# Tensor Fields for Use in Fractional-Order Viscoelasticity*

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**Abstract**

To be able to construct viscoelastic material models from fractional-order differintegral equations that are applicable for 3D finite-strain analyses requires definitions for fractional derivatives and integrals of symmetric tensor fields, like stress and strain. Here we define these fields in the body manifold. We then map them into spatial fields, expressed in terms of an Eulerian or Lagrangian reference frame where most analysts prefer to solve boundary value problems.

## 1 Definitions

Liouville and Riemann[1] defined fractional-order integration as an analytic continuation of Cauchy's $n$-fold integral by writing [4, ¶5, Eqn. A]

$$J^{\alpha}\mathbf{y}(x) := \frac{1}{\Gamma(\alpha)} \int_0^x \frac{1}{(x-x')^{1-\alpha}}\, \mathbf{y}(x')\, dx', \quad \alpha, x \in \mathbb{R}_+, \tag{1}$$

where $J^{\alpha}$ is the Riemann-Liouville integral operator of order $\alpha$.

From this single definition for fractional integration, one can construct several definitions for fractional differentiation (cf. e.g., [10, 13]). The special operator $D^{\alpha}_{\star}$ that we choose to use, which requires the dependent variable $\mathbf{y}$ to be continuous and $\lceil\alpha\rceil$-times differentiable in the independent variable $x$, is defined by

$$D^{\alpha}_{\star}\mathbf{y}(x) := J^{\lceil\alpha\rceil-\alpha} D^{\lceil\alpha\rceil}\mathbf{y}(x), \tag{2}$$

such that

$$D^{\alpha}_{\star} J^{\alpha}\mathbf{y}(x) = \mathbf{y}(x) \quad \text{and} \quad J^{\alpha} D^{\alpha}_{\star}\mathbf{y}(x) = \mathbf{y}(x) - \sum_{k=0}^{\lfloor\alpha\rfloor} \frac{x^k}{k!}\, \mathbf{y}^{(k)}_{0+}, \quad \alpha \in \mathbb{R}_+, \tag{3}$$

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[1]Riemann's pioneering work in the field of fractional calculus was done during his student years, but published posthumous—forty-four years after Liouville first published in the field [12].

with $\mathbf{y}_{0+}^{(k)} := D^k\mathbf{y}(0^+)$ wherein $D^n$, $n \in \mathbb{N}$, is the classical differential operator. It is accepted practice to call $D_\star^\alpha$ the Caputo[2] differential operator of order $\alpha$, since Caputo [1] was the amoung the first to use this operator in applications and to study some of its properties.

## 2 Continua

Body-tensor fields, space-tensor fields, Cartesian space-tensor fields, and the mappings between them have all been carefully documented by Lodge in [5, 6, 7].

A continuum consists of an infinite set of point particles $\{\mathfrak{P}\}$, also referred to as mass elements, filling a region $\mathbb{B}$ in 3-space (i.e., $\mathbb{B} \subset \mathbb{R}^3$). We call this set the body $\mathbb{B}$. In any admissible body-coordinate system $\mathcal{B}$, defined over $\mathbb{B}$, each particle $\mathfrak{P}$ in $\mathbb{B}$ is assigned a unique set of body coordinates, $\mathfrak{x} = (\xi^1, \xi^2, \xi^3)$, $\xi^i \in \mathbb{R}$, that are independent of time (i.e., $\mathcal{B}\colon \mathfrak{P} \to \mathfrak{x}$, cf. Lodge [6]); they convect with the body.

This same continuum $\mathbb{B}$ can also be represented by an infinite set of point places $\{\mathfrak{X}_0\}$ occupying a connected region in space $\mathbb{S}$ at some arbitrary time $t_0$ denoting its reference state. Each place $\mathfrak{X}_0$ relates to a unique particle $\mathfrak{P}$ in $\mathbb{B}$ and is given a label of $\boldsymbol{X}$, which corresponds to the spatial position of $\mathfrak{X}_0$ (and therefore of $\mathfrak{P}$) in this reference configuration. Given an admissible, rectangular-Cartesian, coordinate system $\mathcal{C}$ defined over $\mathbb{S}$, each place $\mathfrak{X}_0$ in $\mathbb{S}$ is thereby assigned a unique set of spatial coordinates $\mathbb{X} = (\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3)$, $\mathrm{X}_i \in \mathbb{R}$, that are fixed in space (i.e., $\mathcal{C}\colon \mathfrak{X}_0 \to \mathbb{X}$).

Later, at current time $t$ $(t > t_0)$, continuum $\mathbb{B}$ coincides with another infinite set of point places $\{\mathfrak{X}\}$ that now occupies a different region in space $\mathbb{S}$. Each place $\mathfrak{X}$ relates to a unique particle $\mathfrak{P}$ in $\mathbb{B}$ and is given a label of $\boldsymbol{x}$ with coordinates $\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3)$ in $\mathcal{C}$, which corresponds to the spatial position of $\mathfrak{X}$ (and therefore of $\mathfrak{P}$) in this current configuration. Because $\mathcal{C}\colon \mathfrak{X}_0 \to \mathbb{X}$ and $\mathcal{C}\colon \mathfrak{X} \to \mathrm{x}$, the location of a particle in space varies over time.

Particle $\mathfrak{P}$ moves through space $\mathbb{S}$ with a velocity $\boldsymbol{v}(t)$ of

$$\boldsymbol{v} := \frac{\partial \boldsymbol{x}}{\partial t}, \tag{4}$$

whose components $\mathrm{v}_i = \partial \mathrm{x}_i / \partial t$, $i = 1, 2, 3$, are quantified in the Cartesian coordinate system $\mathcal{C}$.

The fundamental hypothesis of Cartesian continuum mechanics is that the motion of each particle in the body is assumed to be sufficiently smooth in the sense that mappings

$$\delta \boldsymbol{x} = \boldsymbol{F} \cdot \delta \boldsymbol{X} \quad \text{and} \quad \delta \boldsymbol{v} = D\boldsymbol{F} \cdot \delta \boldsymbol{X} = D\boldsymbol{F} \cdot \boldsymbol{F}^{-1} \cdot \delta \boldsymbol{x} = \boldsymbol{L} \cdot \delta \boldsymbol{x} \tag{5}$$

exist, where $\boldsymbol{F}(t_0, t) := \partial \boldsymbol{x} / \partial \boldsymbol{X}$ defines the deformation-gradient tensor and $\boldsymbol{L}(t) := \partial \boldsymbol{v} / \partial \boldsymbol{x}$ the velocity-gradient tensor, neither of which is symmetric. The deformation gradient $\boldsymbol{F}$ is positive definite because, from the conservation of mass,

$$0 < \frac{\varrho_0}{\varrho} = \det \boldsymbol{F} < \infty, \tag{6}$$

[2] Actually, Liouville introduced the operator in his historic first paper on the topic [4, ¶6, Eqn. B]. Still, nothing in Liouville's works suggests that he ever saw any difference between $D_\star^\alpha = J^{\lceil\alpha\rceil-\alpha} D^{\lceil\alpha\rceil}$ and $D^\alpha = D^{\lceil\alpha\rceil} J^{\lceil\alpha\rceil-\alpha}$, $D^\alpha$ being his accepted definition [4, first formula on pg. 10]—the Riemann-Liouville differential operator of order $\alpha$. Liouville freely interchanged the order of integration and differentiation, because the class of problems that he was interested in happened to be a class where such an interchange is legal, and he made only a few terse remarks about the general requirements on the class of functions for which his fractional calculus works [8]. Rabotnov [11, pg. 129] also introduced this differential operator into the Russian viscoelastic literature a year before Caputo's paper was published.

and consequently $\boldsymbol{F}^{-1}(t_0, t) = \partial \boldsymbol{X}/\partial \boldsymbol{x}$ exists. In contrast, $\boldsymbol{L}$ is not positive definite, in general, and as such $\boldsymbol{L}^{-1}$ need not exist. Here $\varrho = \varrho(t)$ denotes mass density with $\varrho_0 = \varrho(t_0)$.

# 3 Field Transfer: Body To Cartesian Space

The operation of field transfer makes it very plain as to whether a particular spatial field is Eulerian or Lagrangian; it is a consequence of the time when field transfer takes place. Eulerian fields result from a transfer of field at current time $t$ given by the mapping: *body field* $\overset{t}{\Longmapsto}$ *space field*; whereas, Lagrangian fields result from a transfer of field at reference time $t_0$ (which is often taken to be zero) given by: *body field* $\overset{t_0}{\Longmapsto}$ *space field*. Details of the mathematics underlying the mappings $\overset{t}{\Longmapsto}$ and $\overset{t_0}{\Longmapsto}$ can be found in the texts of Lodge [5, 7]. An important property of the field transfer operator is that the resulting spatial fields are objective (i.e., frame invariant).

Let us consider ($i$) an arbitrary, symmetric, covariant, body tensor $\boldsymbol{\mu}(\mathfrak{P}; t)$, and ($ii$) an arbitrary, symmetric, contravariant, body tensor $\boldsymbol{\eta}(\mathfrak{P}; t)$, whose mappings into Cartesian space are known; specifically, let the covariant field map as

$$\boldsymbol{\mu}(\mathfrak{P}; t) \begin{cases} \overset{t}{\Longmapsto} & \boldsymbol{M}(\mathfrak{X}; t), \\ \overset{t_0}{\Longmapsto} & \boldsymbol{N}(\mathfrak{X}_0; t_0, t), \end{cases} \quad \text{such that} \quad \boldsymbol{N} = \boldsymbol{F}^{\mathrm{T}} \cdot \boldsymbol{M} \cdot \boldsymbol{F}, \tag{7a}$$

and the contravariant field map as

$$\boldsymbol{\eta}(\mathfrak{P}; t) \begin{cases} \overset{t}{\Longmapsto} & \boldsymbol{G}(\mathfrak{X}; t), \\ \overset{t_0}{\Longmapsto} & \boldsymbol{H}(\mathfrak{X}_0; t_0, t), \end{cases} \quad \text{such that} \quad \boldsymbol{H} = \boldsymbol{F}^{-1} \cdot \boldsymbol{G} \cdot \boldsymbol{F}^{-\mathrm{T}}, \tag{7b}$$

where $\boldsymbol{M}$ and $\boldsymbol{G}$ are some arbitrary (but known), symmetric, Eulerian, tensor fields, with $\boldsymbol{N}$ and $\boldsymbol{H}$ designating their respective, symmetric, Lagrangian counterparts. In these transformations of field, the deformation gradient $\boldsymbol{F}$ serves as a Jacobian of transformation between the two Cartesian frames that pulls the known Eulerian field backwards, out of the Eulerian frame and into the Lagrangian frame.

## 3.1 Rates of Tensor Fields

Lodge [5, pp. 321–327] has shown that the time rate-of-change $D$ $(= \partial/\partial t)$ of covariant tensor $\boldsymbol{\mu}$ maps into Cartesian space as

$$D\boldsymbol{\mu} \begin{cases} \overset{t}{\Longmapsto} & \overset{\triangledown}{\boldsymbol{M}}, \\ \overset{t_0}{\Longmapsto} & D\boldsymbol{N}, \end{cases} \quad \text{such that} \quad D\boldsymbol{N} = \boldsymbol{F}^{\mathrm{T}} \cdot \overset{\triangledown}{\boldsymbol{M}} \cdot \boldsymbol{F}, \tag{8a}$$

while the time rate-of-change of contravariant tensor $\boldsymbol{\eta}$ maps as

$$D\boldsymbol{\eta} \begin{cases} \overset{t}{\Longmapsto} & \overset{\vartriangle}{\boldsymbol{G}}, \\ \overset{t_0}{\Longmapsto} & D\boldsymbol{H}, \end{cases} \quad \text{such that} \quad D\boldsymbol{H} = \boldsymbol{F}^{-1} \cdot \overset{\vartriangle}{\boldsymbol{G}} \cdot \boldsymbol{F}^{-\mathrm{T}}, \tag{8b}$$

wherein

$$\overset{\triangledown}{\boldsymbol{M}} := D\boldsymbol{M} + \boldsymbol{v} \cdot \nabla \boldsymbol{M} + \boldsymbol{L}^{\mathrm{T}} \cdot \boldsymbol{M} + \boldsymbol{M} \cdot \boldsymbol{L} \quad \text{and} \quad \overset{\vartriangle}{\boldsymbol{G}} := D\boldsymbol{G} + \boldsymbol{v} \cdot \nabla \boldsymbol{G} - \boldsymbol{L} \cdot \boldsymbol{G} - \boldsymbol{G} \cdot \boldsymbol{L}^{\mathrm{T}} \tag{9}$$

denote the lower- and upper-convected derivatives, respectively, of Oldroyd [9], which reduce to Lie derivatives taken with respect to velocity $\boldsymbol{v}$ whenever $D\boldsymbol{M} = \boldsymbol{0}$ or $D\boldsymbol{G} = \boldsymbol{0}$. The vector operator $\boldsymbol{\nabla}$ represents the spatial gradient $\partial / \partial \boldsymbol{x}$.

Even though Cartesian tensor fields cannot distinguish between covariance and contravariance, rates-of-change of Cartesian fields in the Eulerian frame do have an intrinsic dependence upon this property.

### 3.2 Fractional Rates of Tensor Fields

The Caputo derivative $D^\alpha_\star$ (defined in Eq. 2) of covariant tensor $\boldsymbol{\mu}$ maps into Cartesian space as

$$D^\alpha_\star \boldsymbol{\mu} \begin{cases} \stackrel{t}{\Longmapsto} & D^{\alpha\triangledown}_\star \boldsymbol{M}, \\ \stackrel{t_0}{\Longmapsto} & D^\alpha_\star \boldsymbol{N}, \end{cases} \quad \text{such that} \quad D^\alpha_\star \boldsymbol{N} = \boldsymbol{F}^{\mathrm{T}} \cdot D^{\alpha\triangledown}_\star \boldsymbol{M} \cdot \boldsymbol{F}, \tag{10a}$$

while the Caputo derivative of contravariant tensor $\boldsymbol{\eta}$ maps as

$$D^\alpha_\star \boldsymbol{\eta} \begin{cases} \stackrel{t}{\Longmapsto} & D^{\alpha\vartriangle}_\star \boldsymbol{G}, \\ \stackrel{t_0}{\Longmapsto} & D^\alpha_\star \boldsymbol{H}, \end{cases} \quad \text{such that} \quad D^\alpha_\star \boldsymbol{H} = \boldsymbol{F}^{-1} \cdot D^{\alpha\vartriangle}_\star \boldsymbol{G} \cdot \boldsymbol{F}^{-\mathrm{T}}, \tag{10b}$$

wherein

$$D^{\alpha\triangledown}_\star \boldsymbol{M} := \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{1}{(t-t')^\alpha} \boldsymbol{F}^{-\mathrm{T}}(t',t) \cdot \overset{\triangledown}{\boldsymbol{M}}(t') \cdot \boldsymbol{F}^{-1}(t',t)\, dt', \quad 0 < \alpha < 1, \tag{11a}$$

and

$$D^{\alpha\vartriangle}_\star \boldsymbol{G} := \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{1}{(t-t')^\alpha} \boldsymbol{F}(t',t) \cdot \overset{\vartriangle}{\boldsymbol{G}}(t') \cdot \boldsymbol{F}^{\mathrm{T}}(t',t)\, dt', \quad 0 < \alpha < 1, \tag{11b}$$

are objective rates of order $\alpha$, with operator $D^{\alpha\triangledown}_\star$ being affiliated with covariant-like fields and $D^{\alpha\vartriangle}_\star$ with contravariant-like fields. Unlike $D^\alpha_\star \boldsymbol{N}$ and $D^\alpha_\star \boldsymbol{H}$, which are actual Caputo derivatives, derivatives $D^{\alpha\triangledown}_\star \boldsymbol{M}$ and $D^{\alpha\vartriangle}_\star \boldsymbol{G}$ are not true Caputo derivatives, in a strict sense of the definition, which is why they are given different notations.

A rigorous derivation of these results is given in Ref. [3].

### 3.3 Fractional Integrals of Tensor Fields

The Riemann-Liouville integral $J^\alpha$ (defined in Eq. 1) of covariant tensor $\boldsymbol{\mu}$ maps into Cartesian space as

$$J^\alpha \boldsymbol{\mu} \begin{cases} \stackrel{t}{\Longmapsto} & J^{\alpha\triangledown} \boldsymbol{M}, \\ \stackrel{t_0}{\Longmapsto} & J^\alpha \boldsymbol{N}, \end{cases} \quad \text{such that} \quad J^\alpha \boldsymbol{N} = \boldsymbol{F}^{\mathrm{T}} \cdot J^{\alpha\triangledown} \boldsymbol{M} \cdot \boldsymbol{F}, \tag{12a}$$

while the Riemann-Liouville integral of contravariant tensor $\boldsymbol{\eta}$ maps as

$$J^\alpha \boldsymbol{\eta} \begin{cases} \stackrel{t}{\Longmapsto} & J^{\alpha\vartriangle} \boldsymbol{G}, \\ \stackrel{t_0}{\Longmapsto} & J^\alpha \boldsymbol{H}, \end{cases} \quad \text{such that} \quad J^\alpha \boldsymbol{H} = \boldsymbol{F}^{-1} \cdot J^{\alpha\vartriangle} \boldsymbol{G} \cdot \boldsymbol{F}^{-\mathrm{T}}. \tag{12b}$$

The Eulerian tensors $J^{\alpha\triangledown}\boldsymbol{M}$ and $J^{\alpha\vartriangle}\boldsymbol{G}$ are objective integrals of order $\alpha$ defined by

$$J^{\alpha\triangledown}\boldsymbol{M} := \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{1}{(t-t')^{1-\alpha}}\,\boldsymbol{F}^{-\mathrm{T}}(t',t)\cdot\boldsymbol{M}(t')\cdot\boldsymbol{F}^{-1}(t',t)\,dt' \tag{13a}$$

and

$$J^{\alpha\vartriangle}\boldsymbol{G} := \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{1}{(t-t')^{1-\alpha}}\,\boldsymbol{F}(t',t)\cdot\boldsymbol{G}(t')\cdot\boldsymbol{F}^{\mathrm{T}}(t',t)\,dt'. \tag{13b}$$

Unlike $J^{\alpha}\boldsymbol{N}$ and $J^{\alpha}\boldsymbol{H}$, which are actual Riemann-Liouville integrals, integrals $J^{\alpha\triangledown}\boldsymbol{M}$ and $J^{\alpha\vartriangle}\boldsymbol{G}$ are not true Riemann-Liouville integrals, in a strict sense of the definition, which is why they are given different notations.

Again, a rigorous derivation of these results is given in Ref. [3].

# 4 Some Viscoelastic Models

To illustrate the process of analytically continuing a known constitutive equation to one of fractional order, we consider three examples: the rubberlike liquid, the convected Maxwell fluid, and the convected Kelvin-Zener solid (cf. Lodge [5]). Expressed in terms of body tensor fields, the rubberlike liquid is

$$\boldsymbol{\pi}(t) + \wp\,\boldsymbol{\gamma}^{-1}(t) = \mu\int_0^t M(t-t')\,\boldsymbol{\gamma}^{-1}(t')\,\mathrm{d}t', \tag{14a}$$

the convected Maxwell fluid is

$$\tau\,D\boldsymbol{\pi}(t) + \boldsymbol{\pi}(t) + \wp\,\boldsymbol{\gamma}^{-1}(t) = -\eta\,D\boldsymbol{\gamma}^{-1}(t), \tag{14b}$$

and the convected Kelvin-Zener solid is

$$\tau\,D\boldsymbol{\pi}(t) + \boldsymbol{\pi}(t) + \wp\,\boldsymbol{\gamma}^{-1}(t) = \mu\,\boldsymbol{\gamma}^{-1}(t_0) - \eta\,D\boldsymbol{\gamma}^{-1}(t), \tag{14c}$$

where $\boldsymbol{\pi}(\mathfrak{P};t)$ is the symmetric, contravariant, stress tensor, $\boldsymbol{\gamma}(\mathfrak{P};t)$ is the symmetric, postive-definite, covariant, metric tensor, $M$ is the memory function (usually taken to be a decaying exponential), and $\tau$, $\eta$ and $\mu$ are material constants. All three of these models are assumed to be incompressible, with $\wp$ being a Lagrange multiplier introduced to force this constraint: $\det\boldsymbol{\gamma}(t) = \det\boldsymbol{\gamma}(t_0)$.

In much the same way that Caputo and Mainardi [2] analytically continued models in 1D, we analytically continue the 3D models in Eq. (14) by writing the fractional-order rubberlike liquid as

$$\boldsymbol{\pi}(t) + \wp\,\boldsymbol{\gamma}^{-1}(t) = \mu\,J^{\alpha}\boldsymbol{\gamma}^{-1}(t), \tag{15a}$$

the fractional-order convected Maxwell fluid as

$$\tau^{\alpha}D_{\star}^{\alpha}\boldsymbol{\pi}(t) + \boldsymbol{\pi}(t) + \wp\,\boldsymbol{\gamma}^{-1}(t) = -\eta^{\alpha}D_{\star}^{\alpha}\boldsymbol{\gamma}^{-1}(t), \tag{15b}$$

and the fractional-order convected Kelvin-Zener solid as

$$\tau^{\alpha}D_{\star}^{\alpha}\boldsymbol{\pi}(t) + \boldsymbol{\pi}(t) + \wp\,\boldsymbol{\gamma}^{-1}(t) = \mu\,\boldsymbol{\gamma}^{-1}(t_0) - \eta^{\alpha}D_{\star}^{\alpha}\boldsymbol{\gamma}^{-1}(t). \tag{15c}$$

The memory function in Eq. (15a) is now considered to be an Abel (power law) kernel instead of a Boltzmann (exponential) kernel. The fact that the fractional derivatives on both sides of the equation in Eqs. (15b & 15c) are of the same order (viz., $\alpha$, $0 < \alpha < 1$) ensures that stress waves propagate with finite velocities in these models, which is important both physically and numerically.

In an Eulerian transfer of field, Lodge [5, 7] has shown that $\boldsymbol{\pi}(t) \overset{t}{\Longmapsto} \boldsymbol{T}(t)$, $\gamma^{-1}(t) \overset{t}{\Longmapsto} \boldsymbol{I}$, $\gamma^{-1}(t_0) \overset{t}{\Longmapsto} \boldsymbol{B}(t_0, t)$, and $D\gamma^{-1}(t) \overset{t}{\Longmapsto} \overset{\vartriangle}{\boldsymbol{I}} = -2\boldsymbol{D}(t)$, where $\boldsymbol{T}$ is the Cauchy stress tensor, $\boldsymbol{I}$ is the identity tensor, $\boldsymbol{B}$ ($:= \boldsymbol{F} \cdot \boldsymbol{F}^{\mathrm{T}}$) is the Finger deformation tensor, and $\boldsymbol{D}$ ($:= \frac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^{\mathrm{T}})$) is the rate-of-deformation tensor. Using these results, along with Eqs. (10 & 12), one can apply the field transfer operator $\overset{t}{\Longmapsto}$ to Eq. (15) and get the following fractional-order viscoelastic models in the Eulerian frame. The fractional-order rubberlike liquid becomes

$$\boldsymbol{T} + \wp \boldsymbol{I} = \mu \, J^{\alpha\vartriangle} \boldsymbol{I} = \frac{\mu}{\Gamma(\alpha)} \int_0^t \frac{1}{(t - t')^{1-\alpha}} \, \boldsymbol{B}(t', t) \, \mathrm{d}t', \tag{16a}$$

the fractional-order upper-convected Maxwell fluid becomes

$$\tau^\alpha D_\star^{\alpha\vartriangle} \boldsymbol{T} + \boldsymbol{T} + \wp \boldsymbol{I} = 2\eta^\alpha D_\star^{\alpha\vartriangle} \boldsymbol{D}, \tag{16b}$$

and the fractional-order upper-convected Kelvin-Zener solid becomes

$$\tau^\alpha D_\star^{\alpha\vartriangle} \boldsymbol{T} + \boldsymbol{T} + \wp \boldsymbol{I} = \mu \, \boldsymbol{B} + 2\eta^\alpha D_\star^{\alpha\vartriangle} \boldsymbol{D}, \tag{16c}$$

where now the constraint for incompressiblity is $\det \boldsymbol{F} = 1$ in accordance with Eq. (6). Their Lagrangian counterparts can be written straightaway, if needed.

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