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TECHNICAL NOTE

D-772

TRAJECTORY CONTROL IN RENDEZVOUS PROBLEMS USING PROPORTIONAL NAVIGATION

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON April 1961

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588284
46p.

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USING PROPORTIONAL NAVIGATION

By Luigi S. Cicolani

SUMMARY

The rendezvous problem is defined by the end conditions that the position and velocity of a vehicle and its target are to be matched. In its present form proportional navigation theory allows the interception of a target by the vehicle, that is, the matching of positions. This report extends the theory to include the full rendezvous end conditions. Trajectory constraint equations are derived and the method of computing the required thrust program in idealized problems is outlined. The thrust program obtained results entirely from the trajectory and does not consider the dynamics of the vehicles or errors inherent in a real system. The properties of the thrust program, such as its variation both in magnitude and direction, are examined. An acceleration forcing function is also derived and its properties examined. The theory is applied to the satellite rendezvous problem as an example and some computations are presented.

INTRODUCTION

A rendezvous between a maneuverable vehicle and its target requires that the position and velocity of the vehicle be matched simultaneously with those of the target. Such a maneuver is potentially useful in several commonly mentioned astronautic operations, for example, satellite rendezvous and planetary landings. There are also many ordinary instances of maneuvers requiring the rendezvous end conditions, for example, airplane landings, vehicle parking, refueling operations, etc.

The control of both position and velocity to effect the rendezvous implies complex terminal maneuvering for which the utility of a terminal guidance system located in either the vehicle or target is evident. A number of terminal guidance systems, both manual (e.g., ref. 1 and references therein) and automatic (refs. 2, 3, 4), have already been proposed in connection with the satellite rendezvous problem.

Reference 3 views the problem as one of orbital transfer and the dynamic equations of motion are solved to find the single impulse transfer which intercepts the target satellite. At interception a second impulse reduces the relative velocity to zero. This work has been amplified in references 5, 6, and elsewhere, but is essentially restricted to the special case in which the target is in a circular orbit. The effect of gravity has been linearized in this work so that the solution is in error if long transfer times are used, and in practice a number of additional corrective impulses will be required.

Reference 4 proposes a forcing function for a continuous thrust automatic feedback system. This type of solution for the rendezvous problem has also been followed by others (ref. 7 and elsewhere). A number of cases have been integrated to show that the rendezvous does occur. But it is not clear that a rendezvous is always produced or what effect the external forces have on the operation of such a system, nor is it clear what limits must be imposed on the sensitivities of the forcing functions. This type of solution suffers from the analytical difficulty that all information on the nature of the solution can be obtained only by numerical integration of the dynamic equations of motion over the whole range of cases of interest in each rendezvous problem. Once it becomes necessary to integrate the dynamic equations in this problem, the analysis will generally yield nothing further except by an extensive computational program.

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There is, however, an alternative approach to the rendezvous problem which circumvents this analytical difficulty. It should be recognized at this point that the general rendezvous requirements are simply end conditions on the kinematics and that any practicable maneuver which achieves these end conditions is of possible use. The problem must therefore be capable of a general kinematic solution and the complete nature of the solution should then yield to analysis. Therefore, the line of investigation taken in this report is to consider first the kinematics and to seek a trajectory control technique which, in general, produces a rendezvous, leaving until later the matter of forces required to constrain the trajectory in the prescribed manner.

The automatic trajectory control technique of proportional navigation has been applied successfully to aeronautical interception problems, that is, problems in which only the positions of a vehicle and its target are to be matched. Now the proportional navigation control equation implies an interception but does not of itself specify the type of interception any further. Thus, the homing missile maneuver of aeronautics is a special class of proportional navigation interceptions having the additional constraint that the relative velocity remain nearly constant during the maneuver. Such interceptions may also be used in rendezvous problems, provided a terminal burst of retrothrust is applied to reduce the relative velocity to zero. But the point here is that a rendezvous is an interception having the constraint that the relative velocity be zero at the time of interception, and that there may be a general class of proportional navigation interceptions having this constraint. Specifically, the objective of this work is to find the class of proportional navigation interception trajectories in which the velocities of the maneuverable vehicle and the target are matched at the point of interception. The key to finding this class is to replace the constraint of constant relative velocity used in homing missile work with the velocity end condition of rendezvous.

Proportional navigation is, of course, not the only trajectory control technique available, but the extensive experience with homing missile systems may be of direct benefit in rendezvous guidance systems.

The use of proportional navigation in satellite rendezvous operations was originally suggested by Wrigley in reference 2 where the basic homing missile technique was applied directly to rendezvous. This involved, therefore, the assumption of constant relative velocity and also of small lead angle. Both assumptions will be eliminated; the first since, as explained above, we seek a more general class of rendezvous maneuvers, and the second because it is necessary that the initial conditions be unrestricted if the solution is to be of practical value.

The properties of the thrust program required for the vehicle to perform the maneuvers are also investigated. The method for computing the thrust program is outlined. Once the initial conditions are established, the entire desired rendezvous trajectory may be computed and this information utilized in the dynamic equations of motion to compute the required thrust. The equation for thrust can be put into a form appropriate to a forcing function for an automatic feedback system. The thrust program obtained in this report, however, results entirely from the trajectory, and does not consider the dynamics of the vehicle nor the measuring and performance errors inherent in a particular real system. Its examination yields an insight into the general properties of the acceleration time histories of a successful rendezvous.

In the last section of the report, the theory is applied as an example to the satellite rendezvous problem. Calculations were made for coplanar rendezvous with satellites in circular orbits and the characteristics of the thrust program are examined. Propellant requirements for the cases investigated are given, but a computational program sufficient to study fuel optimization was outside the scope of this work.

SYMBOLS

$A_1^*, A_2^*, a_1, a_2, a_3, b_1, b_2, d_1, d_2$	ad hoc symbols in derivations
a	location of reference frame or origin of reference frame
b	observed point

c	rocket exhaust velocity
$\bar{f}$	thrust force per unit mass
$\bar{f}_T$	target inertial acceleration
$\bar{f}_V$	external force per unit mass on the vehicle
g_s	acceleration due to gravity at the satellite position
I	inertial origin
L	lead angle
M	vehicle mass
M_f	vehicle mass at the end of the rendezvous maneuver
$\bar{R}$	position vector of observed point
R	range of observed point
S,K	guidance system parameters
t	time from start of guidance
t_f	time of rendezvous
$\bar{T}$	guidance thrust force on the vehicle
$\bar{u}_1,\bar{u}_2,\bar{u}_3$	orthogonal unit vectors defining reference frame
$\bar{u}_R,\bar{u}_\theta,\bar{u}_\Phi$	orthogonal unit vectors based on motion of observed point
$\bar{u}_V$	direction of relative velocity
$\bar{V}$	relative velocity vector
V	relative velocity
$\bar{W}_R$	angular velocity of the line of sight
$\bar{W}_{Ia}$	inertial angular velocity of reference frame
$\bar{W}_V$	angular velocity of relative velocity vector
x_1,x_2,x_3	direction cosines for the position vector, $\bar{u}_R$
α	thrust angle
A_o,β_o,Γ_o	Euler angles

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γ departure of line of sight from initial line of sight

$\dot{\theta}_s$ satellite orbital angular velocity

τ dimensionless time, $1 - \frac{t}{t_f}$

ψ transformation matrix between $\bar{u}_R, \bar{u}_\theta, \bar{u}_\Phi$ frame and reference frame

Ω_R angular velocity of the line of sight

Ω_V angular velocity of the relative velocity vector

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Subscripts and Superscripts

$(\ )_o$ initial conditions

$(\bar{\ })$ vectors

$(\ )'$ derivative with respect to τ

$(\dot{\ })$ time derivative in reference space

$\frac{d}{dt}(\ )$ time derivative in inertial space

$(\ )^*$ quantities made dimensionless with $c\dot{\theta}_s$

THEORY

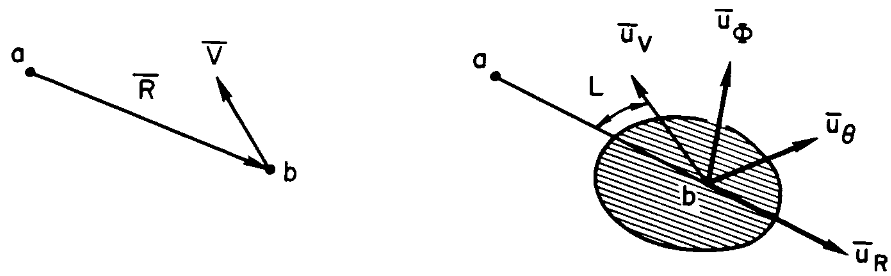
Kinematics and Trajectory Constraint Equations

The condition for a rendezvous is that the positions and velocities of a rendezvous vehicle and its destination, or target, be matched simultaneously. This condition is readily shown to be independent of the reference frame in which velocities are measured. In particular, if the reference frame is based at either vehicle or target, then rendezvous requires that relative velocity and range become zero simultaneously, or:

$$\bar{V} = 0 \quad \text{when} \quad \bar{R} = 0$$

Since any trajectory of the rendezvous vehicle which has these end conditions will produce a rendezvous, it is apparent that one approach to the problem is to specify the trajectory beforehand so that the end conditions are met. This may be done without reference to the forces involved. The thrust program necessary to follow the specified trajectory can be computed later from the required relative acceleration and the external forces.

Proportional navigation.- Consider two vehicles, a and b, one of which may be maneuvered to rendezvous with the other. A reference frame in which relative motion is measured is assumed to be carried by vehicle a. The motion variables may be defined after introducing an auxiliary set of orthogonal unit vectors referred to the observed motion of b.



Sketch (a)

The magnitudes and directions of the position and velocity (the adjective "relative" will be dropped in this section since all motion discussed is relative to the reference frame) of the observed point are defined by

$$\bar{R} = R\bar{u}_R$$

$$\bar{V} = V\bar{u}_V$$

These two vectors establish a plane from which an orthogonal set of unit vectors, $\bar{u}_R$, $\bar{u}_\theta$, $\bar{u}_\Phi$, may be defined:

$$\bar{u}_R = \text{line-of-sight direction}$$

$$\bar{u}_\Phi \equiv \frac{\bar{u}_R \times \bar{u}_V}{\sin L}$$

$$\sin L \equiv |\bar{u}_R \times \bar{u}_V|$$

$$\bar{u}_\theta = \bar{u}_\Phi \times \bar{u}_R$$

The lead angle, L, is the misalinement of the velocity vector from the negative line-of-sight direction. By definition it lies in the range $0 \leq L \leq \pi$ and hence does not indicate the direction of rotation of the line of sight. This information is contained in the definition of the vector $\bar{u}_\Phi$, which is in the positive direction of the line-of-sight angular velocity vector.

The angular velocities of the line-of-sight and velocity vectors are denoted by $\bar{W}_R$ and $\bar{W}_V$. The vector $\bar{W}_R$ is perpendicular to the plane of the line-of-sight and velocity vectors and may therefore be defined as

$$\bar{W}_R = \Omega_R \bar{u}_\Phi$$

The condition that $\bar{R} = 0$ at the end of terminal guidance may be met by the application of the basic proportional navigation constraint equation:

$$\bar{W}_V = S\bar{W}_R \tag{1}$$

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where S is a constant. Such a constraint, for appropriate values of S, continuously reduces the lead angle and thereby alines the relative velocity along the line of sight. The range, R, is also reduced concurrently with the lead angle, and an interception eventually occurs. Furthermore, the motion is now restricted to remain in a single plane relative to the reference frame. Physically, this may be seen by noting that equation (1) requires the velocity vector to rotate only about $\bar{u}_\Phi$, which, by definition, is also the axis of rotation of the line of sight. Thus, the position, $\bar{R}$, its rate of change, $\bar{V}$, and its acceleration, $\dot{\bar{V}}$, are all coplanar, and once terminal guidance begins, the motion can take place only in the plane defined by $\bar{u}_V$ and $\bar{u}_R$ at the start. This is also readily proved by showing that $\bar{u}_\Phi$, which (by definition) is perpendicular to the plane of motion, is a constant direction in the reference space once proportional navigation (eq. (1)) or, indeed, any constraint of the form $\bar{W}_V = \Omega_V(t)\bar{u}_\Phi$ is assumed.

Kinematic relations among the motion variables, which must be satisfied by any motion, are derived from the vector identities:

$$\left.\begin{aligned} \bar{V} &= \dot{\bar{R}} \\ \dot{\bar{V}} &= \ddot{\bar{R}} \end{aligned}\right\} \tag{2}$$

where the vector differentiations are as seen in the reference frame. By application of the Coriolis theorem, we obtain

$$\left.\begin{aligned} V\bar{u}_V &= \dot{R}\bar{u}_R + \bar{W}_R \times \bar{R} \\ \dot{V}\bar{u}_V + \bar{W}_V \times \bar{V} &= \ddot{R}\bar{u}_R + 2\dot{R}\bar{W}_R \times \bar{u}_R + \bar{W}_R \times (\bar{W}_R \times \bar{R}) + \dot{\bar{W}}_R \times \bar{R} \end{aligned}\right\} \tag{3}$$

Equation (1) may be introduced into equations (3) together with the geometrical relation

$$\bar{u}_V = -(\cos L)\bar{u}_R + (\sin L)\bar{u}_\theta \tag{4}$$

The coefficients of $\bar{u}_R$ and $\bar{u}_\theta$, after the indicated operations are carried out, then yield the following three independent kinematic equations:

$$\dot{R} = -V \cos L \tag{5}$$

$$R\Omega_R = V \sin L \tag{6}$$

$$-\dot{V} \cos L - S\Omega_R V \sin L = \ddot{R} - R\Omega_R^{\,2} \tag{7}$$

Equation (6) together with the derivative of equation (5) may be introduced into equation (7) to give:

$$\dot{L} + (S-1)\Omega_R = 0 \tag{8}$$

The velocity V may be eliminated between equations (5) and (6) and the resulting expression for Ω_R introduced into equation (8). Subsequent integration yields:

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$$\frac{\sin L}{\sin L_o} = \left(\frac{R}{R_o}\right)^{S-1} \tag{9}$$

The subscript zero denotes the initial value of the variable. Equation (9) shows the functional relation between range and lead angle which typifies proportional navigation. It follows from (9) that S must exceed 1 for an interception to occur. Other functional relations among the variables may be obtained by substitutions among equations (5) through (9); for example,

$$\frac{V}{V_o} = \frac{\Omega_R}{\Omega_{R_o}}\left(\frac{R}{R_o}\right)^{-(S-2)} \tag{10}$$

or

$$\frac{V}{V_o} = \frac{\dot{L}}{\dot{L}_o}\left(\frac{\sin L}{\sin L_o}\right)^{-\frac{S-2}{S-1}}$$

<u>Rendezvous constraint</u>.- Of the set of kinematic relations among the four variables, R, V, L, and Ω_R, only three are independent, and another or second constraint is necessary before the motion is completely specified. We seek a constraint that will allow the complete motion to satisfy the full rendezvous end conditions within the assumption of proportional navigation. Expressed mathematically, the conditions on the choice are:

(a) Equations (5) through (10) may not be violated and an interception must occur:

$$S > 1$$

$$L(t_f) = 0$$

$$\frac{dL}{dt} \leq 0 \qquad \text{for} \quad 0 \leq t \leq t_f$$

$$\frac{R}{R_o} = \frac{V}{V_o} = \frac{L}{L_o} = \frac{\Omega_R}{\Omega_{R_o}} = 1 \qquad \text{at} \quad t = 0$$

where t_f is defined as the time at which the range becomes zero.

(b) A rendezvous must occur:

$$V(t_f) = 0 \ , \qquad t_f > 0$$

($V(\tau \to 0^+) = V_f > 0$, implying an impulse at interception, is not excluded)

These are minimum restrictions and others could be added, provided they are consistent with the above set. In particular, no restraints due to thrust limitations have been imposed. These limitations generally cannot be expressed as simply as the kinematic conditions. However, the conditions above are always practical rendezvous maneuvers in the sense that the maneuver will be completed with a finite mass.

The second equation of constraint may be chosen variously, for example, forms giving any variable as a function of time, or functional relations among two or more variables or their derivatives. However, functional relations among a set of variables for which a relation is already specified by the preceding kinematics cannot be used since a new relation of this form cannot be independent without being contradictory.

It should be noted that an investigation of only one of these forms will suffice to cover the motion produced by a second constraint chosen in any other form since the choice of a second equation of constraint will completely specify the motion. Our main interest is to satisfy the requirement $V(t_f) = 0$ so it seems natural to study velocity constraints. However, it is much simpler to deal with the lead angle because all motion variables may be given directly in terms of the lead angle and its derivative. Thus, a condition that is to be applied to any other variable has a simple equivalent condition on the lead angle or its derivative. The following form is to be studied:

$$\frac{L}{L_o} = f(\tau) \tag{11}$$

where

$$\tau = 1 - \frac{t}{t_f}$$

The restrictions in (a) and (b) above will delineate the allowable class of constraining functions of this form. Applying these restrictions to $f(\tau)$ through equations (8), (9), and (10) yields:

$$\left.\begin{aligned}
&f(0) = 0 \\
&f'(\tau) \geq 0 \quad \text{for} \quad 0 \leq \tau \leq 1 \\
&f(1) = 1 \\
&f'(1) = (S-1)t_f \frac{V_o \sin L_o}{R_o L_o} \\
&\lim_{\tau \to 0} \frac{f'(\tau)}{\left[f(\tau)\right]^{\frac{S-2}{S-1}}} = 0
\end{aligned}\right\} \tag{12}$$

($\lim_{\tau \to 0^+} \frac{f'(\tau)}{\left[f(\tau)\right]^{\frac{S-2}{S-1}}} = V_f^* > 0$ plus an impulse at $\tau = 0$ is also allowable)

where the prime indicates differentiation with respect to τ.

These conditions describe a simple class of functions. The value of $f(\tau)$ is prescribed at the end points, $\tau = 0$ and $\tau = 1$, and its derivative is specified at $\tau = 1$. Further, $f'(\tau)$ is never negative between the end points. An additional restriction must be imposed on grounds that step changes in range are physically impossible. Therefore, step discontinuities in $f(\tau)$ are proscribed. Any function in this restricted class which also satisfies the last condition above will be satisfactory.

The simplest function that suggests itself is

$$f(\tau) = \tau^m \tag{13}$$

(See appendix A.) So that the continuity of the present theory from the standard proportional navigation may be clearly exposed, a new parameter, K, is defined to replace the exponent, m, by the expression K(S-1). The two parameters, S and K, become the fundamental constraint parameters. Applying the restrictions of (12) to equation (13) yields as the second equation of constraint:

$$\frac{L}{L_o} = \tau^{K(S-1)}$$

$$\left.\begin{array}{l} S > 1 \\ K \geq 1 \end{array}\right\} \tag{14}$$

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From (12) there also follows:

$$t_f = K\frac{R_o}{V_o}\frac{L_o}{\sin L_o} \tag{15}$$

The complete motion may now be derived directly from equations (8), (9), and (10).

$$\frac{R}{R_o} = \left(\frac{\sin L}{\sin L_o}\right)^{\frac{1}{S-1}} \tag{16}$$

$$\frac{\Omega_R}{\Omega_{R_o}} = \tau^{K(S-1)-1} \tag{17}$$

$$\frac{V}{V_o} = \left(\frac{R}{R_o}\right)^{-(S-2)}\tau^{K(S-1)-1} \tag{18}$$

The angle γ will denote the angle between the line of sight $\bar{u}_R(\tau)$ and the initial line-of-sight direction $\bar{u}_{R_o}$ (see sketch (c)). It is readily determined by noting that

$$\dot{\gamma} = \Omega_R$$

which, from equation (17), gives:

$$\gamma = \frac{L_o}{S-1}\left[1 - \tau^{K(S-1)}\right] \tag{19}$$

<u>Characteristics of the rendezvous maneuvers</u>.- Equation (14), as required, specifies that the lead angle decrease monotonically toward zero during the rendezvous maneuver. From equation (16) the range may therefore increase beyond its initial value before decreasing toward zero if the initial lead angle is greater than $\pi/2$, but it never increases if the initial lead angle is less than $\pi/2$. The limiting behavior of the range near rendezvous ($\tau \rightarrow 0$ and therefore L is sufficiently small

to assume $\sin L \cong L$) is readily found to be

$$\frac{R}{R_0} \cong \left(\frac{L_0}{\sin L_0}\right)^{\frac{1}{S-1}} \tau^K$$

which decreases at least linearly with τ. (We note that τ is really the dimensionless time to go.)

The angular velocity of the line of sight (line-of-sight rate) given by (17) exhibits varying behavior according to the value of the parameter, S. In particular

$$\Omega_R(t_f) = 0 \qquad \text{for} \quad S > \frac{K+1}{K}$$

$$\Omega_R(t_f) \to \infty \qquad \text{for} \quad 1 < S < \frac{K+1}{K}$$

While motion having the latter characteristic is of doubtful practical interest, a valid rendezvous is still achieved. It will be seen in the next paragraph that the total line-of-sight angular change remains finite for these values of S but indefinite spiraling motion is approached as $S \to 1$.

Equation (19) indicates how far the line of sight moves from its initial position $\bar{u}_{R_0}$, as observed in the reference frame. The maximum departure is

$$\gamma_{max} = \frac{L_0}{S-1}$$

and occurs at rendezvous. This angle increases as S decreases, becoming arbitrarily large for $S \to 1$ where an indefinite spiraling motion is approached. For values of $S > 2$ the maximum departure is always less than the initial lead angle.[1]

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[1]While off hand "loose" trajectories (S near 1) do not appear to be desirable, these may present some advantages in a special case. Suppose it is desired to rendezvous with a point on a surface so that the maneuver ends tangent to the surface; that is, a landing. If the motion of the approaching vehicle is in the plane perpendicular to the landing surface, with the initial line of sight at an angle Δ_0 above the surface and the relative velocity vector at an angle L_0 below the line of sight, then the required value of S for a tangential landing is given from

$$\gamma_{max} = \Delta_0 = \frac{L_0}{S-1}$$

where values of $S < 2$ are required if $L_0 < \Delta_0$. A vertical landing may be treated similarly.

The character of the velocity time history is obscure in equation (18), which contains two uncombinable terms that may in general have opposing or concurrent trends depending on the values of L_o, S, and K. However, some information may be obtained from the relative acceleration, which, after some algebra, becomes:

$$\dot{V} = \frac{V^2}{R} \cos L \left[(S - 2) - \left(S - \frac{K+1}{K} \right) \frac{\tan L}{L} \right]$$

Examination of the terms in this equation reveals that velocity always decreases during guidance if $S > 2$. For $S < 2$, velocity increases if the lead angle exceeds a certain value, L_1, and decreases below L_1. The value of L_1 depends on S and K and is given by:

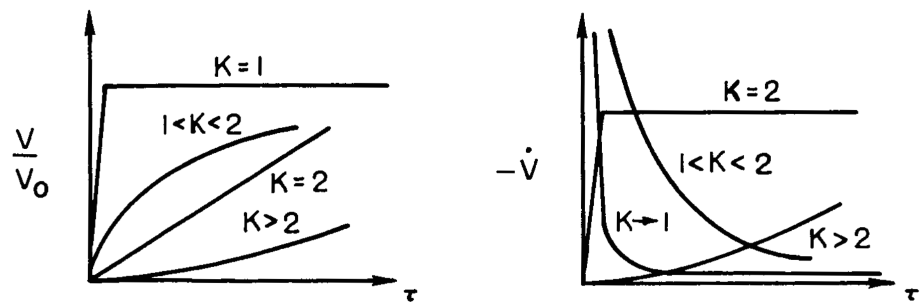
$$\frac{\tan L_1}{L_1} = \frac{S-2}{S - \frac{K+1}{K}}$$

The limiting behavior of velocity near rendezvous (from eq. (18)) and its derivative are:

$$\frac{V}{V_o} \cong \left(\frac{\sin L_o}{L_o} \right)^{\frac{S-2}{S-1}} \tau^{K-1}$$

$$\dot{V} \cong - \frac{K-1}{K} \frac{V_o^2}{R_o} \left(\frac{\sin L_o}{L_o} \right)^{\frac{2S-3}{S-1}} \tau^{K-2}$$

Near rendezvous the behavior of these quantities is therefore dependent on K, as illustrated in sketch (b):



Sketch (b)

Velocity approaches zero continuously for values of $K > 1$, and at $K = 1$ a step decrease of velocity occurs at $\tau = 0$. It may be anticipated that the thrust requirements near rendezvous will exhibit the same behavior as the relative acceleration, $\dot{V}$, whose behavior depends on K. In particular, if $K > 2$, $\dot{V}$ decreases continuously to zero, and if $K = 2$, $\dot{V}$ exhibits a step decrease to zero. In the range $1 < K < 2$, $\dot{V}$

becomes arbitrarily large at rendezvous, and for $K = 1$, an impulse occurs corresponding to the step decrease in velocity. In the cases with singular behavior it can be shown that the mass ratio across the singular region is not zero since the integral of $\dot{V}$ remains finite over this region.

For a given set of initial conditions (R_o, V_o, L_o), the maneuver time, t_f, (eq. (15)) varies directly with K but is independent of S. Maneuver time is therefore a minimum for $K = 1$, which requires impulsive retrothrust at rendezvous. For $K > 2$ the thrust requirements near rendezvous are continuous, but at least twice as much time is required for this type of maneuver as is necessary using impulsive terminal thrusting.

Equation (15) also indicates that t_f becomes arbitrarily large for $L_o \to \pi$ (guidance begins with the vehicle receding along the line of sight from the target) or for $V_o \to 0$. The occurrence of these unfavorable initial conditions may be obviated where necessary by providing the guidance system with a means of choosing the most favorable point on the unguided approach trajectory at which to initiate the rendezvous maneuver. However, the choice of initial point should consider possible fuel optimization as well as time requirements.

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If $K = 1$, then equation (14) is the same as the corresponding result of reference 2 and, as must be the case, the complete motion is that of reference 2 generalized to include large values of lead angle. Thus, for this value of K the present theory reduces to the standard proportional navigation theory up to the point of interception. At interception the two differ in that the rendezvous maneuver requires a step decrease of velocity to zero.

The rendezvous maneuvers derived in this section are illustrated in figures 1, 2, and 3. Figure 1 is a dimensionless polar plot of paths in the plane of relative motion (R/R_o vs. γ). The origin is the origin of the reference frame and the vertical radial line is the initial line-of-sight direction. The equation governing these paths is found from substitutions among equations (14), (16), and (19):

$$\frac{R}{R_o} = \left\{\frac{\sin[L_o - (S-1)\gamma]}{\sin L_o}\right\}^{\frac{1}{S-1}}$$

The paths therefore depend only on L_o and S. Since the maneuver time is independent of S, a set of values of S will give a set of paths of constant maneuver time for given values of L_o, R_o, V_o, and K. (Changes in K do not affect the path for a given value of S but change only the time required to traverse the path.)

Polar plots of V/V_o versus L for various values of S and the initial condition $L_o = \pi/2$ are shown in figure 2. Here, the equation may be found from equations (14) and (18).

$$\frac{V}{V_o} = \left(\frac{\sin L}{\sin L_o}\right)^{-\frac{S-2}{S-1}} \left(\frac{L}{L_o}\right)^{S-\frac{K+1}{K}\Big/ S-1}$$

Plots of lead angle versus velocity for several values of K are given in figure 3.

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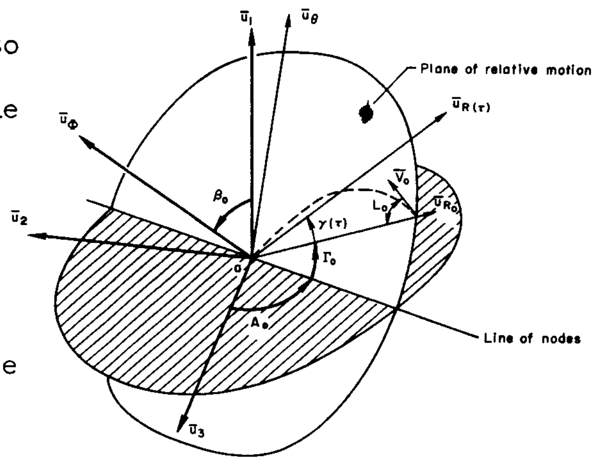
Reference frame.- The command trajectories now have been largely determined except to indicate the necessary transformations of vectors given in the the $\bar{u}_R$, $\bar{u}_\theta$, $\bar{u}_\Phi$ frame to their components in the reference space of the command system. A reference space is indicated in sketch (c) by the orthogonal unit vectors $\bar{u}_1$, $\bar{u}_2$, $\bar{u}_3$. At this point it is not necessary to specify this reference frame further; it could be, for example, an inertial or a local vertical frame. The plane of motion of the observed point and its initial line of sight are established in the reference space by means of the Euler angles A_o, β_o, Γ_o, which necessarily have constant values during guidance. The angle, γ, indicates the departure of the line of sight from its initial direction, $\bar{u}_{R_o}$, as observed in the reference space.

To eliminate ambiguity, A_o is to be measured from the positive $\bar{u}_3$ axis to the nearest branch of the line of nodes in a right-hand sense about $\bar{u}_1$. Similarly Γ_o is measured from the same branch of the line of nodes in a right-hand sense about $\bar{u}_\Phi$.

The commuter coordinate vectors, $\bar{u}_R$ (line of sight), $\bar{u}_\theta$, $\bar{u}_\Phi$, may be given in the satellite reference space by the transformation:

$$\begin{bmatrix} \bar{u}_R \\ \bar{u}_\theta \\ \bar{u}_\Phi \end{bmatrix} = \psi \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{bmatrix} \qquad (20)$$



Sketch (c)

$$\psi = \begin{bmatrix} \sin\beta_o\sin(\Gamma_o+\gamma) & -[\sin A_o\cos(\Gamma_o+\gamma) + \cos A_o\sin(\Gamma_o+\gamma)\cos\beta_o] & \cos A_o\cos(\Gamma_o+\gamma) - \sin A_o\sin(\Gamma_o+\gamma)\cos\beta_o \\ \sin\beta_o\cos(\Gamma_o+\gamma) & \sin A_o\sin(\Gamma_o+\gamma) - \cos A_o\cos(\Gamma_o+\gamma)\cos\beta_o & -[\cos A_o\sin(\Gamma_o+\gamma) + \sin A_o\cos(\Gamma_o+\gamma)\cos\beta_o] \\ \cos\beta_o & \cos A_o\sin\beta_o & \sin A_o\sin\beta_o \end{bmatrix}$$

In general, the components of a vector in the reference space may be given from its components in the motion frame by the transformation:

$$A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \psi^T \begin{bmatrix} A_R \\ A_\theta \\ A_\Phi \end{bmatrix}$$

where ψ^T indicates the transpose of ψ. It will be useful to introduce the notation

$$\bar{u}_R = x_1\bar{u}_1 + x_2\bar{u}_2 + x_3\bar{u}_3$$

where the x_i are the first row of ψ, and to note that $\bar{u}_\theta = (\Omega_R)^{-1}\dot{\bar{u}}_R$ or A 4 9 4

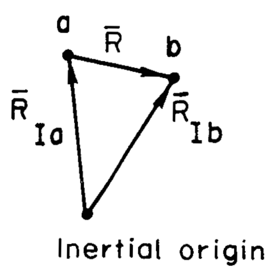
$$\bar{u}_\theta = (\Omega_R)^{-1}(\dot{x}_1\bar{u}_1 + \dot{x}_2\bar{u}_2 + \dot{x}_3\bar{u}_3)$$

where the $\dot{x}_i/\Omega_R$ are the second row of ψ.

Maneuver Forces

The trajectory constraint investigation thus far has not required the specification of forces involved and is generally applicable to any problem requiring the rendezvous end conditions. However, before a maneuver can be carried out in any given problem, the required thrust program must be computed. The external forces on both vehicle and target and the inertial rotational motion of the reference frame must be known.

Maneuver accelerations.- The inertial acceleration required during the maneuver may be found from Newton's law.



Sketch (d)

The reference frame is at a which may be either target or vehicle. If a is the target, then applying Newton's law to the vehicle we obtain:

$$\bar{f}_v + \bar{f} = \frac{d^2}{dt^2}(\bar{R}_{Ia} + \bar{R})$$

or

$$\bar{f} = \frac{d^2\bar{R}}{dt^2} + (\bar{f}_T - \bar{f}_v) \qquad (21)$$

where

$\bar{f}$ acceleration of the vehicle due to thrusting or thrust force per unit vehicle mass (to be referred to as the thrust program)[2]

$\bar{f}_V$ external force per unit mass on the vehicle (gravity, aerodynamic, etc.)

$\bar{f}_T$ acceleration of the target, $\frac{d^2\bar{R}_{Ia}}{dt^2}$

The inertial relative acceleration, $d^2\bar{R}/dt^2$, may be given in terms of the motion observed in the reference frame by the application of Coriolis' theorem. Equation (21) becomes:

$$\bar{f} = \dot{\bar{V}} + 2\bar{W}_{Ia} \times \bar{V} + \bar{W}_{Ia} \times (\bar{W}_{Ia} \times \bar{R}) + \dot{\bar{W}}_{Ia} \times \bar{R} + (\bar{f}_T - \bar{f}_V) \qquad (22)$$

where $\bar{W}_{Ia}$ is the inertial angular velocity of the reference frame and $(\dot{\ })$ indicates time derivatives as seen in the reference frame. Since the desired motion is one of the rendezvous maneuvers, then the appropriate expressions for $\dot{\bar{V}}$, $\bar{V}$, and $\bar{R}$ from the previous text are to be introduced in (22), together with $\bar{W}_{Ia}$ and the external forces. The required thrust program is then computed.

The commanded relative acceleration, $\dot{\bar{V}}$, for the rendezvous maneuvers may be written in terms of the measured motion variables:

$$\dot{\bar{V}} = \frac{V^2}{R}\left[(S-2)\cos L - \left(S - \frac{K+1}{K}\right)\frac{\sin L}{L}\right]\bar{u}_V + S\bar{W}_R \times \bar{V} \qquad (23)$$

Utilizing this form in equation (22) yields an expression for $\bar{f}$ which is appropriate as the basic forcing function in automatic feedback systems since that portion of the required thrust which depends on the relative motion may now be computed directly from the observed current motion.

<u>Characteristics of the maneuver acceleration</u>.- The relative acceleration of the maneuvers is seen from equation (23) to consist of two terms. The first, directed along $\bar{u}_V$, is largely responsible for satisfying the velocity end condition. The second term, $S\bar{W}_R \times \bar{V}$, is the usual proportional navigation control function which forces an interception. Mathematically, $\dot{\bar{V}}$ is apparently well behaved except possibly near rendezvous ($\tau \to 0$) where both V and R become zero, and Ω_R may sometimes be singular. Utilizing the previous results approximated for small values of τ gives the limiting expression for $\dot{\bar{V}}$ near rendezvous:

---

[2] If a is the vehicle, the result is:

$$\bar{f} = -\frac{d^2\bar{R}}{dt^2} + (\bar{f}_T - \bar{f}_V)$$

$$\dot{\bar{V}} = \frac{V_o}{R_o}\left(\frac{\sin L_o}{L_o}\right)^{\frac{2S-3}{S-1}}\left(- \frac{K-1}{K}\,\tau^{K-2}\,\bar{u}_V + SL_o\tau^{KS-2}\,\bar{u}_\Phi \times \bar{u}_V\right) \qquad (24)$$

The first term, due to $\dot{V}$, varies as $(K-1)\tau^{K-2}$ and was previously discussed; it reduces continuously to zero for $K > 2$, becomes constant for $K = 2$, and is singular in the range $1 < K < 2$. At $K = 1$ it is zero except at $\tau = 0$ where it becomes an impulse function corresponding to a step decrease in velocity of

$$\Delta V = V_o\left(\frac{\sin L_o}{L_o}\right)^{S-2/S-1}$$

Near rendezvous, $\bar{u}_V$ is approximately along the negative line-of-sight direction so that this part of the limiting thrust program is directed along $\bar{u}_R$.

The second term, due to $\Omega_R V$ and directed nearly perpendicular to $\bar{u}_R$ along $\bar{u}_\Phi \times \bar{u}_V$, varies as τ^{KS-2}; therefore reduces to zero continuously for $S > 2/K$ and is singular for lower values of S. The singular behavior is possible within the restriction $S > 1$ only if K is less than 2. The singular behavior of the term is due to Ω_R modified by the behavior of V which tends to weaken the singularity and reduce the range of values of S over which it may occur. In fact, for $K > 2$ the singular influence of Ω_R on this term is removed.

Because of the restriction $K \geq 1$, the maximum value of S for which the second term in (24) may be singular is 2. Furthermore, a number of possible peculiarities of the maneuver motion were noted in the previous text for values of S less than 2. A valid rendezvous is not precluded by any of these peculiarities but there is apparently little advantage to be derived from them. The remainder of this report will limit consideration to the main body of proportional navigation maneuvers having fairly uniform characteristics by assuming the restriction

$$S > 2 \qquad (25)$$

As a consequence, the second term in (24) will always reduce to zero more rapidly than $\tau^{2(K-1)}$ near rendezvous.

If $\bar{W}_{Ia}$ and $\bar{f}_T - \bar{f}_V$ are assumed to be well behaved, as must generally be true in practical cases, then the first term in $\dot{\bar{V}}$ will be the dominant term in equation (22) near rendezvous. For K near 1, this dominance is delayed to values of τ very close to zero because of the presence of $(K-1)$ in this term; that is, the system approximates the constant relative velocity behavior. In these cases the behavior prior to the time at which the first term is dominant will be given by some other term in equation (22), possibly the term $\bar{W}_{Ia} \times \bar{V}$ (if $W_{Ia} \neq 0$) or by $\bar{f}_T - \bar{f}_V$.

Propellant requirements.- The mass of the vehicle will vary during terminal guidance because of the expenditure of propellant in providing thrust. If the thrust system is assumed to consist of a single-stage rocket system of exhaust velocity c, then from Newton's second law the mass ratio at any time, is:

$$\frac{M(t)}{M_o} = \exp\left(-\frac{1}{c}\int_o^t |\bar{f}|dt\right) \qquad (26)$$

where M_o is the initial vehicle mass and only the rendezvous maneuver forces are considered. The actual thrust at any time is then given by:

$$\bar{T} = M(t)\bar{f}(t) \qquad (27)$$

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At this point, outside of the context of a specific problem, it is only possible to indicate those factors which affect propellant requirements and which therefore might be controlled so as to reduce the required fuel. The mass ratio over the maneuver depends on the integral $\int_o^{t_f} |\bar{f}|dt$ which, from (22) and (23), depends on the system parameters, initial conditions and reference frame. If all other parameters are specified in a given problem then the system parameters, S and K, could be chosen so as to optimize fuel consumption. However, the latitude of choice of these parameters will likely be limited by other considerations; for example, maneuver time, trajectory constraint, and thrust program characteristics. As previously mentioned, it is possible to choose the point on the approach trajectory at which guidance is begun such that the fuel requirements are a minimum.

In equation (22) there are several terms which depend on the choice of reference frame, that is, $\bar{W}_{Ia}$. If the rendezvous maneuver is carried out in two different reference frames for an otherwise completely specified problem, then the two maneuvers as viewed from inertial space are different. It then follows that $|\bar{f}|$ and $\int_o^{t_f} |\bar{f}|dt$ will, in general, differ for the two frames and that the fuel requirements will vary accordingly, some reference frames requiring less fuel than others in each particular case. While $\bar{W}_{Ia}$ has been assumed arbitrary thus far, the number of useful frames is limited in practice, for example, inertial and local vertical frames. Of course, guidance may be carried out in any imaginary frame from measurements taken in one of the simpler frames by means of appropriate transformation relations, so that in theory an arbitrary $\bar{W}_{Ia}$ is still available. For our purpose, such a procedure is unnecessarily complicated. Moreover, there is no simple means of finding the optimum $\bar{W}_{Ia}$. Nevertheless, for a reasonable problem (restricted range of expected initial conditions) it should be feasible with direct computations to determine if there is any significant advantage in the choice of $\bar{W}_{Ia}$ from among a few simple frames.

As a final point, it should be noted that the thrust requirements for the proposed rendezvous maneuvers are, in general, variable both in magnitude and direction. This is illustrated in the example that follows dealing with the satellite rendezvous problem.

THE SATELLITE RENDEZVOUS PROBLEM

Application of the Force Equation

In what follows the "target" is a satellite in planetary orbit and contains the command guidance system, including the reference frame. The reference frame is taken to be a local vertical system with $\bar{u}_1$ along the outward local vertical and $\bar{u}_3$ in the positive direction of the satellite orbital angular velocity, that is, perpendicular to the orbital plane. The angular velocity of the reference frame, $\bar{W}_{Ia}$, is then the orbital angular velocity of the satellite.

The directions of all vectors in the thrust equation (22), except $\bar{f}_T - \bar{f}_V$, are already given. The following vector relations are repeated here for convenience:

$$\bar{R} = R\bar{u}_R$$

$$\bar{V} = V\bar{u}_V$$

$$\bar{u}_V = -\cos L\bar{u}_R + \sin L\bar{u}_\theta$$

$$\dot{\bar{V}} = \dot{V}\bar{u}_V + S\Omega_R V\bar{u}_\theta$$

and also

$$\bar{W}_{Ia} = \dot{\theta}_S\bar{u}_3$$

The transformation relations of equation (20) now allow all terms of (22) to be expressed in the reference frame, giving the components of $\bar{f}$ along $\bar{u}_1$, $\bar{u}_2$, $\bar{u}_3$ as:

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & 0 \\ -a_2 & a_1 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \frac{1}{\Omega_R} \begin{bmatrix} b_1 & b_2 & 0 \\ -b_2 & b_1 & 0 \\ 0 & 0 & b_1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + (\bar{f}_T - \bar{f}_V) \qquad (28)$$

where

$$a_1 = S\Omega_R V \sin L + \dot{V} \cos L + R\dot{\theta}_S{}^2$$

$$a_2 = R\ddot{\theta}_S - 2\dot{\theta}_S V \cos L$$

$$a_3 = S\Omega_R V \sin L + \dot{V} \cos L$$

$$b_1 = sV\Omega_R \cos L - \dot{V} \sin L$$

$$b_2 = 2\dot{\theta}_S V \sin L$$

It remains to express the external forces, $\bar{f}_T - \bar{f}_V$, in the reference frame. For a satellite orbit outside the planetary atmosphere, the difference in external forces is a result of the gravitational field of the planet. In a central force field, if the vehicles are sufficiently close during guidance, the gravity difference term may be linearized as shown in appendix B; whence

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$$\bar{f}_T - \bar{f}_V = g_s \frac{R}{R_S} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{29}$$

where R_S is the radial distance of the satellite from the center of the force field and g_S is the gravitational acceleration at the position of the satellite. The complete time histories of R, V, L, Ω_R, and the x_i are computed from equations (14) through (20) once the initial conditions and the system parameters, S and K, are given. If the orbit and g_S are also given, then the complete thrust program may be computed.

Calculations were carried out for the simple case in which the initial motion, and hence the entire maneuver, is coplanar with the orbital plane of a satellite in circular orbit around the Earth. When equations (28) and (29) are specialized to this simple case, then:

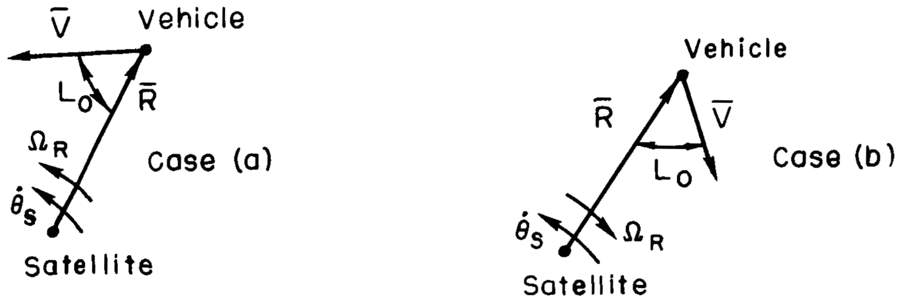
$$\left.\begin{aligned} \ddot{\theta}_S &= 0 \\ \frac{g_S}{R_S} &= \dot{\theta}_S{}^2 \\ x_3 = \dot{x}_3 &= 0 \end{aligned}\right\} \tag{30}$$

For coplanar motion the same initial position may be represented by two sets of Euler angles which correspond to opposite directions of rotation of the line of sight. These are

Case a: $A_O = \pi/2$, $\beta_O = \pi/2$, Γ_O

Case b: $A_O = \pi/2$, $\beta_O = -\pi/2$, $-\Gamma_O$

The rotation of the line of sight is, respectively, clockwise and counterclockwise about the satellite axis of orbital rotation, $\bar{u}_3$. The difference between the two cases depends on the orientation of the relative velocity vectors (sketch (e)).



Sketch (e)

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For cases (a) and (b) the position during guidance is given by:

$$\left.\begin{aligned} x_1 &= \sin[\Gamma_o + (\pm)\gamma] \\ x_2 &= -\cos[\Gamma_o + (\pm)\gamma] \\ \bar{u}_\Phi &= (\pm)\bar{u}_3 \end{aligned}\right\} \qquad (31)$$

The factor $(\pm)$, which henceforth will be denoted by n, is to be taken as $(+1)$ or (-1) for cases (a) and (b), respectively. This procedure allows both cases to be included at once in the following derivation.

Equations (27), (29), (30), and (31) may be introduced in (28) to form an expression for the thrust program. This expression is made dimensionless with the factor $c\dot{\theta}_s$ which depends only on the rocket system and the particular circular orbit. The calculations will then apply to all circular orbits and guidance rockets. In general, these computational benefits are not obtained if the orbit is elliptical or if the gravity difference term is not linearized.

$$\begin{Bmatrix} f_1^* \\ f_2^* \end{Bmatrix} = \begin{bmatrix} A_1^* - 3\dfrac{R_o\dot{\theta}_s}{c}\dfrac{R}{R_o} & A_2^* \\ A_2^* & -A_1^* \end{bmatrix} \begin{Bmatrix} \sin(\Gamma_o + n\gamma) \\ \cos(\Gamma_o + n\gamma) \end{Bmatrix} \qquad (32)$$

$$f_3^* = 0$$

where

$$A_1^* = \left(\frac{V_o}{c}\right)^2 \left(\frac{c}{R_o\dot{\theta}_s}\right) \left[\left(S - \frac{K+1}{K}\right)\frac{\sin L_o}{L_o}\frac{V}{V_o}\frac{\cos L}{\tau} - (S-2)\left(\frac{V}{V_o}\right)^2 \frac{R_o}{R}\cos L - S\sin^2 L_o \frac{R}{R_o}\left(\frac{\Omega_R}{\Omega_{R_o}}\right)^2\right] - 2n\frac{V_o}{c}\sin L_o \frac{R}{R_o}\frac{\Omega_R}{\Omega_{R_o}}$$

$$A_2^* = -2\left(\frac{V_o}{c}\right)\frac{V}{V_o}\cos L - n\left(\frac{V_o}{c}\right)\frac{c}{R_o\dot{\theta}_s}\left[2\sin L_o\frac{V}{V_o}\frac{\Omega_R}{\Omega_{R_o}}\cos L + \left(S - \frac{K+1}{K}\right)\frac{\sin^2 L_o}{L_o}\frac{R}{R_o}\frac{\Omega_R}{\Omega_{R_o}}\frac{1}{\tau}\right]$$

Quantities made dimensionless with $c\dot{\theta}_s$ are denoted by the asterisk. The parameters $R_o\dot{\theta}_s/c$, Γ_o, V_o/c, n, and L_o completely specify the initial conditions in dimensionless form. Computations may be carried out in any given problem by substituting the time histories of the motion variables in equation (32) from equations (14) through (19).

Equations (26) and (27) become:

$$\left.\begin{aligned} \frac{M(\tau)}{M_o} &= \exp\left(-\dot{\theta}_s t_f \int_\tau^1 [f^*]\,d\tau\right) \\ \frac{\bar{T}}{c\dot{\theta}_s M_o} &= \frac{M(\tau)}{M_o}\,\bar{f}^* \end{aligned}\right\} \qquad (33)$$

The factor $\dot{\theta}_s t_f$ is the geocentric orbit angle passed through by the satellite during the maneuver, or, alternatively, a dimensionless maneuver time. It is readily given in terms of the dimensionless initial conditions from equation (15):

$$\dot{\theta}_s t_f = K\left(\frac{R_o\dot{\theta}_s}{c}\right)\left(\frac{c}{V_o}\right)\frac{L_o}{\sin L_o}$$

Computations

The computations covered the following ranges of initial conditions:

$$n = \pm 1$$

$$L_o = 0 \text{ to } 3\pi/4$$

$$\frac{V_o}{c} = 0.01 \text{ to } 0.1$$

$$\Gamma_o = 0 \text{ to } 2\pi$$

$$\frac{R_o\dot{\theta}_s}{c} = 0.005 \text{ to } 0.05$$

The range of values of $R_o\dot{\theta}_s/c$ corresponds to initial ranges from 10 to 100 miles for near-earth orbits ($\dot{\theta}_s \cong 10^{-3}$ radians/sec) and $c = 10{,}000$ fps. The following values of the system parameters were taken:

$$S = 5$$

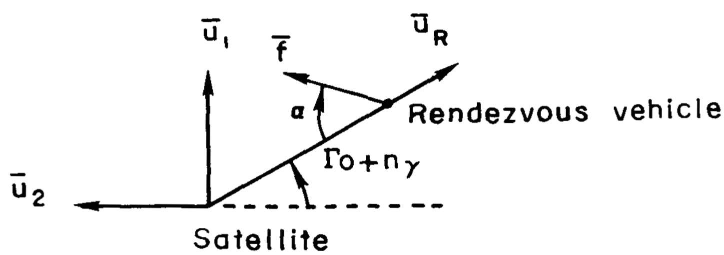
$$K = 1, 2, 3$$

These values of S and K represent fairly "tight" maneuvers ($\gamma_{max} = L_o/4$) and will illustrate the main effects of K on the thrust program. Higher values of K are considered unlikely because of the corresponding increases in maneuver time.

The computed mass ratios for a number of sets of initial conditions and system parameters are given in table I. This table is intended to give typical values. Questions concerning values of S and K for minimum fuel requirements (the propellant ratio is given by $M_p/M_o = 1 - (M_f/M_o)$) demand a more detailed computational program outside the scope of the present work. It appears generally from the table that if the fuel requirements are severe, then the variation in fuel requirements with S and K will be large ~13 percent over the range of S and K covered for the case ($R_o\dot{\theta}_s/c = 0.05$, $V_o/c = 0.01$) of table I(a). In such cases consideration must be given to minimizing fuel requirements by the proper choice of S and K.

Two cases were computed to compare fuel requirements using the proportional navigation maneuvers and the two-impulse method of reference 3. The mass ratios, M_f/M_o, for the two methods are given in table II. These two isolated cases indicate comparable fuel requirements for the two methods, but it would be hazardous to extend this conclusion to apply generally.

In figures 4 through 6, a number of thrust programs (magnitude and direction) are plotted versus the dimensionless time, τ. The thrust per unit mass is given in the dimensionless form $f/c\dot{\theta}_s$. Noting that for near-earth orbits $\dot{\theta}_s \cong 10^{-3}$ radian/sec and taking $c = 10{,}000$ fps, then the order of magnitude of the required acceleration, f, in feet per sec^2, is given by multiplying the value of f^* in the graphs by 10. The thrust direction is denoted by α which is the angle between the negative line-of-sight direction and the thrust vector, as shown in sketch (f)



Sketch (f)

$$\alpha = \Gamma_o + n\gamma + \arc\tan \frac{f_1^*}{f_2^*} \tag{34}$$

Before discussing the thrust equation and the results, it is convenient to rearrange equation (22), utilizing (23), (29), (30), and (31) as before, except that it is now helpful to make the result dimensionless with $V_o\dot{\theta}_s$ rather than $c\dot{\theta}_s$. The result is then:

$$\frac{\bar{f}}{V_o\dot{\theta}_s} = \frac{V_o}{R_o\dot{\theta}_s}\left(\frac{V}{V_o}\right)^2\left(\frac{R}{R_o}\right)^{-1}\left[(S-2)\cos L - \left(S - \frac{K+1}{K}\right)\frac{\sin L}{L}\right]\bar{u}_V$$

$$+ S\,\frac{V_o}{R_o\dot{\theta}_s}\sin L_o\,\frac{V}{V_o}\,\frac{\Omega_R}{\Omega_{R_o}}\,n\bar{u}_3 \times \bar{u}_V + 2\,\frac{V}{V_o}\,\bar{u}_3 \times \bar{u}_V$$

$$- 3\,\frac{R_o\dot{\theta}_s}{V_o}\,\frac{R}{R_o}\sin(\Gamma_o + n\gamma)\bar{u}_1 \qquad (35)$$

The first term is due to $\dot{V}$ and is directed along $\bar{u}_V$. The second and third terms are directed perpendicular to $\bar{u}_V$ in the same direction if $n = +1$ and in opposite directions if n is -1. These two terms are due, respectively, to $S\bar{W}_R \times \bar{V}$ and $2\bar{W}_{Ia} \times \bar{V}$. The first three terms are independent of the line-of-sight direction $(\Gamma_o + n\gamma)$ and therefore for cases in which these terms are dominant there will be very little variation of the thrust program with different directions of the line of sight.

The fourth term is the combined effect of $\bar{W}_{Ia} \times (\bar{W}_{Ia} \times \bar{R})$ and the gravity difference and is the only term to vary with Γ_o. The importance of this term depends on the value of the dimensionless parameter $V_o/R_o\dot{\theta}_s$; if this parameter is sufficiently large (large initial velocity, low initial range), then the first three terms are dominant and the fourth term is negligible. If the parameter is small (low V_o, large R_o), then the fourth term contributes important position-dependent effects to the thrust program. Equivalently, if $V_o/R_o\dot{\theta}_s$ is large, then the required maneuver acceleration is large compared to the gravity difference, while low $V_o/R_o\dot{\theta}_s$ indicates that the required acceleration is very low and of the same order as the gravity difference. The effect of $V_o/R_o\dot{\theta}_s$ on the relative importance of terms is complicated by effects of L_o, n, and K, but the general significance of $V_o/R_o\dot{\theta}_s$ is as stated above.

Table III illustrates a case of large $V_o/R_o\dot{\theta}_s$ for two different positions, Γ_o. A comparison of the two shows that the mass ratio and both the magnitude and the direction of thrust are very little affected by the difference in Γ_o. Figure 4(a) illustrates a case with low $V_o/R_o\dot{\theta}_s$ for two position angles. Strong effects of position and very low values of acceleration (about 0.6 and 0.1 ft/sec^2 for the two cases at $\tau = 1$) should be noted. An intermediate case is given in figure 4(b).

Figure 5(a) illustrates the effects of K. In the thrust program the value of K affects primarily the importance of the term $\dot{V}\bar{u}_V$ at low lead angles. This effect is always seen for low values of τ since the lead angle will have been reduced to small values; that is, it is always seen in the limiting behavior of the thrust program. The case plotted in figure 5(a) is one with a low initial lead angle (10°) so that the influence of K is pronounced throughout most of the maneuver. For $K = 1$, the chief function of the term $\dot{V}\bar{u}_V$ is to remove the relative velocity at $\tau = 0$ by means of an impulse, noted by a break in the f^* plot of figure 5(a) and the point $\alpha = 180^\circ$ in the α plot. For $K = 2$ and 3, the first term becomes dominant very early in the program, reflecting the constant value of $\dot{V}$ for $K = 2$, the linearly decreasing behavior of $\dot{V}$ for $K = 3$, and the thrust direction $\alpha = 180^\circ$ of $\dot{V}\bar{u}_V$ for both values of K.

It was previously noted that the second and third terms of equation (35)

$$\left(S\,\frac{V_o}{R_o\dot{\theta}_s}\sin L_o\,\frac{V}{V_o}\,\frac{\Omega_R}{\Omega_{R_o}}\right)n\bar{u}_3\times\bar{u}_V + 2\,\frac{V}{V_o}\,\bar{u}_3\times\bar{u}_V$$

tend to cancel for $n = -1$. These terms are due to $S\bar{W}_R\times\bar{V}$ and $2\bar{W}_{Ia}\times\bar{V}$ and are directed perpendicular to $\bar{u}_V$. If $n = -1$ and the initial magnitude of $S\bar{W}_R\times\bar{V}$ is larger than that of $2\bar{W}_{Ia}\times\bar{V}$, which is usually the case for large values of S and $V_o/R_o\dot{\theta}_s$, then the combined terms are initially perpendicular to $\bar{u}_V$ in the direction $-\bar{u}_3\times\bar{u}_V$. It is clear that the magnitude of $S\bar{W}_R\times\bar{V}$ disappears much more rapidly than that of $2\bar{W}_{Ia}\times\bar{V}$, so that at some point the two terms will cancel each other after which the term $2\bar{W}_{Ia}\times\bar{V}$ will be larger than $S\bar{W}_R\times\bar{V}$ and the direction of the combined terms will shift by 180° to $\bar{u}_3\times\bar{u}_V$. In figure 5(a) this cancellation occurs near $\tau = 0.6$, and for $K = 1$ a very rapid and large shift in α is noted. The curves for $K = 1$ follow closely the behavior of the combined second and third terms modified by small effects of the first and fourth terms of equation (35). The first term is, of course, seen mainly in the impulse at $\tau = 0$. For $K = 2$ and 3, the first term is dominant and the curves follow closely the behavior of the first term modified by the combined effect of the second and third terms so that the cancellation effect is overpowered. Thus the early portions of all the curves reflect the rapid demise of the terms $S\bar{W}_R\times\bar{V}$ (with τ^3 for $K = 1$, τ^8 for $K = 2$, and τ^{13} for $K = 3$). After the cancellation, the curves for $K = 1$ reflect the rise of f^* to the constant value of $2\bar{W}_{Ia}\times\bar{V}$ but for $K = 2$ and 3 the term $2\bar{W}_{Ia}\times\bar{V}$ is overpowered by the first term.

If $n = +1$, both $\bar{W}_{Ia}$ and $\bar{W}_R$ are in the same direction (see sketch (e)). In this case the two terms add and no cancellation effects occur. The case plotted in figure 5(b) has identical initial conditions to those for figure 5(a) except that $n = +1$ for figure 5(b). As expected, the cancellation effects do not occur.

The procedure of making $\bar{f}$ dimensionless with $V_o\dot{\theta}_s$ also simplifies the discussion of propellant requirements. By specifying the parameter $V_o/R_o\dot{\theta}_s$, Γ_o, n, L_o, S, and K the ratio of the characteristic

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velocity ΔV to the initial relative velocity may be given:

$$\frac{\Delta V}{V_o} = - \frac{c}{V_o} \ln \frac{M_f}{M_o} = \dot{\theta}_s t_f \int_o^1 \frac{|\bar{f}| d\tau}{V_o \theta_s}$$

Several plots of $\Delta V/V_o$ versus $V_o/R_o\dot{\theta}_s$ are given in figures 6(a) and 6(b) for stated values of Γ_o, n, L_o, S, and K. In figure 6(b) an additional curve is given by a dotted line for a different value of Γ_o but otherwise represents the same initial conditions and system parameters. As expected, the two curves do not differ except at low values of $V_o/R_o\dot{\theta}_s$.

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In summary, important influences on the thrust program from four parameters should be noted; namely, those of $V_o/R_o\dot{\theta}_s$, S, K, and n. Equation (35) consists of four terms, the fourth of which contributes the only position-dependent influence on the magnitude of the required thrust. For large $V_o/R_o\dot{\theta}_s$ the fourth term may be neglected; for small $V_o/R_o\dot{\theta}_s$ the required acceleration for the maneuver is of the same order of magnitude as the gravity difference and the fourth term must be retained.

The second term in (35), due to $S\bar{W}_R \times \bar{V}$, is the largest term at the start of guidance for sufficiently large values of S and $V_o/R_o\dot{\theta}_s$. For usual values of S it is also the most rapidly disappearing term in the equation, and this behavior is usually seen in the early portion of the thrust programs.

The parameter, K, primarily influences the final behavior of the thrust program through the first term $(\dot{V}\bar{u}_V)$ in equation (35).

For n = -1 the second and third terms $(S\bar{W}_R \times \bar{V})$ tend to cancel each other, causing the large and rapid shift in thrust angle observed for cases having K = 1. For n = +1 the cancellation does not occur.

Finally, it may be noted that some of these effects are due to the rotation of the reference frame, $\bar{W}_{Ia}$. If an inertial frame were used, the Coriolis terms of equation (22) would not appear, in which case there would be no cancellation effects for n = -1, and the limiting behavior prior to the impulse for K = 1 would be different. In general, there would be an increased uniformity among the thrust programs.

CONCLUSIONS

It has been recognized that the rendezvous problem is capable of general kinematic solution and the theory of proportional navigation for interception has been extended to rendezvous by modification of the end conditions. A simple navigation function has been selected and the resulting solutions of the rendezvous problem have been investigated. The investigation indicates the following conclusions:

1. The extension of proportional navigation theory to produce rendezvous requires the additional parameters, K_1, which, in the simple function considered, reduce to the single parameter K whose primary effect is to control velocity so as to meet the rendezvous end conditions. This parameter also strongly influences the maneuver time.

2. The thrust program requires variable thrust, both in magnitude and direction. The maximum value and the range of variation of the thrust depend on the initial conditions of relative motion and the constraint parameters, S and K. The behavior of the thrust program is strongly influenced by the value of S in the early portion, and by the value of K in the later portion near rendezvous.

3. An acceleration forcing function appropriate for use in automatic feedback guidance systems has been derived from the kinematics.

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Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., Feb. 14, 1961

APPENDIX A

NOTES ON THE SECOND TRAJECTORY CONSTRAINT EQUATION

The function taken in equation (13) of the text as the second constraint equation has, for the purposes of the present report, the advantage of simplicity. However, it may be generalized to the polynomial

$$f(\tau) = \sum_{i=1}^{\infty} a_i \tau^{K_i(S-1)} \tag{A1}$$

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where the K_i are taken as an ascending sequence of arbitrary constants. The conditions of equation (12) applied to this polynomial produce the following restrictions:

$$\left.\begin{array}{c} S > 1 \\[2ex] K_i \geq 1 \\[2ex] \displaystyle\sum_{i=1}^{\infty} a_i = 1 \\[4ex] t_f = \left(\displaystyle\sum_{i=1}^{\infty} K_i a_i\right) \dfrac{R_o}{V_o} \dfrac{L_o}{\sin L_o} \\[4ex] \displaystyle\sum_{i=1}^{\infty} a_i K_i \tau^{K_i(S-1)-1} \geq 0 \qquad \text{for} \quad 0 \leq \tau \leq 1 \end{array}\right\} \tag{A2}$$

Without exploring fully the range of such functions, it is evident from (A1) and (A2) that the permissible sequences a_i at least include all sequences of any length made up of positive numbers whose sum is 1. For example, one such set is given by:

$$f(\tau) = \frac{1}{N} \sum_{i=1}^{N} \tau^{K_i(S-1)} \tag{A3}$$

which might be described as an "average value" system.

Equation (A1) indicates that a host of solutions of the rendezvous problem must exist within the confines of proportional navigation alone. The analysis is not carried farther here.

APPENDIX B

GRAVITY CONTRIBUTION TO THE REQUIRED THRUST FOR THE SATELLITE RENDEZVOUS PROBLEM

For the satellite rendezvous problem the effects of gravitation appear in equation (28) as the difference in the acceleration due to gravity on the satellite and vehicle $\bar{f}_{g_s} - \bar{f}_{g_v}$.

A central force field will be assumed and the gravity forces will be expressed in the geocentric reference space described in the text. The positions of the satellite and vehicle from the center of the earth are $\bar{R}_s$ and $\bar{R}_v$, and the acceleration due to gravity for a central force field at the satellite is g_s.

$$\left.\begin{aligned} \bar{f}_{g_s} &= -g_s\bar{u}_1 \\ \bar{f}_{g_v} &= -g_s\left(\frac{R_s}{R_v}\right)^3 \frac{\bar{R}_v}{R_s} \end{aligned}\right\} \qquad \text{(B1)}$$

Noting that

$$\frac{\bar{R}_v}{R_s} = \bar{u}_1 + \frac{R}{R_s}\left(x_1\bar{u}_1 + x_2\bar{u}_2 + x_3\bar{u}_3\right) \qquad \text{(B2)}$$

then

$$\left(\frac{R_v}{R_s}\right)^{-3} = \left[1 + 2\frac{R}{R_s}x_1 + \left(\frac{R}{R_s}\right)^2\right]^{-3/2}$$

During terminal guidance the two vehicles are sufficiently close together to assume

$$\frac{R}{R_s} \ll 1$$

$$2\frac{R}{R_s}x_1 + \left(\frac{R}{R_s}\right)^2 \ll 1$$

and therefore $(R_v/R_s)^{-3}$ may be expanded binomially. The gravity difference then becomes:

$$\bar{f}_{g_s} - \bar{f}_{g_v} = g_s\frac{R_o}{R_s}\left(\frac{R}{R_o}\right)\left(-2x_1\bar{u}_1 + x_2\bar{u}_2 + x_3\bar{u}_3\right) + 3g_s\left(\frac{R_o}{R_s}\right)^2\left(\frac{R}{R_o}\right)\left[\left(\frac{3}{2}x_1^2 - \frac{1}{2}\right)\bar{u}_1 - x_1x_2\bar{u}_2 - x_1x_3\bar{u}_3\right] + \text{Order}\left(\frac{R_o}{R_s}\right)^3 \qquad \text{(B3)}$$

Only the first-order terms will be retained in the computations for the satellite rendezvous:

$$\overline{f}_{g_s} - \overline{f}_{g_v} = g_s \frac{R_o}{R_s} \frac{R}{R_o} (-2x_1\overline{u}_1 + x_2\overline{u}_2 + x_3\overline{u}_3) \tag{B4}$$

The successive terms in (B3) differ in magnitude by the factor R_o/R_s, which in the computations is restricted to values less than 0.02. This corresponds to values of initial range, R_o, up to 100 miles for near-earth orbits. Greater ranges may be included in the computations by retaining the second-order terms in (A3). It may be noted that the first term to appear due to oblateness is of order $J(R_E/R_s)^2(R_o/R_s)g_s$ where R_E is the earth's equatorial radius and J, the oblateness constant, is about 10^{-3}. This term will be less than 10 percent of the second-order term in (B3) for values of R_o/R_s of 0.02 or somewhat greater, and therefore the gravity difference may be given up to second-order terms assuming a central force field.

The accuracy with which the gravity difference need be computed depends on its magnitude relative to the magnitude of the remaining terms in the expression for the thrust program (eq. (28)), that is, the magnitude of the inertial relative acceleration due to external forces compared to the magnitude of the inertial relative acceleration required by the commanded maneuver. Three regimes may therefore be recognized: The commanded relative acceleration may be much greater than the gravity difference which may then be neglected; the commanded relative acceleration may be of the same order of magnitude as the gravity difference which must then be retained to first order; or the commanded maneuver may be so close to the free-fall motion that the commanded acceleration and the first-order gravity terms cancel each other, in which case the second-order gravity terms become the largest.

For the problem discussed in the text (coplanar rendezvous with a satellite in circular orbit), it was seen that the regime of operation in any case depends on the initial conditions; in particular, on the dimensionless parameter $V_o/R_o\dot{\theta}_s$. If this parameter is large, then the case is in the first regime and the gravity difference may be neglected. If the parameter is small, then the case is in the second regime and the first-order gravity difference must be retained. In this case the required thrust per unit mass computed for the maneuver is necessarily small, that is, of the order of the gravity difference. The computations involved cases in both these regimes, and hence the gravity difference term was retained to first order.

The third regime does not occur except under the fortuitous circumstance that the commanded relative acceleration is the same, both in magnitude and direction, as the free-fall acceleration. This event, if it should occur, will be of such short duration and the acceleration error sufficiently small that no significant motion errors can build up.

The second-order gravity difference term may also be important in the second regime from the point of view that a small thrusting error maintained over a sufficiently long maneuver time will build up significant position errors. However, for the proportional navigation system this effect is obviated by fairly short maneuver times and by the feedback nature of the system. The second-order gravity difference terms may therefore be neglected in general.

REFERENCES

1. Brissenden, Roy F., Burton, Bert B., Foudriat, Edwin C., and Whitten, James B.: Analog Simulation of a Pilot-Controlled Rendezvous. NASA TN D-747, 1961.

2. Wrigley, Walter: Performance of a Linear Accelerometer. Mass. Inst. of Tech., Dept. of Aero. Engr., Orbital and Satellite Vehicles, vol. 1. Notes for a Special Summer Program, Summer Session, Aug. 6-17, 1956, Ch. 9, pp. 9-1 - 9-14.

3. Clohessy, W. H., and Wiltshire, R. S.: Terminal Guidance System for Satellite Rendezvous. IAS Paper No. 59-93, June 1959.

4. Sears, N. E., and Felleman, P. G.: Terminal Guidance for a Satellite Rendezvous. Mass. Inst. of Tech., Instrumentation Lab., Rep. R-232, April 1959.

5. Spradlin, Louis W.: The Long-Time Satellite Rendezvous Trajectory. Proc. National Specialists Meeting on Guidance of Aerospace Vehicles, (Sponsored by IAS), May 1960, pp. 21-27.

6. Eggleston, John M.: Optimum Time to Rendezvous. ARS Jour. vol. 30, no. 11, Nov. 1960, pp. 1089-1091.

7. Harrison, Eugene: Some Considerations of Guidance and Control Techniques for Coplanar Orbital Rendezvous. Proc. National Specialists Meeting on Guidance of Aerospace Vehicles, (Sponsored by IAS), May 1960, pp. 14-20.

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TABLE I.- MASS RATIOS (M_f/M_o) FOR VARIOUS SETS OF INITIAL CONDITIONS AND SYSTEM PARAMETERS

(a) $\Gamma_o = 3\pi/2$, $S = 5$, $n = -1$, $L_o = \pi/2$

$\frac{R_o\dot{\theta}_s}{c}$	0.005			0.01			0.02			0.03			0.04			0.05		
$\frac{V_o}{c}$ \ K	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
0.01	0.973	0.981	0.982	0.955	0.960	0.958	0.884	0.870	0.858	0.780	0.738	0.713	0.655	0.586	0.550	0.523	0.435	0.394
.03	.929	.948	.951	.927	.951	.953	.907	.930	.931	.870	.884	.878	.819	.820	.807	.758	.743	.722
.06	.861	.895	.898	.863	.899	.903	.859	.904	.908	.846	.893	.896	.823	.866	.865	.793	.828	.823
.1	.776	.827	.833	.779	.832	.838	.782	.841	.847	.779	.845	.852	.770	.842	.848	.756	.828	.832

(b) $R_o\dot{\theta}_s/c = 0.01$, $\Gamma_o = 3\pi/2$, $V_o/c = 0.05$, $S = 5$

K \ nL_o	$-3\pi/4$	$-\pi/2$	$-\pi/4$	$-\pi/18$	$\pi/18$	$\pi/4$	$\pi/2$	$3\pi/4$
1	0.901	0.897	0.916	0.937	0.922	0.892	0.870	0.870
2	.917	.917	.934	.952	.943	.916	.894	.890
3	.919	.920	.937	.953	.945	.920	.899	.894

TABLE II.- COMPARISON OF MASS RATIOS FOR TWO-IMPULSE AND PROPORTIONAL NAVIGATION METHODS

Initial conditions					Two-impulse method		Proportional navigation (S=5)		
$\frac{R_o\dot{\theta}_s}{c}$	Γ_o	$\frac{V_o}{c}$	n	L_o	$\dot{\theta}_s t_f$,* radian	$\frac{M_f}{M_o}$	K	$\dot{\theta}_s t_f$, radian	$\frac{M_f}{M_o}$
0.005	$\frac{\pi}{2}$	0.01	-1	$\frac{\pi}{18}$	4.0	0.985	1.0	0.5	0.981
							2.0	1.0	.989
							3.0	1.5	.990
.01	$\frac{\pi}{2}$	.01	-1	$\frac{\pi}{18}$	4.0	.975	1.0	1.0	.966
							2.0	2.0	.975
							3.0	3.0	.976

*Computations for the two-impulse theory may be made dimensionless in the same manner as the proportional navigation results. In that case it is again unnecessary to specify the orbit or a value of c. The free parameter of the two-impulse method is no longer the maneuver time, t_f, but the portion of the circular orbit that the satellite passes through during the maneuver, $\dot{\theta}_s t_f$, which may also be regarded as a dimensionless maneuver, time.

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TABLE III.- EFFECTS OF POSITION ON THE THRUST PROGRAM FOR

LARGE $V_o/R_o\dot{\theta}_s$; S = 5, K = 1

Initial conditions: $n = -1$ $\quad \frac{R_o\dot{\theta}_s}{c} = 0.01$ $\quad L_o = \frac{\pi}{2}$

$\frac{V_o}{c} = 0.1$ $\quad \frac{V_o}{R_o\dot{\theta}_s} = 10$

τ	Dimensionless thrust-to-mass ratio		Thrust direction α, deg	
	$\Gamma_o = 0$	$\Gamma_o = \frac{3\pi}{2}$	$\Gamma_o = 0$ (a)	$\Gamma_o = \frac{3\pi}{2}$ (b)
1.0	5.166	5.194	21.70	21.58
.9	2.898	2.913	44.24	43.75
.8	1.797	1.798	61.36	60.56
.7	1.110	1.103	73.32	72.16
.6	.6379	.6280	81.09	79.51
.5	.3092	.2992	85.70	83.16
.4	.0903	.0820	87.79	80.75
.3	.0428	.0505	270.84	278.88
.2	.1117	.1165	270.34	272.67
.1	.1378	.1402	270.19	271.16
0	.1425 (c)	.1425 (c)	270 180	270 180

[a] $\frac{M_f}{M_o} = 0.7795$

[b] $\frac{M_f}{M_o} = 0.7794$

[c] Impulse at $\tau = 0$

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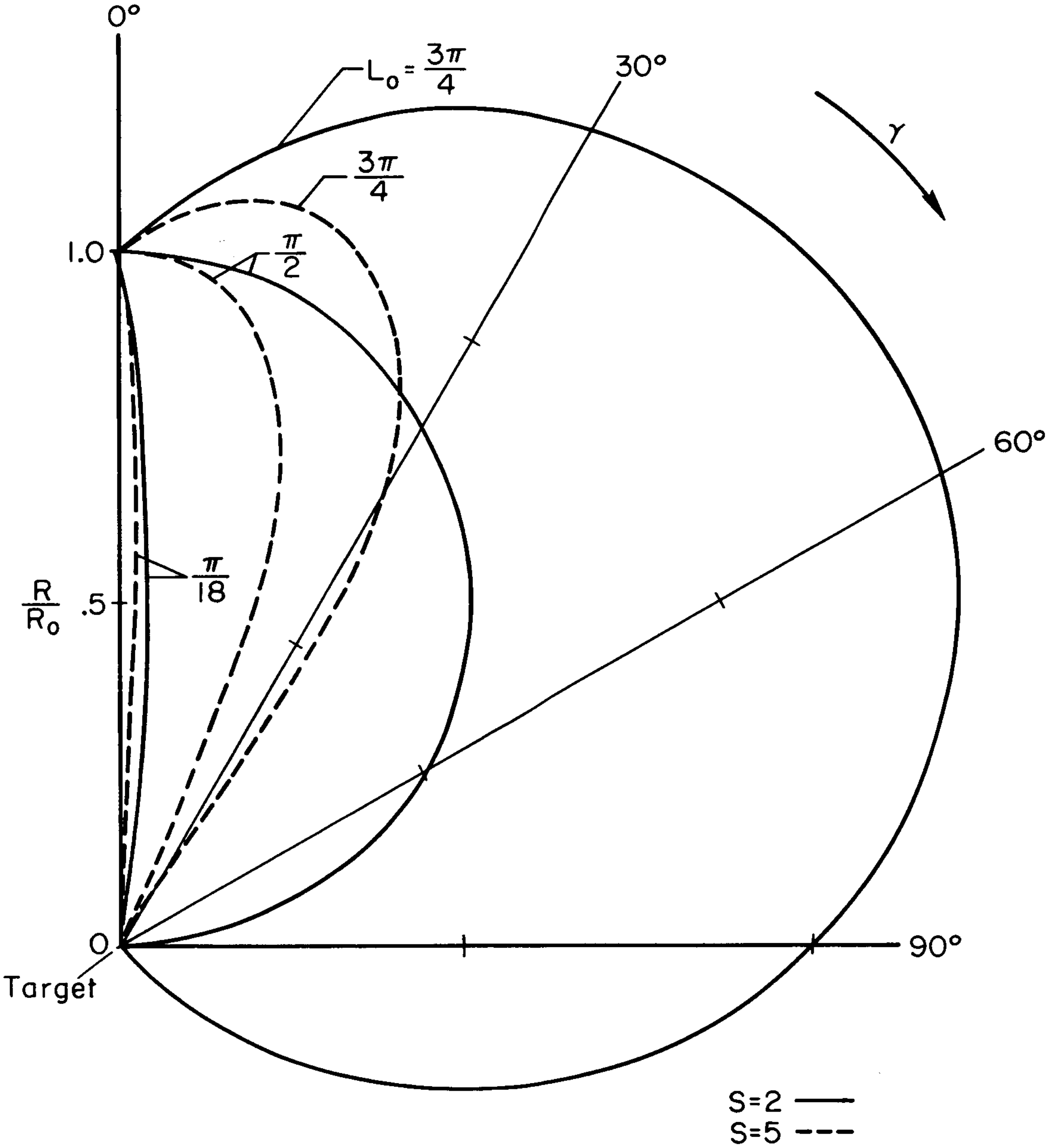

Figure 1.- Maneuver paths: range versus angular departure from the initial line of sight.

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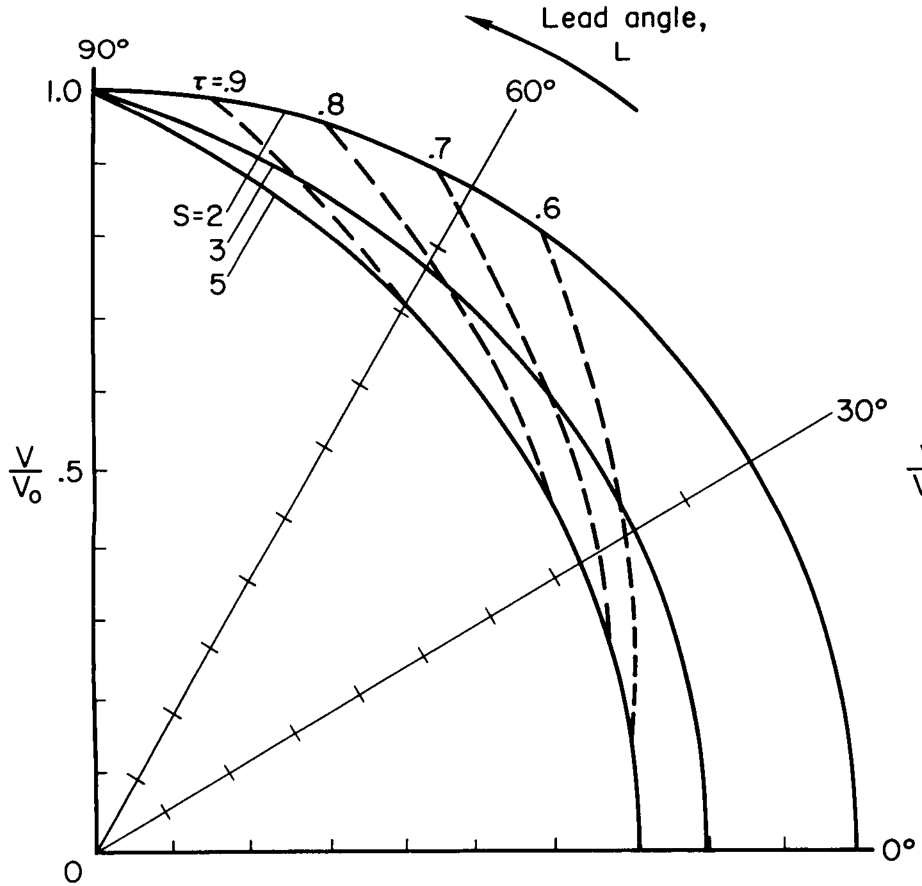
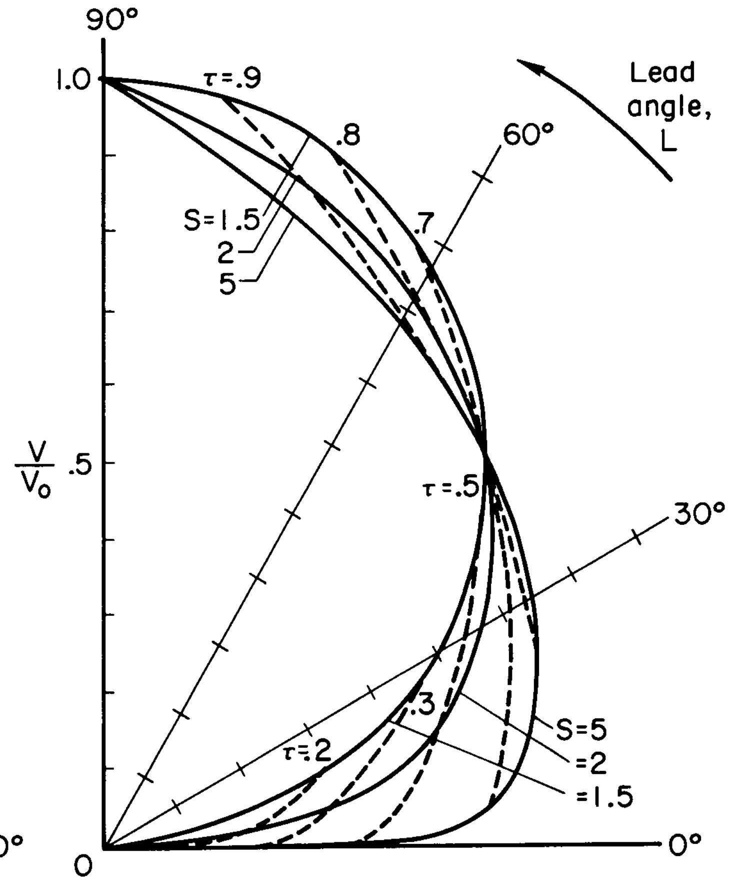
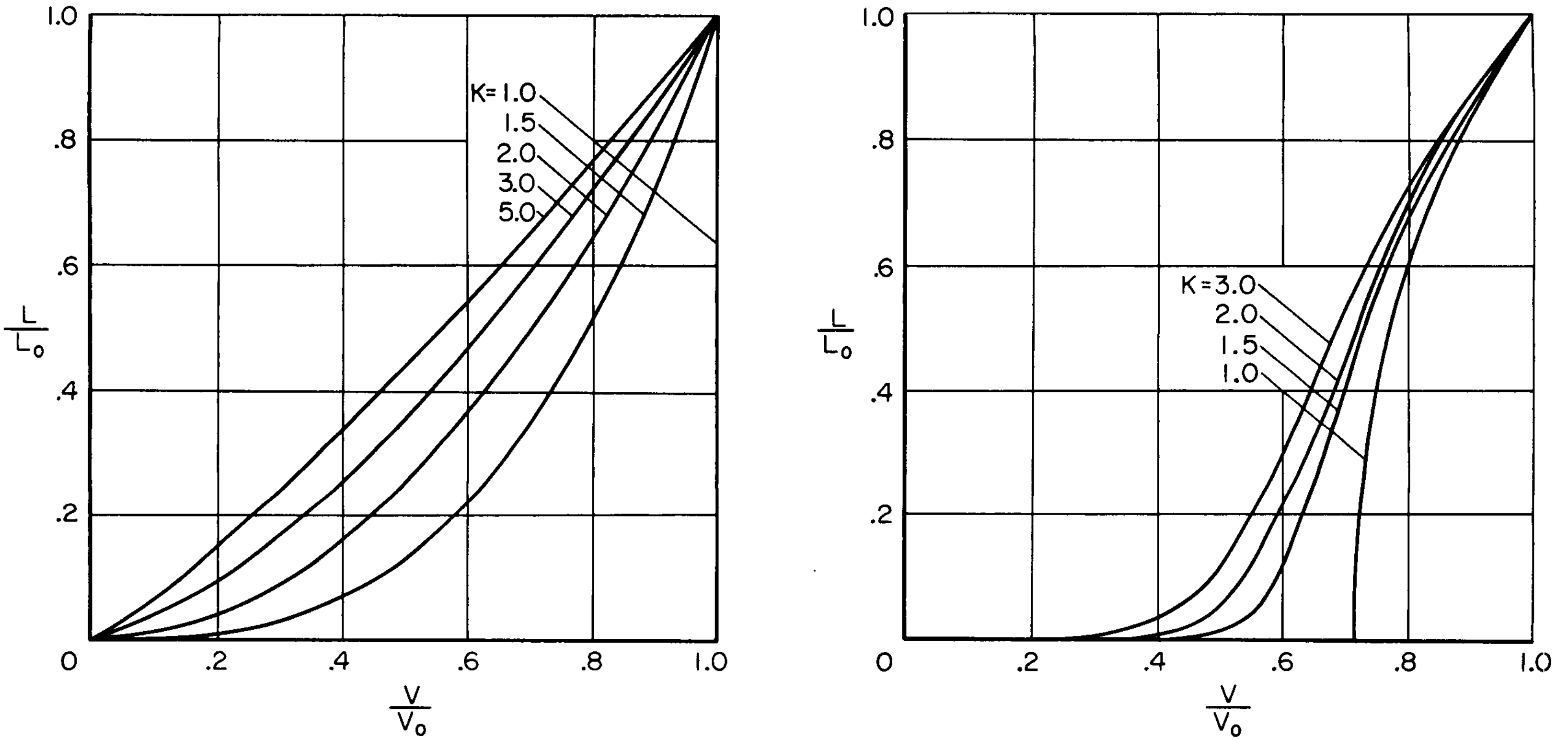


(a) K = 1



(b) K = 2

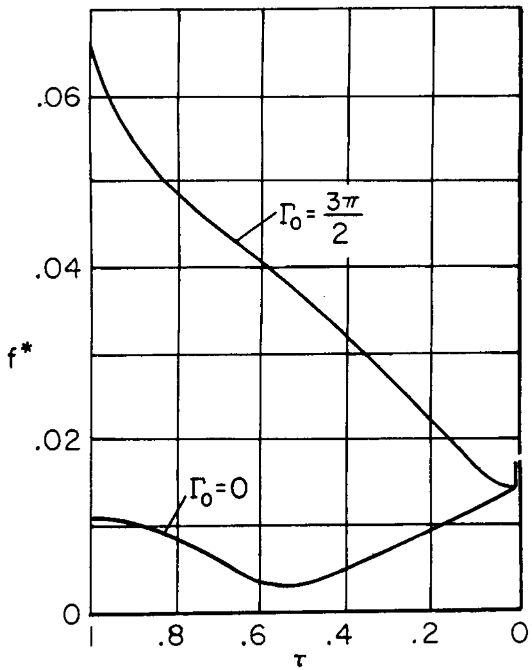
Figure 2.- Lead angle versus velocity, $L_o = \frac{\pi}{2}$.

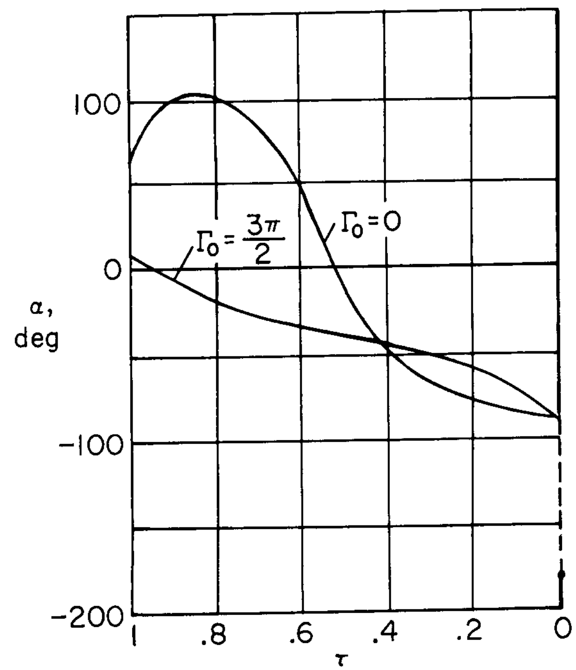


(a) S = 2

(b) S = 5

Figure 3.- Lead angle versus velocity, $L_o = \frac{\pi}{2}$.





System parameters

S = 5

K = 1

Mass ratio:

$\frac{M_f}{M_o} = .9666$ for $\Gamma_o = 0$

$\frac{M_f}{M_o} = .8840$ for $\Gamma_o = \frac{3\pi}{2}$

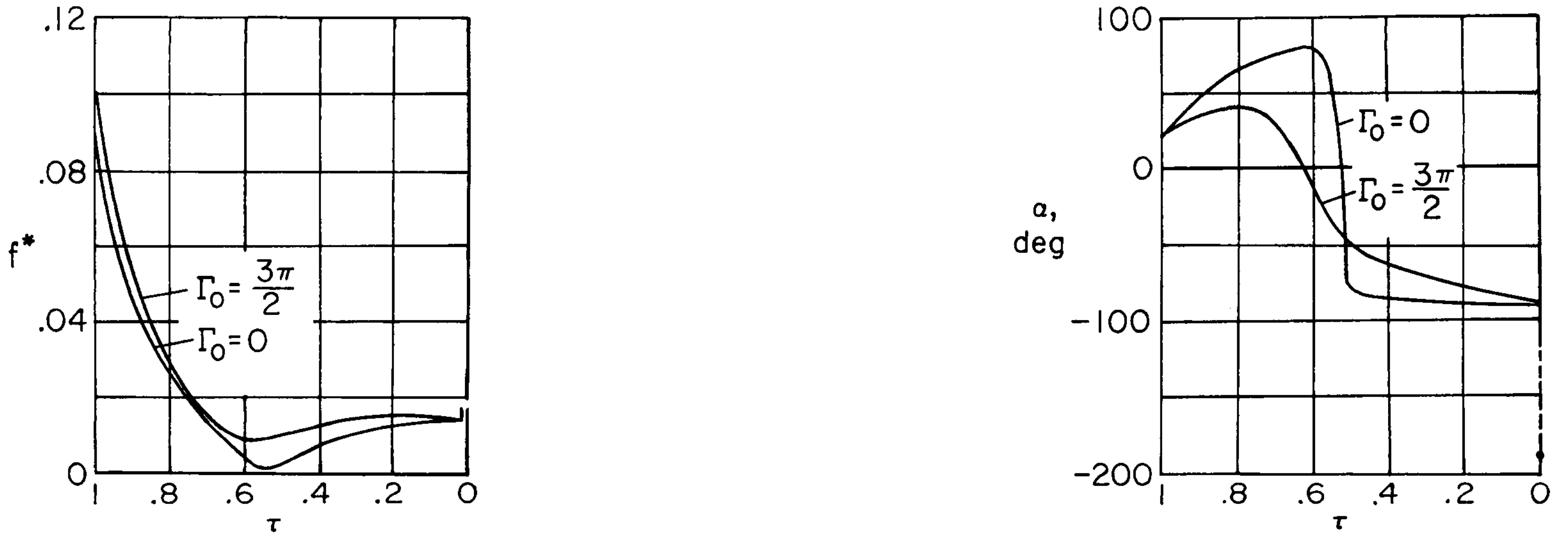
Initial conditions

$n = -1$ $\quad \frac{V_o}{C} = .01$

$L_o = \frac{\pi}{2}$ $\quad \frac{R_o \dot{\theta}_s}{C} = .02$

(a) $\frac{V_o}{R_o \dot{\theta}_s} = 0.5$

Figure 4.- Effect of position on the thrust program.



System parameters

$S = 5$

$K = 1$

Mass ratio:

$\frac{M_f}{M_o} = .9758$ for $\Gamma_o = 0$

$\frac{M_f}{M_o} = .9724$ for $\Gamma_o = \frac{3\pi}{2}$

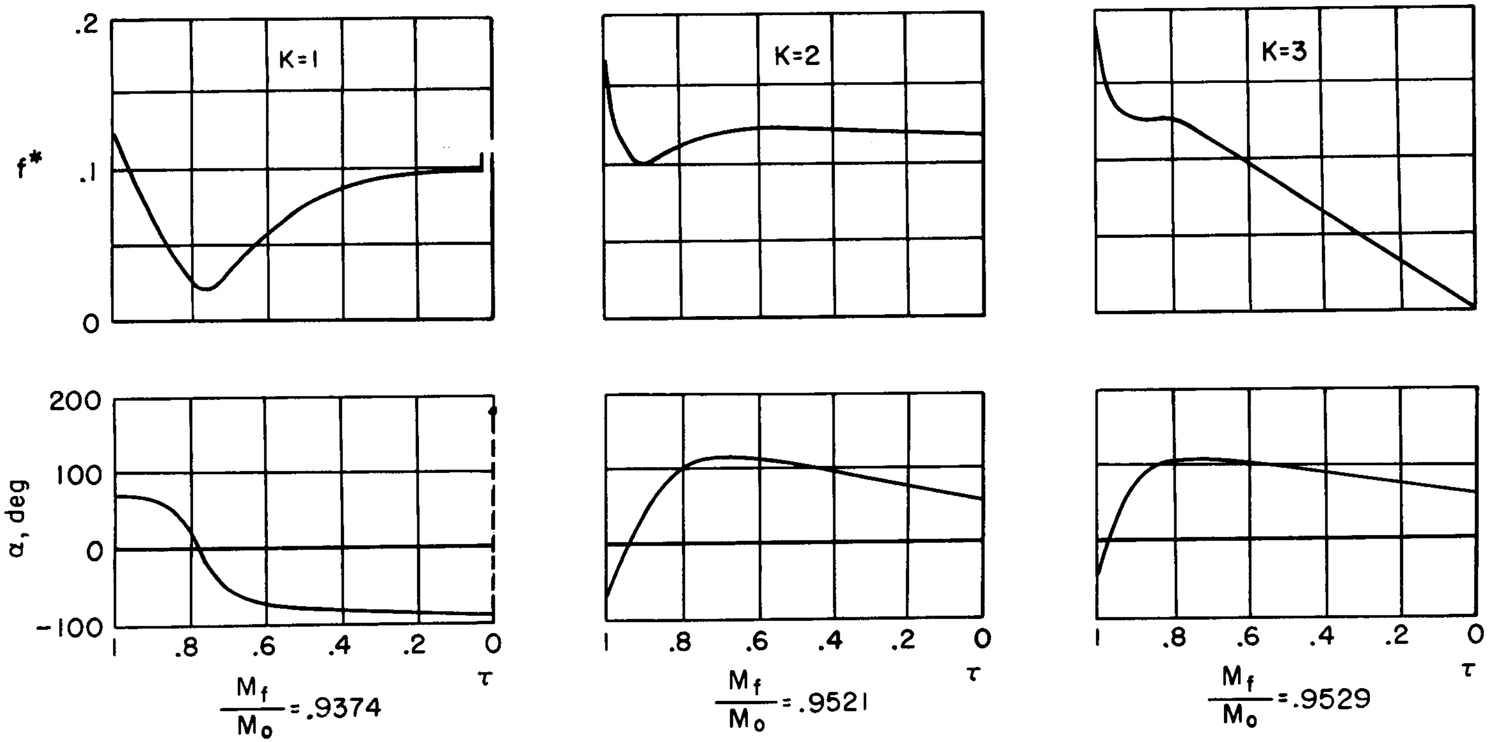
Initial conditions

$n = -1$ $\qquad \frac{V_o}{C} = .01$

$L_o = \frac{\pi}{2}$ $\qquad \frac{R_o \dot{\theta}_s}{C} = .005$

(b) $\frac{V_o}{R_o \dot{\theta}_s} = 2$

Figure 4.- Concluded.

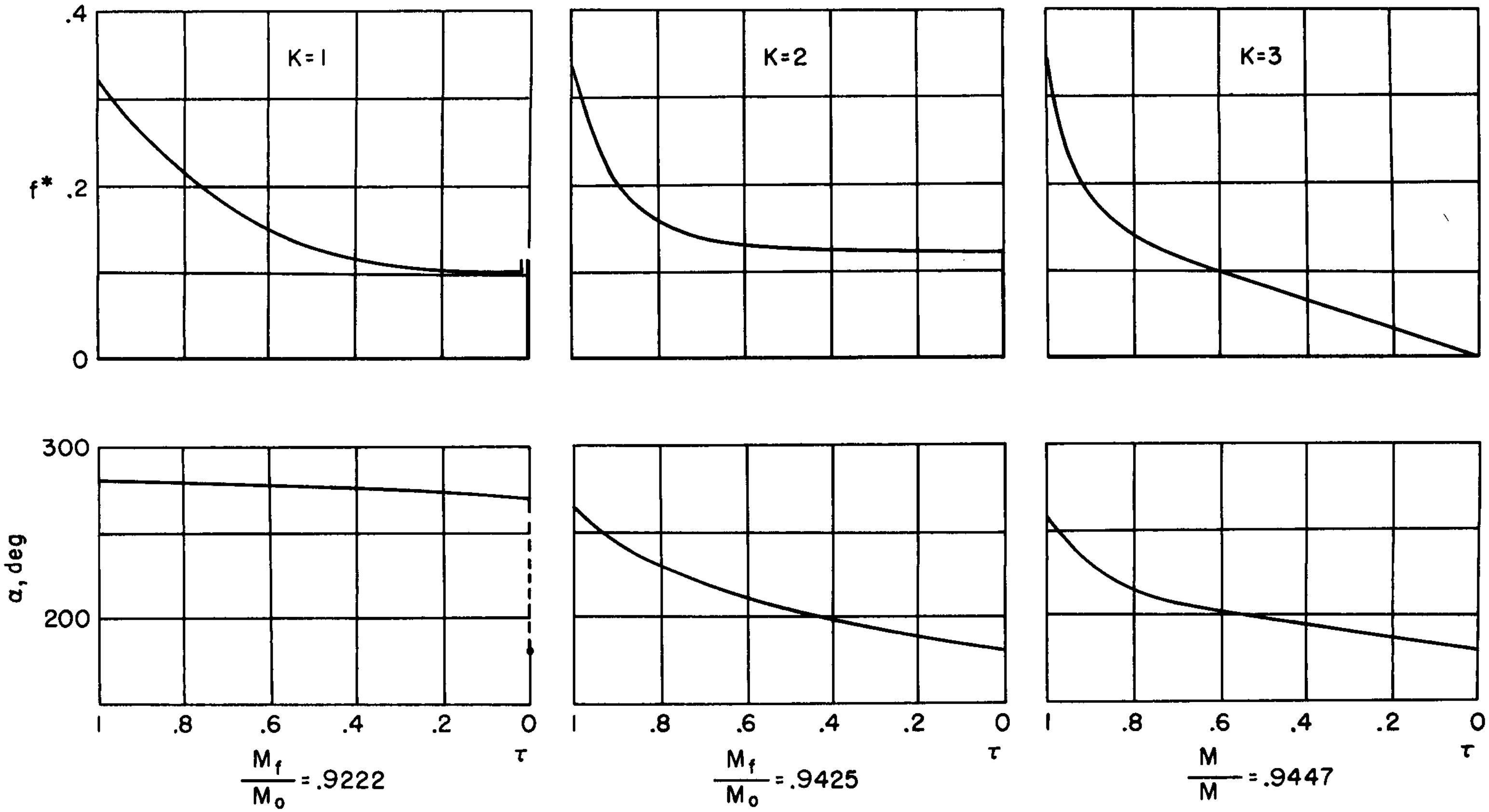


Initial conditions: $\frac{R_o \dot{\theta}_s}{C} = .01 \quad \Gamma_o = \frac{3\pi}{2}$

$\frac{V_o}{C} = .05 \quad L_o = \frac{\pi}{18}$

(a) n = -1

Figure 5.- Effects of n and K on the thrust program, S = 5.

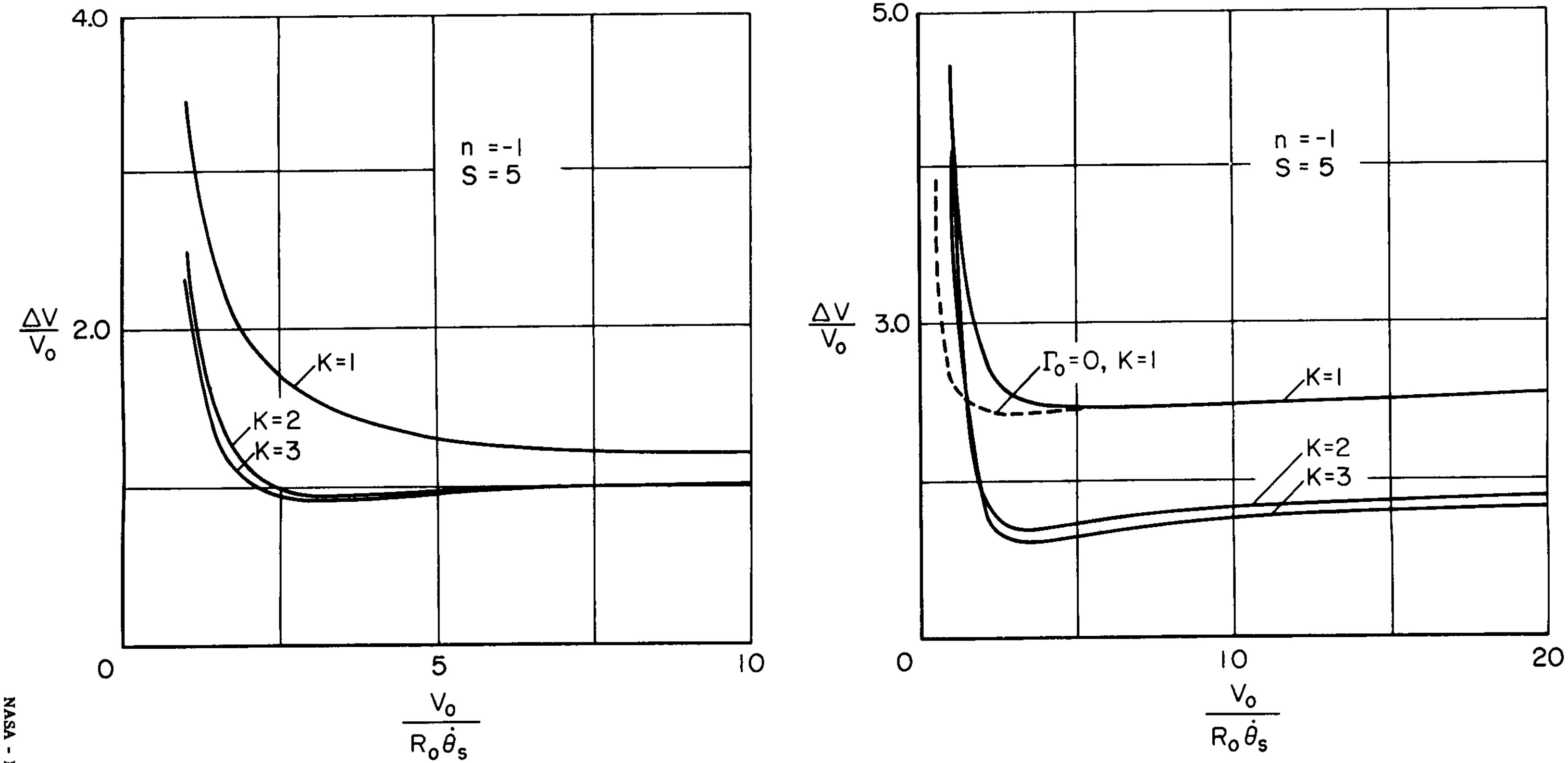


Initial conditions: $\frac{R_o \dot{\theta}_s}{C} = .01 \quad \Gamma_o = \frac{3\pi}{2}$

$\frac{V_o}{C} = .05 \quad L_o = \frac{\pi}{18}$

(b) n = 1

Figure 5.- Concluded.



(a) $L_o = \frac{\pi}{8}$

(b) $L_o = \frac{\pi}{2}$

Figure 6.- Characteristic velocity requirements, $\Gamma_o = \frac{3\pi}{2}$.

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