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As long as the velocities of airplanes were relatively small aerodynamics was a science of steady motion in an incompressible fluid. It was found that neglecting the compressibility of the air and considering only steady motion led to results which, for small velocities of the airplanes, could be applied in practice. The considerable increase, particularly in recent times, of the speeds of airplanes has forced attention on unsteady motion in incompressible and compressible fluids.

The most important effect on the development of nonstationary aerodynamics was that exerted by the phenomenon of flutter which appears in certain cases at large flight speeds. Failure of the airplane structure due to flutter have occurred in all countries. Survivors of flutter catastrophies and casual observers have reported that on increasing the speed of the airplane a vibration of the wing or tail surfaces began after which the wing or tail broke suddenly, the speed and force of the rupture being similar to those of an explosion. The phenomenon of flutter may be readily observed also in a wind tunnel in which is placed a thin wooden flat strip. If through a window in the wall of the tunnel this strip is moved crosswise to the tunnel on increasing the velocity of the flow, the strip at first begins to flutter and then at a certain flow velocity its amplitude of vibration increases so rapidly that the inexperienced or nonalert experimenter rarely succeeds in saving the strip from failure by removing it from the flow in time. The phenomenon of flutter is more energetic than that of resonance. To obtain flutter there is not required any outside vibratory motions for the wing or tail surfaces. The cause of the flutter of the wing or tail surfaces is evidently a purely aerodynamic one but to understand this phenomenon qualitatively may be possible even from a consideration of the motion along a straight line of a material point. Let us assume that the material point with mass m moves along the axis O_x under the action of forces the projection X of which on the O_x axis is equal to

$$X = -mk^2x - m2\mu \frac{dx}{dt}$$

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Then the equation of motion of the material point will be

$$\frac{d^2x}{dt^2} + 2\mu \frac{dx}{dt} + k^2x = 0$$

The integral of this equation, if μ is small, is of the form

$$x = ae^{-\mu t} \sin(\sqrt{k^2 - \mu^2}t + \epsilon)$$

where a and ϵ are arbitrary constants. We see that if the quantity μ is positive the material point will be in harmonic vibratory motion with damping. We shall now assume that μ itself depends on a certain parameter w so that $\mu = \mu(w)$, where for $w = w_{cr}$ $\mu = 0$ and for $w > w_{cr}$ μ is negative, that is, equal to $-\lambda$ (where $\lambda > 0$). Then for $w = w_{cr}$ the above equation assumes the form

$$x = a \sin(kt + \epsilon)$$

that is, we shall have the usual harmonic vibration of a point, but for $w > w_{cr}$ we obtain

$$x = ae^{\lambda t} \sin(\sqrt{k^2 - \lambda^2}t + \epsilon)$$

that is, the amplitude of the vibration begins to increase exponentially. Thus, whereas in the case of resonance the amplitude of vibration has, as is known, the form at , in the case of flutter it has the form $ae^{\lambda t}$. Table I shows the increase of the amplitude with time in resonance and in flutter for different values of the quantity λ .

t	at	$ae^{0,1t}$	$ae^{0,5t}$	ae^t
0	0	a	a	a
1	a	1,105a	1,649a	2,718a
2	2a	1,221a	2,718a	7,389a
3	3a	1,351a	4,482a	20,086a
4	4a	1,492a	7,389a	54,598a
5	5a	1,649a	12,183a	148,413a

We note that whatever the value of the positive number λ the ratio $e^{\lambda t}/t$ approaches infinity with increasing t . For the airplane the parameter w is the velocity of flight where the velocity $w = w_{cr}$, the value at which the coefficient μ becomes zero, is called the critical flutter velocity.

It is not difficult to understand the physical cause of the phenomenon of flutter. Let us assume the wing of the airplane for any reason whether, for example, a gust of air, acquires a very small vibration. The elastic forces evidently are always damping forces which tend to extinguish the vibrations of the wing or tail. The aerodynamic forces however can be both damping forces for certain types of vibrations and forces which assist the vibrations. As long as the velocity of the airplane is small the elastic damping forces are predominant over the aerodynamic forces. With increase in the flight speed the elastic forces remain unchanged in magnitude but the aerodynamic forces increase, as is known proportionally to the square of the velocity of flight. There must therefore be an instant at which the aerodynamic forces, which assist the vibrations, become predominant over the elastic forces and this is the instant at which flutter begins. The problem of the designer is to construct the wing such that the critical velocity of the flutter is higher than the velocities at which the airplane will fly and which is sufficiently strong to withstand the vibratory action of the aerodynamic forces. From this it is clear that even before constructing the airplane it is necessary to be able to determine theoretically the critical velocity of flutter of the airplane only from its design and by changing the design to raise the critical flutter velocity.

It is not difficult with the aid of simple examples to show that the aerodynamic forces due to the deformations of the wing or tail surfaces depend on the magnitude of these deformations and the rate of change of these deformations and this explains the necessity for the presence of terms in x and dx/dt in the differential equation of the motion of the material point, which was taken above as an example to illustrate the phenomenon of flutter. Let us assume first that the wing of the airplane received only a twist deformation so that the angle of attack of the wing increased somewhat. Since the aerodynamic force supporting the wing increases with increase (up to a certain limit) of the angle of attack, as also the torsional aerodynamic moment, we conclude that the additional aerodynamic force obtained on twisting the wing depends on the magnitude of this wing deformation and increases together with this magnitude. We assume next that the wing of the airplane received only a bending deformation occurring at velocity v (fig. 1). As long as there was no bending deformation the angle of attack of the wing was $\alpha = \angle MAB$ where $w = AB$ is the flight speed of the airplane. With the flight velocity $w = AB$ and the velocity of bending deformation $v = AC$ we shall have a total velocity equal to AD and the angle of attack becomes equal to $\angle MAD = \alpha - \Delta\alpha$ where $\tan \Delta\alpha = v/w$ or on account of the smallness of $\Delta\alpha$ approximately $\Delta\alpha = v/w$. To the decrease

of the angle of attack α by the amount $\Delta\alpha = v/w$ corresponds a decrease in the aerodynamic forces depending as, we have seen, on the velocity v of the deformation and not on the magnitude of this deformation. For downward bending of the wing with velocity v the angle of attack α changes into the angle $\alpha + v/w$ as a result of which the aerodynamic forces increase. From what has been said it follows that the aerodynamic forces obtained in the deformation of the wing depend on the velocity v of this deformation and are damping forces so that they oppose the development of the deformation.

In computing the critical flutter velocity it is necessary to deal both with the elastic properties of the wing or tail surfaces and with the magnitudes tending to produce flutter. As regards the elastic forces, experience shows that up to 60 percent of the breaking loads the deformation of the wing is proportional to the force, that is, the wing behaves like a beam. For this reason the matter is simple as regards the elastic forces of the wing and the results of the theory of elastic beams can be applied.

As regards the aerodynamics of the fluttering wing the matter is not however so simple. There is as yet no definite clearness in this field of investigation. It is evident that the aerodynamics of nonsteady flow by the essential nature of the phenomena must in addition to flutter have application in the theory of flight acrobatics, in studying the effects of gusts of air, in the theory of flapping wings and in the flight of birds, in which Leonardo da Vinci had already been interested, etc. As a result of this it can be understood why the great interest which at the present time appears throughout the world in the study of the aerodynamics of nonsteady flow. In Germany, for example, about 10 years ago a special division of the Prandtl Institute was set apart to devote itself entirely to nonstationary aerodynamics, the institute being headed by Küssner. This institute has already issued a series of important investigation results. The American Glen-Martin company applies nonstationary aerodynamics to compute the airfoil flutter. In the closing remarks of one of its publications the company points out how important it is at the present time to direct all efforts toward the development of the aerodynamics of nonsteady flow.

The aerodynamics of steady flow was first applied to the nonsteady motions as for example, to wing flutter, that is, it was assumed that at each instant at every position of the wing the flow about the latter was such as though the wing were in this position an infinitely long time. As it turned out, the application of the aerodynamics of steady flow to nonsteady conditions of motion gives

in a number of cases results that are useful for application, and this has permitted the use of the aerodynamics of steady flow up to the present time. With further investigations, however, it became clear that in a number of other cases the application of the aerodynamics of steady flow to nonsteady motions gives wrong results.

The air forces acting on various types of wing profiles under steady flow conditions were studied by Joukovsky, Kutta, Prandtl, Chaplygin and others in investigations many of which even now are of fundamental importance. The study of the forces acting on wings under nonsteady flow conditions presents for the general case of the nonsteady motion as expressed by Prandtl, "a problem of transcendental difficulty". But it is precisely these particular cases of nonsteady flow which present the greatest interest for aeronautics that have all been found amenable to investigation.

Thus was explained that the motion of an airplane in a circle with constant angular velocity, as approximately occurs in acrobatic flights on entering a dive, proceeds with constant circulation as a result of which there may not be a vortex wake behind the wing. The first one to study this type of motion was S. A. Chaplygin in 1926 and after him independently Glauert in 1929, completing his investigations with the addition of a vortex wake behind the wing. On the other hand in the case of the phenomenon of flutter of the wings and tail surfaces of an airplane it was found sufficient to investigate the phenomenon only for small vibrations and this enables the investigation of problems with variable circulation.

The first investigator who laid the foundation for the present day study of the aerodynamic forces acting on a vibrating wing was Birnbaum (1924), a student of Prandtl. He introduced the important concepts of free and bound vortices and vortex wake. Kussner developed the initial ideas of Birnbaum and thus advanced the problem of the airfoil in nonsteady motion. Beginning in 1935 there appeared a number of Russian papers the authors of which were mainly Keldish, Sedov, and Lavrentyev who indicated ways in which we can apply the theory of the functions of a complex variable to the problems of the nonstationary dynamics of an airfoil. In 1941 N. E. Kochin gave a strict solution of the problem of the vibrations of a wing of circular plan form, making only the assumption of linearity of the equations. The character of the solution in closed form as given by Kochin appears the only one at present for a wing of finite span. In 1938 Karman and Sears in the United States concerned themselves with problems of nonsteady flow and gave a new method of approach to the problem, thereby founding the American school of investigators. In following this school we study as it were the instantaneous photograph of the aerodynamic phenomena as they occur. The methods of the

German school on the contrary deal as it were with the kinematographic picture of the phenomena since they deal with the development of the phenomena in time. In 1938 Cicala in Italy made the first attempt to extend the theory of the lifting vortex of Prandtl to the case of a wing of finite span in a nonsteady flow. Recently Kussner, starting from the method of acceleration potential introduced by Prandtl returned to the problem of Cicala. In the same year there appeared in Italy the work of Possio who attempted to solve the problem of the nonsteady aerodynamics of a profile for the case of a compressible fluid. In his investigations Possio makes use of the acceleration potential of Prandtl. Thus at the present time there are three independent schools occupying themselves with the problems of the aerodynamics of unsteady flow: the American, Russian, and German.

As in the usual aerodynamics of steady motion the problem of the nonsteady motion of the infinite span airfoil is most advanced while there hardly exists any complete theory of the unsteady motion of a wing of finite span of arbitrary plan form even for the incompressible fluid although for a wing of circular plan form there exists the completed solution of Kochin.

By the nonsteady motion of a wing there is understood in all these theories the following. It is assumed that the wing has a certain finite velocity w constant in magnitude and direction; in addition to the velocity w the wing possesses, as an absolutely rigid body, arbitrary infinitely small deviations from this velocity. Finally, the wing need not be absolutely rigid but may have infinitely small deformations varying with time.

In order to render the problems under consideration linear it is assumed that the wings are infinitely thin, the profile in the absence of deformations coinciding with a linear segment. The angles of attack of the wing are assumed infinitesimal, in the fundamental equations or in the boundary conditions there being kept only the terms of first order smallness. Only with these restrictions imposed does the problem of the unsteady motion of a wing become solvable at the present time in general form. What is required to find in the solution of these problems is the magnitude of the forces and moments which act on the wing in its unsteady motion.

It was found by Joukovsky that the existence of a lift force on a profile is explained by the presence around the profile of circulation on the magnitude of which the lifting force depends. If this profile moves with constant velocity the circulation is constant

but as a result of the nonuniformity of the motion of the profile the circulation around the profile must be variable since the lift force for nonuniform motion of the profile varies. Since the total circulation by the laws of hydrodynamics cannot change there must appear behind the profile in nonuniform motion a vortex wake consisting of the continuously shed vortices. As has been said, Birnbaum was the first to turn attention to the existence of the vortex wake. The vortices forming the wake are not displaced behind the moving profile but remain motionless in the fluid at rest, if we neglect their small displacement at right angles to the direction of the velocity of the airfoil.

In all these theories there are considered principally periodic motions since it can be shown that an aperiodic phenomenon may be represented as the sum of an infinite number of infinitely small periodic phenomena with different frequencies. Let ω be the angular frequency of vibration of the airfoil, c half the profile chord and w the forward velocity of motion of the airfoil. The nondimensional number $k = \omega c/w$ is called the Strouhal number which is constantly applied in these theories.

It is first of all necessary to solve the problem to what extent the rectilinear vortex wake can be considered stable since all theories assume its rectilinearity. The investigation of this problem is rather a delicate one but in any case it may be carried out for a single particular case where the profile is not absolutely rigid. In this case it is found that if the vertical velocity on the profile is small the amplitudes of the vibrations of the vortex wake about its rectilinear form will likewise be small, the value of the velocity of the wake perpendicular to its length being propagated as a wave along the wake with the velocity $w = \omega c/k$ and on the vortex wake there will be waves with these velocities and frequency ω .

According to the views of the American school there exists at each instant on the profile a quasi-stationary vortex intensity γ_0 which is the vortex intensity on the profile if at a given instant all the vortex wake were rolled into a point at infinity. Since for each instant of time there will be a value of this intensity although in all cases we imagine the vortex wake rolled up into a point at infinity the intensity is called a quasi-stationary vortex intensity in contrast to the stationary vortex intensity which occurs in the absence of a vortex wake and whose magnitude is constant in time. Since the vortex wake at all times exists directly behind the trailing edge of the moving airfoil up to infinity, the total vortex intensity on the airfoil is equal to $\gamma_0 + \gamma_1$ where γ_1 is due to

the vortex wake. Karman showed that knowing the functions γ_0 and γ_1 there may be found the vortex intensity in the wake. From this it is possible to find the momentum and angular momentum of a fluid due to the entire system of vortices on the profile and in the vortex wake. Knowing the momentum and angular momentum of the fluid surrounding the profile it is possible to find by the theory of Euler the pressure force L and the moment M of the pressure forces on the airfoil. Karman showed that the force L and the moment M may be represented as the sum of three components:

$$L = L_0 + L_1 + L_2, \quad M = M_0 + M_1 + M_2$$

The force L_0 is the quasi-stationary force of Joukovsky to which corresponds the moment M_0 . The magnitudes L_0 and M_0 may exist alone only in the limiting case when the airfoil moves without acceleration and there is no vortex wake behind the airfoil. The force L_1 and moment M_1 are due to the associated mass of the profile. To obtain them it is necessary to assume that the profile moves nonuniformly but without the presence of circulation about it so that there is no vortex wake behind the profile. Finally, the force L_2 and the moment M_2 are due to the vortex wake, where knowing the expressions for these it is not difficult to prove the following theorem due to Glauert: The resultant of the pressure forces on an infinitely thin straight profile that is due to the vortex wake is applied at the forward focus of the profile, that is, at one quarter chord from the leading edge.

In the theory of Karman and in the other theories there is used the function $C(k)$ of the Strouhal number k introduced by the American investigator Theodorsen. The function $C(k)$ of Theodorsen is of the form

$$C(k) = F(k) + iG(k) = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + iH_0^{(2)}(k)}$$

where the functions $H_0^{(2)}(k)$ and $H_1^{(2)}(k)$ are the Hankel functions satisfying the equation of Bessel and having the form

$$H_0^{(2)}(k) = \frac{2i}{\pi} \int_1^{\infty} \frac{e^{-ik\tau}}{\sqrt{\tau^2 - 1}} d\tau, \quad H_1^{(2)}(k) = -\frac{2}{\pi} \int_1^{\infty} \frac{k\tau e^{-ik\tau}}{\sqrt{\tau^2 - 1}} d\tau$$

A table of functions of Theodorsen for the real values of the Strouhal number k has been set up where $C(0) = 1$. Although there have been objections against dividing the force L and moment M in which it was pointed out that this splitting up was artificial these objections do not appear to us as well founded since such a splitting up appears also possible in the work of the German and Russian school and has moreover an entirely concrete physical basis.

Proceeding to the consideration of the Russian school on the theory of the airfoil in a nonsteady flow it is first of all necessary to dwell on the introduction of functions of a complex variable into this theory. As is known, the description of steady motion of an incompressible fluid by functions of a complex variable is entirely natural and has been productive of important results. Joukovsky and Chaplygin conducted their work on airfoil theory in a steady flow using the theory of functions of a complex variable. Thus the work of the Russian school on airfoil theory in nonsteady flow may be considered as the natural continuation and extension of the work of Joukovsky and Chaplygin. In this field however we have considerable difficulties since the functions describing the unsteady phenomena depend not only on the complex variable $z = x + iy$ but also on a real variable, namely, the time t . It was possible to overcome these rather important difficulties and obtain an extension of the theorem of Joukovsky and Chaplygin on the lift force and the moment of the pressure forces for an arbitrarily deformed airfoil in a non-incompressible fluid for the cases of both fixed and moving coordinate axes. Expanding the derivative the complex potential in powers of the complex variable z in the most general form, assuming that the coefficients of this expansion depend on the time t and introducing a certain auxiliary function $g(z)$ it was possible with the aid of the extended theorems of Joukovsky and Chaplygin to obtain expressions for the forces and moments acting on the airfoil in unsteady motion. The obtained expressions for the force L and moment M as in the theory of the American school were obtained by following the same principle, that is, splitting up into three components:

$$L = L_0 + L_1 + L_2, \quad M = M_0 + M_1 + M_2$$

The expressions for the quantities L_2 and M_2 obtained by the Russian school were found to be identical with the expressions obtained for the analogous quantities by the American school. As regards the quantities L_0 , L_1 and M_0 , M_1 obtained by the Russian and American schools there is however an essential difference. In the formulas of the American school these quantities are expressed

in terms of the quasi-stationary intensity γ_0 whereas in the formulas of the Russian school these components are expressed in terms of the component $v_y(x, t)$ of the velocity of the fluid near the airfoil perpendicular to the forward velocity w of the airfoil. The transformation of the formulas of one school into those of the other has as yet not been effected.

The work of N. E. Kochin on the aerodynamics of a vibrating wing of circular plan form could not of course be carried out in terms of functions of a complex variable since it is not a two-dimensional but a three-dimensional problem. This work appeared as a result of an earlier work by Kochin on the steady motion of such a wing. It is to be noted in these works by Kochin among others that Kochin was able to construct a function possessing definite properties and singularities by which it was possible to express the solution in closed form without making use of series. The form of the circle was taken for the reason that only for it, has it as yet been possible to find the above mentioned special function. All restrictions and assumptions which Kochin introduced in his investigations reduce to those by which the problems are made linear. For this reason the solutions of Kochin both for the steady and nonsteady flow are the most accurate of the existing solutions. A comparison of the results obtained for concrete problems on steady flow by the theory of Kochin and according to general approximate theories shows that the difference may amount to several tens percent of the magnitude to be determined. From this it is clear how important it is to have exact solutions of the type of the solution of Kochin. It is desirable therefore that by the same method it be possible to solve the problem of the elliptic wing which resembles more nearly an actual wing than does the circular wing but at the same time it is impossible to close our eyes to the fact that such a solution presents extremely great difficulties and requires great mathematical aptitude.

Although behind the profile in unsteady motion there should as a general rule exist a vortex wake it may be proven that even for periodic motion of a profile there may be particular cases of its motion where there will exist no vortex wake behind the profile. This will be the case for example when the projection of the velocity at the profile at its rear focus, that is, at one quarter chord from its trailing edge in a direction at right angles to forward velocity w of the profile is equal to zero, and also in the case where the absolutely rigid airfoil moves over a circle with constant angular velocity. In the first case the circulation about the profile must be equal to zero and in the second case it must in general be constant. In 1926 Chaplygin considered a number of profiles with constant

circulation about them in circular motion as absolutely rigid bodies. In 1935 the work of Chaplygin was continued by Sedov by the extension of the range of profiles and by a detailed analysis of the analytic structure of the expressions for the associated mass and associated moment of inertia. As has already been said above, in 1929 Glauert arrived at the same results independently of the work of Chaplygin but then Glauert supplemented his work by considering elliptic profiles with vortex wakes behind them making use of the results already available at that time to take account of the effect of the vortex wake.

For studying the cases of motion of an airfoil with constant circulation about it both Chaplygin and Glauert make use of a moving system of coordinate axes rigidly fixed to the airfoil and obtain the equations of hydrodynamics with the integral of Lagrange determining the pressure of the fluid at any point. From this, after rather complicated computations, it is found possible to find the expressions for the projections on the axes of coordinates of the resultant pressure and the resultant moment of the pressure forces exerted by the fluid on the airfoil. In particular, Chaplygin applied the results obtained by him to the elliptic airfoil and to the rectangular airfoil moving over a circle with constant angular velocity. This problem is an example of unsteady motion differing essentially from rectilinear and not differing only by infinitely small deviations from the latter. Thus, assuming the circulation constant it is found possible to reject the condition of infinitely small disturbances from a uniform rectilinear motion.

We now proceed with the presentation of the results of the German school of investigators. If, around a sufficiently thin rectangular airfoil ($-c, +c$) there exists a constant circulation, the vortex intensity η on the airfoil is constant, that is, independent of time, and by the usual formulas of Helmholtz it is possible to compute the velocity at any point of the fluid due to the vortex intensity distribution on the airfoil. The case will be otherwise however if this intensity η depends on the time t since in the first place there will exist behind the airfoil a vortex wake which will have an effect on the value of the required velocity and in the second place the free vortices forming the vortex wake behind the airfoil will exist on the airfoil itself so that to the intensity η of the vortices on the airfoil must be added the intensity ϵ of the free vortices at the profile. The formation of the free vortices at the airfoil is explained by the fact that the increase $\Delta\eta$ during each infinitesimal element of time of the intensity η on the airfoil must be accompanied by the formation during the same time interval of free vortices with the intensity $-\Delta\eta$. The free vortices forming about each point of the airfoil, which remain fixed in space at the point at which they

are formed, will continuously maintain their intensity until the airfoil passes by them. When the airfoil in its displacement will coincide with them the latter, remaining stationary, will continue the vortex wake behind the airfoil. Thus also for unsteady motion the projections of the velocity at any point of the fluid can be computed by the formulas of Helmholtz but their expressions will consist of three components depending respectively on the vortex intensity η in the wake, the vortex intensity ϵ of the free vortices at the airfoil and the vortex intensity ϵ' of the free vortices in the vortex wake. It is not difficult to find the analytical expressions that connect the functions ϵ and ϵ' with the function η . Hence the projection v_y of the velocity of a fluid on a direction perpendicular to the vector w can be expressed in terms of the function η , the expression consisting of three components containing the function η under the integral sign. Such expression for the projection v_y of the velocity of the fluid was first obtained by Birnbaum. Assuming that the motion of the airfoil and of the fluid are periodic with frequency ω , that is, introducing for the functions depending on the time the exponential factor $e^{i\omega t}$ we can divide all the obtained expressions for the projection v_y by the quantity $e^{i\omega t}$. It is evident that this expression for the projection v_y can be applied to the fluid particles in the immediate neighborhood of the airfoil. Since the projection v_y for the particles at the airfoil can also be determined from the kinematic considerations while the vortex intensity η on the airfoil is an unknown function, the obtained equation may be looked upon as an integral equation for the function η . The latter is called the integral equation of Birnbaum. The solution of this equation is required since the lifting force, that is, the resultant force L of the pressures on the airfoil and the resultant moment M of the pressure forces exerted by the fluid on the airfoil are expressed in terms of the function η . The solution of the integral equation of Birnbaum is obtained in two forms. In the first place there are found the relations by which the coefficients of the series for the required function η are expressed in terms of the coefficients of the series of the given function v_y . In the second place there is obtained a solution of the equation of Birnbaum in closed form, that is, a formula is found expressing the function η in terms of the function v_y .

In order to give an idea of the analytical complexities which are here encountered we present both the equation of Birnbaum and its solution in closed form. We assume that the airfoil $(-c, +c)$ is placed along the Ox axis with its center at the point O . We set

$$x = c \cos \theta \quad \text{or} \quad x = c \cos \mu; \quad \eta = f(\theta) e^{i\omega t}, \quad v_y = w(\theta) e^{i\omega t}$$

The integral equation of Birnbaum is of the form

$$2\pi w(\mu) = \int_0^\pi \frac{f(\theta) \sin \theta d\theta}{\cos \mu - \cos \theta} - ik \left\{ \int_0^\pi \frac{e^{-ik \cos \theta} \sin \theta d\theta}{\cos \mu - \cos \theta} \right. \\ \left. + \int_0^\pi f(\varphi) e^{ik \cos \varphi} \sin \varphi d\varphi + \int_0^\infty \frac{e^{-ik \operatorname{ch} \lambda} \operatorname{sh} \lambda d\lambda}{\cos \mu - \operatorname{ch} \lambda} \right. \\ \left. + \int_0^\pi f(\varphi) e^{ik \cos \varphi} \sin \varphi d\varphi \right\}$$

where k is the Strouhal number, $k = \omega c/w$. The solution of this equation in closed form is

$$f(\theta) = \frac{2}{\pi} \int_0^\pi \left\{ [-\cos \epsilon + C(k) (1 + \cos \epsilon)] \frac{1 - \cos \theta}{\sin \theta} + \right. \\ \left. + \frac{ik}{2} \sin \epsilon \operatorname{lg} \left[\frac{1 - \cos (\epsilon + \theta)}{1 - \cos (\epsilon - \theta)} \right] \right\} w(\epsilon) d\epsilon + \frac{2}{\pi} \\ \int_0^\pi [w(\theta) - w(\epsilon)] \frac{\sin \theta d\epsilon}{\cos \theta - \cos \epsilon}$$

where $C(k)$ is the function of Theodorsen.

Having the expressions for the force L and the moment M in terms of the function η and making use of the solution of the equation of Birnbaum it was found possible to obtain the expressions for the force L and the moment M in terms of the coefficients of the series of the function v_y . The magnitudes L and M can be decomposed into three components following the ideas of the American school. It may be noted that the lift force $L = L_0 + L_1 + L_2$ depends only on the three first coefficients and the moment $M = M_0 + M_1 + M_2$ depends on the coefficients $A_0, A_1, A_2, A_3, A_4, A_5 \dots$ of the expansion of the function v_y in a trigonometric series. In particular, for an absolutely rigid rectangular airfoil in an unsteady motion it was found that the lift force of the fluid is reduced to three definite forces applied at the forward focal point, at the center and at the rear focal point of the airfoil and to a pair of forces with definite moment.

If the airfoil is a rectangular absolutely rigid segment the mathematical analysis in determining the resistance can be brought to completion. It is found that the resistance force is due in this case not only to the formation of the Karman vortex street behind the airfoil but to the formation behind the airfoil of a vortex wake. Whereas the vortex street of Karman always leads to a resistance, the force due to the formation of a vortex wake may be positive as well as negative, that is, a thrust or a drag. In the case of a thrust it is important of course to know the efficiency. The efficiency η_{tr} for the translational motion alone and the efficiency η_r for the rotational motion alone can in any case be determined. In these computations it is necessary to take into account the suction force obtained as a result of the pressure of great reduction in pressure at the leading edge of the rectangular wing. In figure 2 are shown the values of these coefficients η_{tr} (the top curve) and η_r (the lower curve) as functions of the Strouhal number k .

We see that the coefficient η_{tr} which is important for small values of the Strouhal number k drops to half its value on increasing the number k ; the coefficient η_{tr} is always positive for all values of k , that is, for only translational fluctuations the airfoil always possesses a thrust force. The coefficient η_r however increases with increase in the Strouhal number. At $k = 0.954$ approximately the coefficient η_r becomes zero. Thus, for $k < 0.954$ the rotating wing has only a resistance force but for $k > 0.954$ it has a thrust force, that is, the latter is obtained only for rapid rotational motion of the airfoil. Although these conclusions were obtained for an airfoil of infinite span it is useful to compare them with the flight data of birds whose wings perform predominantly translational fluctuations in flight (table 2).

	$S,$ M^2	$L,$ M	$\bar{c},$ M	ωc^{-1}	$v,$ M/CEK	k
Stork	0,500	2,03	0,123	12,6	11	0,134
Pigeon	0,075	0,67	0,045	31,4	25	0,056
Rook	0,130	0,95	0,068	19,0	12	0,108
Seagull	0,126	0,96	0,066	13,9	12	0,076

In the above table S is the area of the wings, L their span, c the mean half chord equal to $S/2L$, ω the angular frequency, v the flight speed, k the Strouhal number. We see

that birds fly at small Strouhal numbers for which their motion has a considerable efficiency. Recently V. Golubev has worked out a method of computing the thrust force of an airfoil in translational oscillatory motion on the basis of the existence of two vortex streets whose origin is at the extreme positions of the translational fluctuations of the airfoil and extending behind the airfoil. The direction of rotation of the vortices in these vortex streets is opposite to the rotation of the Karman vortices. Thus the vortices of Golubev push the fluid behind the airfoil and thereby move the airfoil forward. In this case infinitely small oscillations of the airfoil are not required.

The methods developed by the German school may also be applied to a composite airfoil consisting of an absolutely rigid airfoil, aileron and trimmer. It is then necessary to compute a large number of magnitudes. These computations were carried out twice by two different methods by Klüssner in Germany and Theordorsen in America. From the agreement between the obtained results it follows that these results can be reliably used.

The results obtained for periodic phenomena may, on applying the Fourier integral, be extended to phenomena which are not periodic in time. The mathematical analysis is here rather complicated, a fundamental part being played by the function $k_1(\sigma)$ given by

$$k_1(\sigma) = \frac{1}{2\pi i} \int_{\gamma} C(z) \frac{e^{i\sigma z}}{z} dz$$

where the path of integration consists of the entire real axis with the origin bypassed below over an infinitely small semicircle. In order to give an example of how rapidly these aperiodic phenomena are developed we assume that the velocity v_y at the airfoil at the instant $t = 0$ has received a certain instantaneous increase. Then the force L and the moment M likewise change their values discontinuously but only by half their final value. If the chord of the airfoil is 4 meters and the velocity $w = 100$ meters per second then at the end of 2 seconds the force L is 99 percent of its final increment. The total increase of the force L and the moment M will occur after an infinitely long time, that is, to attain the remaining 1 percent an infinite time is required.

In the same as the phenomena which are aperiodic with respect to time we can consider the phenomena which are aperiodic with respect to space. It is possible for example to consider the case where the airfoil during its motion enters a rising current of air. We can obtain the solution by considering the system of infinitely

many solutions with different Strouhal numbers. In this theory the function

$$k_3(\sigma) = \frac{1}{2\pi i} \int_{\gamma \rightarrow} [J_0(z) - iJ_1(z)] C(z) \frac{e^{iz\sigma}}{z} dz$$

where $J_0(z)$ and $J_1(z)$ are Bessel functions and $C(z)$ the Theodorsen function, plays an important part. In order to obtain an idea as to how rapidly the airfoil reacts to the effect of the rising air current we may point out that after the airfoil has moved a distance equal to fifty times its chord in the rising current the increase in the lift force attains 99 percent of its total value.

It has been pointed out above that the extension of the problem of the unsteady motion of an airfoil to the case of an incompressible fluid was made by the Italian Possio. In addition to general restrictions by which the problems are rendered linear Possio introduced into the problem a further restriction namely that the velocity of sound a in the gas is always a constant quantity whereas for adiabatic processes the velocity of sound depends on the square of the velocity of the gas and a constant value for it is only approximate. To solve the problem Possio makes use of the potential acceleration of Prandtl, the partial derivatives with respect to the coordinates of which are equal to the corresponding projections of the accelerations. It may be shown that for the assumed restrictions the acceleration potential satisfies the same second order partial differential equation as the velocity potential. At infinity the acceleration potential must be equal to zero and on the airfoil have a certain discontinuity. From this it follows that the acceleration potential may be represented as a potential of dipoles distributed along the airfoil with axes perpendicular to the latter. We may observe that this dipole does not in its dimensions correspond to an actual dipole since the word is here used to denote a particular solution of the equation for the acceleration potential. The analytical expression for the latter will contain under the integral sign the intensity of these dipoles. From this analytical expression it is possible to find the accelerations by simple differentiation and then the velocities of the compressible fluid, the transition from accelerations to velocities, under the assumptions and restrictions made, being effected through the integration of the linear equations. We can then find the projection of the velocity of the gas on the normal to the vector w in the immediate neighborhood of the airfoil and the expression for this projection will contain under the integral sign the unknown intensity of the dipoles. On the other hand this projection may be determined directly on the basis of kinematic

considerations. Equating the two expressions we obtain an integral equation for determining the unknown dipole intensity. Since the lifting force L and total moment M are expressed in terms of this intensity the solution of this integral equation furnishes the solution of the problem. Possio succeeded in obtaining this integral equation but neither he nor anyone else has succeeded as yet in solving this equation. The equation has the following form

$$v_y(x) = - \frac{\sqrt{1-\mu^2}}{2\pi} \int_{-c}^{+c} q(x') e^{-p(x'-x)} dx' \int_{-\infty}^{x-x'} \frac{i p \eta}{e^{1-\mu^2}} \rightarrow$$

$$\rightarrow \left[\frac{\pi}{2i} H_1(2) \left(\frac{\mu p |\eta|}{1-\mu^2} \right) \frac{\mu p |\eta|}{1-\mu^2} \right] \frac{d\eta}{\eta^2}$$

where μ is the Mach number equal to w/a , $p = \omega/w$, the function $H_1(2)$ is a Hankel function and the symbol $|\eta|$ represents the absolute value of η . From this equation it is required to determine the intensity $q(x')$ of the dipoles on the airfoil for given value of $v_y(x)$ of the projections of the velocity on the airfoil. From the mere appearance of the equation it is clear how difficult this problem must be.

The theory of the wing of finite span in an unsteady flow was treated by two different methods, the method of Küssner, using the acceleration potential and the method of Cicala introducing a special system of vortices consisting of one infinite rectilinear lifting vortex and three systems of infinitely many rectilinear and rectangular vortices. After complicated explanations, Cicala, for determining the circulation $\Gamma(y_1)$ at any perpendicular section y_1 of the wing, arrives at the following integro-differential equation

$$v_z(y_1) = \frac{\mu(k) \Gamma(y_1)}{2\pi\sigma} + \frac{1}{\pi} \int_{-s}^{+s} \frac{d\Gamma}{dy} S(p|y - y_1|) \frac{dy}{y_1 - y}$$

where

$$S(\sigma) = -\sigma \int_0^{\infty} \frac{dt}{t} \frac{d}{dt} \int_0^{\infty} \frac{e^{-itu}}{\sqrt{u^2 + 1}} du$$

$2s$ being the span of the wing, $2c$ its chord, $k = \omega c/w$, the Strouhal number and $v_z(y_1)$ the component of the velocity of the fluid normal to the wing at the section y_1 , the component by

assumption being constant along the entire section y_1 . Küssner, applying the acceleration potential of Prandtl also arrived at an integral equation which resembles that of Cicala. If p equals 0, that is, $\mu(k) = 1$, $S(p|y - y_1|) = 1$ and both the equations of Cicala and Küssner become the equation of Prandtl which holds for the case of the steady motion of the wing. No solutions of the equations of Cicala and Küssner have as yet been found although to obtain these forms of the equations the authors introduced into the analysis many simplifying assumptions without which the equations would have been still more complicated. In Russia a paper on the theory of the wing of finite span in an unsteady flow has been published by M. Keldish but the latter's paper contains the results of several stages of his work rather than a detailed presentation of his analysis.

We see that notwithstanding the rather large number of papers devoted to the theory of the unsteady motion of an airfoil it must be confessed that in this domain the phenomena are still not completely investigated. In addition to continuing purely theoretical investigations, strictly scientific experiments are required both in order to explain the physical side of the phenomena and for the continued comparison of the results of theory and experiment. At the present time it is still far from evident that the physical picture which is assumed as the basis of the theory of the phenomena describes the latter completely. In particular, for example, it is entirely not clear how this physical picture must be changed for oscillations with finite amplitudes. Scientific experiments should aid in dealing with these problems. It is desirable also to carry out a comparison of as many as possible different applications of stationary and nonstationary aerodynamics to the same phenomena. The theory of the unsteady motion of an airfoil is a field in which many scientific conquests yet remain to be made.

Translation by S. Reiss,
National Advisory Committee
for Aeronautics.

Table 1

t	at	$ac^{0.1t}$	$ac^{0.5t}$	ac^t
0	0	a	a	a
1	a	1.105 a	1.649 a	2.718 a
2	2 a	1.221 a	2.718 a	7.389 a
3	3 a	1.351 a	4.482 a	20.086 a
4	4 a	1.492 a	7.389 a	54.598 a
5	5 a	1.649 a	12.183 a	148.413 a

Table 2

	$S,$ M ²	$L,$ M	$\bar{c},$ M	ac^{-1}	$v,$ M/SEC	k
stork	0.500	2.03	0.123	12.6	11	0.134
pigeon	0.075	0.67	0.045	31.4	25	0.056
rook	0.130	0.95	0.068	19.0	12	0.108
seagull	0.126	0.96	0.066	13.9	12	0.076

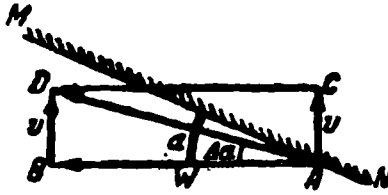


Fig. 1

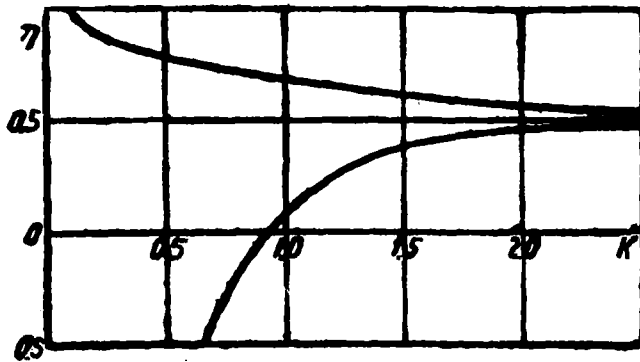


Fig. 2

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