A Comparison of Trajectory Optimization Methods for the Impulsive Minimum Fuel Rendezvous Problem

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In this paper we present a comparison of trajectory optimization approaches for the minimum fuel rendezvous problem. Both indirect and direct methods are compared for a variety of test cases. The indirect approach is based on primer vector theory. The direct approaches are implemented numerically and include Sequential Quadratic Programming (SQP), Quasi-Newton, and Nelder-Meade Simplex. Several cost function parameterizations are considered for the direct approach. We choose one direct approach that appears to be the most flexible. Both the direct and indirect methods are applied to a variety of test cases which are chosen to demonstrate the performance of each method in different flight regimes. The first test case is a simple circular-to-circular coplanar rendezvous. The second test case is an elliptic-to-elliptic line of apsides rotation. The final test case is an orbit phasing maneuver sequence in a highly elliptic orbit. For each test case we present a comparison of the performance of all methods we consider in this paper.

INTRODUCTION

The minimum fuel rendezvous problem has received considerable attention in the literature and numerous approaches to both posing and solving the problem have been developed. The basic objective is to find a minimum solution to a two-point boundary value problem (TPBVP) that has multiple feasible solutions. The primary difference between approaches is the choice of independent variables and how the TPBVP is posed in terms of these variables. Methods also differ on how constraints are handled. In the next few paragraphs we give a brief overview of the methods considered in this paper. We also discuss the test problems we used to allow a comparison of each method we consider for different flight regimes.

This paper deals with the problem of optimizing trajectories where the maneuvers can be modeled impulsively. In general, two approaches are utilized for optimization: direct and indirect.[1] Both approaches will be compared in this investigation. A survey of the rendezvous problem is provided by Jezewski et al.[2]

A direct optimization method tries to minimize the cost function by directly varying the *control (independent) variables.* The methods employed for the optimization process are mathematical programming techniques.[3] The mathematical programming techniques utilized in this paper include Sequential Quadratic Programming (SQP), Quasi-Newton, and Nelder-Meade Simplex. (Other techniques, such as Genetic Algorithms, are under current investigation but are not discussed here). For direct optimization

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the choice of independent variables and constraints is extremely important. For example, suppose that there are n impulses in the trajectory of interest. Then, two possible choices for independent variables are (1) the actual ΔV's, or (2) the impulse locations and times. In this paper, we find the option of using the impulse locations and times more effective. This is consistent with previous work by Brusch.[4] Hence, we utilize a Lambert-type approach with implicit position continuity. Thus, as the impulse locations change, new arcs are computed utilizing either Lambert arcs (in the two-body model) or differentially corrected arcs (allows for more complex models than the two-body model). The result of computing the lambert arcs is a path continuous trajectory with velocity discontinuities. The velocity discontinuities are then utilized to formulate the cost function. Other approaches are possible but include the addition of position continuity as constraints.[5]

An indirect optimization method tries to minimize the cost function by considering its variations, thus it involves Calculus of Variations (COV). It leads to two-point boundary value problems in the *co-state variables*. Once the co-state variables are obtained, the optimal control variables are usually readily available. Primer vector theory can be considered to be a byproduct of applying the Calculus of Variations to the problem of minimizing the fuel usage of impulsive trajectories. Perhaps more importantly, primer vector theory can be utilized to "find" the optimal number of impulses. A survey of primer vector theory is presented by Guzman[6] *et al.*

The parameterization of the cost function chosen in this work ensures that for every cost function evaluation, the rendezvous constraints are satisfied implicitly. Hence, from the point-of-the-view of a numerical optimizer, the problem is unconstrained. Several numerical optimization routines are considered. Three of the routines are products of The Mathworks and are available in their Optimization Toolbox. The Mathworks' routines that we employ are fmincon, fminunc, and fminsearch. The fmincon function is an SQP algorithm, fminunc is a Quasi-Newton algorithm, and fminsearch is a Nelder-Meade Simplex algorithm. For a detailed discussion of The Mathworks' routines and their implementations, we refer the reader to The Mathworks Optimization Toolbox[7] documentation. We also use SNOPT, an SQP algorithm developed by Gill[8] *et. al.* For details on SNOPT we refer the reader to the user's guide for SNOPT[8] 5.3. For the indirect method we develop software in Matlab that uses a fully automated algorithm which implements the primer vector theory first developed by Lawden.[9]

We employ three test cases to compare the performance of each trajectory optimization method in different flight regimes. The first test case is a simple circular-to-circular, coplanar rendezvous. The second test case is an elliptic-to-elliptic, line of apsides rotation. The final test case is a phasing maneuver in a highly elliptic orbit. For each test case we generate numerous initial guesses. The initial guesses are generated using simple intuition. The times and locations of the initial and final maneuvers are chosen based on experience. Given the initial and final burn locations we generate a two-burn rendezvous sequence. Multiple maneuver sequences are generated by placing small maneuvers, equally spaced in time, along two maneuver initial guesses. Sequences of two, three, and four burns are considered. The performance of all methods for the test cases described above is presented in the final section. It is difficult to provide a single metric that can conclude which method is the best. Hence we have provided a variety of statistics to illustrate the performance of all of the trajectory optimization approaches we consider. It should be noted that we have not considered the rate of convergence to be a measure of importance. Since all methods are acceptably fast, we use metrics based on the total ΔV of the converged solution to be the primary measures of performance.

To solve the minimum fuel rendezvous problem we must first develop a mathematical model. In the next section we discuss the distinction between an orbit transfer and an orbit rendezvous and develop a model for the rendezvous problem.

PROBLEM STATEMENT

The minimum fuel rendezvous problem can be posed in numerous ways. The primary difference between approaches is the choice of independent variables, how the boundary value problem is posed in terms of the independent variables, and how the constraint functions are handled. It is useful at this point to make a clear distinction between a rendezvous problem and a transfer problem. In an orbit transfer problem we are only concerned with finding a maneuver sequence that will take a spacecraft from some position in its initial orbit, to some position in the desired final orbit. For a transfer, the initial maneuver can occur anywhere along the initial orbit, and the final maneuver can occur anywhere along the final orbit, without concern for the orbit phasing. However, for a rendezvous problem, we have the additional constraints determined by the time and place of the spacecraft in the initial orbit, and the time and place of the target spacecraft in its orbit. In this section we develop a mathematical model of the rendezvous problem. In later sections the model is used to develop optimization algorithms to find minimum fuel solutions.

A diagram to aid in illustrating the general impulsive rendezvous problem is shown in Figure 1. The arc denoted $\mathcal{P}_o$ represents the path of the spacecraft in its initial orbit. Similarly, the arc denoted $\mathcal{P}_f$ represents the path of path of the spacecraft in its final orbit. Throughout the paper a superscript '+' sign is used to denote a condition immediately prior to an impulsive maneuver, and a superscript '-' sign is used to denote a condition immediately following an impulsive maneuver. According to these conventions, we can write the conditions for the i^{th} burn as,

$$\mathbf{r}_i^+ = \mathbf{r}_i^- \tag{1}$$

$$t_i^+ = t_i^- \tag{2}$$

$$\Delta\mathbf{V}_i = \mathbf{V}_i^- - \mathbf{V}_i^+ \tag{3}$$

where $\mathbf{r}_i^+$ and $\mathbf{r}_i^-$ are the positions immediately before and after the i^{th} burn respectively, t_i^+ and t_i^- are the times immediately before and after the i^{th} burn respectively, and $\mathbf{V}_i^+$ and $\mathbf{V}_i^-$ are the velocity vectors immediately before and after the i^{th} burn respectively. The constraints in Eqs.(1) and (2) are a result of the definition of an impulsive maneuver.

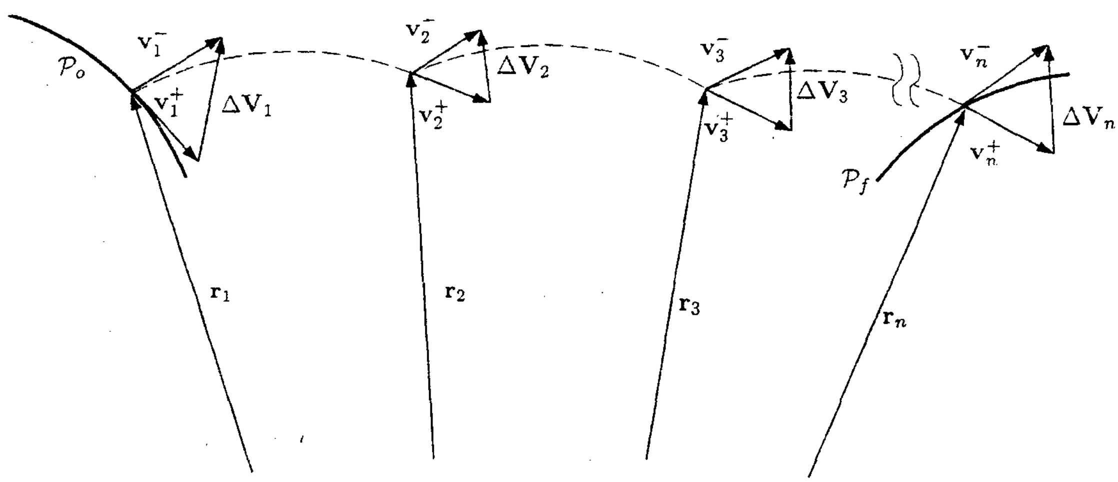

Figure 1: Impulsive Orbit Transfer Diagram

There are additional constraints that must be satisfied at the boundaries, and at the interior impulses which are imposed by the orbit dynamics. We assume that the dynamics for the entire maneuver sequence are given by

$$\ddot{\mathbf{r}} = -\mu\frac{\mathbf{r}}{r^3} \tag{4}$$

Let $\mathbf{r}_f = \mathbf{f}(\mathbf{r}_o, \mathbf{v}_o, t_o, t_f)$ and $\mathbf{v}_f = \mathbf{g}(\mathbf{r}_o, \mathbf{v}_o, t_o, t_f)$ be the solution to Eq. (4), with the initial conditions $(\mathbf{r}_o, \mathbf{v}_o, t_o)$ and the final time of t_f. For an interior impulse $i \neq 1$, $i \neq n$, and $n > 2$. The additional constraints that must be satisfied at an interior impulse are given by

$$\mathbf{r}_i^+ = \mathbf{f}(\mathbf{r}_{i-1}, \mathbf{v}_{i-1}, t_{i-1}, t_i) \tag{5}$$

$$\mathbf{v}_i^+ = \mathbf{g}(\mathbf{r}_{i-1}, \mathbf{v}_{i-1}, t_{i-1}, t_i) \tag{6}$$

The constraints that must be satisfied at the initial boundary are given by

$$\mathbf{r}_1^+ = \mathbf{f}(\mathbf{r}_o, \mathbf{v}_o, t_o, t_1) \tag{7}$$

$$\mathbf{v}_1^+ = \mathbf{g}(\mathbf{r}_o, \mathbf{v}_o, t_o, t_1) \tag{8}$$

where $(\mathbf{r}_o, \mathbf{v}_o, t_o)$ are the position and velocity of the initial orbit at the reference epoch t_o, and t_1 is the time of the first maneuver. A similar set of constraints apply at the final boundary and are give by

$$\mathbf{r}_n^- = \mathbf{f}(\mathbf{r}_f, \mathbf{v}_f, t_f, t_n) \tag{9}$$

$$\mathbf{v}_n^- = \mathbf{g}(\mathbf{r}_f, \mathbf{v}_f, t_f, t_n) \tag{10}$$

where $(\mathbf{r}_f, \mathbf{v}_f, t_f)$ are the position and velocity of the initial orbit at the reference epoch t_f, and t_n is the time of the last maneuver. Without loss of generality, we can appropriately formulate the optimal control problem to ensure that the constraints $\mathbf{r}_i^+ = \mathbf{r}_i^-$ and $t_i^+ = t_i^-$ are satisfied implicitly. For the remainder of this work we drop the superscripts on $\mathbf{r}$ and t, and assume

$$\mathbf{r}_i = \mathbf{r}_i^+ = \mathbf{r}_i^- \tag{11}$$

$$t_i = t_i^+ = r_i^- \tag{12}$$

In summary, the problem is to find the set $(\mathbf{r}_i, \mathbf{v}_i^+, \mathbf{v}_i^-, t_i)$, where $i = 1, 2...n$ and $n \geq 2$, that satisfies the constraints

$$\mathbf{r}_1 = \mathbf{f}(\mathbf{r}_o, \mathbf{v}_o, t_o, t_1) \tag{13}$$

$$\mathbf{v}_1^- = \mathbf{g}(\mathbf{r}_o, \mathbf{v}_o, t_o, t_1) \tag{14}$$

$$\mathbf{r}_i = \mathbf{f}(\mathbf{r}_{i-1}, \mathbf{v}_{i-1}, t_{i-1}, t_i) \tag{15}$$

$$\mathbf{v}_i^- = \mathbf{g}(\mathbf{r}_{i-1}, \mathbf{v}_{i-1}, t_{i-1}, t_i) \tag{16}$$

$$\mathbf{r}_n = \mathbf{f}(\mathbf{r}_f, \mathbf{v}_f, t_f, t_n) \tag{17}$$

$$\mathbf{v}_n^+ = \mathbf{g}(\mathbf{r}_f, \mathbf{v}_f, t_f, t_n) \tag{18}$$

and that is a minimum solution to the function

$$J = \sum_{i=1}^{n} \|\Delta \mathbf{V}_i\| \tag{19}$$

where n is the number of maneuvers, the set $(\mathbf{r}_o, \mathbf{v}_o, t_o)$ defines the initial orbit and the set $(\mathbf{r}_f, \mathbf{v}_f, t_f)$ defines the final orbit.

There are two main methods available for solving the optimal control problem, including direct and indirects approaches. Within each method there are numerous possible implementations. Often methods are actually a combination of indirect and direct techniques. For direct methods, the parameterization of the problem depends on the type of optimizers available. Indirect methods are sensitive to the constraints and often require complete reformulation if new constraints are imposed. In the next two sections we discuss several approaches for solving the minimum fuel rendezvous problem using both indirect and direct methods.

DIRECT APPROACH

The success of a direct method is intimately dependent on the choice of independent variables and the constraint function formulation. In this section we discuss three parameterizations for solving the minimum fuel rendezvous problem using direct methods. There are numerous possible parameterizations and comparing all approaches is beyond the scope of this work. We categorize the parameterizations we consider into two groups called the Feasible Iterate Approach and the Infeasible Iterate Approach. We present a brief comparison of the parameterizations, and the most promising method is chosen for further development. For this work we choose a Feasible Iterate Approach for reasons explained later. This section is concluded with a discussion of several numerical optimization packages used in this work.

Objective Function Parameterizations

There are numerous approaches to solving the minimum fuel rendezvous problem using direct methods. For convenience we break down the direct approaches considered here into two categories called the Feasible Iterate Approach and the Infeasible Iterate Approach. In the Feasible Iterate Approach, each evaluation of the objective function is also a feasible solution that satisfies the rendezvous conditions. Hence, from the point of view of the optimizer, the problem is unconstrained. The Feasible Iterate Approach requires that for each objective function evaluation TBPVP is solved. The Feasible Iterate Approach also requires a careful selection of independent variables for the objective function parameterization. In the Infeasible Iterate Approach, there is less restriction on the choice of independent variables, however, the constraints must be defined carefully, and the optimizer must be able to handle nonlinear constraints. For continuity, we present the discussion of alternative parameterizations in Appendix 1. Here we present the parameterization chosen for this work.

Our Choice of independent variables is as follows:

$$\text{Given: } (\mathbf{r}_o, \mathbf{v}_o, t_o) \text{ and } (\mathbf{r}_f, \mathbf{v}_f, t_f) \tag{20}$$

Choose the independent variables:

$$t_i \qquad i = 1, 2 \ldots n \tag{21}$$

$$\mathbf{r}_j \qquad j = 2, 3 \ldots n-1 \tag{22}$$

where t_i are the times of the maneuvers, and $\mathbf{r}_j$ are the positions of the intermediate burns. Given the independent variables defined in Eqs.(21) and Eqs.(22), we must develop an algorithm to determine the total ΔV of the maneuver sequence. The boundary conditions $\mathbf{r}_1$, $\mathbf{v}_1^+$, $\mathbf{r}_n$, and $\mathbf{v}_n^-$ are determined using Eqs.(13),(14),(17),and (18), where t_1 and t_n are known from Eq.(21). After solving for the boundary conditions we know the times and positions of all the maneuvers. Therefore, we can solve for the velocities before and after each maneuver, by solving Lambert's problem for each of the $n-1$ trajectory segments. There are numerous well-known approaches to solving Lambert's problem. We choose a method developed by Howell and Pernicka[10] and further developed by Guzman.[11] The method developed by Howell is chosen because it can solve Lambert's problem with perturbations included, as well as when there is more than one significant gravitational body. Perhaps more importantly, Howell demonstrated that using the method described below, we can provide analytic approximations for the gradient of the total ΔV with respect to the independent variables chosen in Eqs. (21) and Eqs.(22). The algorithm is described as follows. Given an initial position $\mathbf{r}_{i-1}$, and an initial velocity $\mathbf{v}_{i-1}$ both at time t_{i-1}, find $\delta\mathbf{v}_{i-1}$ applied at time t_{i-1} so that we achieve $\mathbf{r}_i$ at t_i. Figure 2 illustrates the problem. The dark black arc denotes the path a spacecraft would follow if no $\delta\mathbf{v}_{i-1}$ is applied. For this arc, the final position denoted by $\mathbf{r}_a$ is not equal to the desired final position $\mathbf{r}_i$. Hence $\delta\mathbf{r}_i \neq 0$. The dashed arc denotes the trajectory that is the solution to Lambert's problem. For this arc $\mathbf{r}_i = \mathbf{r}_a$, or $\delta\mathbf{r}_i = 0$. To solve for $\delta\mathbf{v}_{i-1}$ such that $\delta\mathbf{r}_i = 0$ first define $\mathbf{x}$ as

$$\mathbf{x} = \begin{bmatrix} \mathbf{r} \\ \mathbf{v} \end{bmatrix} \tag{23}$$

then $\delta\mathbf{x}_i$ is given by

$$\delta\mathbf{x}_i = \mathbf{\Phi}\left(t_i, t_{i-1}\right)\delta\mathbf{x}_{i-1} \tag{24}$$

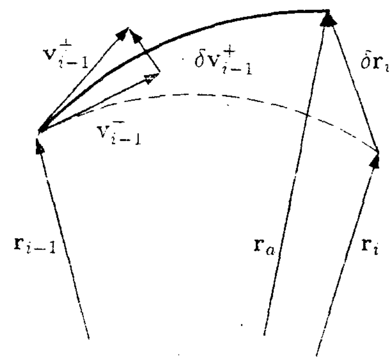

Figure 2: Illustration of One Trajectory Arc

where t_i and t_{i-1} are fixed. We can write $\mathbf{\Phi}(t_i, t_{i-1})$ as

$$\mathbf{\Phi}(t_i, t_{i-1}) = \begin{bmatrix} \mathbf{A}_{i,i-1} & \mathbf{B}_{i,i-1} \\ \mathbf{C}_{i,i-1} & \mathbf{D}_{i,i-1} \end{bmatrix} \tag{25}$$

Using the fact that $\delta\mathbf{r}_{i-1} = 0$ we can write

$$\delta\mathbf{r}_i = \mathbf{B}_{i,i-1}\delta\mathbf{v}_{i-1} \tag{26}$$

Solving for $\delta\mathbf{v}_{i-1}$ we see that

$$\delta\mathbf{v}_{i-1} = \mathbf{B}^-_{i,i-1}\delta\mathbf{r}_i \tag{27}$$

From inspection of Fig. 2 we can write

$$\mathbf{r}_i + \delta\mathbf{r}_i = \mathbf{r}_a \tag{28}$$

Solving for $\delta\mathbf{r}_i$ we have

$$\delta\mathbf{r}_i = \mathbf{r}_a - \mathbf{r}_i \tag{29}$$

Substituting Eq.(29) into Eq.(27) yields

$$\delta\mathbf{v}_{i-1} = \mathbf{B}^-_{i,i-1}(\mathbf{r}_a - \mathbf{r}_i) \tag{30}$$

We can solve for $\delta\mathbf{v}_{i-1}$ such that $\delta\mathbf{r}_i$ is zero by iterating on Eq.(30) until $(\mathbf{r}_a - \mathbf{r}_i)$ meets a user defined tolerance. It is important to note that this approach assumes that $\mathbf{B}^-_{i,i-1}$ exists. For cases when $\mathbf{B}^-_{i,i-1}$ is not invertible we use a method by Gedeon[12] to solve the TPBVP. We solve all $n-1$ trajectory arcs using the algorithm defined above, then the ΔV is calculated using Eq.(19). Note that although the algorithm above can be extended to provide a gradient approximation, we have used finite differencing for gradients in this work. Analytic approximations for the gradient are a topic of current research.

There are several other concerns to address to completely define the objective function parameterization. We also must choose appropriate time and coordinate systems to express the independent variables given in Eqs. (21) and (22). There are several issues in choosing a time parameterization that must be considered. The first issue is to select appropriate units, the second is to select an appropriate reference epoch. We choose the units of seconds for time, and reference each time t_i to the reference epoch t_0 given in Eq. (20). The positions in Eq. (22) are expressed in cartesian coordinates in the Mean of J2000 Earth Equatorial system and the units are in *km*.

With the parameterization of the cost function described above, the rendezvous conditions are satisfied implicitly for every cost function evaluation. Hence, the problem is essentially an unconstrained optimization problem and we are free to use unconstrained as well as constrained optimization techniques to find a minimum ΔV solution. It is exceedingly rare that both TPBVP methods described above fail to converge. In the case that neither method converges we must "inform" the optimizer. This is discussed in a subsequent section. In the next subsection we discuss the numerical optimization packages we use in this work.

Direct Optimizers

There are numerous optimization packages available that solve unconstrained optimization problems. We investigate the performance of four routines for the parameterization of the minimum fuel rendezvous problem chosen above. Three of the routines are products of The Mathworks and are available in their Optimization Toolbox.[7] The first routine is an SQP algorithm called fmincon. The second routine is a quasi-Newton algorithm called fminunc. The third routine is a Nelder-Meade simplex algorithm and is called fminsearch. For a detailed discussion of the optimization routines developed by The Mathworks we refer the reader to the Optimization Toolbox documentation.[7] The fourth numerical optimization routine we employ is an SQP routine developed by Gill[8] *et al.* called SNOPT. For a detailed discussion of SNOPT we refer the reader to the SNOPT[8] 5.3 documentation.

Numerical Issues

To avoid numerical difficulties the independent variables and the objective function values are normalized to be on the order of one. Secondly, all derivatives are calculated using finite differencing. Providing analytic derivatives for gradient based methods is a topic of current research. The final numerical issue occurs when both TPBVP solvers fail to converge. This is exceedingly rare. However when it occurs we must "inform" the optimizer. SNOPT has a built in capability to step back from a poorly conditioned state vector. Hence, for SNOPT, if both TPBVP solvers fail, the appropriate message is sent to SNOPT and the SNOPT steps away from the poorly conditioned state. However, The Mathworks routines do not have this capability yet. As a temporary solution, when using a Mathworks' routine, in the event of a poorly conditioned state vector we pass back a crude approximation of the total ΔV.

To solve the minimum fuel rendezvous problem using direct methods we must choose an appropriate parameterization of the problem, and an appropriate numerical optimization routine. In this section we discussed some choices involved in choosing a specific parameterization. We chose one method that performed well in a preliminary comparison. We also discussed several numerical optimization routines we employ and some numerical issues one must deal with to ensure adequate performance. In the next section we discuss an indirect approach to solve the minimum fuel rendezvous problem.

INDIRECT APPROACH

In this section, we present a review of the primer vector theory as well as the main challenges involved in its implementation.[13] Primer vector theory[14] is an optimization technique based on calculus of variations. The theory has several appealing features including the indication of when and where to add an impulse to a non-optimal trajectory, and a visual assessment of the optimality of neighboring solutions.

Primer Vector Theory

Primer vector theory provides a set of first order necessary conditions, which a trajectory must meet to be locally optimal. The necessary conditions, first derived by Lawden, are expressed in terms of the primer vector, which is defined as the adjoint to the velocity vector in the variational Hamiltonian formulation.[14] If any of Lawden's conditions are violated, the rendezvous trajectory is not optimal and we can use the primer vector history to obtain information on how to improve its ΔV cost. In his initial work, Lawden solved a fixed-time rendezvous and his theory was further extended by Lion and Handelsman and later, by Jezewski and Rozendaal to solve the n-impulse optimal rendezvous problem.[15,16] A more detailed derivation of the primer vector theory is provided by Hiday.[17] For the rendezvous problem considered in this paper, we can express the primer vector equations using calculus of variations theory.

The initial trajectory or first-guess, is labeled as the reference trajectory. Since primer vector is a first-order theory based on local variations, it will converge to local optimal neighboring trajectories of the reference trajectory. Therefore, the optimal solution is highly dependent on the reference trajectory but it will also depend on other design parameters discussed later in this section. The primer vector obeys the following equation, also known as the second order canonical form of the Euler-Lagrange equation,

$$\ddot{\mathbf{p}} = \mu\left(-\frac{\mathbf{p}}{r^3} + 3\frac{\mathbf{r}}{r^5}\left(\sum_{i=1}^{3} p_i r_i\right)\right) \tag{31}$$

where $\mathbf{r}$ is the satellite position vector on the reference trajectory, $\mathbf{p}$ is the primer vector and μ is the Earth's gravitational constant. The satellite position is found using Eq. (4). As evident in Eq. (31), the primer vector state can not be derived simultaneously with the spacecraft state. The spacecraft trajectory must be propagated first for the primer vector history to be computed. Using calculus of variations, Lawden derived four necessary conditions expressed in terms of the primer vector defined by Eq. (31) for an optimal rendezvous trajectory:

1. The primer vector must be C^1 (i.e. the primer vector and its first derivative are continuous) for the entire history.
2. Between impulses, $||\mathbf{p}|| = p <= 1$. Impulses occur when $p = 1$.
3. At an impulse, $\mathbf{p} = \hat{\mathbf{p}} = \hat{\mathbf{u}}_T^*$, where $\hat{\mathbf{u}}_T^*$ is the optimal thrust direction.
4. At all interior impulses (not at the initial or final times) $\mathbf{p} \cdot \dot{\mathbf{p}} = 0$. This condition has implications on the slope of the primer vector magnitude since

$$\frac{d||\mathbf{p}||}{dt} = \frac{d}{dt}(\mathbf{p}\cdot\mathbf{p})^{1/2} = \frac{\dot{\mathbf{p}}\cdot\mathbf{p}}{||\mathbf{p}||}.$$

Therefore, $\frac{d||\mathbf{p}||}{dt} = 0$ at the intermediate impulses. Also, for convenience, let $\dot{p} = \frac{d||\mathbf{p}||}{dt}$.

These conditions are necessary (but not sufficient) for an optimal trajectory. In this paper, a trajectory that meets the above conditions will be called an optimal trajectory. When solving a rendezvous problem using primer vector theory, we first need to evaluate the primer vector history along the reference trajectory or initial guess. Solving for the primer vector history is equivalent to solving a TPBVP. Let's first assume that we are solving a two-burn rendezvous problem. We know from Lawden's conditions that, at an impulse, the primer vector is in the direction of the thrust vector. Thus, we can express the initial and final primer vector as :

$$\mathbf{p}_o = \frac{\Delta\mathbf{V}_o}{|\Delta\mathbf{V}_o|}, \tag{32}$$

$$\mathbf{p}_f = \frac{\Delta\mathbf{V}_f}{|\Delta\mathbf{V}_f|}. \tag{33}$$

The initial and final primer vectors and the time-of-flight $(t_f - t_o)$ are known. However, to propagate Eq. (31), we need the complete primer initial state $(\mathbf{p}_o, \dot{\mathbf{p}}_o)$. To obtain $\dot{\mathbf{p}}_o$, we can either use a shooting method (TPBVP solver) or use the STM formulation from Eq. (34) below.

$$\begin{pmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{pmatrix} = \mathbf{\Phi}(t, t_o) \cdot \begin{pmatrix} \mathbf{p}_o \\ \dot{\mathbf{p}}_o \end{pmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \cdot \begin{pmatrix} \mathbf{p}_o \\ \dot{\mathbf{p}}_o \end{pmatrix} \tag{34}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are 3x3 matrices, partitions of $\mathbf{\Phi}(t, t_o)$. In general, using the STM is faster and although not as accurate as the TPBVP solver, it is sufficient. Using Eq. (34), the initial derivative of the primer vector can be expressed as:

$$\dot{\mathbf{p}}_o = \mathbf{B}^{-1} \cdot (\mathbf{p}_f - \mathbf{A} \cdot \mathbf{p}_o) \tag{35}$$

It is worthwhile noting that if $\mathbf{B}^{-1}$ is singular, $\dot{\mathbf{p}}_o$ cannot be computed using Eq. (35). In the two-body problem, there are three known singularities.[18] Eq. (35) can be generalized for a n-impulse trajectory by examining each individual two-burn transfer. Once the primer vector history is computed, we can determine the optimality of the trajectory using Lawden's necessary conditions. If any of Lawden's conditions are violated, the rendezvous trajectory is not optimal and we can use the primer vector history to obtain information on how to improve its ΔV cost. Improving non-optimal trajectories using primer vector theory is the main contribution of Lion and Handelsman.[15] Essentially, first order variations of the total ΔV are considered for different perturbed trajectories.[15-17] For a non-optimal primer vector history, two types of actions are possible to lower the ΔV cost: (I) moving the initial or final impulse and (II) adding and/or moving an interior impulse. If the epoch of the endpoints is unconstrained, an action of type I is performed whenever the initial and/or the final primer vector magnitude slope is different from zero. The general expression for time variations at both endpoints is given by Hiday[17] as

$$\delta J = -\dot{p}_o ||\Delta \mathbf{v}_o|| dt_o - \dot{p}_f ||\Delta \mathbf{v}_f|| dt_f, \tag{36}$$

where the initial and final time variations are given by dt_o and dt_f respectively. Furthermore, the initial and final primer magnitude slopes are given by $\dot{p}_o$ and $\dot{p}_f$, respectively. To have a lower cost, δJ must be less than zero. If the following terminology is defined, 1) initial coast ($dt_o > 0$), 2) early departure ($dt_o < 0$), 3) final coast ($dt_f < 0$), and, 4) late arrival ($dt_f > 0$): then, the following four cases cover all the non-zero primer slope combinations,

- If $\dot{p}_o > 0$ and $\dot{p}_f < 0 \Rightarrow$ Apply Initial Coast and Final Coast.
- If $\dot{p}_o > 0$ and $\dot{p}_f > 0 \Rightarrow$ Apply Initial Coast and Late Arrival.
- If $\dot{p}_o < 0$ and $\dot{p}_f < 0 \Rightarrow$ Apply Early Departure and Final Coast.
- If $\dot{p}_o < 0$ and $\dot{p}_f > 0 \Rightarrow$ Apply Early Departure and Late Arrival.

See Hiday[17] for details. Of course, if the slopes are zero, no further improvement is achieved by varying the endpoints times. If the primer vector magnitude goes above one during a coasting phase on a given segment, an impulse is added to the trajectory (action type II) to improve the overall ΔV budget. This impulse is examined by considering a neighboring path with an additional impulse. The two-impulse arc is perturbed with the addition of a mid-impulse at some time, t_m, and position $\mathbf{r}_m + \delta\mathbf{r}_m$. (The position $\mathbf{r}_m$ is the position along the unperturbed path at t_m.) The initial and the final times and positions, (t_o, $\mathbf{r}_o$) and (t_f, $\mathbf{r}_f$), are fixed. The cost function variation is defined as $\delta J = J' - J$, where J' is the fuel cost on the perturbed trajectory. Then, considering first order terms only, it can be shown that

$$\delta J = c(1 - \mathbf{p}_m^T \hat{\eta}), \tag{37}$$

where the following are mid-impulse parameters: c is the magnitude of the impulse, $\mathbf{p}_m$ is the primer vector at t_m and $\hat{\eta}$ is an unit vector in the direction of the impulse. Then, for an improvement in cost, $\delta J < 0$. Thus, using the definition of a dot product, if $||\mathbf{p}_m|| > 1$ at any time, a third impulse is beneficial. Furthermore, the greatest decrease in the cost function will be achieved if the impulse is applied at the maximum of $||\mathbf{p}_m||$ at time t_m and in the direction of $\hat{\eta}$. The position along the perturbed path and the magnitude of the impulse are yet to be determined. Consider finding the position of the impulse. Utilizing the fact that the position across an impulse is continuous, using the STM before and after the impulse, and utilizing the information obtained from Eq. (37), gives the following variational equation,

$$\delta\mathbf{r}_m = c\mathbf{A}^{-1}\frac{\mathbf{p}_m}{||\mathbf{p}_m||}, \tag{38}$$

where

$$\begin{aligned} \mathbf{A} \;=\; & \mathbf{D}(t_m, t_f)\mathbf{B}(t_m, t_f)^{-1} \\ & -\mathbf{D}(t_m, t_o)\mathbf{B}(t_m, t_o)^{-1}. \end{aligned}$$

This equation, of course, is valid only for non-singular $\mathbf{A}$ and determines the position of the mid-impulse. To estimate the magnitude, c, of the impulse, the expression for the cost on the perturbed path, J', can be expressed as a function of c and minimized. The mid-impulse should decrease the cost but might not produce an optimal trajectory in the sense of Lawden. The subsequent optimization of the three-impulse trajectory is presented next. For a three-impulse arc, if the *time and position* of the mid-impulse are allowed to vary, it can be shown that,

$$\delta J = (\Delta \dot{\mathbf{p}}_m)^T \, d\mathbf{r}_m + (\Delta H_m) \, dt_m \tag{39}$$

where $\Delta \dot{\mathbf{p}}_m$ and ΔH_m are, respectively, the primer vector and the Hamiltonian differences at the impulse. If the trajectory is indeed optimal, the first cost variation must vanish. Thus, it is required that, $\Delta \dot{\mathbf{p}}_m = \mathbf{0}$ and, $\Delta H_m = 0$. It can be shown that $\Delta H_m = \dot{\mathbf{p}}_m^{-T} \mathbf{v}_m^- - \dot{\mathbf{p}}_m^{+T} \mathbf{v}_m^+$. In an interesting "mix", a direct optimization method can be applied to vary the mid-impulse time and position to meet the above conditions. In this investigation, following Hiday,[17] the Broyden-Fletcher-Goldfarb-Shanno, (BFGS), algorithm is used.

Primer Vector Implementation

The primer vector theory described above was implemented in Matlab in tool we call PVAT. PVAT has a fully automated algorithm, which iterates following the primer vector principles to optimize a non-optimal reference trajectory. Most of the time multiple actions for improvements are possible. However, there is yet no mathematical theory that determines the optimal sequencing of the actions and, in general, different sequencing will lead to different neighboring solutions. One way of solving this problem is to combined Eqs. (36) and (39) to form a unique cost function gradient. This gradient is then used in the BFGS optimization algorithm to simultaneously move the endpoints times and the midcourse impulses position and time. A separate check for the addition of an impulse is then required. In this paper, we choose to implement a sequential algorithm where the endpoint times and the midcourse impulses are varied with separate PVAT iterations. Each time the underlying trajectory is changed, the algorithm recomputes the primer vector data and re-iterates until no improvement can be made. A conceptual flowchart of PVAT is shown in Fig. 3.

First, the initial and final endpoint gradients are evaluated to check whether their values are greater than zero within some specified tolerance ϵ_t. Note that only one endpoint is moved at a time for a given iteration and the priority is given to the boundary with the highest gradient value. To determine the departure/arrival state which corresponds to a zero initial slope, we implemente a bissection method. Both endpoint algorithms being identical, only the initial boundary is discussed here. Whenever the primer magnitude slope, $\dot{p}_o$, is positive the departure state is propagated forward, otherwise it is propagated backward. To quickly recompute the trajectory data, we use a Lambert solver. The primer magnitude slope is then updated for the new trajectory and the process is repeated until the value of the slope is below the tolerance. Whenever there are internal impulses, the norm of the internal gradient is evaluated. The midcourse burn(s) are moved whenever the norm exceeds a specified tolerance ϵ_m. To move the midcourse impulses states, we use the BFGS minimization technique. This variant of the Quasi-Newton method uses an approximation of the Hessian matrix instead of its direct evaluation. The approximation formula is called a BFGS update. Finally, if the primer vector is above some maximum value, $p_{\,threshold}$, during a coasting arc, an impulse is added to the trajectory. Note that theoretically the threshold value is one but for numerical implementation, the value is typically chosen to be slightly higher than one. The value of perturbation impulse is computed using an estimate derived by Jezewski.[16] However, we need to ensure that its magnitude remains small for the theory to be valid. In this paper, we implemented two different algorithms labeled PVAT1 and PVAT2 respectively. For PVAT1, the algorithm was not permitted to add a burn in spite of indication of potential improvements by the primer vector history. The second implementation permitts the addition of burns to the rendezvous trajectory to take full advantage of the primer vector theory. However, for practical reasons,we limited the number of burns to a maximum of six. For this first version of the code, the equations where not non-dimensionalized which means that the choices of the various tolerance are specific to the problem solved and that the

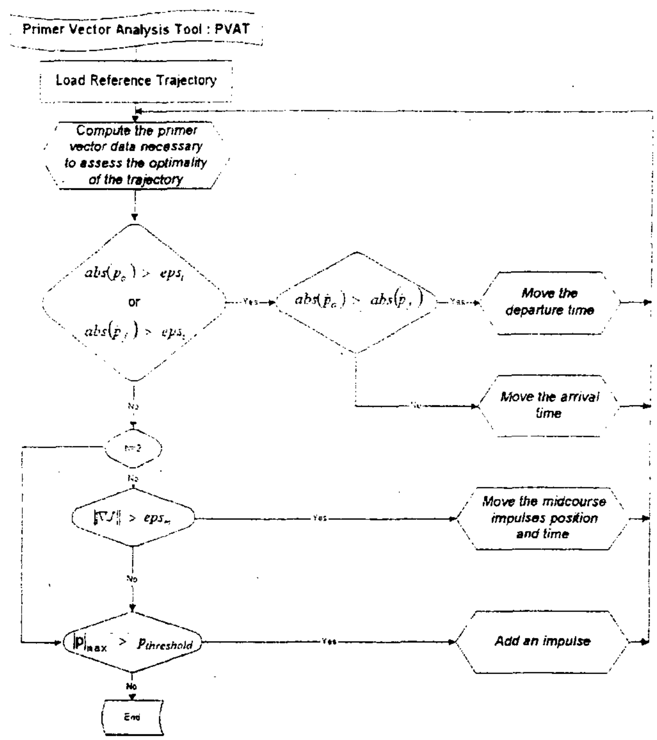

Figure 3: **PVAT** Conceptual Flowchart

final optimal solution will depend on their value.

TEST CASES

In order to compare the performance of each optimization method we utilize three test cases. The test cases are chosen to investigate the performance of the each method in different flight regimes. The first test case is a low-Earth, circular-to-circular, coplanar rendezvous. The second test case is an elliptic-to-elliptic line of apsides rotation. The third case is a highly elliptic orbit phasing sequence. For each test case there are numerous initial guesses. To generate different initial guesses for a particular test case we first define an initial and final orbit. Next, we vary the true anomaly and the epoch for the initial and final maneuvers. Given the times and positions of the initial and final maneuvers for a particular initial guess, we solve Lambert's problem to yield a two burn solution. A simple algorithm is used to place small maneuvers along the two-burn trajectory arc to generate multiple maneuver solutions. In the next three subsections we discuss the three test cases in more detail.

Case One

Test Case One is a simple circular-to-circular coplanar transfer. The optimal solution is simply the Hohmann ΔV. The orbital elements for this test case are shown in Eqs.(40) and (41). The format for the orbital elements is [a e i ω Ω ν] unless specified otherwise. Distance is measured in km, and angles are measure in degrees.

$$\text{Initial Orbit:} \quad oe_o = [7000\ 0\ 0\ 0\ 0\ 0], \quad T = 0 \tag{40}$$

$$\text{Final Orbit:} \quad oe_f = [7500\ 0\ 0\ 0\ 0\ M_f], \quad T = T_f \tag{41}$$

The ΔV for a Hohmann Transfer for Case One is 255.8 m/s. A set of initial guesses for case one is generated by varying M_f and T_f. We vary M_f from 90^o to 270^o in increments of 20^o. For each value of M_f we choose two values for T_f. For each pair of M_f and T_f we solve Lambert's problem to obtain a two-maneuver rendezvous for the initial to the final orbit. The values of T_f are chosen so that the ΔV for the initial guess is less than 1.5 km/s. Note this is considerably higher than the known optimal solution. For each two burn solution we add small interior burns to create similar three and four burn solutions. In summary, there are nine different values for M_f, two different values of T_f for each M_f, and a two, three and four burn solution for each pair of M_f and T_f. Hence there are fifty four initial guesses for Case One. A figure showing a sample of the initial guesses for Case One is shown in Figure (4).

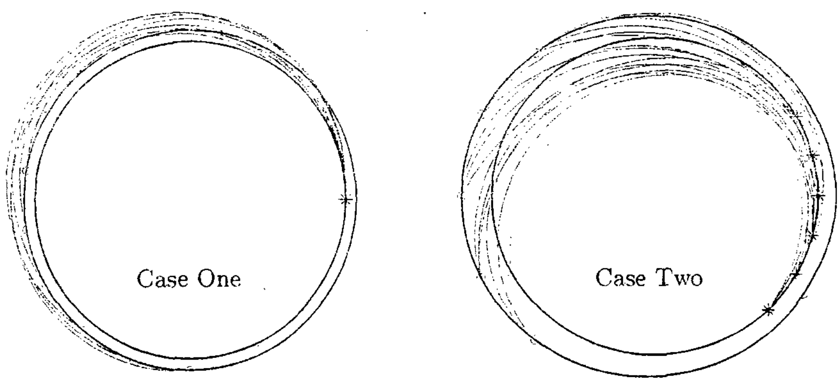

Figure 4: Sample of Initial Guesses for Case One and Two

Case Two

The second test case involves a combined maneuver that changes both the semimajor axis and the inclination of the orbit. The orbital elements for Case Two are shown in Eqs. (42) and (43). The eccentricity of both orbits is 0.3. The semimajor axis for the initial orbit is 7000 km, and the semimajor axis for the final orbit is 8000 km. The initial orbit is equatorial, and has an argument of periapsis of 0^o. Hence the five degree plane change is a rotation about the line of apsides. We generate 180 sets of initial conditions by varying the position and epoch of the initial maneuver, ν_o and T_o respectively, as well as the position and epoch of the final maneuver, νf and T_f. We vary ν_o from 300^o to 40^o in increments of 20^o. We vary ν_f from 140^o to 220^o in increments of 20^o. These ranges are chosen because we know the initial maneuver should occur near periapis, and the final maneuver should occur near apoapsis. For each possible pair of ν_o and ν_f we choose two times of flight and compute two transfer trajectories that result in a two-burn maneuver sequence with a total ΔV of less than 2 km/s. Three and four burn maneuver sequences are generated from each two burn solution by putting small maneuvers, equally spaced in time along the initial two burn sequence. Hence, there are thirty combinations of ν_o and ν_f, two times of flight for each pair of ν_o and ν_f, and two, three and four burn sequence for each time of flight resulting in one hundred and eighty initial guesses. A representative sample of the initial guesses for Case Two are shown in Figure (4). Positions of the initial maneuvers are label by black asterisks. Positions of the final maneuvers are marked by red circles.

$$\text{Initial Orbit:} \quad oe_o = [7000\ 0.3\ 0\ 0\ 0\ \nu_o], \quad T = T_o \tag{42}$$

$$\text{Final Orbit:} \quad oe_f = [8000\ 0.3\ 5.0\ 0\ 0\ \nu_f], \quad T = T_f \tag{43}$$

Case Three

Case three is a phasing maneuver sequence in a highly elliptic orbit. The orbital elements for the initial and final orbits are given in Eqs.(44) and (45) where E_o and E_f are the eccentric anomalies of the initial orbit and final orbit at times T_o and T_f respectively. To generate a set of initial guesses for Case Three we vary E_o from 0^o to 345^o in increments of 15^o. For each value of E_o we calculate two values of E_f such that there is a 30 minute separation in time between the initial and final orbits. The first value of E_f, for a given E_o, is calculated so that the final position is thirty minutes ahead of the initial position. The second value of E_f, for a given E_o, is calculated so that the final position is thirty minutes behind the initial position. For each E_f and E_o pair we generate a two maneuver sequence by solving Lambert's problem for an E_f and E_o pair knowing that Δt is thirty minutes. Three and four burn cases are generated by inserting small maneuvers along the two burn sequence. For Case Three there is a total of 96 initial guesses.

$$\text{Initial Orbit:} \quad oe_o = [42000\ 0.8\ 10\ 0.0\ 0.0\ E_o], \quad T = T_o \tag{44}$$

$$\text{Final Orbit:} \quad oe_f = [42000\ 0.8\ 10\ 0.0\ 0.0\ E_f], \quad T = T_f \tag{45}$$

The test cases developed above are intended to allow a comparison of the methods for rendezvous sequences in different flight regimes. In the next section we present results that compare the converged solutions for the six methods investigated in this paper.

RESULTS

Comparing the performance of the trajectory optimization methods studied in this work is non trivial. To begin, we must develop some metrics that allow us to compare the solutions provided by the different techniques. The ultimate goal is to provide an analyst with some insight into which technique to employ for a given problem. So ultimately, we want to know which technique is most likely to converge to the lowest ΔV cost. However, this is not the only important statistic. We are also interested in determining the relative performance of different methods. For example, if one approach converges to a known optimal 70% of the time, and a second approach converges to a known optimal 25% of the time, the second approach may still be useful for several reasons. Firstly, for a given problem the hypothetical method 1 will not converge to the known optimal 30% of the time. Hence we will need another approach. Secondly, the hypothetical method 2 might converge to a slightly higher solution in terms of the ΔV cost, but it might provide a solution that is better when other mission constraints are taken into account.

We employ two statistics to capture the relative performance of the different trajectory optimization techniques. The first statistic is simply the percentage of initial guesses where a particular method yields the lowest ΔV cost. We denote this statistic $\mathcal{S}_1$. For example, there are 54 initial guesses for Case One. If fmincon converges to the lowest ΔV cost of all the techniques, for 10 of the initial guesses, then $\mathcal{S}_1$ for fmincon is 17.54%. According to this definition, the sum of the $\mathcal{S}_1$ statistic for for all of the techniques for a particular test case is 100%.

The second statistic we employ is intended to provide more insight into the relative performance of the different techniques. For example, if SNOPT converges to a solution of 260 m/s for a particular initial guess, and PVAT1 converges to a solution of 261 m/s for the same initial guess, then although PVAT1 has a higher solution, it still performed well relative to SNOPT. This is not captured in the $\mathcal{S}_1$ statistic. We define a second statistic $\mathcal{S}_2$, and define it as the percentage of initial guesses that converge to within 5% of the lowest solution yielded by the six techniques for the particular initial guess. An example makes this clear. Suppose we are comparing solutions from fminunc and fminsearch for a set of three initial guesses. Suppose fminunc converges to 100 m/s for the first initial guess, and fminsearch converges to 102 m/s for the first initial guess. For the second initial guess fminunc converges to 104 m/s and fminsearch converges to 120 m/s. For the third initial guess fminunc and fminsearch converge to 110, and 101 m/s respectively. For this set of solutions S_2 for fminsearch is 66.66%, which means that 66.66% of the time fminsearch converged to within the 5% of the lowest solution of all the methods.

Both the S_1 and S_2 statistic can be applied to the results in several ways. For example, if only four-burn initial guesses are used in generating an S_2 statistic, we add an additional subscript and call the statistic S_{2_4}. If all of the initial guesses are used for determining an S_2 statistic, there is no additional subscript. This convention is also used for the S_1 statistic.

In the next three subsections we present a performance comparison of the six techniques for each of the three test cases. We use both the statistics described above, as well as graphical methods to illustrate how the techniques compare with one another.

Case One

Recall that Case One is a simple circular-to-circular coplanar rendezvous problem. Hence the optimal solution is simply the Hohmann ΔV, which for the orbits defined in Eqs. (40) and (41) is 255.8 m/s. A figure illustrating the results for all six optimization techniques we investigate is shown in Fig. 5. The top subplot contains the solutions for all two burn maneuver sequences. The middle and bottom subplots contain the solutions for all three and four burn solutions respectively. On each subplot the x-axis is the initial guess number. The initial guesses have been numbered in order of increasing ΔV. This is done to enable one to draw conclusions as to the performance of each method as the ΔV of initial guess increases. Furthermore, the initial guesses between the subplots are similar. For example, the initial guess number one in the two-maneuver subplot is similar to initial guess number one in the three-maneuver subplot. The difference being only that a small mid-course maneuver is placed along the two-maneuver sequence to create a three maneuver sequence. The same is true for initial guess one of the four-maneuver subplot. On all subplots the y-axis is the ΔV in km/s. Solutions for all direct methods are plotted in black. Solutions for indirect methods are plotted in red. Statistics that aid in comparing the results for each of method are found in Table1. By inspection of Fig. 5 we see that the direct methods outperformed the indirect methods. For all but one initial guess, the direct methods always converged to within a few m/s of the known optimal solution for two-maneuver sequences. For three and four burn sequences, fminsearch consistently had the poorest performance of the direct methods. In comparisons between PVAT1 and PVAT2, PVAT2 performed better. Examining the S_1 statistics for Case One we see SNOPT found the lowest solution most often at 35.09% of the time. The method with the second best performance was fmincon, which found the lowest solution 31.58 % of the time. Examining the S_2 statistics we see that both SQP methods converged to within 5% of the lowest solutioin at least 93% of the time. It is also important to note that although other methods performed poorly in comparison to SQP, about 33% of the time the lowest solution was found by a method other than SQP. This suggests that although SPQ is the single best performing method for Case One, it is not safe to rely only on SQP as an optimization approach. It is also important to note that although SPQ found the lowest solution most often, solutions from other methods were often close to the solution found by SQP methods.

Table 1: Statistics: Case One Results

	S_1	S_{1_2}	S_{1_3}	S_{1_4}	S_2	S_{2_2}	S_{2_3}	S_{2_4}
fmincon	**31.58**	21.05	15.79	57.89	**92.98**	100	78.95	100
SNOPT	**35.09**	57.89	42.11	5.263	**94.74**	100	89.47	94.74
fminunc	**10.53**	5.263	5.263	21.05	**78.95**	94.74	52.63	89.47
fminsearch	**7.018**	10.53	0.00	10.53	**43.86**	100	21.05	10.53
PVAT1	**3.509**	0.00	5.263	5.263	**71.93**	94.74	42.11	78.95
PVAT2	**12.28**	5.263	31.58	0.00	**66.67**	57.89	57.89	84.21

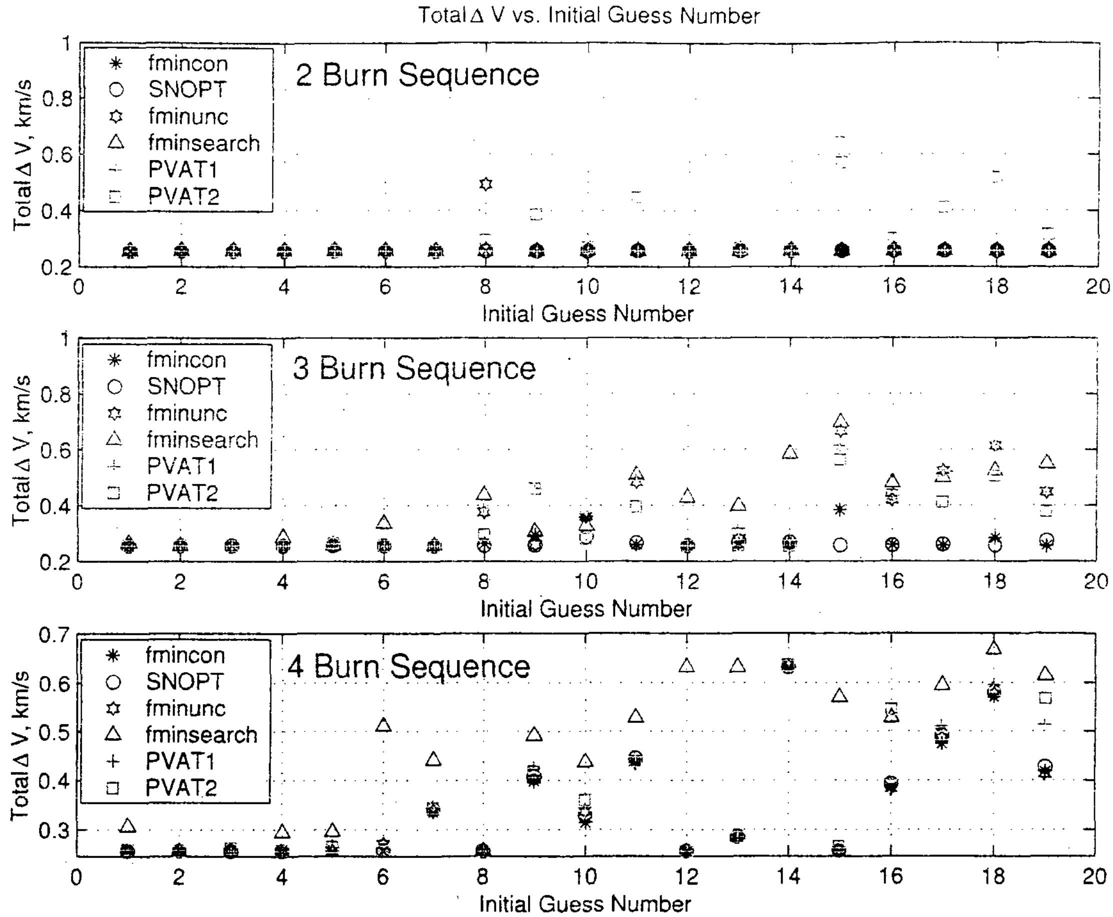

Figure 5: Results for Case One

Case Two

The maneuver sequence in Case Two involves raising the semi-major axis and rotating about the line of apsides, as well as satisfying the rendezvous conditions. Hence, two burn solutions will often perform poorly. Figure 6 shows all of the converged solutions for Case Two. The results are plotted using conventions identical to the results for Case One and we refer the reader to the previous subsection for details. Table 2 summarizes the selected statistics for each method for Case Two. For the two-burn sequences seen in the first subplot, PVAT2 yielded the lowest ΔV about 48% of the time and converged 71% of time within 5% of the lowest solution. PVAT2 higher performance can be explained by its unique ability compared to the other methods to add maneuvers to lower the ΔV cost. For the two-burn sequences all methods, with the exception of PVAT2, frequently converged to the same local minimum. Note that PVAT1 did not perform as well as the direct methods for the two-burn sequences as it converged only 29% of the time within 5% of the best solution. For three-burn sequences, SNOPT and fmincon yielded to the lowest ΔV about 42% of the time and converged about 90% and 81% of the time, respectively, within 5% of the lowest solution. For this set of initial guesses, PVAT2 yielded the worst results. Finally, for the four-burn sequence, fmincon out-performed all the other methods. fmincon found the lowest solution 58% of the time and 93% of its converged solutions were within 5% of the lowest ΔV cost. fminunc is the second best with about 16% and 64% for S_1 and S_2 respectively. SNOPT, fminsearch and PVAT1 have comparable statistics for this initial guess sequence. PVAT2 performed poorly compared to the other methods. Overall, we can see that, for Case Two, when there are more than two-impulses, fmincon appears to be consistently better. However, for the two-burn sequence, PVAT2 dominates over the other method because of its ability to add impulses to the trajectory.

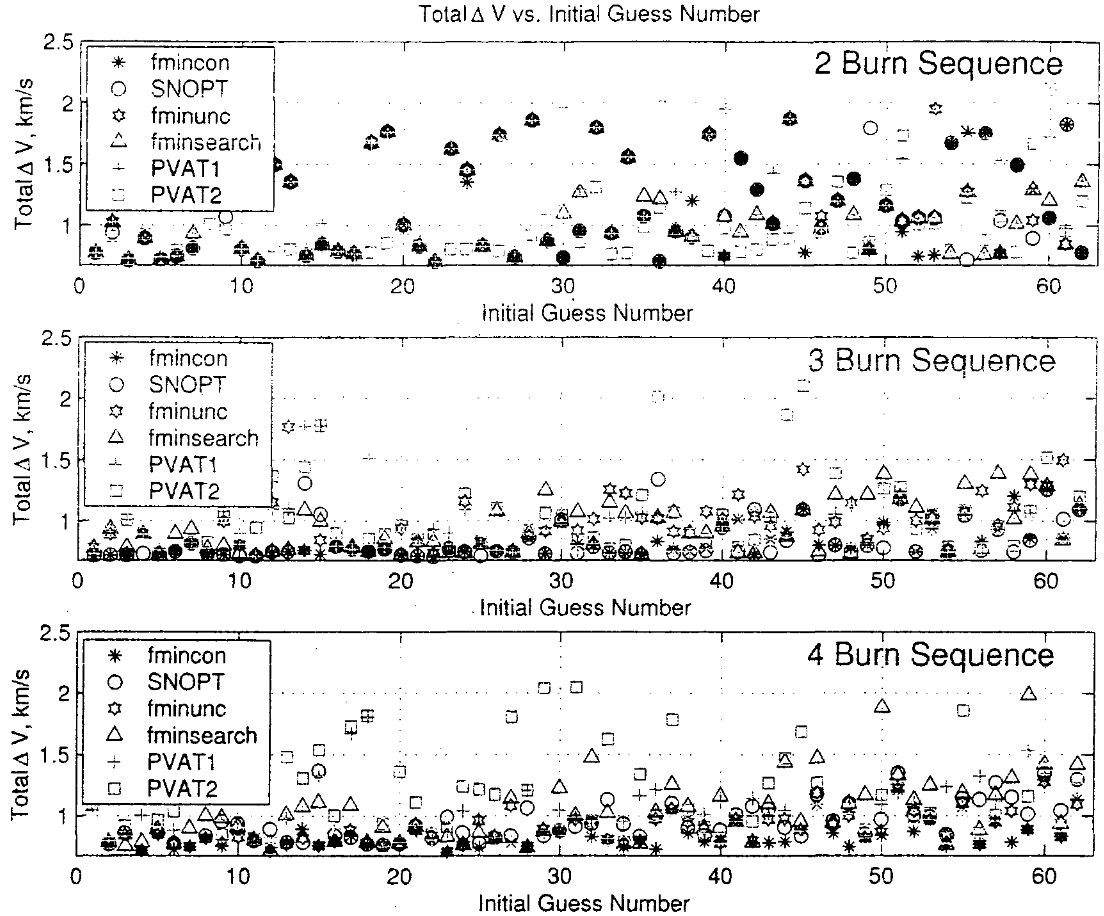

Figure 6: Results for Case Two

Table 2: Statistics Case Two Results

	S_1	S_{1_2}	S_{1_3}	S_{1_4}	S_2	S_{2_2}	S_{2_3}	S_{2_4}
fmincon	**31.58**	9.677	41.94	58.06	**75.81**	53.23	80.65	93.55
SNOPT	**24.73**	22.58	41.94	9.677	**56.99**	48.39	90.32	32.26
fminunc	**6.452**	1.613	1.613	16.13	**50.00**	48.39	37.1	64.52
fminsearch	**8.065**	8.065	8.065	8.065	**39.25**	43.55	45.16	29.03
PVAT1	**7.527**	9.677	4.839	8.065	**34.41**	29.03	41.94	32.26
PVAT2	**16.67**	48.39	1.613	0.00	**33.33**	70.97	17.74	11.29

Case Three

Recall that Case Three involves a phasing maneuver in a highly eccentric orbit which implies that the initial guess is going to be critical in defining which optimal solution is achievable. This sequence is a pure phasing maneuver where the number of revolutions is not to exceed one. Because of the high eccentricity of the orbit, moving the initial and final epochs of the rendezvous will in some cases change the trajectory dramatically making it more difficult to set the proper tolerances for the finite differencing to compute accurate gradients. Figure 6 shows all of the converged solutions for Case Three. Table 3 summarizes the selected statistics for each method for Case Three. For both the three-burn and four-burn sequences seen in the first subplot, all the indirect methods performed very poorly. fmincon outperformed all other methods with about 69% and 66.7% of the lowest solutions for the three-burn sequence and the four-burn sequence respectively. For the three and four burn sequences, S_2 for fmincon was 89.6% and 91.67% respectively. fminsearch and SNOPT were the direct methods with the lowest performance rate for both burn sequences. Note that fminunc converged 62.5% of the time within 5% of the best solution but was only the best method 10.42% of the time for the three-burn case. However, for the four-burn sequence, fminunc statistics improved to a convergence rate of 91.67% within 5% of the best solution and it found the lowest ΔV 22.92% of the time. Overall, fmincon and fminunc seem to be the preferable methods for solving a Case Three type of rendezvous. All the other methods performed very poorly.

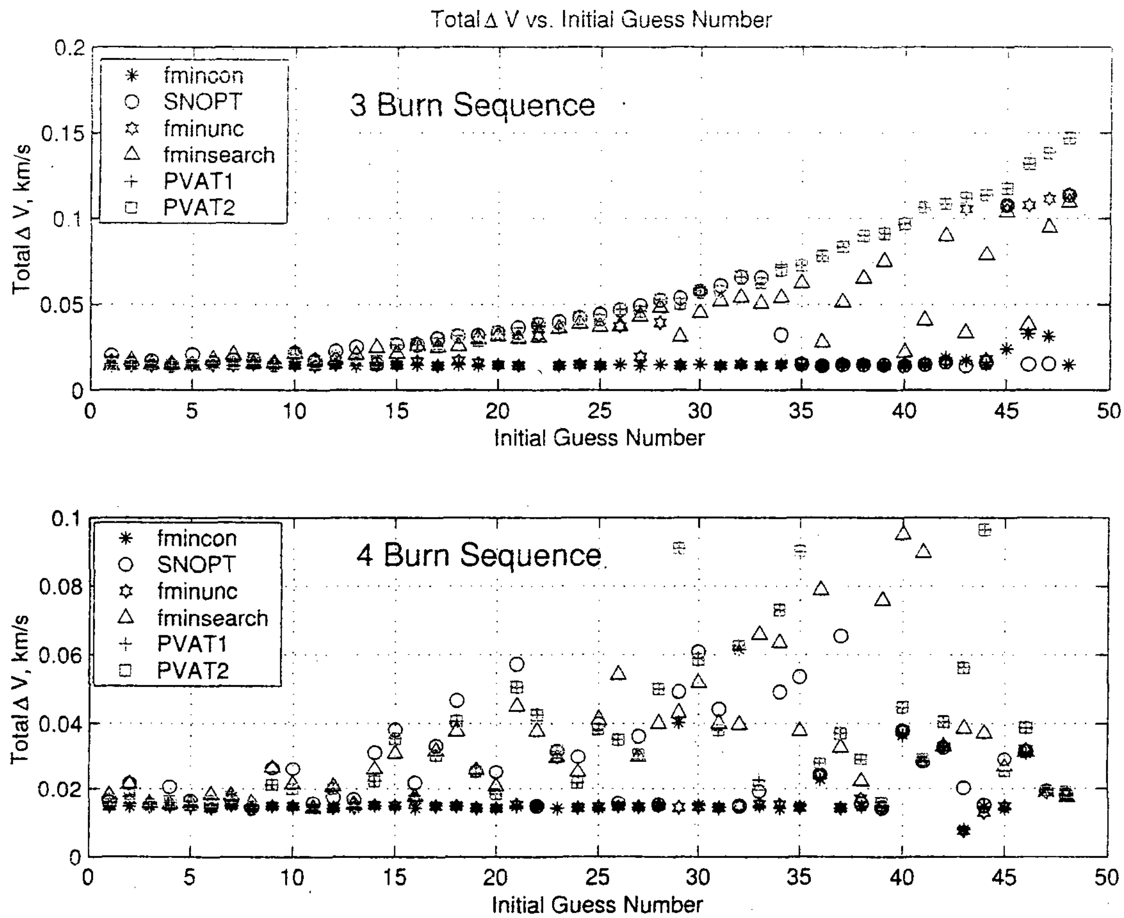

Figure 7: Results for Case One

Table 3: Statistics: Case Three Results

	S_1	S_{1_3}	S_{1_4}	S_2	S_{2_3}	S_{2_4}
fmincon	**67.71**	68.75	66.67	**90.63**	89.58	91.67
SNOPT	**11.46**	16.67	6.25	**25.00**	25.00	25.00
fminunc	**16.67**	10.42	22.92	**75.00**	62.50	87.50
fminsearch	**4.167**	4.167	4.167	**9.375**	6.25	12.50
PVAT1	**0.00**	0.00	0.00	**10.42**	8.33	12.50
PVAT2	**0.00**	0.00	0.00	**10.42**	8.33	12.50

CONCLUSIONS AND FUTURE WORK

The minimum fuel rendezvous problem has received extensive attention in the literature. There are two primary methods to solving the problem: direct and indirect. However, many methods are actually a hybrid between a direct and indirect approach.

In this work we compared several approaches for finding optimal rendezvous solutions. We compared several parameterizations for an objective function for a direct approach, and chose the one that appeared the best after some preliminary comparisons. We applied four numerical optimization routines to the direct method parameterization. It is important to note that finite differencing was used to calculate gradients for the gradient-based direct methods. Providing analytic gradients is a topic of current research and will likely improve both the rate of convergence, and the final values of the final converged solutions.

We also developed two indirect approaches. The indirect approaches were based on primer vector theory. The first approach does not allow additional impulses higher than the number of impulses contained in the initial guess. The second approach will include an additional impulse if the optimality conditions suggest that it will result in a ΔV reduction.

Ideally, from a software maintenance point of view, it is desirable for one method to always excel over competing methods. However, in this work we found that no single method was always the best. Yet, there are certain methods that outperform rival methods a significant amount of time. In general, in comparing direct approaches, SQP outperformed other methods such as Quasi-Newton, and Nelder-Meade Simplex. However, these results are intimately dependent on the parameterization of the objective function. For alternative parameterizations, SQP might not necessarily outperform other methods.

In comparisons between the direct and indirect approaches we implemented, we found that the direct approaches tended to find lower solutions more often. However, there were a significant number of cases where indirect methods outperformed direct approaches. Therefore, we cannot simply discard the indirect approaches in favor of the direct approaches.

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APPENDIX 1

There are numerous considerations to take into account when selecting an objective function parameterization for a direct method. In this work we break down the types of parameterizations into two categories called the Feasible Iterate Approach, and the Infeasible Iterate Approach. In the Feasible Iterate Approach the cost function is parameterized in such a way that the rendezvous constraints are satisfied implicitly for each cost function evaluation. Using a Feasible Iterate Approach allows one to choose between both constrained and unconstrained numerical optimization packages. However, we must provide a robust way to solve the TPBVP. In the Infeasible Iterate Approach the rendezvous constraints are not necessarily satisfied for each cost function evaluation. The rendezvous conditions are satisfied upon convergence of the numerical optimization routine. The strength of the Infeasible Iterate Approach is that we do not have to provide a robust TPBVP algorithm. However, it is necessary that we use only constrained optimization packages. Hence, in the Feasible Iterate Approach we are in a sense applying a change of variables to convert a constrained problem into an unconstrained problem. In this appendix we present one possible parameterization for each category to illustrate some differences in the methods. One possible parameterization of the Infeasible Iterate Approach is

$$\text{Given: } (\mathbf{r}_o, \mathbf{v}_o, t_o) \text{ and } (\mathbf{r}_f, \mathbf{v}_f, t_f) \tag{46}$$

Choose the independent variables:

$$t_i \qquad i = 1, 2 ... n \tag{47}$$

$$\mathbf{r}_j \qquad j = 1, 2, 3 .... n \tag{48}$$

require the optimizer to satisfy the following constraints

$$\mathbf{r}_1 - \mathbf{f}(\mathbf{r}_o, \mathbf{v}_o, t_o, t_1) = 0 \tag{49}$$
$$\mathbf{r}_n - \mathbf{f}(\mathbf{r}_f, \mathbf{v}_f, t_f, t_n) = 0 \tag{50}$$

Given Eqs. (47) and (48), the entire maneuver sequence is defined. By solving Lambert's problem for each trajectory segment we can calculate the total ΔV.

Although many parameterizations of the Feasible Iterate Approach are possible, we only present one here. One possible parameterization is

$$\text{Given: } (\mathbf{r}_o, \mathbf{v}_o, t_o) \text{ and } (\mathbf{r}_f, \mathbf{v}_f, t_f) \tag{51}$$

Choose the independent variables:

$$t_i \qquad i = 1, 2, ..., n \tag{52}$$
$$\Delta \mathbf{V}_j \qquad j = 1, 2, ..., n-2 \tag{53}$$

For the this parameterization the entire maneuver sequence is determined and we can solve for the total ΔV and satisfy the boundary conditions simultaneously. We first determine $\mathbf{r}_1$ and $\mathbf{v}_1^-$ from Eqs.(13) and (14). The next step is to solve $n-2$ initial value problems by iterating on the following algorithm

for $i = 1$ to $n-2$

$$\mathbf{v}_i^+ = \mathbf{v}_i^- + \Delta V_i \tag{54}$$
$$\mathbf{r}_{i+1} = \mathbf{f}(\mathbf{r}_i, \mathbf{v}_i^+, t_i, t_{i+1}) \tag{55}$$
$$\mathbf{v}_{i+1}^- = \mathbf{g}(\mathbf{r}_i, \mathbf{v}_i^+, t_i, t_{i+1}) \tag{56}$$

end

Finally we ensure the rendezvous conditions are satisfied by solving Lambert's problem for $\mathbf{r}_{n-1}$, $\mathbf{r}_n$, and $\Delta t = t_n - t_{n-1}$. With the solution for Lambert's problem we can solve for ΔV_{n-1} and ΔV_n and we then solve Eq. (19) for the total ΔV. There are many more possible choices for independent variables that fall under the Feasible Iterate Approach. Presenting all of the possibilities is beyond the scope of this work. However, it is worth mentioning that other Feasible Iterate Approaches are likely to be a hybrid of the method described by Eqs.(20), (21) and (22) or the method described by Eqs.(51), (52) and (53).

Choosing a specific parameterization for the objective function is nontrivial. Each of the parameterizations discussed above have some strengths and some weaknesses. Although a detailed comparison of the Feasible Iterate Approach and the Infeasible Iterate Approach is beyond the scope of this work, preliminary comparisons suggest that the Feasible Iterate Approach performs better. This is expected because it is often better to convert a constrained problem to an equivalent unconstrained problem if an appropriate change of variables is possible. Therefore we have chosen to consider only Feasible Iterate parameterizations. Choosing a specific parameterization of the Feasible Iterate Approach is also nontrivial. For this paper we choose to parameterize the objective function using the independent variables given in Eqs.(20), (21) and (22). The justification for choosing this parameterization is discussed in a previous section.