.$$
The scheduling design technique suffers from slow varying parameter requirements. The resulting gain-scheduling controller could render the closed-loop system unstable when the rotor speed changes rapidly. Rotor flexibility was considered using the \( \mu \)-synthesis technique. It was shown that the \( \mu \) controller exhibited much greater stability robustness to variations in rotor mass. However, the controller caused significant performance degradation when the actual rotor speed differs from the pre-specified design condition. One way to address this problem is to design a series of controllers for each operating speed and then interpolate between these controllers. However, this approach does not provide any stability and performance guarantee. In Tsiotras and Mason and Tsiotras and Knospe, the LPV control theory was applied to address variable rotor operating speeds using a rigid rotor model and conservatism of LPV control design was recognized. Similar controllers were also developed by Sivrioglu and Nonami.

### Rotor Dynamics Modeling

Owing to the linear dependence of the rotor speed in the plant dynamics, the nonlinear gyroscopic equations of AMB can be simplified to a set of linear, time-varying differential equations. However, the rotor dynamics are inherently unstable; that is, even small unbalanced masses can create large synchronous disturbances with the same frequency as the rotor’s. Therefore it is necessary to develop a gain-scheduled controller capable of rejecting the periodic disturbance as the rotor speed changes.

The rotational motion of a magnetic bearing can be derived from its rigid body dynamics, which is

\[
\begin{align*}
\dot{\theta} &= -\frac{\rho J_o}{J_r} \dot{\psi} + \frac{\ell}{J_r} (f_{r1} - f_{r2} + f_{a2} - f_{a1} + f_{dB}) \\
\dot{\psi} &= \frac{\rho J_o}{J_r} \dot{\theta} + \frac{\ell}{J_r} (f_{t3} - f_{t4} + f_{a4} - f_{a2} + f_{d\psi})
\end{align*}
\]

where \( \theta, \psi \) are the Euler angles denoting the orientation of the rotor centerline, \( J_o, J_r \) are the moment of inertia of the rotor in axial and radial directions, respectively. The parameter \( \rho \) denotes the rotor speed. The magnetic forces generated by four pairs of electromagnets are denoted by \( f_{ri}, f_{ti} \) for \( i = 1, 2, 3, 4 \). \( f_{dB}, f_{d\psi} \) are disturbance forces caused by gravity, modeling errors, imbalances, etc.

The electromagnetic force \( f_j \) is related to the voltage \( e_j \) across the \( jth \) coil through the magnetic flux \( \psi_j \) by the equations

\[
\begin{align*}
f_j &= k \phi_j^2 \left( 1 + \frac{2 \theta_j}{\pi h} \right) \\
e_j &= N \frac{d\phi_j}{dt} + \frac{2R}{\nu_0 AN} g_j \phi_j
\end{align*}
\]

where \( k \) is a constant.

The following system parameters are chosen for the active magnetic bearing example (Table 1).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A ) area of each pole</td>
<td>1531.79mm²</td>
</tr>
<tr>
<td>( h ) pole width</td>
<td>40.00mm</td>
</tr>
<tr>
<td>( G_0 ) nominal gap</td>
<td>0.55mm</td>
</tr>
<tr>
<td>( J_r ) radial moment of inertia</td>
<td>0.333kg · m²</td>
</tr>
<tr>
<td>( J_a ) axial moment of inertia</td>
<td>0.0136kg · m²</td>
</tr>
<tr>
<td>( \ell ) half the length of the shaft</td>
<td>0.13m</td>
</tr>
<tr>
<td>( k )</td>
<td>4.6755576 × 10^8</td>
</tr>
<tr>
<td>( N ) number of coil turns</td>
<td>400</td>
</tr>
<tr>
<td>( R ) coil resistance</td>
<td>14.60ohm</td>
</tr>
<tr>
<td>( \Phi_0 ) nominal airgap</td>
<td>2.09 × 10^-4Wb</td>
</tr>
</tbody>
</table>

### Table 1 Magnetic bearing parameters.

In the absence of disturbances and modeling errors, the above equation specifies an equilibrium. Linearizing the nonlinear equations at the equilibrium, we obtain

\[
\begin{align*}
\ddot{\theta} &= -\frac{\rho J_o}{J_r} \dot{\theta} + \frac{\ell}{J_r} (-4c_2 \ell \theta + 2c_1 \phi_\theta + f_{dB}) \\
\ddot{\psi} &= \frac{\rho J_o}{J_r} \dot{\psi} + \frac{\ell}{J_r} (-4c_2 \ell \psi + 2c_1 \phi_\psi + f_{d\psi}) \\
N_0 \dot{\dot{\phi}} &= c_\theta + 2d_2 \ell \theta - d_1 \phi_\theta \\
N_0 \dot{\dot{\psi}} &= c_\psi + 2d_2 \ell \psi - d_1 \phi_\psi
\end{align*}
\]

where \( \phi_\theta, \phi_\psi \) are the differential magnetic flux from electromagnetic pairs, and \( c_\theta, c_\psi \) are the corresponding differences of electric voltage. The constants \( c_1, c_2, d_1, d_2 \) depend on \( \Phi_0, G_0, R, A, N, \nu_0 \) and the geometry of the bearings as follows

\[
\begin{align*}
c_1 &= 2k \Phi_0 \left( 1 + \frac{2G_0}{\pi h} \right), \quad c_2 = \frac{2k \Phi_0^2}{\pi h}, \\
d_1 &= \frac{2RG_0}{\nu_0 AN}, \quad d_2 = \frac{2RG_0}{\nu_0 AN}, \quad m = \frac{J_r}{\ell}.
\end{align*}
\]

The imbalance forces \( f_{dB} \) and \( f_{d\psi} \) are typically modeled as sinusoidal disturbances, and are given by

\[
\begin{align*}
f_{dB} &= (J_r - J_a) \rho^2 r \cos(pt), \\
f_{d\psi} &= \frac{J_r - J_a}{\ell} \rho^2 r \sin(pt).
\end{align*}
\]

In automatic balancing design, the imbalance forces will be treated as a sinusoidal sensor noise \( n^T = [\ell \tau \sin(pt + \lambda) \ell \tau \cos(pt + \lambda)] \) on the measured rotor displacement, which can be written in a state-space form of

\[
\begin{align*}
\dot{x}_{di} &= [\sigma -\rho \sigma] x_{di} + [1 \ 0] \bar{d} \\
n &= [1 \ 0 \ 1] \ 
\end{align*}
\]

where \( \sigma = -0.001 \).

Let \( x^T = [\ell \theta \ \ell \psi \ \ell \dot{\theta} \ \ell \dot{\psi} \ \phi_\theta \ \phi_\psi \ \phi_\theta \ x_{d1} \ x_{d2}] \), \( w = \bar{d} \) and \( u^T = [e_\theta \ e_\psi] \). Combining rotor dynamics and the
disturbance equation, the linearized equation can be written as

\[ \dot{x} = A(\rho)x + B_1 w + B_2 u \]
\[ z = C_1 x + D_{11} w + D_{12} u \]
\[ y = C_2 x + D_{21} w + D_{22} u \]

with

\[
A = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-\frac{\ell_2}{m} & 0 & 0 & -\frac{\ell_2}{D} & \frac{\ell_2}{m} & 0 & 0 & 0 \\
0 & -\frac{\ell_2}{m} & \frac{\ell_2}{D} & 0 & 0 & -\frac{\ell_2}{m} & 0 & 0 \\
\frac{2\ell_2}{N} & 0 & 0 & 0 & -\frac{\ell_2}{N} & 0 & 0 & 0 \\
0 & \frac{2\ell_2}{N} & 0 & 0 & 0 & -\frac{\ell_2}{N} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sigma & -\rho \\
0 & 0 & 0 & 0 & 0 & 0 & \rho & \sigma
\end{bmatrix}
\]

\[
B_1 = \begin{bmatrix} 0 & x_1 \\ 1 \\ 0 \end{bmatrix}, \quad B_2 = \frac{1}{N} \begin{bmatrix} 0 & x_2 \\ I_2 \\ 0 & x_2 \end{bmatrix}
\]
\[
C_1 = \begin{bmatrix} I_2 & 0_{2\times 6} & 0_{2\times 8} \\ 0_{2\times 6} & 0_{2\times 8} & I_2 \end{bmatrix}, \quad D_{11} = 0_{4\times 1}, \quad D_{12} = \begin{bmatrix} 0_{2\times 2} \\ I_2 \end{bmatrix}, \quad D_{21} = 0_{2\times 1}, \quad D_{22} = 0_{2\times 2}
\]

Note that the linearized rotor dynamic equation is in the form of a linear parameter-varying system with the rotor speed serving as the parameter. For gain-scheduled control, the rotational speed is assumed to be available in real-time for controller use.

**Interpolating LPV Control Design**

The design objectives of the LPV controller are to asymptotically stabilize the system over the whole range of rotor speeds and to minimize an error signal representing a weighted sum of the forces at the bearings, the gap displacement at the bearing locations, and the control input used.

Although the scheduling variable is time-varying in the LPV dynamics, it is simply treated as a fixed parameter in the design stage. The design objective is quantified from a frozen parameter design viewpoint by weighting functions, and the weighted open-loop interconnection is given in Figure 1.

In Figure 1 the weighting functions are chosen as

\[
W_e(s) = \frac{200(s+100)}{s+0.01} - I_2, \quad (57)
\]
\[
W_u(s) = \frac{0.001 s}{0.05 s + 1} I_2, \quad (58)
\]
\[
W_n(s) = 0.001 \quad (59)
\]

The rotor speed is assumed to change freely between 315rad/s to 1100rad/s. The rotor dynamics exhibit significant gyroscopic effects in this speed range. The LPV synthesis problem can be solved using either a single or parameter-dependent quadratic Lyapunov function over all gridding points in a one-dimensional parameter space. The performance obtained through a single quadratic Lyapunov function (SQLF) approach is \( \gamma = 320.78 \), which is quite conservative. Also, multiple LPV control syntheses are conducted with the whole parameter space divided into two subsets \( [315,720] \cup [700, 1100] \) and four subsets \( [315,520] \cup [500,720] \cup [700,920] \cup [900, 1100] \), respectively. Then five points are used to grid each parameter subspace uniformly. The free parameters \( R_1 \) and \( Q_S \) are chosen as 0.1111 for \( \rho \) in \( \mathcal{P}_i \cap \mathcal{P}_j \) or zero otherwise. In each partitioned parameter space case, two sets of identical basis functions are used to parameterize the functional space,

\[
f_i(\rho) = 1, \quad (60)
\]
\[
g_i^1(\rho) = 1, \quad g_i^2(\rho) = \rho \quad (61)
\]

for each \( i \). Note that the basis function for \( R_i(\rho) \) is chosen as a constant over each parameter subset. This will only cause small performance degradation compared with more complicated basis function selections. Since both \( R \) and \( S \) become functions of the scheduling parameters after interpolation, the interpolated LPV controller gain will depend on both the parameter and its derivative. The performance level of the interpolating LPV controller versus number of parameter subsets under different parameter variation rates is shown in Table 2. Note that the \( \gamma \) value represents the “worst-case” interpolating LPV control performance over different parameter subsets. The actual performance over each parameter subset could be less than the “worst-case” performance.

For comparison, a single parameter-dependent Lyapunov function is also considered to demonstrate the sharpness of the newly proposed LPV synthesis condition. The calculated performance bound is about the same as the “worst-case” performance of interpolated LPV cases.
<table>
<thead>
<tr>
<th>Variation rate</th>
<th>PDLF</th>
<th>Interpolating LPV</th>
<th>Interpolating LPV</th>
</tr>
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<tr>
<td></td>
<td></td>
<td>2 subsets</td>
<td>4 subsets</td>
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<tr>
<td>1</td>
<td>9.863</td>
<td>9.936</td>
<td>9.954</td>
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<tr>
<td>100</td>
<td>10.064</td>
<td>10.107</td>
<td>10.149</td>
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<tr>
<td>10000</td>
<td>141.24</td>
<td>144.43</td>
<td>143.50</td>
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</table>

Table 2 Achievable induced-$\mathcal{L}_2$ performance using single and multiple parameter-dependent Lyapunov functions (PDLFs).

Simulation Results

Next, the LPV controller synthesized using two subsets is used for comparison study and simulation work.

To compare the LPV controller with the optimal $\mathcal{H}_\infty$ controller, eight points are chosen from the parameter set $\mathcal{P}$ and $\mathcal{H}_\infty$ controller is synthesized for each operating condition. Then their $\mathcal{H}_\infty$ norm values are compared with the sub-optimal control performance that is achieved by LPV controllers evaluated at fixed parameter values. These are provided in the Table 3.

<table>
<thead>
<tr>
<th>$\rho$ (rad/s)</th>
<th>$\mathcal{H}_\infty$</th>
<th>Interpolating LPV</th>
</tr>
</thead>
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<tr>
<td>315</td>
<td>9.778</td>
<td>9.833</td>
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<td>427.1</td>
<td>7.487</td>
<td>8.147</td>
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<td>539.3</td>
<td>6.149</td>
<td>6.719</td>
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<td>651.4</td>
<td>5.289</td>
<td>5.772</td>
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<td>763.6</td>
<td>4.696</td>
<td>4.988</td>
</tr>
<tr>
<td>875.7</td>
<td>4.264</td>
<td>4.831</td>
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<tr>
<td>987.9</td>
<td>3.936</td>
<td>4.600</td>
</tr>
<tr>
<td>1100</td>
<td>3.678</td>
<td>4.383</td>
</tr>
</tbody>
</table>

Table 3 Frozen optimal/LPV closed-loop $\mathcal{H}_\infty$ norm.

Since the LPV controller is designed for a range of parameters, it is not surprising that they are only sub-optimal for each fixed parameter value. However, it is observed that the optimal performance level using $\mathcal{H}_\infty$ is very close to the achievable performance derived from switching the LPV controller at each frozen parameter. However, the optimal controller is highly tuned to its designated rotor speed. When a magnetic bearing operates at a different rotational speed, significant performance degradation or even loss of stability is expected. For example, the optimal $\mathcal{H}_\infty$ controller designed for $\rho = 700\text{rad/s}$ results in a performance level $\gamma = 1.16 \times 10^6$ when the rotor is actually rotating with the speed of $1100\text{rad/s}$.

Finally, we compare the closed-loop step response for both optimal and LPV controllers. The time responses for the optimal controller and the LPV controller are shown in Figures 2 and Figure 3, respectively. It can be seen that the step responses of rotor displacement from both controllers are quite similar, whereas the control actions are slightly different due to sub-optimality of LPV controllers.

After analyzing the frozen point LPV controller property, we are ready to do some nonlinear simulations for AMB with time-varying rotational speed. A time-varying rotor speed profile is chosen as

$$
\begin{align*}
    d(t) = \begin{cases}
        \sin(680t) & 0 \leq t < 0.5s \\
        \sin(\omega t), \quad \omega = 655 + 50t & 0.5s \leq t < 1.5s \\
        \sin(730t) & 1.5s \leq t < 2s
    \end{cases}
\end{align*}
$$

Note that the rotor speed trajectory is deliberately chosen to cross the intersection of two subsets $[315, 720]$ and $[700, 1100]$ of the scheduling parameter to illustrate LPV controller interpolating effect. Disturbance $\vec{d} = 1.3 \times 10^{-5}$ is used for the simulation purpose. Except for small glitches during the controller interpolation period, the simulation results demonstrate good performance of interpolating LPV control as shown in Figure 4.

Concluding Remarks

In this paper, an LPV control interpolation algorithm was proposed to achieve high-performance of gain-scheduled control. The proposed LPV control approach unified the systematic LPV control theory with interpolation technique. The stability of the
interpolated LPV control was achieved by enforcing continuity of Lyapunov function over entire parameter space. With modest increase of computational effort, the proposed LPV interpolation scheme improved the LPV control performance considerably. The newly developed LPV control synthesis method was applied to the magnetic bearing control problem to reject unbalancing sinusoidal disturbances and accommodate changing rotor speed. Promising simulation results were obtained.

For future research, an F-16 nonlinear model has been acquired from the NASA Langley Research Center and the proposed LPV control technique is currently being applied to a highly maneuverable aircraft to improve its performance over the expanded flight envelope into nonlinear unsteady flight regimes. Because aerodynamic controls may be insufficient for control at post-stall conditions, we would like to augment the conventional aerodynamic surface controls using thrust vectoring (TV) control effectors. The LPV interpolated control design approach is capable of unifying the aerodynamic force and thrust force control law development in a unified framework, and of providing guaranteed stability and performance for large flight envelope.

References
10 S. Lim and J.P. How, “Control of LPV systems using a


