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THE NAVIER-STOKES STRESS PRINCIPLE FOR VISCOUS FLUIDS

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THE NAVIER-STOKES STRESS PRINCIPLE FOR VISCOUS FLUIDS*

By Ernst Mohr

The Navier-Stokes stress principle is checked in the light of Maxwell's mechanism of friction and in connection herewith the possibility of another theorem is indicated.

SUMMARY

The Navier-Stokes stress principle is in general predicated upon the conception of the plastic body. Hence the process is a purely phenomenological one, which Newton himself followed with his special theorem for one-dimensional flows. It remained for Maxwell to discover the physical mechanism by which the shear inflow direction is developed: According to it, this shear is only "fictitious" as it merely represents the substitute for a certain transport on macroscopic motion quantity, as conditioned by Brown's molecular motion and the diffusion, respectively. It is clear that this mechanism is not bound to the special case of the one-dimensional flows, but holds for any flow as expression of the diffusion, by which a fluid differs sharply from a plastic body. If it is remembered, on the other hand, that the cause of the stresses on the plastic body lies in a certain cohesion of the molecules, it appears by no means self evident that this difference in the mechanism of friction between fluid and plastic body should not prevail in the stress principle as well, although it certainly is desirable in any case, at least subsequently, to establish the general theorem in the sense of Maxwell. Actually, a different theorem is suggested which, in contrast to that by Navier-Stokes, has the form of an unsymmetrical matrix. Without anticipating a final decision several reasons are advanced by way of a special flow which seem to affirm this new theorem. To make it clear that the problem involved here still awaits its final solution, is the real purpose behind the present article.

*"Über den Navier-Stokesschen Spannungsansatz für zähe Flüssigkeitsstromungen." Luftfahrtforschung, vol. 18, no. 9, Sept. 20, 1941, pp. 327-330.

THE PHENOMENOLOGICAL THEOREM OF NAVIER-STOKES

The present analysis deals with so-called laminar flows of fluids, which also includes gases to the extent that their density variations are negligible. No limitations are suggested by the restriction of the study to two-dimensional flows which, however, must always be envisaged as being three-dimensional, so that the conventional three-dimensional identifications can be maintained.

Figure 1 shows in the usual notation the stresses $\underline{p}_x$, $\underline{p}_y$ which in vector notation lead to the stress matrix

$$\begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{pmatrix} \tag{1}$$

whence the easily understood differentiation process

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{pmatrix} = \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}, \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y}\right) \tag{2}$$

affords the related force by volume. Then with $\underline{w} = (u,v)$ as the flow velocity of the fluid, the Navier-Stokes theorem states that the stress matrix is proportional to the matrix of the deformation speeds

$$\begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{pmatrix} = 2\mu \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} \end{pmatrix} \tag{3}$$

or else

$$\begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{pmatrix} = \mu \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix} + \mu \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \tag{3a}$$

where μ is a characteristic constant for the fluid, termed "viscosity." On computing the force by volume of (2) by means of the right-hand side of equation (3a), it is seen that owing to the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4}$$

the second matrix in (3a) at the right, contributes nothing. Equation (3) specially postulates the symmetry of the stress matrix. As the most important boundary condition "the adhesion on fixed walls due to the viscosity" is cited.

The same principle (3) is applied to plastic bodies, with the difference that μ is no longer a constant.

To illustrate:

1. For the flow indicated in figure 2, termed for short, plate flow, the shear inflow direction is according to (3):

$$\tau = \mu \frac{\partial u}{\partial y} = \mu \frac{U}{d} \tag{5}$$

in which form it had been originally expressed by Newton.

2. On the "rigid" rotational flow, (fig. 3, ω = speed of rotation), the matrix of the deformation speeds disappears identically; hence no shear exists.

MECHANISM OF FRICTION ACCORDING TO MAXWELL

The Newton or Navier-Stokes equation is, as seen, purely phenomenological: other than that, such as the manner in which shear inflow direction of figure 2 comes into being, it says nothing. And we are indebted to Maxwell who made the important discovery for the case that the fluid is a real (ideal) gas, that on the plate flow the related shear inflow direction is a result of the transport on macroscopic momentum, as it arises by the diffusion: since the same number of molecules pass from below and above through each unit area of the shaded surface $y = y_0$ in figure 2; while the upper ones involve on the average a greater macroscopic velocity, the slower layer is subjected per unit time to a certain increment of motion quantity, which is proportional to $(u_2 - u_1)$, that is, proportional to the speed increase at that point and hence according to Newton's fundamental law (according to which, the change in motion quantity per unit time corresponds to the action of a force) equals the effect of a shear.

The calculation involved is briefly reproduced.

In this Maxwell conception, the shear (similar to pressure) refers to the entrained surface element. The result is the flow (fig. 4), in which the speed increase is the same as before.

With the notation

c amount of the microscopic molecule velocity

n number of molecules per cubic centimeter

Δn_c number of molecules per cubic centimeter, with a speed at interval $c \ldots\ldots c + \Delta c$

m mass of a molecule

ρ gas density; hence $m\, n$

the argument is as follows:

To define the transport from above, we concentrate for the first, on the molecules having a speed at interval $c \ldots\ldots c + \Delta c$, hereinafter called c molecule, for short. From these we eliminate those which arrive at an angle in the interval $\vartheta \ldots\ldots \vartheta + \Delta\vartheta$ toward the y axis (fig. 5), and call these c, ϑ molecules. Such molecules per cubic centimeter are

$$\frac{\Delta n_c}{2} \sin\vartheta\ \Delta\vartheta \tag{6}$$

This number, multiplied by the content $c \cos\vartheta$ of the cylinder of surface length c indicated in figure 5, then gives the number ΔN_c of c, ϑ molecules striking the unit surface per second:

$$\Delta N_c = \frac{\Delta n_c}{2} \sin\vartheta\ \Delta\vartheta\ c \cos\vartheta$$

If such a molecule had its last collision at the (oblique) distance λ, it transports at its free passage the macroscopic motion quantity

$$mu = m\left\{u_o + \lambda \cos \vartheta \left(\frac{\partial u}{\partial y}\right)_o\right\} \tag{8}$$

where the value of the pertinent quantities for the dashed layer, $u_o = 0$, is indicated by subscript 0. In figure 5, where such a molecule is plotted, it should be noted that of c, ϑ and λ only their related projections c', ϑ', and λ' are shown in plan view. All the c, ϑ molecules together then transport

$$\sum_{\sigma}^{\Delta N_c} m\left\{u_o + \lambda \cos \vartheta \left(\frac{\partial u}{\partial y}\right)_o\right\} \tag{9}$$

or, using an average value λ_c for λ

$$\Delta N_c\, m\left\{u_o + \lambda_c \cos \vartheta \left(\frac{\partial u}{\partial y}\right)_o\right\} \tag{10}$$

Thus the c, ϑ molecules advancing from above give the transport $\tau_{c,\vartheta}^{+}$:

$$\tau_{c,\vartheta}^{+} = \frac{\Delta n_c}{2} \sin \vartheta\, \Delta\vartheta\, c \cos \vartheta\, m\left\{u_o + \lambda_c \cos \vartheta \left(\frac{\partial u}{\partial y}\right)_o\right\} \tag{11}$$

whence the integration along ϑ from 0 to $\frac{\pi}{2}$ gives the transport τ_c^{+} from above of all c molecules at

$$\tau_c^{+} = \frac{\Delta n_c}{2} cm\left\{\frac{1}{2} u_o + \frac{1}{3} \lambda_c \left(\frac{\partial u}{\partial y}\right)_o\right\} \tag{12}$$

and from below at

$$\tau_c^{-} = \frac{\Delta n_c}{2} cm\left\{\frac{1}{2} u_o - \frac{1}{3} \lambda_c \left(\frac{\partial u}{\partial y}\right)_o\right\} \tag{13}$$

hence leaving an excess τ_c:

$$\tau_c = \tau_c^+ - \tau_c^- = \frac{1}{3} \Delta n_c \; m \; c \; \lambda_c \left(\frac{\partial u}{\partial y}\right)_0 \tag{14}$$

Integrating with respect to all c molecules and substituting the average value $\overline{c\lambda_c}$ for $c\lambda_c$, the looked-for transport or shear τ is:

$$\tau = \frac{1}{3} \, n \, m \, \overline{c \, \lambda_c} \left(\frac{\partial u}{\partial y}\right)_0 = \frac{1}{3} \rho \overline{c\lambda_c} \left(\frac{\partial u}{\partial y}\right)_0 \tag{15}$$

that is, the Newton formula again (5), and by comparison with it for the viscosity the explicit expression

$$\mu = \frac{1}{3} \rho \, \overline{c\lambda_c} \tag{16}$$

In this calculation Maxwell's distribution law for velocities at rest is used, an assumption which is certainly permitted so long, as in this case, the macroscopic velocity u is small relative to the microscopic c (reference 1). The calculation further indicates that the same shear is transported by the hypothetical static unit surface of the shaded layer $y = y_0$ in figure 2. It also is not tied to the condition that the velocity profile as here has a constant rise: instead of (8) it would then afford

$$mu = m\left\{u_0 + \lambda \cos \vartheta \left(\frac{\partial u}{\partial y}\right)_0 + \ldots \right\} \tag{17}$$

the dots denoting negligible terms of lower order.

PROBLEM

In this Maxwellian concept the shear τ is therefore merely "fictitious," since it simply forms the substitute for a certain transport of momentum, which in turn is caused by the diffusion, as exemplified in figure 6: during time unit 0 . . . . 1 a certain number of blank molecules depart and a corresponding number of white ones arrive. As a corollary the method of continuum mechanics is for the present inapplicable because the particle defined according to the continuum no longer contains all earlier molecules in the state $t = 1$! To make the continuum mechanics applicable requires first the application of the correction due to the transition of molecules and which exactly results in the fictitious stress. Such an analysis is obviously impossible for the corresponding flow condition of a plastic body, since in contrast to gas it has no diffusion.

Remembering that the conventional foundation of the Navier-Stokes theorem rests on the concept of plastic body it seems, in any event, desirable to interpret this general equation similar to Newton's special case in the sense of Maxwell. But, on applying the argument associated with the greatly schematized (fig. 6) to the arbitrary flow (fig. 7), for stress there is obtained in respect to the entrained surface element, with $\partial/\partial n$ as normal differentiation:

$$\mu \frac{\partial \underline{w}}{\partial n} \tag{18}$$

hence the unsymmetrical stress matrix

$$\mu \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{pmatrix} \tag{19}$$

which exactly agrees with the first matrix in (3a) at the right. According to this there is obtained for this rotation in contrast to the former a shear inflow direction

$$\tau = \mu \left\{ \text{velocity increase} \right\} = \mu\ \omega \tag{20}$$

which, similar to the plate flow, every faster layer exerts on its slower layer below it.

Which of the matrices is preferable - the symmetrical (3) or the unsymmetrical (19)? Undoubtedly the diffusion is the controlling characteristic by which a gas differs from a plastic body and it would almost be a miracle if this difference had no effect in the stress formula.

The problem is outlined as follows:

Plastic body	Gases
Cohesion of molecules	Diffusion rather than cohesion.
Method of continuum mechanics directly applicable	Inapplicable.
Cause of stresses in the cohesion of molecules	In the case of the plate flow the stresses according to Maxwell are a result of the transport of macroscopic motion quantity attending the diffusion.
The stress matrix is proportional to the matrix of the deformation speeds	?

So, if it succeeds in extending Maxwell's argument to the rotational flow, the question can be answered. This is attempted herewith.

FINDING THE ANSWER

The procedure is as follows: instead of computing the excess transport or shear τ per unit surface in motion as indicated by the shaded area in figure 8, try directly for the total shear $2\pi r\ \tau$, which this moving surface undergoes. But since this area constantly changes into itself by the macroscopic motion, the total shear is obviously identical with that total shear to which the enclosed shaded fluid mass of figure 8 is subjected by the outer fluid; and in this interpretation, however, the shaded area may be treated as quiescent! In other words, the problem simply involves the excess transport through this quiescent area; if such occurs, it corresponds to a shearing effect on the shaded fluid mass. Further, visualize identical coordinate systems x,y,z as indicated in figure 8 (z axis at right angles to plane of drawing) plotted along the shaded area in close succession for each unit surface. If the transport can be computed for such a surface element fixed in space, the task is finished;

in the event that the transport is altogether positive, the unsymmetrical equation (19) should be applicable.

For the actual calculation of such a surface element, that of the plate flow is reverted to, even to the extent of using the same notation. Figure 9 shows in plan view the arrival of a c, ϑ molecule from distance λ (the projected ϑ', λ' are shown only): the same conveys the motion quantity $mu(P)$, where $u(P)$ is the speed in point P and hence slopes slightly downward. Then the two subsequent facts permit the reduction to the previous case of plate flow:

1. Put

$$u(P = u(P'') + \left\{u(P) - u(P'')\right\} \tag{21}$$

for $u(P)$, since the braces on the average cancel for all molecules from the symmetry with respect to the yz plane, and it is seen that the calculation can be made as if only $mu(P'')$ were transported from the particular molecule.

2. Since, further,

$$\overline{OP''} = \overline{OP'} + \ldots = \lambda \cos \vartheta + \ldots \tag{22}$$

for the lengths $\overline{OP''}$ and $\overline{OP'}$ (the dots denoting terms of the order λ^2, hence are negligible),

$$u(P'') = u_o + \lambda \cos \vartheta \left(\frac{\partial u}{\partial y}\right)_o + \ldots \tag{23}$$

Then, the calculation is obviously the same as before, while the speed increase $\left(\frac{\partial u}{\partial y}\right)_o$ now equals the speed of rotation ω.

It is shown herewith that the shaded fluid mass per unit area in figure 8 is subjected to the shear $\tau = \mu \omega$ in flow direction by the outside fluid. The similarity of plate flow and rotational flow itself is also plain; for both are practically the same in vicinity of the moving wall, and in both cases this wall continues to perform work to the extent that the molecules advancing at a slightly lower macroscopic speed are speeded up again to full wall speed and then expelled into the fluid; this work then appears in a corresponding heat. Lastly, the equation according the unsymmetrical matrix (19) is no longer in agreement with the so-called Boltzmann axiom (reference 2).

FURTHER REMARKS

Having recognized the shear τ as fictitious, the same is suspected for the pressure p, and found to be a fact. It is proved in two stages: first in quiescent, then in moving gas.

In the case of quiescent gas the pressure p on a wall is, as is known, the result of the continual impacts of the molecules: a molecule flying with speed c at angle ϑ against the wall transmits to it altogether the vertical motion quantity $2\,c\cos\vartheta$, and these impacts produce, according to the kinetic gas theory, the mean pressure

$$p = \frac{1}{3}\rho\,\overline{c^2} \tag{24}$$

with ρ density and $\overline{c^2}$ the average value of c^2. A hypothetical cut within the gas then leaves the same pressure (24) on either side, but in this instance, in the Maxwellian sense as transport of molecular or microscopic speed. In fact, each molecule emerging at angle ϑ with speed c together with the reflective incoming molecule furnish an increment of perpendicular quantity of motion of the same amount $2\,c\cos\vartheta$ as before, hence result in the same pressure as on the fixed wall.

If the gas is in motion our previous concept of shear can be supplemented by inclusion of the pressure as follows: the transport of the molecules is accomplished in first approximation solely by the microscopic speed; but transported are 1) the microscopic and 2) the macroscopic speed. The second case then affords the previously known shear, the first, the pressure. Herein is embodied the expression for the free displaceability of the molecules which Euler recognized as representative of the pressure and Maxwell of the shear.

Since the new theorem leads to the same motion equations, hence also to the same pressure field, the pressure resistance in both instances is the same. This also holds for the frictional resistance so far as the body is at rest (or in uniform motion, respectively) and the condition of adhesion is satisfied. It is sufficient to perceive the proposition for a surface element of the body. Locating the origin of the coordinates including the x axis in it, the y axis perpendicularly outward, the adhesion condition gives

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial v}{\partial x} = 0 \quad \text{for} \quad y = 0 \tag{25}$$

hence, because of the continuity equation

$$\frac{\partial v}{\partial y} = 0 \quad \text{for } y = 0 \tag{26}$$

wherefrom the proposition follows immediately. Differences in frictional resistance, therefore, occurs first on arbitrarily moving surfaces such, as for instance, in the case of our rotational flow.

Lastly, it is to be observed that, if the new theorem proves correct, all transitions, of which the two extreme cases have been considered here, prevail between the plastic body and the fluid.

Translation by J. Vanier,
National Advisory Committee
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2. Hamel, G.: Elementare Mechanik. 2nd ed. Teubner. (Leipzig and Berlin), 1922, pp. 325-326.

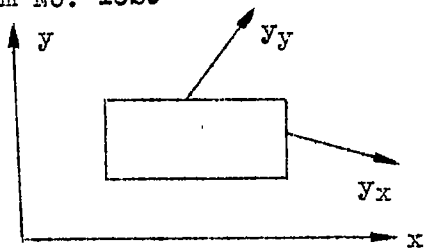

Figure 1.- Stress pattern.

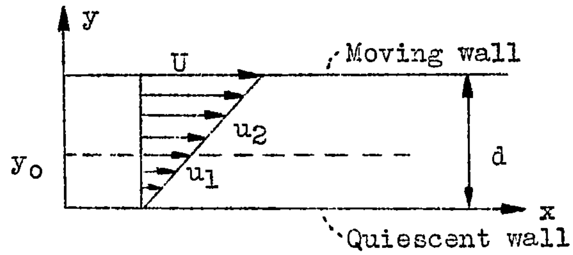

Figure 2.- Plate flow.

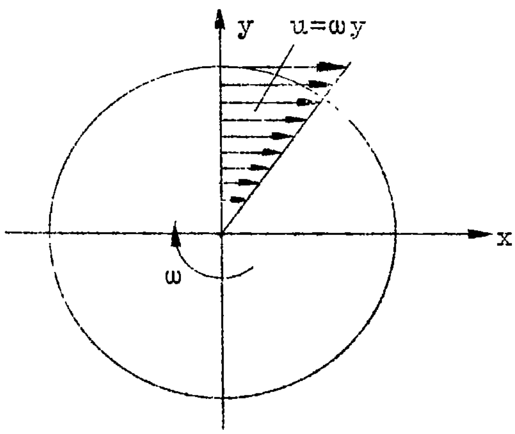

Figure 3.- "Rigid" rotational flow.

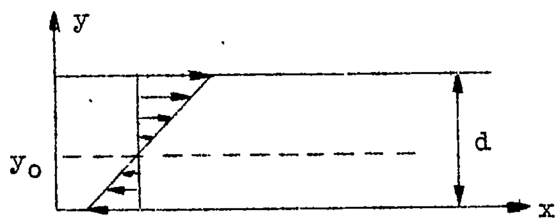

Figure 4.- Plate flow from a reference system, in which the dashed layer is at rest.

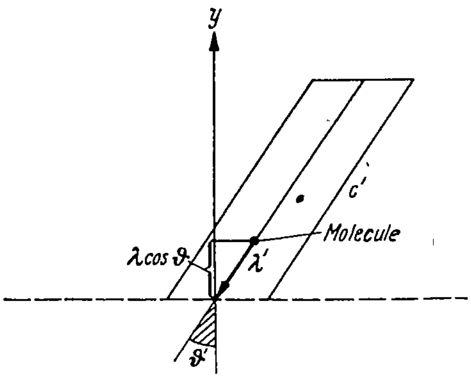

Figure 5.- Definition of shear as excess of transport of macroscopic motion quantity (plate flow).

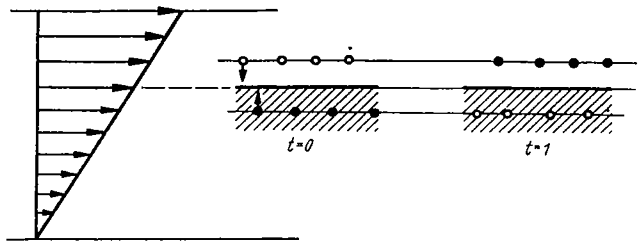

Figure 6.- Exaggerated diagrammatic representation of the formation of the stresses in the case of the plate flow.

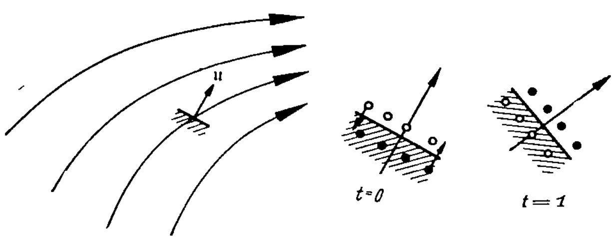

Figure 7.- Application of figure 6 to any selected flow.

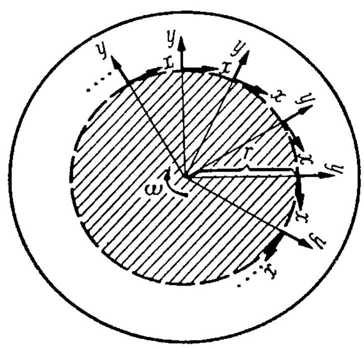

Figure 8.- The shear effect on the shaded fluid mass in the case of the rigid rotational flow.

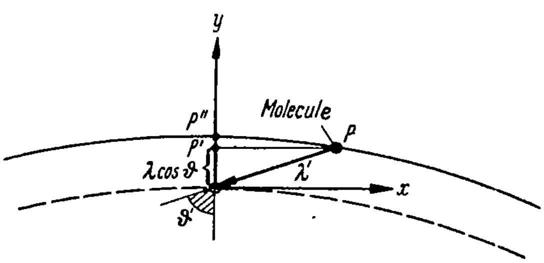

Figure 9.- Shear as excess of transport of macroscopic motion quantity (rigid rotational flow).