NASA TN D-814

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TECHNICAL NOTE

D-814

NOMOGRAPHIC SOLUTION OF THE MOMENTUM EQUATION FOR VTOL-STOL AIRCRAFT

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

WASHINGTON April 1961

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SUMMARY

A general nomographic solution for the induced velocities and wake skew angle is presented for use with VTOL-STOL systems.

INTRODUCTION

VTOL-STOL aircraft are characterized in general by the fact that in some portion of their flight envelope the wake is sharply inclined to the free stream. Under such conditions, the usual small-angle assumptions used in determining the induced velocities, and consequently, the power required, are no longer valid. Indeed, the use of small-angle assumptions leads to such anomalous results as infinite induced velocities and required power in the extreme case of hovering.

The aforementioned difficulties may be avoided by a more complete examination of the horizontal and vertical momentum imparted to the air by the aircraft at low speeds. The resulting equation is a quartic in the induced velocity, and, as such, is difficult to apply. On the other hand, this quartic can be solved in its most general terms and the resulting solution then can be derived and presented in the form of a chart, or nomograph, from which the required induced velocities may be read directly. This paper presents such a chart.

The problem of estimating the induced velocities becomes of additional importance since the induced velocities combine with the forward velocity to determine the wake skew angle upon which, for example, wind-tunnel corrections and ground effect have been shown to depend. (See refs. 1 and 2.) The wake skew angle also can be read directly from the nomograph presented herein.

SYMBOLS

A_m	momentum area of aerodynamic force-generating system, sq ft
D_i	induced drag, or horizontal component of force, lb
L	lift, lb
n	ratio of initial to final induced velocity
q	free-stream dynamic pressure, $\frac{1}{2}\rho V^2$, lb/sq ft
q_s	slipstream dynamic pressure, $\frac{1}{2}\rho\left(nu_0 + V\right)^2$, lb/sq ft
T	thrust of a rotor, lb
u_0	horizontal induced velocity, at the force-generating system, required to produce a given horizontal force, positive rearward, ft/sec
V	free-stream velocity, ft/sec
V_R	resultant velocity at force-generating system, ft/sec
w_0	vertical induced velocity, at the force-generating system, required to produce a given vertical force, positive upward, ft/sec
w_h	hovering induced velocity, vertical induced velocity when $V = 0$ and $D_i = 0$ (see eqs. (8) and (23)), positive upward, ft/sec
X,Z	lift and drag axes, perpendicular and parallel, respectively, to the horizontal or free-stream direction (see fig. 1)
α	angle of attack of tip-path plane, positive when leading edge of disk is high, deg
θ_n	net wake deflection angle, angle between the wake and the X-axis, or free-stream direction, $90^\circ - X$, deg (see fig. 1)
v_0	mean induced velocity, normal to the tip-path plane of a rotor, positive downward, ft/sec

v_h hovering mean-induced velocity, normal to the tip-path plane of a rotor, positive downward, ft/sec

ρ mass density of air, slugs/cu ft

χ wake skew angle, angle between the wake and the negative Z-axis, positive when measured rearward, deg (see fig. 1)

THEORY

Induced Velocities

Because of its lift and drag forces, the machine creates the velocities u_0 and w_0 at its own location and the velocities nu_0 and nw_0 in the far wake. The force and velocity vectors are shown in figure 1. From this figure, it may be seen that the resultant velocity V_R at the force-generating device is

$$V_R = \sqrt{(V + u_0)^2 + (-w_0)^2} \tag{1}$$

Furthermore, the lift and induced drag are

$$L = \rho A_m V_R(-nw_0) \tag{2}$$

$$D_i = \rho A_m V_R(-nu_0) \tag{3}$$

where A_m is the area of the cylinder of air affected by a wing, the actual area of a propeller or rotor, or the duct-exit area of a ducted fan.

Notice that a forward thrust is merely considered equivalent to a negative drag in equation (3).

Dividing equation (3) by equation (2) yields

$$\frac{D_i}{L} = \frac{u_0}{w_0} \tag{4}$$

provided, of course, that L is not zero.

Now substitute equation (4) into equation (1) to obtain

$$V_R = \sqrt{V^2 + 2\left(\frac{D_i}{L}\right) Vw_0 + \left[1 + \left(-\frac{D_i}{L}\right)^2\right](-w_0)^2} \tag{5}$$

or

$$\frac{V_R}{-w_0} = \sqrt{\left(\frac{V}{-w_0}\right)^2 - 2\left(\frac{D_i}{L}\right)\left(\frac{V}{-w_0}\right) + 1 + \left(-\frac{D_i}{L}\right)^2} \tag{6}$$

and from equation (2)

$$w_0{}^2 = \frac{L}{n\rho A_m\left(\frac{V_R}{-w_0}\right)} \tag{7}$$

Now, if the force-generating device could hover with the same lift, the same momentum area, zero forward speed, and zero induced drag, the vertical induced velocity in hovering w_h would be found from equations (6) and (7) as

$$w_h = \pm\sqrt{\frac{L}{n\rho A_m}} \tag{8}$$

where the negative sign corresponds to a positive or upward lift and the positive sign corresponds to a negative or downward lift. Note that w_h is purely fictitious for a wing, but that it can still be calculated for use as a reference velocity.

Combining equations (6), (7), and (8) yields

$$\left(\frac{w_0}{w_h}\right)^2 = \frac{-w_0}{V}$$

$$= \frac{1}{\sqrt{1 + \left(\frac{V}{w_0} + \frac{D_i}{L}\right)^2}} \tag{9}$$

or, finally, squaring both sides of equation (9)

$$\left(\frac{w_0}{w_h}\right)^4 = \frac{1}{1 + \left(\frac{V}{w_0} + \frac{D_i}{L}\right)^2} \tag{10}$$

Equation (10) is essentially the same as the equivalent expression derived for a rotor in reference 3 except that in the present case equation (10) gives the vertical component of induced velocity in terms of an arbitrary value of the induced drag-lift ratio, whereas reference 3 solves for a total induced velocity which is assumed to be normal to the plane of the rotor.

Equation (10) is, of course, a quartic in w_0 and, as such, is difficult to solve for any one given case. Points for use in the construction of a general nomographic solution, however, may be obtained rapidly by substituting a series of values for V/w_0 into equation (10), solving for w_0/w_h, and then noting that

$$\frac{V}{w_h} = \frac{V}{w_0}\frac{w_0}{w_h} \tag{11}$$

The results of this calculation can then be plotted with w_0/w_h as a function of V/w_h for specified values of D_i/L. Such a chart is presented in figure 2; however, a full discussion of it will be deferred to a later section of this paper.

With w_0 obtained from a general nomograph of equations (10) and (11), u_0 can be obtained from equation (4).

Wake Skew Angle

The remaining item of interest is the wake skew angle, or its complement, the angle by which the wake is deflected from the horizontal or free-stream direction. This angle may be obtained by examining figure 1 which indicates that

$$\tan X = \frac{V + u_0}{-w_0}$$

$$= -\frac{V}{w_0} - \frac{D_i}{L} \tag{12}$$

so that equation (9) may be rewritten as

$$\left(\frac{w_0}{w_h}\right)^2 = \frac{1}{\sqrt{1 + \tan^2\chi}} \tag{13}$$

or, solving equation (13) for χ

$$|\chi| = \cos^{-1}\left(\frac{w_0}{w_h}\right)^2 \tag{14}$$

where, from equation (12), χ is positive if $\frac{V}{-w_0} > \frac{D_i}{L}$ and negative if $\frac{V}{-w_0} < \frac{D_i}{L}$.

The net wake deflection angle, being the complement of χ, is then immediately given as

$$\theta_n = \sin^{-1}\left(\frac{w_0}{w_h}\right)^2 \tag{15}$$

and, similarly, θ_n is in the first quadrant if $\frac{V}{-w_0} > \frac{D_i}{L}$ and in the second quadrant if $\frac{V}{-w_0} < \frac{D_i}{L}$.

The Case L = 0

When L is zero, the foregoing treatment is not necessary. Note that for $L = 0$, $w_0 = 0$, so that equation (1) becomes

$$V_R = V + u_0 \tag{16}$$

Equation (3) may then be solved immediately in the form

$$u_0 + \frac{V}{2} = \sqrt{\left(\frac{V}{2}\right)^2 - \frac{D_i}{n\rho A_m}} \tag{17}$$

It is of interest to square both sides of equation (17) and then multiply both sides by 2ρ to obtain

$$\frac{1}{2}\rho(2u_0 + V)^2 = \frac{1}{2}\rho V^2 - \frac{2}{n}\frac{D_i}{A_m} \tag{18}$$

which for the large class of devices characterized by $n = 2$ (i.e., wings, rotors, unshrouded propellers) becomes

$$q_s = q - \frac{D_i}{A_m} \tag{19}$$

Equation (19) is, of course, quite familiar since it yields the slipstream dynamic pressure used in nondimensionalizing many VTOL test results. (Notice that the thrust of a propeller at zero angle of attack is equal to $-D_i$.) It is evident that equation (19) holds for all wings, rotors, and propellers where n is equal to 2. On the other hand, consider, for example, a ducted fan where n is not equal to 2. In this case, equation (17) cannot be simplified to the form of equation (19). Note that, in general,

$$\begin{aligned} q_s &= \frac{1}{2}\rho(nu_0 + V)^2 \\ &\neq q - \frac{D_i}{A_m} \end{aligned} \tag{20}$$

It is, of course, obvious for the case of $L = 0$ that $\chi = 90^\circ$ and $\theta_n = 0^\circ$. (See fig. 1.)

Equivalence of Reference 3

Reference 3 derives an induced-velocity equation which corresponds to equation (10) for the special case of a rotor. Since the work of reference 3 has been confirmed experimentally, both with regard to induced velocity (ref. 4) and wake skew angle (ref. 5) it is advisable to examine the equivalence of reference 3 and the present analysis.

Reference 3 assumes a thrust force T on the rotor which is normal to the rotor disk and a corresponding induced velocity ν also normal to the disk. Thus, from figure 1(b)

$$\frac{D_i}{L} = \tan\alpha \tag{21}$$

$$\nu_0 = \sqrt{w_0{}^2 + u_0{}^2}$$

$$= w_0\sqrt{1 + \left(\frac{D_i}{L}\right)^2}$$

$$= \frac{-w_0}{\cos\alpha} \tag{22}$$

$$\nu_h = \frac{-w_h}{\cos\alpha} \tag{23}$$

Substituting the values of w_0, w_h, and D_i/L as obtained from equations (21), (22), and (23) into equation (10), expanding, and then simplifying leads to

$$\left(\frac{\nu_0}{\nu_h}\right)^4 = \frac{1}{1 - 2\frac{V}{\nu_0}\sin\alpha + \left(\frac{V}{\nu_0}\right)^2} \tag{24}$$

which, noting that α in this paper is identical to $-\alpha$ in reference 3, is identical to the expression derived in reference 3.

In view of the complete equivalence of reference 3 and the present analysis, the experimental confirmation of references 4 and 5 may be considered to apply to the present analysis as well.

PRESENTATION OF RESULTS

A general nomograph for solution of the momentum equation, based on the preceding equations, is presented in figure 2. This figure is primarily a set of curves giving w_0/w_h as a function of V/w_h for specified values of the induced drag-lift ratio. As such w_0/w_h may be read directly on the ordinate for any given value of V/w_h. It is understood, of course, that the velocity w_h is obtained from equation (8).

In consequence of equation (11), lines of constant V/w_0 are radial lines in figure 2. A number of such lines are shown. These may be used as a guide in determining the values of V/w_0 without recourse to equation (11). In this regard, for $V/w_0 \geqq 6$, a line may be drawn through both the origin and any point determined by specified values of V/w_h and D_i/L. The required value of V/w_0 may then be read directly from the V/w_h scale at the intersection of the radial line and the top of the figure $\left(w_0/w_h = 1.0\right)$.

According to equations (14) and (15), χ and θ_n are functions of w_0/w_h only. Therefore scales have been added to figure 2 in order to allow these quantities to be read directly. These scales are extremely compressed for wake angles near the vertical. For greater ease of reading, figure 3 has been prepared. This figure presents directly the wake angles as a function of w_0/w_h. Note in particular the signs and quadrants of these angles as pointed out following equations (14) and (15).

The horizontal induced velocity u_0 can be obtained by using w_0/w_h, as obtained from figure 2, in equation (4). In the case $L = 0$, u_0 may be obtained from equation (17).

The same chart may be used in rearward flight as well provided that V/w_h, V/w_0, χ, and D_i/L are all simultaneously multiplied by -1.

DISCUSSION

It will be observed in figure 2 that, for all cases of forward thrust $\left(D_i/L < 0\right)$, the vertical induced velocity w_0 decreases continuously as the forward velocity increases. Note that when $D_i/L < 0$ there is a forward-directed component of force and, consequently, a rearward-directed horizontal induced velocity even at zero forward speed. The superposition of a forward velocity always adds to this horizontal velocity, increasing the mass flow through the system and reducing the required vertical induced velocity for a given lift.

On the other hand, when there is a drag or negative thrust $\left(D_i/L > 0\right)$, the vertical induced velocity increases until w_0/w_h is equal to one and then decreases thereafter. The reason for this behavior will become evident upon consideration of the mass flow through the system. In this case, when $D_i/L > 0$, the horizontal force is rearward and the net horizontal velocity is the difference between the forward velocity and the horizontal induced velocity. Thus, as the forward speed is increased from zero, the mass flow is decreased and the vertical induced velocity required for a given lift is increased. At a forward speed of $V/w_h = V/w_0 = -D_i/L$, the forward velocity exactly cancels the horizontal component of induced velocity. Thus, at this speed, the mass flow has its minimum value and the required vertical induced velocity has its maximum value. Further increases in forward speed increase the mass flow so that the vertical induced velocity decreases thereafter.

At very large values of D_i/L, the curves of w_0/w_h against V/w_h become triple valued in the range of values near V/w_h of $-D_i/L$. For these conditions, it is likely that no true wake, as postulated by momentum theory, exists. This situation is similar to the vortex ring state of a propeller or rotor (ref. 6) and probably also to the separated flow of a flat plate at high angles of attack.

It should be noted in particular that the induced velocities and the wake skew angle of the present analysis are always defined in relation to the lift and drag axes of the aircraft. This is not always the case when applying the results of the analysis and suitable adjustments must be made in certain cases. For example, reference 7 gives the flow field of rotors or propellers as a function of wake skew angle where the skew angle is defined in relation to the tip-path-plane axes. In this case, X of reference 7 is equal to $X + \alpha$ from the present paper.

CONCLUDING REMARKS

A nomographic solution for the induced velocities and wake skew angle of an arbitrary aerodynamic force-generating system has been given. These quantities may be obtained directly from a chart which requires a knowledge of only the lift and induced drag of the system.

Langley Research Center,
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Langley Field, Va., February 14, 1961.

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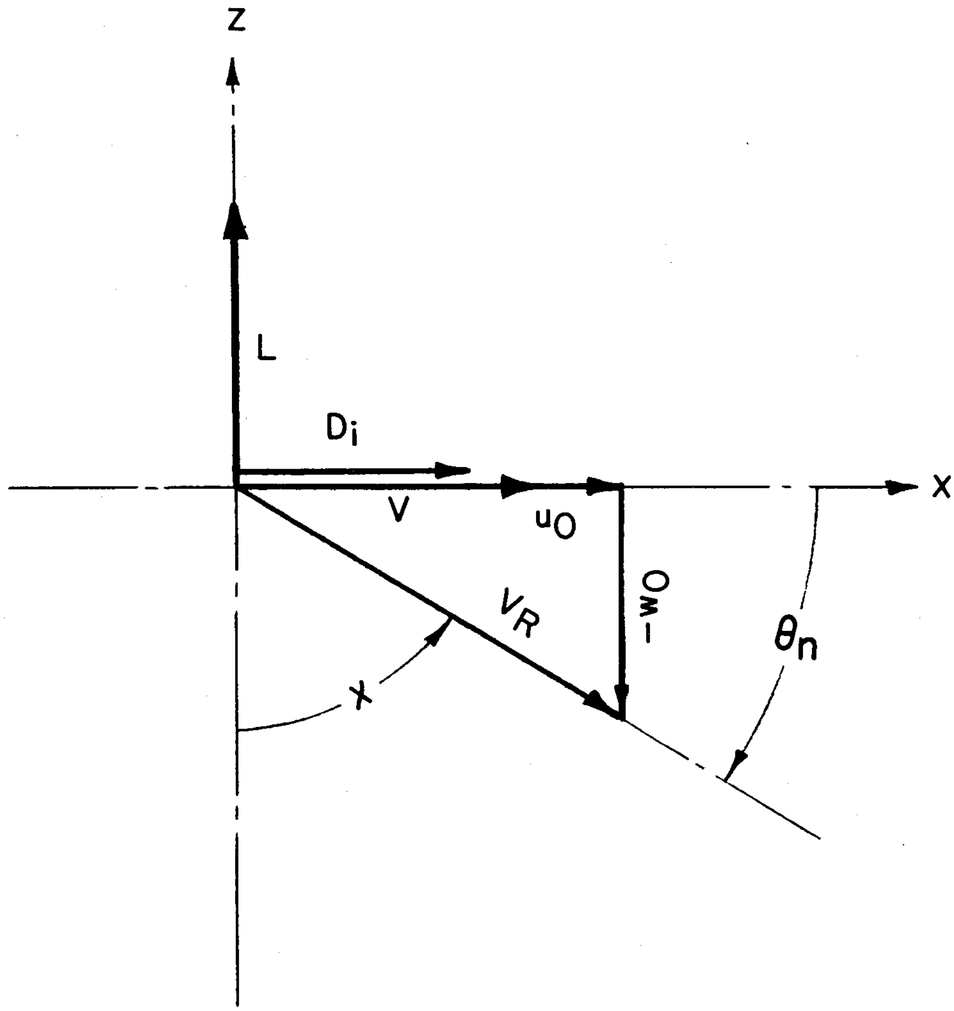
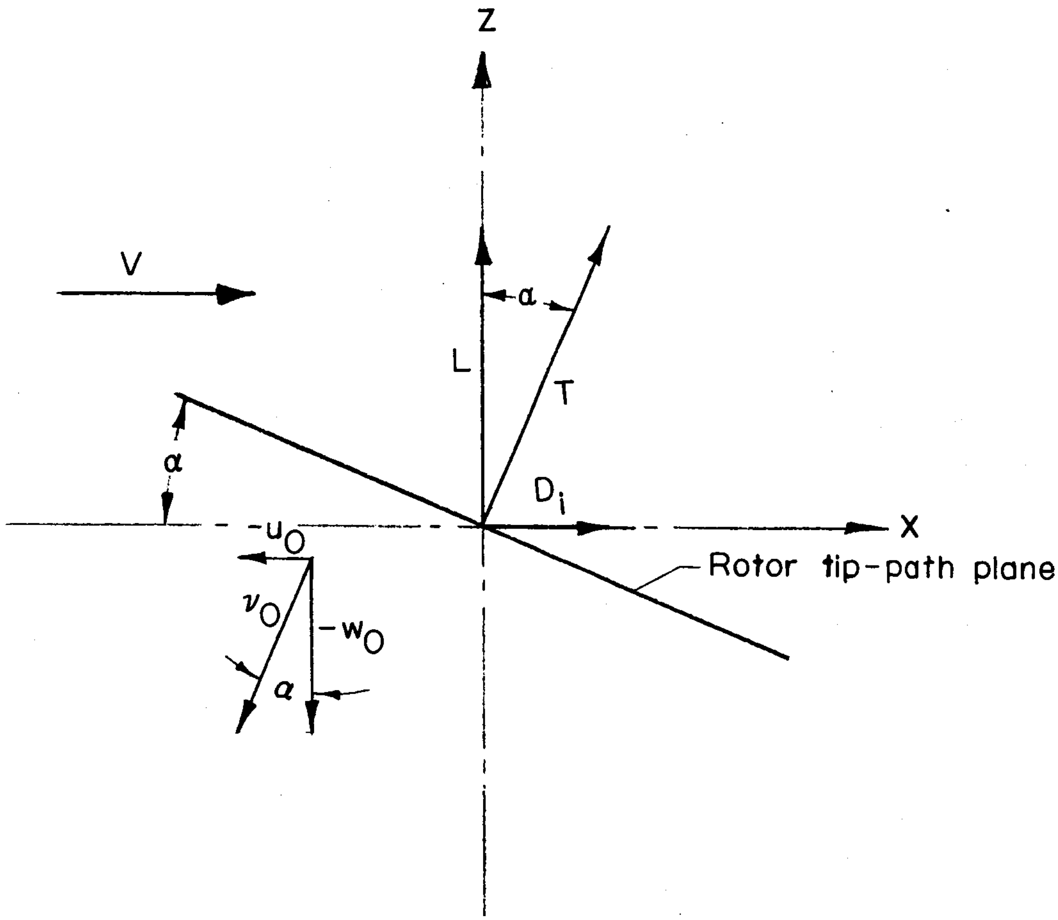


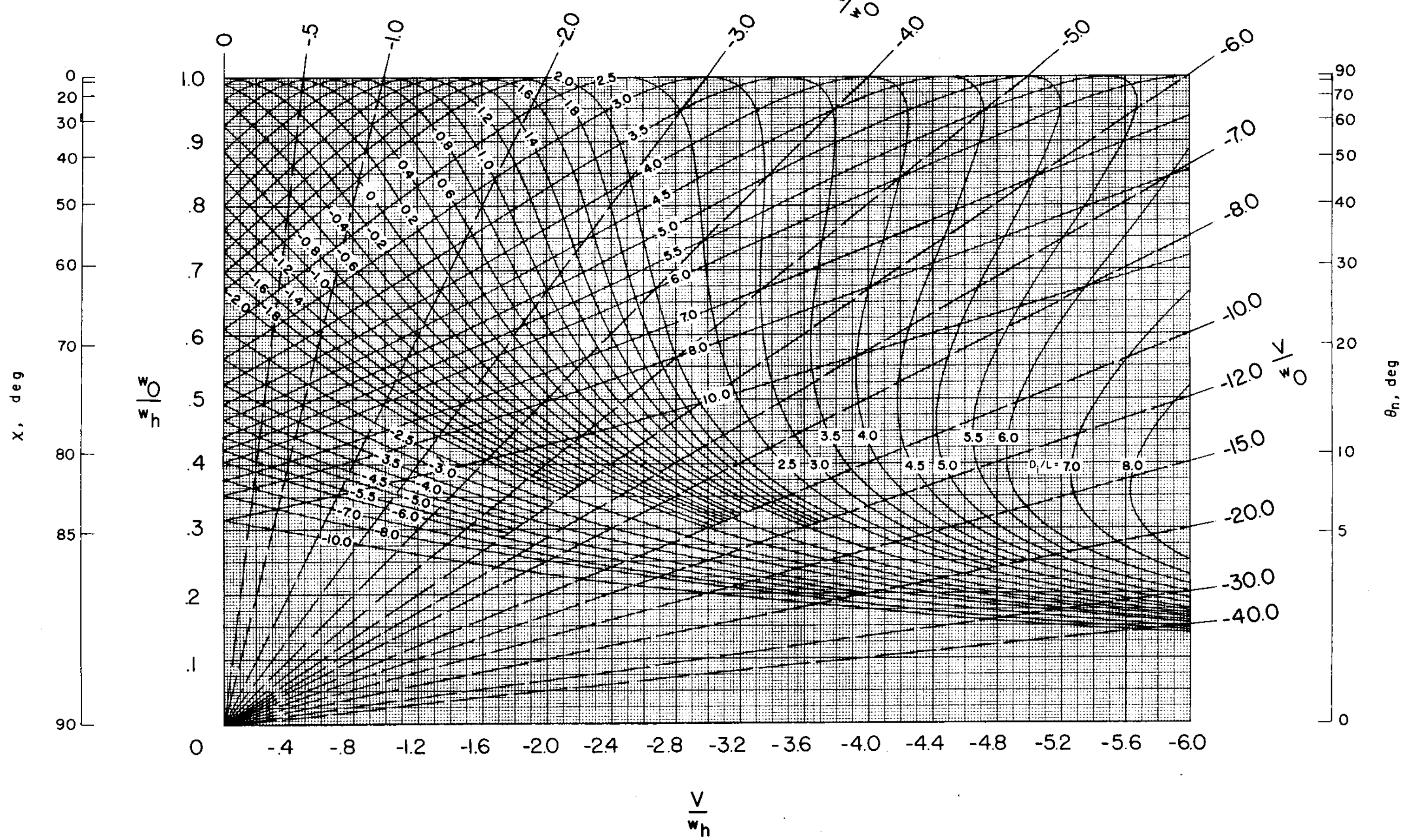
(a) General system.

Figure 1.- Force and velocity vectors at aircraft.



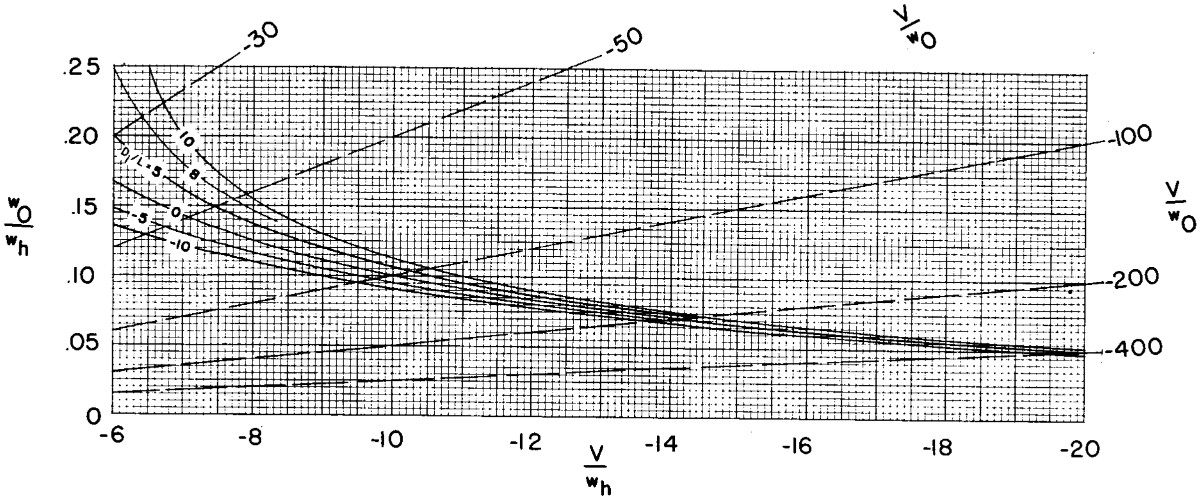
(b) System for rotor.

Figure 1.- Concluded.

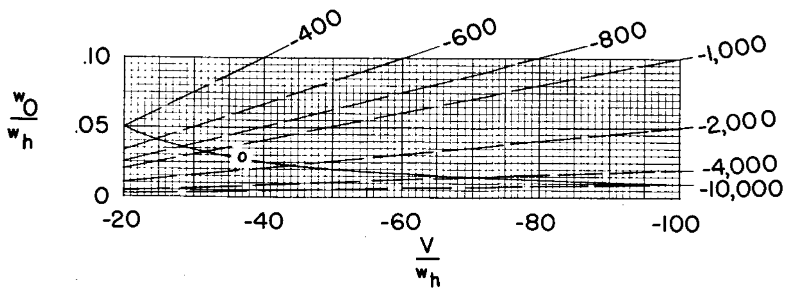


(a) $0 \geqq \frac{V}{w_h} \geqq -6.0$.

Figure 2.- Nondimensional values of the vertical induced velocity for various values of D_i/L as a function of forward velocity.



(b) $-6 \geqq \frac{V}{w_h} \geqq -20.$



(c) $-20 \geqq \frac{V}{w_h} \geqq -100.$

Figure 2.- Concluded.

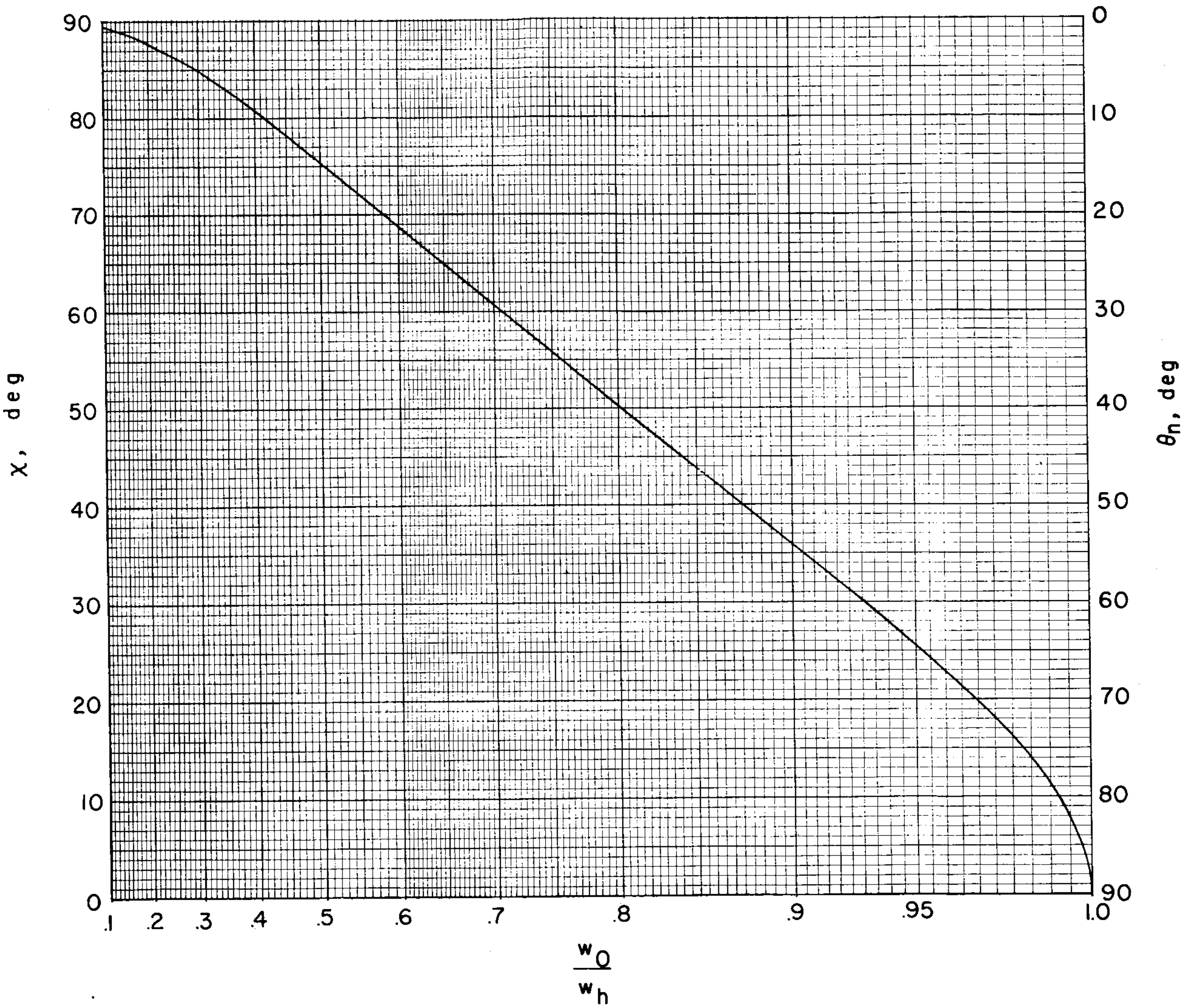


Figure 3.- Skew angle and wake deflection angle as functions of w_0/w_h.

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