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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM

No. 1188

THE ELASTO-PLASTIC STABILITY OF PLATES

By A. A. Ilyushin

TRANSLATION

"Uprugo-plasticheskaya Ustoichivost Plasteen"
Prikladnaya Matematika i Mekhanika X, 1946

Washington
December 1947

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THE ELASTO-PLASTIC STABILITY OF PLATES*

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In this article are developed the results of my work (reference 1) "The Stability of Plates and Shells beyond the Elastic Limit." A significant improvement is found in the derivation of the relations between the stress factors and the strains resulting from the instability of plates and shells. In a strict analysis the problem reduces to the solution of two simultaneous nonlinear partial differential equations of the fourth order in the deflection and stress function, and in the approximate analysis to a single linear equation of the Bryan type. Solutions are given for the special cases of a rectangular plate buckling into a cylindrical form, and of an arbitrarily shaped plate under uniform compression. These solutions indicate that the accuracy obtained by the approximate method is satisfactory.

1. EXPRESSIONS FOR THE FORCES AND MOMENTS IN TERMS OF THE STRAINS IN THE MIDDLE SURFACE

On a moveable Darboux trihedron, relative to which we shall study the element of the shell, we choose the xy plane to be tangent to the middle surface, and the x and y directions along orthogonal curves (fig. 1).

The state of stress of the element is determined by the tensor of the stress S. Its components Z_z, Z_y, Z_x, are small compared to X_x, Y_y, and X_y, that is, each layer of the shell element parallel to the middle surface is in a state of plane stress. The intensity of stress in this layer will be

*"Uprugo-plasticheskaya Ustoichivost Plasteen." Prikladnaya Matematika i Mekhanika X, 1946, pp 623-638.

$$\sigma_i = \sqrt{X_x^2 + Y_y^2 - X_x Y_y + 3X_y^2} \qquad (1.1)$$

The state of strain of the element is determined by the components of the tensor of the strains e_{xx}, e_{yy}, and e_{xy}, since the shears e_{xz}, e_{yz} are small, but the relation to the strain e_{zz} may be found from the condition of constant volume of the element,

$$e_{xx} + e_{yy} + e_{zz} = 0 \qquad (1.2)$$

The intensity of strain in this layer of the material is given by the formula

$$e_i = \frac{2}{\sqrt{3}} \sqrt{e_{xx}^2 + e_{yy}^2 + e_{xx}e_{yy} + \frac{1}{4} e_{xy}^2} \qquad (1.3)$$

In agreement with the laws for the elasticity and plasticity of materials the stresses and strains are connected by the relations

$$S_x = X_x - \frac{1}{2} Y_y = \frac{\sigma_i}{e_i} e_{xx} \qquad S_y = Y_y - \frac{1}{2} X_x = \frac{\sigma_i}{e_i} e_{yy} \qquad X_y = \frac{\sigma_i}{3e_i} e_{xy} \qquad (1.4)$$

Here $\sigma_i = \sigma_i(e_i)$ is determined for each material as a function of e_i. The properties of this function are as follows. Within the elastic limit, that is, for $\sigma_i \leq \sigma'$ where σ' is a physical constant, Hooke's Law $\sigma_i = Ee_i$ always holds. Beyond the elastic limit $\sigma_i = \Phi(e_i)$ is a certain curve (fig. 2). If at a certain instant of time there occur infinitely small variations from the state of strain, that is, the quantities e_{xx} . . . receive increments δe_{xx} . . ., then the increments of stress below the elastic limit are given by the formulae (1.4) by setting $\sigma_i = Ee_i$. Beyond the elastic limit the increments of stress for $\delta e_i > 0$ are given by formulae (1.4) in accordance with the curve $\sigma_i = \Phi(e_i)$, but for $\delta e_i < 0$ in accordance with the law of unloading $\sigma_i = Ee_i$.

The problem of the stability of shells (plates) is stated as follows: Given, a shell under a given system of applied forces and with the states of stress and strain known. Required, the critical value of the external forces for which, at the same time, there is equilibrium with other possible states of strain infinitely close to the original state.

Let the change in the first and second quadratic form of the middle surface of the shell relative to the given equilibrium position be characterized by the parameters ϵ_1, ϵ_2, $2\epsilon_3$, and χ_1, χ_2, τ where ϵ_1, ϵ_2 are length ratios and $2\epsilon_3$ is the shear in the middle surface in the x,y plane, and χ_1, χ_2, τ are changes in curvature and twist. According to the Kirchoff hypothesis the increments in length and shear at a distance z from the middle surface will be

$$\delta e_{xx} = \epsilon_1 - z\chi_1 \qquad \delta e_{yy} = \epsilon_2 - z\chi_2 \qquad \delta e_{xy} = 2\epsilon_3 - 2z\tau \tag{1.5}$$

We seek the stress increments corresponding to the strains (1.5). For this it is necessary to take the variations of relations (1.4). The variation of the intensity of strain may be found by making use of (1.5), but afterward we write for the variation of the work of the internal forces $\sigma_1 \delta\epsilon_1$, in terms of the stress components,

$$\sigma_1 \delta\epsilon_1 = X_x \delta e_{xx} + Y_y \delta e_{yy} + X_y \delta e_{xy} \tag{1.6}$$

We introduce the nondimensional quantities

$$X_x^* = \frac{X_x}{\sigma_1},\ Y_y^* = \frac{Y_y}{\sigma_1},\ X_y^* = \frac{X_y}{\sigma_1},\ \chi_1^* = \frac{h}{2}\chi_1,\ \chi_2^* = \frac{h}{2}\chi_2,\ \tau^* = \frac{h}{2}\tau \tag{1.7}$$

where h is the shell thickness. Then in agreement with (1.1)

$$X_x^{*2} + Y_y^{*2} - X_x^* Y_y^* + 3X_y^{*2} = 1$$

From (1.6) and (1.5) we have

$$\delta e_1 = \epsilon - z^* X^* = \epsilon - zX \tag{1.8}$$

where

$$\left.\begin{aligned} \epsilon = X_x{}^*\epsilon_1 + Y_y{}^*\epsilon_2 + 2X_y{}^*\epsilon_3 \qquad \chi^* &= X_x{}^*\chi_1{}^* + Y_y{}^*\chi_2{}^* + 2X_y{}^*\tau^* \\ \chi &= X_x{}^*\chi_1 + Y_y{}^*\chi_2 + 2X_y{}^*\tau \end{aligned}\right\} \tag{1.9}$$

For the variation of formulae (1.4) we note that

$$\delta \frac{\sigma_i}{e_i} = -\frac{1}{e_i}\left(\frac{\sigma_i}{e_i} - \frac{d\sigma_i}{de_i}\right)\delta e_i$$

in which by the properties of the curve $\sigma_i = \Phi(e_i)$, $\dfrac{\sigma_i}{e_i} \geqslant \dfrac{d\sigma_i}{de_i} \geqslant 0.$

We denote by $z_o = z_o{}^* \dfrac{h}{2}$ the coordinate of the layer for which the intensity of strain is unchanged ($\delta e_i = 0$) during instability. It is clear that

$$z_o = \frac{\epsilon}{\chi}, \quad z_o{}^* = \frac{\epsilon}{\chi^*} \tag{1.10}$$

The variations of formulae (1.4) have the form

$$\left.\begin{aligned} \delta S_x &= \left(\frac{\sigma_i}{e_i} - \frac{d\sigma_i}{de_i}\right) S_x{}^*\chi^*(z^* - z_o{}^*) + \frac{\sigma_i}{e_i}\left(\epsilon_1 - \chi_1{}^*z^*\right) \\ \delta S_y &= \left(\frac{\sigma_i}{e_i} - \frac{d\sigma_i}{de_i}\right) S_y{}^*\chi^*(z^* - z_o{}^*) + \frac{\sigma_i}{e_i}\left(\epsilon_2 - \chi_2{}^*z^*\right) \\ \delta X_y &= \left(\frac{\sigma_i}{e_i} - \frac{d\sigma_i}{de_i}\right) X_y{}^*\chi^*(z^* - z_o{}^*) + \frac{2\sigma_i}{3e_i}\left(\epsilon_3 - \tau^*z^*\right) \end{aligned}\right\} \tag{1.11}$$

All quantities entering into the right-hand side of these equations except the strains and the curvatures are known, since the original state of stress of the shell (whose stability is sought) is supposed given. The quantities E, $E' = \frac{\sigma_1}{e_1}$, $E'' = \frac{d\sigma_1}{de_1}$ are shown in figure 2 as tangents of angles, Young's modulus being constant but E' and E'' depending on the state of stress.

Before instability the shell may find itself wholly beyond the elastic limit, or it may have elastic regions, elasto-plastic regions, and purely plastic regions. If the state of stress is momentless, then the region of elasto-plastic strains, that is, the region where part of the shell thickness is elastic, part plastic, is absent. In this paper we confine ourselves to the detailed stability investigation of compressed plates in which the state of stress is always momentless before instability. Hence we shall suppose that in the shells considered below the region of elasto-plastic strain is missing, before buckling (this assumption is not essential).

After instability the region of the shell where the stress was originally elastic will be, generally speaking, elastically deformed, since the strain variations are assumed infinitesimal. The region of purely plastic strain will be, generally speaking, resolved after buckling into two - one remaining purely plastic, the other elasto-plastic. Figure 3 shows a section normal to a shell with the three designated regions (the plastic region after instability is shaded).

Let the surface $z = z_0$ represent the boundary between the regions, one of which is elastic after instability, the other plastic. For its determination we shall suppose that in the elasto-plastic zone, the plastic zone adjoins the shell surface $z = +\frac{h}{2}$ and the elastic zone which originates as a result of unloading adjoins the surface $z = -\frac{h}{2}$.

In the region of elastic strain and in the zone of unloading $(z \leqslant z_0)$ formulae (1.11) take the form

$$\delta S_x = E(\epsilon_1 - \chi_1 z) \qquad \delta S_y = E(\epsilon_2 - \chi_2 z) \qquad \delta X_y = \frac{2}{3} E(\epsilon_3 - \tau z) \quad (1.12)$$

In the region of plastic strain and in the zone of loading $(z \geqslant z_0)$ of the elasto-plastic region, these formulae may be presented in the form[1]

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[1]From (1.12) and (1.13) it is seen that the variations δS_x . . . on the boundary z_0 are continuous in the case where the original state of stress corresponds to the beginning of flow, and equally so when, as a result of variation, the state of stress changes in proportion to the original state (reference 2).

$$\left.\begin{aligned}\delta S_x &= (E' - E'')\ S_x{*}\chi(z - z_o) + E'(\epsilon_1 - \chi_1 z)\\ \delta S_y &= (E' - E'')\ S_y{*}\chi(z - z_o) + E'(\epsilon_2 - \chi_2 z)\\ \delta X_y &= (E' - E'')\ X_y{*}\chi(z - z_o) + \frac{2}{3}E'(\epsilon_3 - \tau z)\end{aligned}\right\} \qquad (1.13)$$

We proceed to the derivation of expressions for forces and moments arising in the shell during instability. For their determination we have

$$\delta T_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \delta X_x\, dz \quad \delta T_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \delta Y_y\, dz \quad \delta S = \int_{-\frac{h}{2}}^{\frac{h}{2}} \delta X_y\, dz$$

$$\delta M_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \delta X_x z\, dz \quad \delta M_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \delta Y_y z\, dz \quad \delta H = \int_{-\frac{h}{2}}^{\frac{h}{2}} \delta X_y z\, dz$$

<u>In the region of purely plastic strains</u> we obtain for the forces, in agreement with (1.13):

$$\frac{1}{E'h}\left(\delta T_1 - \frac{1}{2}\delta T_2\right) = \epsilon_1 - \lambda' S_x{*}\epsilon$$
$$\frac{1}{E'h}\left(\delta T_2 - \frac{1}{2}\delta T_1\right) = \epsilon_2 - \lambda' S_y{*}\epsilon \qquad \frac{1}{E'h}\delta S = \frac{2}{3}\epsilon_3 - \lambda' X_y{*}\epsilon \qquad (1.14)$$

and for the moments

$$\frac{4}{3D'}\left(\delta M_1 - \frac{1}{2}\delta M_2\right) = -\chi_1 + \lambda' S_x{*}\chi$$
$$\frac{4}{3D'}\left(\delta M_2 - \frac{1}{2}\delta M_1\right) = -\chi_2 + \lambda' S_y{*}\chi \qquad \frac{4}{3D'}\delta H = -\frac{2}{3}\tau + \lambda' X_y{*}\chi \qquad (1.15)$$

where

$$D' = \frac{E'h^3}{9} \qquad \lambda' = \frac{E' - E''}{E'} \tag{1.16}$$

In the region of purely elastic strains, formulae (1.14) and (1.15) hold, only $E' = E'' = E$, $\lambda' = 0$.

Thus in the two regions, the forces are linear functions only of ϵ_1, ϵ_2, and $2\epsilon_3$, the middle surface shear, and the moments are linear functions only of the changes in curvature.

In the region of elasto-plastic strains, the stresses δX_x . . . have different expressions for $z \geqslant z_o$ and for $z \leqslant z_o$. Hence, the integrals in the expressions for the forces and moments must be split into two parts. For example,

$$\delta M_1 - \frac{1}{2}\delta M_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \delta S_x \, z \, dz = \int_{-\frac{h}{2}}^{z_o} \delta S_x \, z \, dz + \int_{z_o}^{\frac{h}{2}} \delta S_x \, z \, dz$$

in which for the region $z_o \geqslant z \geqslant -\frac{h}{2}$ we take δS_x according to (1.12) and for the region $\frac{h}{2} \geqslant z \geqslant z_o$, according to (1.13). As a result of these calculations we obtain for the forces

$$\left.\begin{aligned}
\frac{2}{h}\left(\delta T_1 - \frac{1}{2}\delta T_2\right) &= \left[E + E' + (E - E')\, z_o^*\right]\epsilon_1 + \frac{1 - z_o^{*2}}{2}(E - E')\chi_1^* \\
&\quad + \frac{E' - E''}{2} S_x^* (1 - z_o^*)^2 \chi^* \\
\frac{2}{h}\left(\delta T_2 - \frac{1}{2}\delta T_1\right) &= \left[E + E' + (E - E')z_o^*\right]\epsilon_2 + \frac{1 - z_o^{*2}}{2}(E - E')\chi_2^* \\
&\quad + \frac{E' - E''}{2} S_y^* (1 - z_o^*)^2 \chi^* \\
\frac{2}{h}\delta S &= \frac{2}{3}\left[E + E' + (E - E')z_o^*\right]\epsilon_3 + \frac{1 - z_o^{*2}}{3}(E - E')\tau^* \\
&\quad + \frac{E' - E''}{2} X_y^* (1 - z_o^*)^2 \chi^*
\end{aligned}\right\} \tag{1.17}$$

and for the moments

$$\left.\begin{aligned}
\frac{12}{h^2}\left(\delta M_1 - \frac{1}{2}\,\delta M_2\right) &= -\left[E + E' + (E - E')z_o^{*3}\right]\chi_1^* - \frac{3}{2}(E - E')(1 - z_o^{*2})\epsilon_1 \\
&\quad + \frac{E' - E''}{2}(1 - z_o^*)^2\,(2 + z_o^*)\,S_x^*\chi^* \\
\frac{12}{h^2}\left(\delta M_2 - \frac{1}{2}\,\delta M_1\right) &= -\left[E + E' + (E - E')z_o^{*3}\right]\chi_2^* - \frac{3}{2}(E - E')\left(1 - z_o^{*2}\right)\epsilon_2 \\
&\quad + \frac{E' - E''}{2}(1 - z_o^*)^2(2 + z_o^*)S_y^*\chi^* \\
\frac{12}{h^2}\,\delta H &= -\frac{2}{3}\left[E + E' + (E - E')z_o^{*3}\right]\tau^* - (E - E')\left(1 - z_o^{*2}\right)\epsilon_3 \\
&\quad + \frac{E' - E''}{2}(1 - z_o^*)^2(2 + z_o^*)X_y^*\chi^*
\end{aligned}\right\} \qquad (1.18)$$

The dependency between forces and strains is nonlinear, since z_o^* enters into the formula and from (1.10) it depends on the strains. From this fact proceed all the difficulties of solution of problems in shell stability beyond the elastic limit.

Further, it is essential that the ordinate z_o^* depending on both the changes in curvature χ_1, χ_2, τ and on the strains ϵ_1, ϵ_2, ϵ_3, be expressed only in the changes in curvature and the forces δT_1, δT_2, δS. Multiplying the first equation of (1.17) by X_x^*, the second by Y_y^*, the third by $3X_y^*$ and adding, we get

$$\lambda(1 - z_o^*)^2 + 4z_o^* - 4\,\frac{S_x^*\delta T_1 + S_y^*\delta T_2 + 3X_y^*\delta S}{Eh\chi^*} = 0 \qquad (1.19)$$

By introduction of the notation ζ for the ratio of the thickness h_p of the plastic layer to the thickness of the shell

$$\zeta = \frac{h_p}{h} = \frac{1 - z_o^*}{2} \tag{1.20}$$

and solving equation (1.19) for ζ, we get

$$\zeta = \frac{E - \sqrt{EE''(1 + \varphi)}}{E - E''} = \frac{1 - \sqrt{(1 - \lambda)(1 + \varphi)}}{\lambda} \tag{1.21}$$

where

$$\varphi = \frac{\lambda}{1 - \lambda}\,\frac{S_x^*\delta T_1 + S_y^*\delta T_2 + 3X_y^*\delta S}{Eh\chi^*}, \quad \lambda = \frac{E - E''}{E} = 1 - \frac{\frac{d\sigma_i}{de_i}}{E} \tag{1.22}$$

Formulae (1.17), (1.18) are appreciably simplified (otherwise conserving the principal complications) if we consider only the beginning of flow, that is, we suppose that the shell material before instability exceeds the elastic limit very slightly. In this case

$$E' = E \qquad \lambda' = \lambda = \frac{E - E''}{E}$$

Therefore, in the notation of (1.20) the corresponding formulae have the form for the forces

$$\left.\begin{aligned}
\frac{1}{Eh}\left(\delta T_1 - \frac{1}{2}\,\delta T_2\right) &= \epsilon_1 + \frac{\lambda h}{2}\,S_x^*\zeta^2\chi \\
\frac{1}{Eh}\left(\delta T_2 - \frac{1}{2}\,\delta T_1\right) &= \epsilon_2 + \frac{\lambda h}{2}\,S_y^*\zeta^2\chi \\
\frac{1}{Eh}\,\delta S &= \frac{2}{3}\,\epsilon_3 + \frac{\lambda h}{2}\,X_y^*\zeta^2\chi
\end{aligned}\right\} \tag{1.23}$$

for the moments

$$\left.\begin{aligned}
\frac{4}{3D}\left(\delta M_1 - \frac{1}{2}\,\delta M_2\right) &= -\chi_1 + \lambda S_x{}^*\zeta^2(3 - 2\zeta)\chi \\
\frac{4}{3D}\left(\delta M_2 - \frac{1}{2}\,\delta M_1\right) &= -\chi_2 + \lambda S_y{}^*\zeta^2(3 - 2\zeta)\chi \\
\frac{4}{3D}\,\delta H &= -\frac{2\tau}{3} + \lambda X_y{}^*\zeta^2(3 + 2\zeta)\chi
\end{aligned}\right\} \tag{1.24}$$

where D is the usual stiffness for Poisson's ratio equal to 1/2.

2. THE STABILITY OF COMPRESSED PLATES

Denoting the bending of the plate during instability by $w(x,y)$ and the displacements of points in the middle surface projected in the x,y directions by $u(x,y)$, $v(x,y)$, respectively, we have expressions for the changes in curvature χ_1, χ_2, τ, and the strains ϵ_1, ϵ_2, ϵ_3:

$$\left.\begin{aligned}
\chi_1 &= \frac{\partial^2 w}{\partial x^2} \qquad \chi_2 = \frac{\partial^2 w}{\partial y^2} \qquad \tau = \frac{\partial^2 w}{\partial x\,\partial y} \\
\epsilon_1 &= \frac{\partial u}{\partial x} \qquad \epsilon_2 = \frac{\partial v}{\partial y} \qquad \epsilon_3 = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)
\end{aligned}\right\} \tag{2.1}$$

The forces applied in the middle surface before instability may be written in the following form:

$$T_1 = h\sigma_i X_x{}^* \qquad T_2 = h\sigma_i Y_y{}^* \qquad S = h\sigma_i X_y{}^*$$

and their projection on the Z-axis after instability in the form

$$T_1\chi_1 + T_2\chi_2 + 2S\tau = h\sigma_i\chi$$

Therefore, the condition of equilibrium of all forces applied to an element and projected on the z-axis, gives

$$\frac{\partial^2 \delta M_1}{\partial x^2} + 2\frac{\partial^2 \delta H}{\partial x\, \partial y} + \frac{\partial^2 \delta M_2}{\partial y^2} + h\sigma_i X = 0 \tag{2.2}$$

The condition of equilibrium of the middle surface forces after instability will be

$$\frac{\partial \delta T_1}{\partial x} + \frac{\partial \delta S}{\partial y} = 0 \qquad \frac{\partial \delta T_2}{\partial y} + \frac{\partial \delta S}{\partial x} = 0 \tag{2.3}$$

Finally, the compatibility condition for the strains has the form

$$\frac{\partial^2 \epsilon_1}{\partial y^2} + \frac{\partial^2 \epsilon_2}{\partial x^2} - 2\frac{\partial^2 \epsilon_3}{\partial x\, \partial y} = 0 \tag{2.4}$$

The combination of differential equations (2.2), (2.3), and (2.4) is necessary and sufficient for the solution of the problem of stability, if the corresponding boundary conditions are set up. Indeed, according to (1.14), or to (1.24) and (1.20), the strains ϵ_1, ϵ_2, ϵ_3 may be expressed in terms of the forces δT_1, δT_2, δS and the curvatures X (bending w), following which the moments δM_1, δM_2, δH are functions of these same four arguments. Thus the problem reduces itself to four differential equations with four unknown functions, of which (2.2) is of the Bryan type, and (2.3), (2.4) are of the type of equations in plane problems.

In the region of purely plastic strain of the plate, (that is, such that the whole thickness, plastic before instability, remains plastic after instability), the system of differential equations is resolved into two. For simplicity we consider only the case of the beginning of flow. Substitution of the values of δM_1, δM_2, δH from (1.15) into (2.2) gives a differential equation for w of the Bryan type:

$$\nabla^4 w - \frac{h\sigma_i}{D} X = \frac{3}{4}\left(\frac{\partial^2}{\partial x^2} X_x{}^* + 2\frac{\partial^2}{\partial x\, \partial y} X_y{}^* + \frac{\partial^2}{\partial y^2} Y_y{}^*\right)\lambda X \tag{2.5}$$

where, in agreement with (1.9) and (2.1),

$$\chi = X_x^* \frac{\partial^2 w}{\partial x^2} + 2X_y^* \frac{\partial^2 w}{\partial x\,\partial y} + Y_y^* \frac{\partial^2 w}{\partial y^2} \tag{2.6}$$

The two boundary conditions on w agree with the usual boundary conditions for the Bryan equation.

Solving equations (1.14) for the strains, we get

$$\left.\begin{aligned}
\epsilon_1 &= \frac{1}{Eh}\left(\delta T_1 - \frac{1}{2}\,\delta T_2\right) + \frac{\lambda S_x^*}{(1-\lambda)Eh}\left(S_x^*\delta T_1 + S_y^*\delta T_2 + 3X_y^*\delta S\right)\\
\epsilon_2 &= \frac{1}{Eh}\left(\delta T_2 - \frac{1}{2}\,\delta T_1\right) + \frac{\lambda S_y^*}{(1-\lambda)Eh}\left(S_x^*\delta T_1 + S_y^*\delta T_2 + 3X_y^*\delta S\right)\\
2\epsilon_3 &= \frac{3\delta S}{Eh} + \frac{3\lambda X_y^*}{(1-\lambda)Eh}\left(S_x^*\delta T_1 + S_y^*\delta T_2 + 3X_y^*\delta S\right)
\end{aligned}\right\} \tag{2.7}$$

Equations (2.3) are satisfied if the stress function F is introduced:

$$\frac{\delta T_1}{Eh} = \frac{\partial^2 F}{\partial y^2} \qquad \frac{\delta T_2}{Eh} = \frac{\partial^2 F}{\partial x^2} \qquad \frac{\partial S}{Eh} = -\frac{\partial^2 F}{\partial x\,\partial y} \tag{2.8}$$

following which, analogous to (2.6) we denote

$$t = S_x^* \frac{\partial^2 F}{\partial y^2} + S_y^* \frac{\partial^2 F}{\partial x^2} - 3X_y^* \frac{\partial^2 F}{\partial x\,\partial y} \tag{2.9}$$

we obtain the compatibility condition for strain in the form

$$\nabla^4 F = -\left(\frac{\partial^2}{\partial y^2} S_x^* + \frac{\partial^2}{\partial x^2} S_y^* - 3\frac{\partial^2}{\partial x\,\partial y} X_y^*\right)\frac{\lambda t}{1-\lambda} \tag{2.10}$$

In order to write the boundary conditions for this equation, it is necessary to compute the variations of the normal force δT_ν, and of the tangential force δS_ν on a certain curvilinear contour in the middle surface of the plate.

If the outward normal ν and the tangent s to the contour constitute a coordinate system such that by rotation the positive direction of ν coincides with that of y and the positive direction of s coincides with that of x and if the angle between the normal and the x-axis is denoted by α (fig. 4), then our quantities have the known expressions

$$\left.\begin{aligned}\delta T_\nu &= \frac{\delta T_1 + \delta T_2}{2} + \frac{\delta T_1 - \delta T_2}{2}\cos 2\alpha + \delta S \sin 2\alpha \\ \delta S_\nu &= \frac{\delta T_1 - \delta T_2}{2}\sin 2\alpha - \delta S \cos 2\alpha\end{aligned}\right\} \tag{2.11}$$

The purely plastic region of the plate may be bounded by a contour, part of which coincides with the boundary of the plate, the part adjoins the elasto-plastic region. For the formulation of the stability problem in the first part, the boundary conditions have the form

$$\delta T_\nu = \delta S_\nu = 0 \tag{2.12}$$

and in the second part δT_ν, δS_ν must be continuous.

It is easy to show that during instability the entire plate may not remain in the purely plastic state; that is, an elasto-plastic region may come into being. Indeed, going back we shall have the uniform boundary conditions (2.12) on all external edges of the plate. But the differential equations (2.3) and (2.4) for conditions (2.7) will be also linear and homogeneous and so will have the unique solution

$$\delta T_1 = \delta T_2 = \delta S = 0$$

It follows from (2.7) that $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$, from which on the basis of (1.9) and (1.10), $z_o = 0$. But $z = z_o$ is the boundary between the elastic and plastic zones through the thickness of the plate and the condition $z_0 = 0$ specifies that the middle surface is this boundary. It follows that a given region of a plate is not purely plastic, but elasto-plastic, which contradicts the assumption.

During instability of a plate beyond the elastic limit it will either go completely over to the elasto-plastic state or there will remain purely plastic regions in it, which are not diffused throughout the plate.

In the region of elasto-plastic strains, equation (2.2) on the basis of expressions (1.24) may be presented in the form

$$\nabla^4 w - \frac{h\sigma_i}{D}\chi = \frac{3}{4}\left(\frac{\partial^2}{\partial x^2}X_x{}^* + 2\frac{\partial^2}{\partial x\,\partial y}X_y{}^* + \frac{\partial^2}{\partial y^2}Y_y{}^*\right)\lambda\zeta^2(3 - 2\zeta)\chi \qquad (2.13)$$

in which, as in equations (2.5), (2.10), the operator in parenthesis acts like a multiplier on the quantity to its right.

The condition of compatibility of strain (2.4) on the basis of (1.23) has the form

$$\nabla^4 F = \frac{h}{2}\left(\frac{\partial^2}{\partial y^2}S_x{}^* + \frac{\partial^2}{\partial x^2}S_y{}^* - 3\frac{\partial^2}{\partial x\,\partial y}X_y{}^*\right)\lambda\zeta^2\chi \qquad (2.14)$$

where the stress function F is determined by formulae (2.8). The value of ζ, the ratio of the thickness of the plastic layer to the plate thickness, enters into equations (2.13) and (2.14), therefore they show compatibility; this quantity ζ is expressed by formula (1.21) in which the function φ is, if use is made of the notation (2.9)

$$\varphi = \frac{2}{h}\frac{\lambda}{1 - \lambda}\frac{t}{\chi} \qquad (2.15)$$

Equations (2.13), (2.14) agree with the corresponding equations (2.5) and (2.10) at the boundary of the purely plastic and the elasto-plastic regions. Indeed, at this boundary, besides continuity in the values of the forces δT_ν, δS_ν, the moments δM_ν, $\delta H_\nu{}'$ (where $\delta H_\nu{}'$ is the rotational moment according to the boundary conditions of Kirchoff), the bending w and the slope of the tangent plane, there must also hold the condition

$$h_p = h$$
or
$$\zeta = 1 \tag{2.16}$$

From (1.21) for this condition we have $\varphi = -\lambda$ and $t = -\frac{1 - \lambda}{2} \chi h$, following which the remarked coincidence of the equations is easily shown.

The boundary conditions for equations (2.13), (2.14) on the elasto-plastic part of the contour, coinciding with the plate contour, yield the usual requirement $\delta T_\nu = \delta S_\nu = 0$ and two conditions relating to the bending w.

Condition (2.15) or

$$t = -\frac{1 - \lambda}{2} \chi h \tag{2.17}$$

represents in itself the equation of the boundary between the purely plastic and the elasto-plastic regions.

The possibility of purely plastic regions arising at the same with the elasto-plastic regions follows from the fact that the value of ζ in agreement with (1.21) and (2.15) may take on values not lying in the interval $1 \geqslant \zeta \geqslant 0$. Certain examples are given below of exact solutions of the stability of plates, and, in particular, the problem of the compressed plate freely supported along two sides; the edges of the plate near the free supports, after instability, remain in the purely plastic state.

3. EXAMPLES OF EXACT SOLUTIONS OF PROBLEMS IN THE STABILITY OF PLATES

The integration of the system of differential equations (2.13) and (2.14) in the elasto-plastic region, and of (2.5) and (2.10) in the plastic region with an undetermined boundary between them given by (2.16), is fraught with significant mathematical difficulties. As was shown in 1, the stability problem simplifies when the variations of the forces in the middle surface are zero everywhere. In that case the relative thickness ζ of the plastic layer is a known function of the coordinates, since from (1.22) $\varphi = 0$ and consequently

$$\zeta = \frac{1 - \sqrt{1 - \lambda}}{\lambda} \tag{3.1}$$

If the state of stress of the plate before instability is uniform, the value of ζ will be constant, since in (1.22) $\frac{d\sigma_i}{de_i}$ will be the same for the whole plate.

We call those solutions of stability problems approximate for which the variations δT_1, δT_2, δS of the forces are identically zero. Thus, the equations (2.3) of equilibrium and the boundary conditions (2.12) are satisfied, but, except in special cases, the compatibility condition (2.4) is not satisfied. The simplicity of such a solution arises from the fact that in equation (2.13) the value of ζ is known and given by formula (3.1), as a result of which this equation becomes linear with constant or variable coefficients. It closely resembles the equation for the elastic stability of an anisotropic plate.

The exact solutions of the system (2.13), (2.14) are undoubtedly of interest in their own right, but for us they have significance because they can be made use of to estimate the degree of exactness of approximate solutions.

We discuss a certain class of exact solutions of stability problems for uniformly compressed plates of arbitrary shape and the solution for a rectangular plate in the case when buckling into a cylindrical shape is possible.

a. Stability of a Uniformly Compressed Plate of Arbitrary Shape (Fig. 4)

In this case the state of stress of the plate before instability is uniform and given by the formulae

$$X_x = Y_y = -\sigma_i, \qquad X_y = 0 \tag{3.2}$$

where σ_1 is the compressive stress along the edge and is also the uniform stress intensity at any point in the plate. The resulting stresses according to (1.7) and (1.4) will be

$$X_x^* = Y_y^* = -1 \qquad X_y^* = 0 \qquad S_x^* = S_y^* = -\frac{1}{2} \tag{3.3}$$

For the values of χ and t we have the expressions from (2.6) and (2.9)

$$\chi = -\nabla^2 w \qquad t = -\frac{1}{2}\nabla^2 F \tag{3.4}$$

Equation (2.14) takes the form

$$\nabla^2\left(t - \frac{\lambda h}{8}\zeta^2\chi\right) = 0 \tag{3.5}$$

Neglecting the harmonic function, we obtain a class of exact solutions

$$t = \frac{\lambda h}{8}\zeta^2\chi$$

as a result of which the value of φ in (2.15) is expressed in terms of ζ, and from (1.2) we find

$$\zeta = \frac{4}{3\lambda}\left(1 - \sqrt{1 - \frac{3\lambda}{4}}\right) = \text{const.} \tag{3.7}$$

The fundamental differential equation of stability (2.13) is now linear with constant coefficients and has the simple form

$$\left[1 - \frac{3\lambda}{4}\zeta^2(3 - 2\zeta)\right]\nabla^4 w + \frac{h\sigma_1}{D}\nabla^2 w = 0 \tag{3.8}$$

Its solution has been much studied for different shapes of plates and for different boundary conditions, although in connection with the elastic stability of compressed plates.

The value of ζ (3.7) is little different from the approximation (3.1), and characterizes the degree of deviation of the exact solution from the approximate.

In the general case we have from (3.5)

$$t = \frac{\lambda h}{8}\left(\zeta^2 \chi + \Gamma_1\right) \tag{3.9}$$

where Γ_1 is an arbitrary harmonic function. For continuous circular plates, for example, Γ_1 is a constant. According to (2.15) and (1.21) we now have an expression for ζ in terms of χ

$$\zeta = \frac{4}{3\lambda}\left(1 - \sqrt{1 - \frac{3\lambda}{4} + \frac{3\lambda^2 \Gamma_1}{16\chi}}\right) \tag{3.10}$$

following which equation (2.13), having in the given case the form

$$\nabla^2\left[1 - \frac{3\lambda}{4}\zeta^2(3 - 2\zeta)\right]\chi + \frac{h\sigma_i}{D}\chi = 0 \tag{3.11}$$

has only one unknown function χ. By use of relations (3.4) it may be integrated once

$$\left[1 - \frac{3\lambda}{4}\zeta^2(3 - 2\zeta)\right]\nabla^2 w + \frac{h\sigma_i}{D} w = \Gamma_2 \tag{3.12}$$

where Γ_2 is a new harmonic function, also a constant for continuous circular plates, insofar as w and $\nabla^2 w$ must be finite in the middle. Equation (3.12), in view of (3.10), may be solved for $\nabla^2 w$, after which the problem reduces to the integration of only one linear partial differential equation of the second order (for circular plates)

$$\nabla^2 w = \Phi(w, \Gamma_1, \Gamma_2)$$

The stress function F is now determined, in accordance with (3.9) and (3.4), from the Poisson differential equation

$$\nabla^2 F = -\frac{\lambda h}{4}\left(\zeta^2 \chi + \Gamma_1\right) \tag{3.13}$$

As we see, the problem of the stability of circular plates may be solved in comparatively simple fashion through to the end. The details of a similar calculation will be clarified below for the example of a rectangular plate compressed in one direction.

b. Stability of a Rectangular Plate Under the Condition of Plane Strain (Fig. 5)

Such a case occurs if, when a rectangular plate of length l is compressed in the x-direction, the width b in the y-direction cannot change as a result of walls along the boundaries $y = 0$ and $y = b$. The plane $x = 0$ shown in figure 5, where $C = 2c$ and $L = 2l$, will evidently be a plane of symmetry of strains.

We assume the buckling to result in a cylindrical shape. In such a case, according to the conditions of the problem, we have for the stresses before instability

$$\left.\begin{aligned} &X_x = -p \qquad Y_y = \frac{1}{2} X_x \qquad X_y = 0 \qquad \sigma_i = \frac{\sqrt{3}}{2} p \\ &X_x{}^* = -\frac{2}{\sqrt{3}} \qquad Y_y{}^* = -\frac{1}{\sqrt{3}} \qquad S_x{}^* = -\frac{\sqrt{3}}{2} \qquad S_y{}^* = 0 \end{aligned}\right\} \tag{3.14}$$

After buckling, $w = w(x)$, $\epsilon_2 = \epsilon_3 = 0$.

From equations (1.24) we have

$$\delta S = 0 \qquad \delta T_2 = \frac{1}{2}\,\delta T_1$$

Since, in accordance with the equations of equilibrium, $\delta T_1 = \text{const.}$, and $\delta T_1 = 0$ from the condition at the edge $x = \frac{l}{2}$, then we have the case $\delta T_1 = \delta T_2 = \delta S = 0$. In consequence, the approximate solution, as was noted at the beginning of 3, here becomes exact.

The thickness ratio ζ for the plastic layer is a constant and is determined by formula (3.1). The stability equation (2.13) takes the form

$$\frac{d^4w}{dx^4} + \frac{hp}{D\left[1 - \lambda\zeta^2(3 - 2\zeta)\right]} \frac{d^2w}{dx^2} = 0 \tag{3.15}$$

If the relative Kármán modulus, expressed by

$$K = \frac{4\frac{d\sigma_i}{de_i}}{\left(\sqrt{E} + \sqrt{\frac{d\sigma_i}{de_i}}\right)^2} = \frac{4(1 - \lambda)}{(1 + \sqrt{1 - \lambda})^2} \tag{3.16}$$

is introduced, then we get from (3.1)

$$\lambda = \frac{4(1 - \sqrt{k})}{(2 - \sqrt{k})^2}, \quad \zeta = 1 - \frac{\sqrt{k}}{2} \tag{3.17}$$

following which we may simplify the expression for the parameter in equation (3.15)

$$\gamma^2 = \frac{hp}{D\left[1 - \lambda\zeta^2(3 - 2\zeta)\right]} = \frac{hp}{Dk} \tag{3.18}$$

Since $k = 1$ up to the elastic limit, and $k = 0$ in a small area where there is flow of the material, and since the characteristic value of the parameter γ must be the same in elastic and in plastic problems, then it follows from (3.18) that the critical stress corresponding to the small area of flow, is zero.

It is interesting to note that the Kármán problem may be considered as a limiting case of the stability of a rectangular plate compressed in one direction, of small width b, for which the parameter γ will have the expression

$$\gamma^2 = \frac{4hp}{3Dk}$$

and consequently the critical stress is zero at the small area of flow. As seen from the preceding and following examples of exact solutions, the total loss of load-carrying ability of a plate, predicted in the Kármán problem, does not occur, generally speaking. This circumstance has already been noted (reference 1).

c. The Stability of a Rectangular Plate Compressed In One Direction (Fig. 5)

We shall suppose that the rectangular plate, sufficiently long in the y-direction and compressed only in the x-direction, buckles into a cylindrical shape. In this case

$$\left.\begin{aligned} X_x = -\sigma_1 \qquad Y_y = X_y = 0 \\ S_x{}^* = -1 \qquad S_y{}^* = \frac{1}{2} \qquad X_x{}^* = -1 \qquad Y_y{}^* = X_y{}^* = 0 \end{aligned}\right\} \tag{3.19}$$

By the conditions of the problem, all sections of the plate $y = \text{const.}$ remain plane after buckling and so we have

$$\epsilon_3 = 0 \qquad \epsilon_2 = \text{const.} \tag{3.20}$$

on the basis of which from (1.24), $\delta S = 0$. Besides this, $\delta T_1 = 0$ from the boundary condition at the edges $x = \pm \frac{l}{2}$ and consequently it follows from (2.3) that $\delta T_1 = 0$ everywhere.

Since there are no forces in the y-direction, we must use the condition

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} \delta T_2 \, dx = 0 \tag{3.21}$$

From the second equation of the system (2.14) we have

$$\frac{d^2F}{dx^2} \equiv \frac{\delta T_2}{Eh} = \epsilon_2 - \frac{\gamma h}{4} \zeta^2 \chi_1 \tag{3.22}$$

since $\chi = -\chi_1$. It is not difficult to convince one's self that (3.22) is the integral of equation (2.14). The function φ, by which is found the value of ζ from (1.21), here has the form

$$\varphi = -\frac{\lambda \delta T_2}{(1-\lambda)Eh^2\chi_1} = -\frac{\lambda \epsilon_2}{(1-\lambda)h\chi_1} + \frac{\lambda^2 \zeta^2}{4(1-\lambda)} \tag{3.23}$$

The bending moment in any section is

$$\delta M_1 = -D\left[1 - \frac{3}{4}\lambda\zeta^2(3-2\zeta)\right]\chi_1 \tag{3.24}$$

and so the boundary condition on the edges $x = \pm \frac{l}{2}$ is $\chi_1 = 0$.

It is clear from (3.23) that χ_1 cannot be zero in the elasto-plastic region since $\epsilon_2 \neq 0$ (this follows from the constancy of sign of $\zeta^2\chi_1$, positive along the entire plate, necessitating $\epsilon_2 \neq 0$ to satisfy condition (3.22)). Thus the elasto-plastic region does not go up to the edges of the plate and stops at the section $x = \pm \frac{c}{2}$. The region adjoining this to the edge will be purely plastic. Indeed, since $\zeta^2\chi$ is positive, then ϵ_2 is also positive. It follows from (2.7) that in the purely plastic and in the purely elastic regions the force δT_2 has the same sign as ϵ_2, that is, is a tensile force. But if, to the plate, compressed beyond the elastic limit in the x-direction, there is applied a tensile force in the y-direction, then the plate remains in the plastic state. One may convince one's self of this by formally calculating the value of δe_i according to (1.8), which at the edges is equal to ϵ_1, but the strain ϵ_1 according to (2.7) is negative, and so the value of δe_i will be positive, that is, plastic strains before buckling remain plastic after buckling.

From (1.21) and (3.23) we now have

$$\frac{h\chi_1}{4\epsilon_2} = \frac{1}{p(\zeta)}, \quad p(\zeta) = -4 + 8\zeta - 3\lambda\zeta^2 \tag{3.25}$$

From this we find the lower limit to the value of $\zeta\,(p > 0)$

$$1 \geqslant \zeta > \frac{4}{3\lambda}\left(1 - \sqrt{1 - \frac{3\lambda}{4}}\right) \tag{3.26}$$

The fundamental differential equation of stability (2.13) takes the form

$$\frac{d^2}{dx^2}\left[1 - \frac{3}{4}\lambda\zeta^2(3 - 2\zeta)\right] X_1 + \frac{h\sigma_i}{D} X_1 = 0 \tag{3.27}$$

By introduction of the notation

$$Q(\zeta) = 4 - 9\lambda\zeta^2 + 6\lambda\zeta^3 \qquad \xi = \frac{2x}{l} \qquad \alpha = \frac{c}{l} \tag{3.28}$$

we write equation (3.27) in the form

$$\frac{d^2}{d\xi^2}\frac{Q}{P} + \frac{\mu^2}{P} = 0 \tag{3.29}$$

where μ is the basic parameter determining the critical stress

$$\mu^2 = \frac{hl^2\sigma_i}{D} \tag{3.30}$$

The integral of equation (3.29) may be obtained by quadratures. Through introduction of the notation

$$R(\zeta) = 2\,\frac{4 - 12\zeta + 12\zeta^2 - 3\lambda\zeta^3}{(4 - 8\zeta + 3\lambda\zeta^2)^2} \tag{3.31}$$

we obtain as a result

$$\frac{1}{2}\left(\frac{d\zeta}{d\xi}\right)^2\left(\frac{Q}{P}\right)^2 = \mu^2 C_1 - \mu^2 R, \quad \sqrt{2\mu}\;\xi = C_2 - \int \frac{P\,dR}{\sqrt{C_1 - R}} \tag{3.32}$$

In the purely plastic region we have for the force δT_2 and the moment δM_1, in agreement with the results of 2 and (3.19):

$$\frac{\lambda T_2}{Eh} = \frac{4(1 - \lambda)}{4 - 3\lambda} \qquad \delta M_1 = - D\left(1 - \frac{3\lambda}{4}\right) \chi_1 \tag{3.33}$$

The fundamental differential equation takes the form

$$\frac{d^2 \chi_1}{d\xi^2} + \frac{\mu^2}{4 - 3\lambda} \chi_1 = 0 \tag{3.34}$$

The solution, satisfying the condition $\chi_1 = 0$ at the end $\xi = 1$, is written in the form

$$\chi_1 = C_3 \epsilon_2 \sin \frac{\mu(1 - \xi)}{\sqrt{4 - 3\lambda}} \tag{3.35}$$

in which as a result of symmetry we consider only deflections in the right half of the plate $(x \geqslant 0)$.

For determination of the five undetermined constants namely, the three integration constants C_1, C_2, C_3, the boundary coordinate α and the critical number μ, we may, besides equation (3.21), write four more conditions: Conditions of symmetry

$$\xi = 0 \qquad \frac{d\zeta}{d\xi} = 0 \tag{3.36}$$

conditions at the boundary region

$$\xi = \alpha \qquad \zeta = 1 \tag{3.37}$$

two continuity conditions, of moment and shear force, which in accordance with (3.25) and (3.35) take the form

$$C_3 \sin \frac{\mu(1 - \alpha)}{\sqrt{4 - 3\lambda}} = \frac{4}{h(4 - 3\lambda)}, \quad \frac{\mu C_3}{\sqrt{4 - 3\lambda}} \cos \frac{\mu(1 - \alpha)}{\sqrt{4 - 3\lambda}} = \frac{4P'(1)}{hP^2(1)} \left(\frac{d\zeta}{d\xi}\right)_\alpha \tag{3.38}$$

The constant ϵ_2 is not necessary and does not enter the conditions insofar as they are independent of $\frac{\delta T}{\epsilon_2}$ and $\frac{\chi_1}{\epsilon_2}$.

By making use of the prescribed conditions and introducing a new unknown ζ_o, the relative thickness of the plastic layer at $x = 0$, we get for the values of μ and $1 - \alpha$ (the relative length of the purely plastic part) the following formulae

$$\mu = \frac{\lambda M}{\sqrt{2}(1 - \lambda)}, \quad 1 - \alpha = \frac{4 - 3\lambda}{\lambda} \frac{L}{M} \tag{3.39}$$

where L and M are the integrals

$$L = \int_{R(1)}^{R(\zeta_o)} \frac{1 - 2\zeta + \lambda\zeta^2}{\sqrt{R(\zeta_o) - R(\zeta)}} dR, \quad M = \int_{R(1)}^{R(\zeta_o)} \frac{(1 - \zeta)^2 \, dR}{\sqrt{R(\zeta_o) - R(\zeta)}} \tag{3.40}$$

in which the value of ζ_o is determined by the relation

$$\cot^2\left(\frac{\sqrt{4 - 3\lambda}}{\sqrt{2}(1 - \lambda)} L\right) = 2(4 - 3\lambda)\left[R(\zeta_o) - R(1)\right] \tag{3.41}$$

As was already established, the value of $1 - \alpha$ is positive, therefore the integral L must be positive, and for this it is required that $1 - 2\zeta_o + \lambda\zeta^2{}_o > 0$, that is,

$$\zeta_o < \frac{1 - \sqrt{1 - \lambda}}{\lambda} \tag{3.42}$$

By considering the estimate (3.26), which is also reasonable for ζ_o, we see that this quantity is contained within narrow limits and close to the approximate value (3.1). It follows from this that the critical stress will differ only slightly from the approximate value.

d. Approximate Solution of the Problem for a Plate in a Uniform State of Stress Before Buckling

In this case the stress components X_x, Y_y, and X_y and the stress intensity σ_i are constant everywhere; the quantity λ will also be constant, and hence ζ by (3.1).

The x and y axes in a given case may be so chosen that the X_y stress is zero (principal axes of stress). The fundamental stability equation (2.13) takes the form

$$\left[1 - \frac{3}{4}(1 - k)X_x{*}^2\right] \frac{\partial^4 w}{\partial x^4} + 2\left[1 - \frac{3}{4}(1 - k)X_x{*}Y_y{*}\right] \frac{\partial^4 w}{\partial x^2 \partial y^2} + \left[1 - \frac{3}{4}(1 - k)Y_y{*}^2\right] \frac{\partial^4 w}{\partial y^4} = \frac{h\sigma_i}{D}\left[X_x{*} \frac{\partial^2 w}{\partial x^2} + Y_y{*} \frac{\partial^2 w}{\partial y^2}\right] \tag{3.43}$$

in which the generalized Kármán modulus is introduced in accordance with formulae (3.16) and (3.17), since the relation

$$\lambda\zeta^2(3 - 2\zeta) = 1 - k$$

holds.

The coefficients in equation (3.43) are all positive, since the largest value of each of the quantities $X_x{*}$, $Y_y{*}$ is $\frac{2}{\sqrt{3}}$ and $1 \geqslant k \geqslant 0$.

Hence, the problem may be solved as a linear differential equation of the Bryan type with constant coefficients, and in difficulty is little different from the corresponding elastic case.

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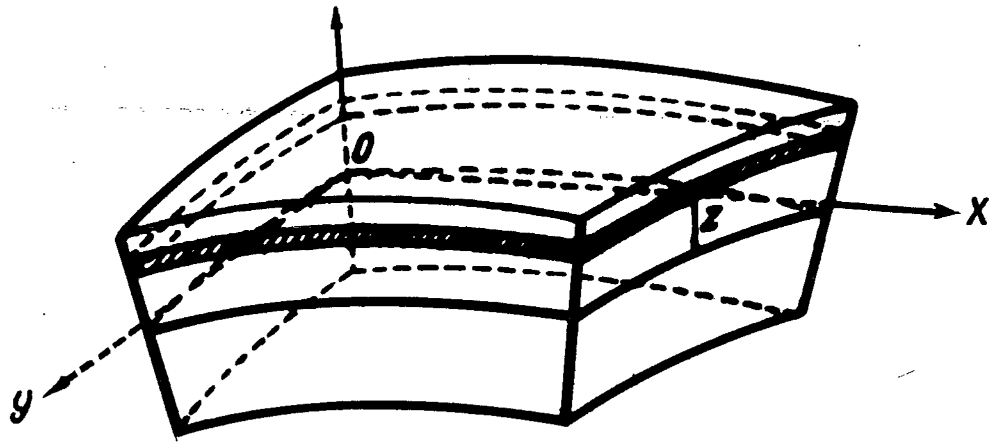

Figure 1.

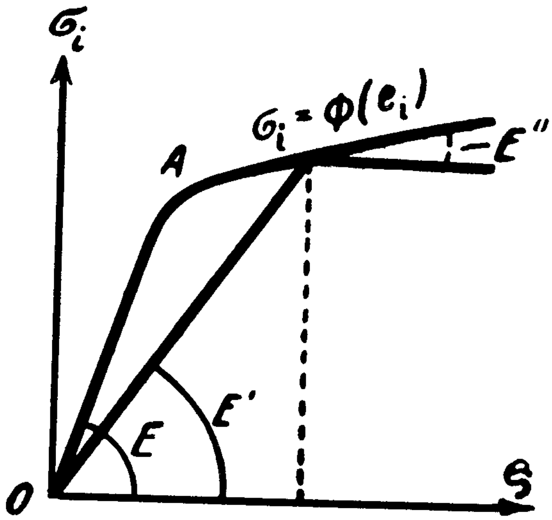

Figure 2.

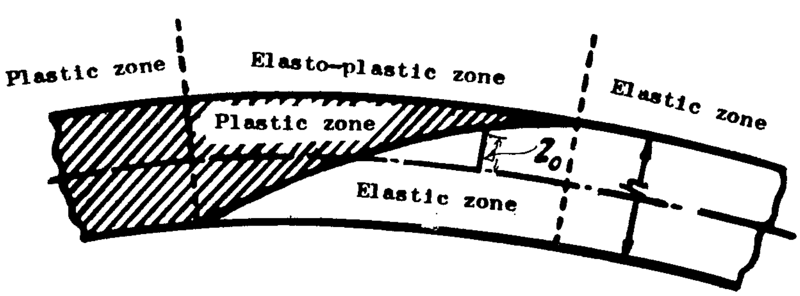

Figure 3.

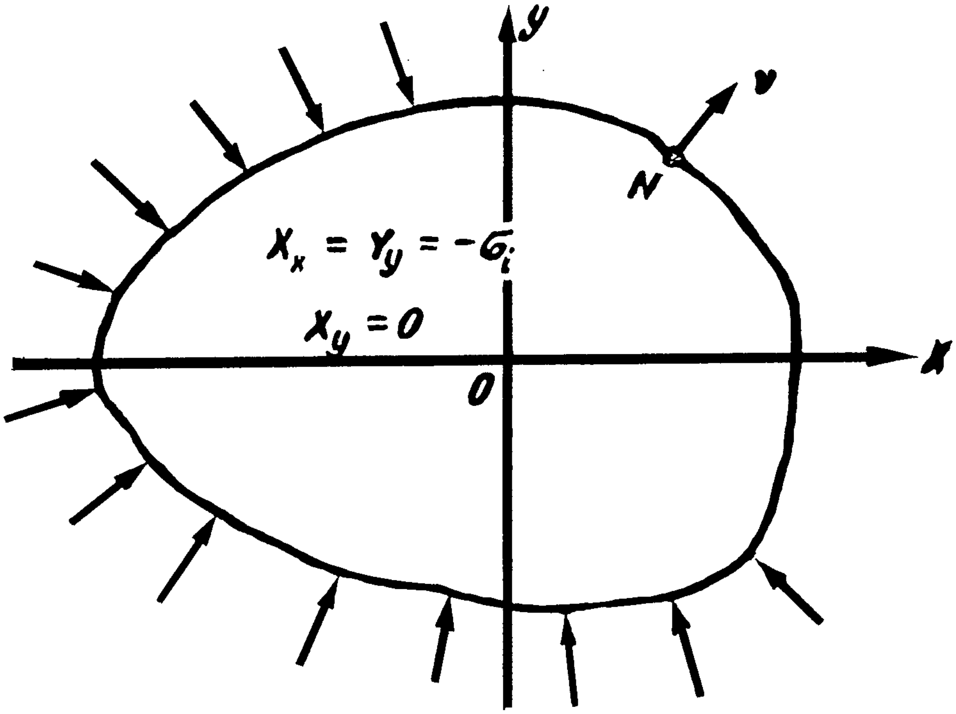

Figure 4.

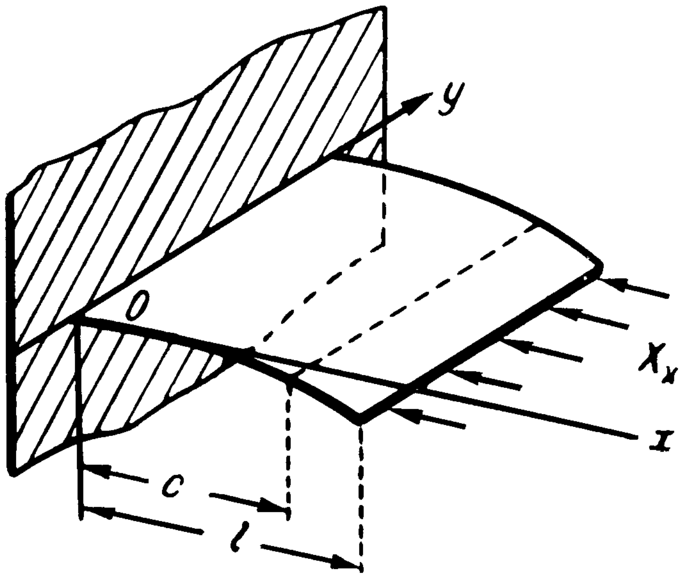

Figure 5.