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AERODYNAMIC FORCES ON A VIBRATING UNSTAGGERED CASCADE

By H. Söhngen

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The unsteady aerodynamic forces, [based on two-dimensional incompressible flow considerations], are determined for an unstaggered cascade, the blades of which are vibrating in phase in an approach flow parallel to the blades.

INTRODUCTION

In the theory of axial turbomachines, the aerodynamic forces acting on a vibrating cascade are of interest for investigations of vibrations. These forces were determined in reference 1 for the case where the spacing of the cascade is small. Information regarding an arbitrary staggering does not yet exist; we shall investigate here the simplest case of this kind, namely an unstaggered cascade with straight profiles, the blades of which vibrate in phase. We do not presuppose to limit our consideration to pure bending or torsional vibrations; rather, arbitrary periodic deformations of the blade are admitted and their lift distribution is determined. This includes also the case where the flow approaching the blades is unsteady. We do assume, however, that the blades are not under static load, that the fluid is incompressible and frictionless, and that the amplitudes of vibration are small.

LIFT DISTRIBUTION AND LIFT

Assume the approach flow toward the cascade to be parallel to the blades with the velocity V. Let the velocity with which the blade point x' moves in the y' direction be given by $g(x') \times e^{i\nu t}$. This velocity is the same for all blades. We visualize the blades of the cascade as covered by vortices of the density $\gamma(x')e^{i\nu t}$. It must be noted, in addition, that free vortices separate from the trailing edges of the individual blades in proportion as the total circulation about

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the blade in question varies with the time, although with the inverse sign. This is stipulated by the theorem of the conservation of circulation in the total space. In a linear theory, for the amplitudes of vibration to which we here limit ourselves, we may assume that these free vortices move away with the basic flow. If we denote accordingly their density by

$$\gamma_0 e^{i\nu\left(t - \frac{x'}{V}\right)} \qquad (x' \geq l/2)$$

there applies at the trailing edge ($x' = l/2$) the relationship

$$\frac{d}{dt}\int_{-l/2}^{l/2} \gamma(x') e^{i\nu t}\, dx' + V\gamma_0 e^{i\nu(t - l/2V)} = 0 \qquad (1)$$

by which the density of the free vortices is connected with the variation of the circulation about the blade.

The vortices situated on the blades $\gamma(x')$ and the free vortices γ_0 have to be determined - with consideration of the relationship (1) - in such a manner that the velocity component in the y' direction, induced by all the vortices together at the blade point x', is equal to that with which the blade point moves in this direction, that is, equal to $g(x')$. A row of vortices, which lie on the blades at the point $x' = 0$ and each of which has the circulation γ, induce a field whose y' component on the blades at the point x' is equal to

$$-\frac{\gamma}{\alpha}\frac{e^{\frac{\pi}{\alpha}x'}}{e^{\frac{\pi}{\alpha}x'} - e^{-\frac{\pi}{\alpha}x'}}$$

This expression is easily obtained by superimposing the fields of the individual vortices and by taking into consideration since this is an axial-flow problem, that there can be no velocity induced far upstream of the cascade, at development. This makes it necessary to superimpose another transverse velocity on the summed row of vortices. Using the expression given above, we then obtain as a boundary condition the integral equation

$$-\frac{1}{\alpha}\oint_{-l/2}^{l/2}\gamma(\xi')\frac{e^{\frac{\pi}{\alpha}(x'-\xi')}\,d\xi'}{e^{\frac{\pi}{\alpha}(x'-\xi')}-e^{-\frac{\pi}{\alpha}(x'-\xi')}}-$$

$$\frac{\gamma_0}{\alpha}\int_{l/2}^{\infty}e^{-i\frac{\nu}{V}\xi'}\frac{e^{\frac{\pi}{\alpha}(x'-\xi')}}{e^{\frac{\pi}{\alpha}(x'-\xi')}-e^{-\frac{\pi}{\alpha}(x'-\xi')}}\,d\xi'=g(x')$$

for which we may also write

$$-\frac{1}{\alpha}\oint_{-l/2}^{l/2}\gamma(\xi')\frac{e^{\frac{2\pi}{\alpha}x'}}{e^{\frac{2\pi}{\alpha}x'}-e^{\frac{2\pi}{\alpha}\xi'}}\,d\xi'-$$

$$\frac{\gamma_0}{\alpha}\int_{l/2}^{\infty}e^{-i\frac{\nu}{V}\xi'}\frac{e^{\frac{2\pi}{\alpha}x'}}{e^{\frac{2\pi}{\alpha}x'}-e^{\frac{2\pi}{\alpha}\xi'}}\,d\xi'=g(x') \qquad (2)$$

From this equation and equation (1), we have to determine the vortex density $\gamma(x')$.

However, it is not so much the vortex density which is physically interesting but the lift distribution $\Delta p(x')$, that is, the pressure difference between positive pressure side and suction side. We conclude from the unsteady-pressure equation that between lift distribution and vortex density there exists the relationship

$$P(x')=\gamma(x')+i\frac{\nu}{V}\int_{-l/2}^{x'}\gamma(\xi')\,d\xi' \qquad (3)$$

where

$$\Delta p(x')=\rho VP(x')e^{i\nu t}$$

Thus, the lift distribution may be calculated as soon as the vortex distribution is known.

In determining the lift distribution for the single blade, one usually starts from the integral equation which is satisfied by the

lift distribution. Therefore, we shall indicate the latter for the present case. It reads

$$\oint_{-l/2}^{l/2} P(\xi')K(x' - \xi')d\xi' = g(x') \qquad (|x'| < l/2) \tag{4}$$

with the kernel

$$K(x') = -\frac{1}{\alpha}\frac{e^{\frac{\pi}{\alpha}x'}}{e^{\frac{\pi}{\alpha}x'} - e^{-\frac{\pi}{\alpha}x'}} + i\frac{\nu}{V\alpha}\int_0^\infty e^{-i\frac{\nu}{V}u'}\frac{e^{\frac{\pi}{\alpha}(x'-u')}}{e^{\frac{\pi}{\alpha}(x'-u')} - e^{-\frac{\pi}{\alpha}(x'-u')}}du' \tag{5}$$

If we note that, because of (3),

$$\gamma(x') = P(x') - i\frac{\nu}{V}\int_{-l/2}^{x'} P(\xi')e^{-i\frac{\nu}{V}(x'-\xi')}d\xi'$$

we can show that the integral equation (4) is identical with (2) if γ_0 satisfies equation (1).

However, since the kernel of the integral equation (4) can be less easily seen through than that of (2), it appears more suitable to determine the lift distribution not directly by inversion of equation (4) but first to solve equation (2) with respect to $\gamma(x')$, with an as yet undetermined constant γ_0. Thus we obtain $\gamma(x')$ as a function of $g(x')$ and γ_0. If we introduce it into equation (1), a linear equation for γ_0 results from which γ_0 may be calculated as a function of $g(x')$. Thereby $\gamma(x')$ also is determined, and the lift distribution can be calculated.

For solving the integral equation (2), one's first impulse is to try to reduce it back to the integral equation

$$\frac{1}{\pi}\oint_{-1}^{1}\frac{F(\xi)}{x - \xi}d\xi = G(x) \tag{6}$$

known from lifting-surface theory. For this purpose, we set

$$x = \frac{2}{l}x' \tag{7}$$

and

$$x_1 = \frac{1}{s}\left[e^{\tau x} - c\right] \tag{8}$$

with

$$\tau = \pi l/\alpha \qquad s = \text{Sinh } \tau \qquad c = \text{Cosh } \tau$$

Furthermore, we stipulate that the primed coordinates shall always be dimensional quantities; those without primes, in contrast, dimensionless, as they result by means of equation (7). Coordinates with the subscript 1 shall always result from the coordinates without primes by means of the substitution (8). In a manner which cannot be misinterpreted, we write

$$g(x') = g(x) = g(x_1)$$

and correspondingly also, other functions. If $f(x)$ is a given function, we always understand by $f_1(x_1)$ the expression

$$f_1(x_1) = \frac{f(x_1)}{sx_1 + c}$$

Thus, the integral equation (2) then is transformed into

$$\frac{1}{2\pi} \oint_{-1}^{1} \frac{\gamma_1(\xi_1)}{x_1 - \xi_1} d\xi_1 = -\gamma_0 w_1(x_1) - g_1(x_1) \tag{9}$$

where $w_1(x_1)$ denotes the function described by

$$w_1(x_1) = \frac{l}{2\alpha s} \int_1^{\infty} e^{-i\omega\xi} \frac{d\xi}{x_1 - \xi_1} \tag{10}$$

and

$$\omega = \nu l/2V$$

is the so-called reduced frequency. Furthermore, the relationship (1) is equivalent to

$$\gamma_0 = -i \frac{\omega s}{\tau} e^{i\omega} \int_{-1}^{1} \gamma_1(x_1) dx_1 \tag{11}$$

From the integral equation (9), there follows, (reference 2)

$$\gamma_1(x_1) = \frac{C}{\pi\sqrt{1 - x_1^2}} + \frac{2}{\pi}\oint_{-1}^{1} \sqrt{\frac{1 - \xi_1^2}{1 - x_1^2}} \frac{\gamma_0 w_1(\xi_1) + g_1(\xi_1)}{x_1 - \xi_1} d\xi_1 \qquad (12)$$

where the constant C at first may still be chosen arbitrarily. The condition of a smooth outflow at the trailing edge requires that

$$C = -\int_{-1}^{1} \sqrt{\frac{1 + \xi_1}{1 - \xi_1}} \left[2\gamma_0 w_1(\xi_1) + 2g_1(\xi_1)\right] d\xi_1 \qquad (13)$$

Hence $\gamma_1(x_1)$ then takes the form

$$\gamma_1(x_1) = \frac{2}{\pi}\sqrt{\frac{1 - x_1}{1 + x_1}}\oint_{-1}^{1} \sqrt{\frac{1 + \xi_1}{1 - \xi_1}} \frac{\gamma_0 w_1(\xi_1) + g_1(\xi_1)}{x_1 - \xi_1} d\xi_1 \qquad (14)$$

If we finally take into consideration that

$$C = \int_{-1}^{1} \gamma_1(x_1) dx_1 \qquad (15)$$

there results for the constant γ_0 the value

$$\gamma_0 = \frac{1}{B(\omega,\tau)} \frac{2s}{\tau} \int_{-1}^{1} g_1(x_1)\sqrt{\frac{1 + x_1}{1 - x_1}} dx_1 \qquad (16)$$

where

$$B(\omega,\tau) = \frac{e^{-i\omega}}{i\omega} - 2\frac{s}{\tau}\int_{-1}^{1} w_1(x_1)\sqrt{\frac{1 + x_1}{1 - x_1}} dx_1 \qquad (17)$$

is a function which depends only on the reduced frequency and the geometrical dimensions of the cascade.

Thus the problem is solved in principle. The function $g(x')$ is given; with it, the constant γ_0 may be calculated from equation (16), equation (14) then yields the vortex distribution $\gamma_1(x_1)$, and the lift distribution results from

$$P(x) = \gamma_1(x_1)(sx_1 + c) + i\omega \frac{s}{\tau}\int_{-1}^{1} \gamma_1(\xi_1)\, d\xi_1 \tag{18}$$

It is now only a matter of writing this expression in a form more suitable for numerical calculation. It will be seen, in particular, that, whereas the influence of the function $w_1(x_1)$ - that is the influence of the free vortices - enters into the vortex distribution in a relatively complicated manner, its effect on the lift distribution is of a considerably simpler structure.

THE FUNCTION $T(\omega,\tau)$

Before deriving a simpler representation of the lift distribution, we shall give a series expansion for the function $B(\omega,\tau)$. From (10) and (17), it follows that

$$B(\omega,\tau) = \frac{e^{-i\omega}}{i\omega} - \frac{1}{\pi}\int_{-1}^{1} \sqrt{\frac{1 + x_1}{1 - x_1}} \int_{1}^{\infty} \frac{e^{-i\omega\xi}}{x_1 - \xi_1}\, d\xi\, dx_1$$

If we interchange the integration sequence and take into account that

$$\int_{-1}^{1} \sqrt{\frac{1 + x_1}{1 - x_1}} \frac{dx_1}{x_1 - \xi_1} = -\pi\left\{\sqrt{\frac{\xi_1 + 1}{\xi_1 - 1}} - 1\right\} \qquad (\xi_1 > 1)$$

there results

$$B(\omega,\tau) = \frac{e^{-i\omega}}{i\omega} + \int_{1}^{\infty} e^{-i\omega\xi}\left[\sqrt{\frac{\xi_1 + 1}{\xi_1 - 1}} - 1\right] d\xi \tag{19}$$

Furthermore it is convenient to split the function $B(\omega,\tau)$ into two parts. We let

$$B_0(\omega,\tau) = \frac{2}{\pi}\int_{1}^{\infty} \frac{e^{-i\omega\xi}}{\sqrt{\xi_1^2 - 1}}\, d\xi \tag{20}$$

and

$$B_1(\omega,\tau) = \frac{2}{\pi}\left\{\frac{e^{-i\omega}}{i\omega} + \int_{1}^{\infty} e^{-i\omega\xi}\left[\frac{\xi_1}{\sqrt{\xi_1^2 - 1}} - 1\right] d\xi\right\} \tag{21}$$

Then

$$B(\omega,\tau) = \frac{\pi}{2}\left\{B_0(\omega,\tau) + B_1(\omega,\tau)\right\} \tag{22}$$

Particularly simple representations exist for these two functions if $\alpha/l = \infty$, that is, $\tau = 0$, so that we have the case of the single blade. Since, $\xi_1 = \xi$, we see immediately that

$$B_0(\omega,0) = -iH_0^{(2)}(\omega)$$

where $H_0^{(2)}$ is the Hankel function. Furthermore, we can easily show that

$$B_1(\omega,0) = -H_1^{(2)}(\omega)$$

We shall now derive representations for these two functions which are convenient especially for moderate values of α/l. We have

$$B_0(\omega,\tau) = \frac{2}{\pi} s \int_1^\infty \frac{e^{-i\omega\xi}}{\sqrt{\left(e^{\tau\xi} - e^{\tau}\right)\left(e^{\tau\xi} - e^{-\tau}\right)}}\, d\xi$$

If we understand by $\underline{C}$ the path of integration represented in figure 2, the integral taken over it vanishes. Since the expression under the square root sign takes equal values on the two parallel paths, there follows, with

$$\omega' = \frac{\omega}{\tau}$$

$$\left(1 - e^{2\pi\omega'}\right)B_0(\omega,\tau) = i\,\frac{2}{\pi}\, s e^{-i\omega-\tau} \int_0^{2\alpha/l} \frac{e^{\omega\eta}}{\sqrt{\left(e^{i\tau\eta} - 1\right)\left(e^{i\tau\eta} - e^{-2\tau}\right)}}\, d\eta$$

$$= i\,\frac{2}{\pi}\,\frac{s}{\tau}\, e^{-i\omega-\tau} \int_0^{2\pi} \frac{e^{\omega'\vartheta}}{\sqrt{\left(e^{i\vartheta} - 1\right)\left(e^{i\vartheta} - e^{-2\tau}\right)}}\, d\vartheta$$

$$= i\,\frac{2}{\pi}\,\frac{s}{\tau}\, e^{-i\omega-\tau} \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n e^{-2n\tau} \int_0^{2\pi} \frac{e^{(\omega'-in)\vartheta}}{\sqrt{\left(e^{i\vartheta} - 1\right)e^{i\vartheta}}}\, d\vartheta$$

However (ref. 3, p. 138)

$$\int_0^{2\pi} \frac{e^{(\omega'-in)\vartheta}}{\sqrt{\left(e^{i\vartheta}-1\right)e^{i\vartheta}}}\,d\vartheta = \int_0^{2\pi} \frac{e^{-i\frac{\pi}{4}+\left(\omega'-in-i\frac{3}{4}\right)\vartheta}}{\sqrt{2\sin\vartheta/2}}\,d\vartheta$$

$$= \frac{2\pi^{3/2}(-1)^{n+1}e^{\pi\omega'}}{\Gamma\left(n+\frac{3}{2}+i\omega'\right)\Gamma\left(-n-i\omega'\right)}$$

and therefore,

$$B_0(\omega,\tau) = \frac{2\omega'\sqrt{\pi}\left(1-e^{-2\tau}\right)e^{-i\omega}}{\tau(1+2i\omega')\Gamma(-n-i\omega')\Gamma\left(n+\frac{3}{2}+i\omega'\right)\operatorname{Sinh}\pi\omega'}\sum_{n=0}^{\infty}\alpha_n e^{-2n\tau}$$

where

$$\alpha_n = \prod_{\nu=1}^{n}\frac{\left(\nu-\frac{1}{2}\right)(\nu+i\omega')}{\nu\left(\nu+\frac{1}{2}+i\omega'\right)} \qquad \alpha_0 = 1$$

Because of $\left|a_n\right| < 1$, this expansion converges so well for the spacing factors customary in turbomachines $\alpha/l \leq 1$, that is, $\tau \geqq \pi$, that we generally can get by with only the two first terms. The function $B_1(\omega,\tau)$ may be calculated in exactly the same manner, and we obtain

$$B_1(\omega,\tau) = -\frac{i\pi^{3/2}e^{-i\omega}}{\tau\Gamma(1-i\omega')\Gamma\left(\frac{1}{2}+i\omega'\right)\operatorname{Sinh}\pi\omega'}\left\{1-\frac{i\omega'}{1+2i\omega'}\sum_{1}^{\infty}\left(2-\frac{1}{n}\right)\alpha_n e^{-2n\tau}\right\}$$

Therefore, for the function which, later on, will be the only one of interest

$$T(\omega,\tau) = \frac{B_1 - B_0}{B_1 + B_0} \tag{23}$$

which, for $\tau = 0$, is transformed into the function known from the theory of the single wing (ref. 4), namely,

$$T(\omega) = \frac{H_1^{(2)}(\omega) - iH_0^{(2)}(\omega)}{H_1^{(2)}(\omega) + iH_0^{(2)}(\omega)}$$

there follows the representation

$$T(\omega,\tau) = 1 - \frac{2i\omega'}{1 + 2i\omega'}\left(1 - e^{-2\tau}\right)\frac{\sum_0^\infty \alpha_n e^{-2n\tau}}{1 - \frac{i\omega'}{1 + 2i\omega'}\sum_1^\infty \left(2 - \frac{1}{n}\right)\alpha_n e^{-2n\tau}} \tag{24}$$

From this we see that for spacing factors $\alpha/l \leq 1$ with an error of not quite 1 percent we may set

$$1 - T(\omega,\tau) = \frac{2i\omega'}{1 + 2i\omega'} \tag{24'}$$

LIFT DISTRIBUTION AND LIFT

If we write the vortex distribution in the form

$$\gamma_1(x_1) = -\frac{2}{\pi}\sqrt{\frac{1 - x_1}{1 + x_1}}\int_{-1}^{1}\frac{\gamma_0 w_1(x_1) + g_1(x_1)}{\sqrt{1 - x_1^2}}\,dx_1 +$$

$$\frac{2}{\pi}\sqrt{1 - x_1^2}\,\oint_{-1}^{1}\frac{\gamma_0 w_1(\xi_1) + g_1(\xi_1)}{(x_1 - \xi_1)\sqrt{1 - \xi_1^2}}\,d\xi_1 \tag{25}$$

there follows (ref. 5, A1)

$$\int_{-1}^{x_1}\gamma_1(\xi_1)\,d\xi_1 = C\left[1 - \frac{1}{\pi}\arccos x_1\right] +$$

$$\frac{2}{\pi}\sqrt{1 - x_1^2}\,\oint_{-1}^{1}\frac{\gamma_0 w_2(\xi_1) + g_2(\xi_1)}{(x_1 - \xi_1)\sqrt{1 - \xi_1^2}}\,d\xi_1 \tag{26}$$

with

$$0 \leqq \arccos x_1 \leqq \pi$$

where

$$w_2(x_1) = \int^{x_1} w_1(\xi_1)\,d\xi_1 \qquad g_2(x_1) = \int^{x_1} g_1(\xi_1)\,d\xi_1 \tag{27}$$

and C has the significance given above; namely

$$C = \frac{\tau}{s}\,\frac{e^{-i\omega}}{i\omega}\,\gamma_0 \tag{28}$$

In order to calculate the function $w_2(x_1)$, we start from the function

$$w_1(x_1,\sigma) = \frac{l}{2\alpha s}\int_1^{\infty} e^{-\sigma\xi}\,\frac{d\xi}{x_1 - \xi_1} \qquad |x_1| < 1$$

For this function, $w_1(x_1,i\omega) = w_1(x_1)$. If we first assume that $\underline{R}\sigma > 0$, there applies

$$\int^{x_1} w_1(u_1,\sigma)\,du_1 = \frac{l}{2\alpha s}\int_1^{\infty} e^{-\sigma\xi}\,\ln(\xi_1 - x_1)\,d\xi$$

$$= \frac{l}{2\alpha s}\left\{\frac{e^{-\sigma}}{\sigma}\ln(1 - x_1) + \frac{1}{\sigma}\int_1^{\infty} e^{-\sigma\xi}\,\frac{1}{\xi_1 - x_1}\,\frac{d\xi_1}{d\xi}\,d\xi\right\}$$

$$= \frac{l}{2\alpha s}\left\{\frac{e^{-\sigma}}{\sigma}\ln(1 - x_1) + \frac{\tau}{s\sigma}\int_1^{\infty} e^{-\sigma\xi}\,\frac{s(\xi_1 - x_1) + sx_1 + c}{\xi_1 - x_1}\,d\xi\right\}$$

Hence it follows that

$$\int^{x_1} w_1(u_1,\sigma)\,du_1 = \frac{l}{2\alpha s\sigma}e^{-\sigma}\ln(1 - x_1) - (sx_1 + c)\,\frac{\tau}{s\sigma}\,w_1(x_1,\sigma)$$

If we let $\sigma \to i\omega$, there follows

$$w_2(x_1) = \frac{l}{2\alpha s}\,\frac{e^{-i\omega}}{i\omega}\ln(1 - x_1) - \frac{\tau}{si\omega}\,(sx_1 + c)\,w_1(x_1)$$

If we further consider the integral relationship

$$\sqrt{1 - x_1^2} \oint_{-1}^{1} \frac{\ln(1 - \xi_1)}{(x_1 - \xi_1)\sqrt{1 - \xi_1^2}} \, d\xi_1 = \pi^2 \left[1 - \frac{1}{\pi} \arccos x_1\right]$$

with

$$0 \leq \arccos x_1 \leq \pi$$

we obtain from (18) for the lift distribution the representation

$$P(x) = \frac{2}{\pi}(sx_1 + \alpha)\sqrt{\frac{1 - x_1}{1 + x_1}} \oint_{-1}^{1} \sqrt{\frac{1 + v_1}{1 - v_1}} \frac{g_1(v_1)}{x_1 - v_1} \, dv_1 +$$

$$i \frac{\omega s}{\tau} \frac{2}{\pi} \sqrt{1 - x_1^2} \oint_{-1}^{1} \frac{g_2(v_1) dv_1}{(x_1 - v_1)\sqrt{1 - v_1^2}} =$$

$$\gamma_0 \left[(sx_1 + c)\sqrt{\frac{1 - x_1}{1 + x_1}} - s\sqrt{1 - x_1^2}\right] \frac{2}{\pi} \int_{-1}^{1} \frac{w_1(v_1)}{\sqrt{1 - v_1^2}} \, dv_1$$

in which only the integral over the function w_1 remains to be calculated. For it there results

$$\frac{2}{\pi} \int_{-1}^{1} \frac{w_1(v_1)}{\sqrt{1 - v_1^2}} \, dv_1 = \frac{l}{\pi \alpha s} \int_{-1}^{1} \frac{1}{\sqrt{1 - v_1^2}} \int_{1}^{\infty} \frac{e^{-i\omega\xi}}{v_1 - \xi_1} \, d\xi \, dv_1$$

$$= -\frac{l}{\alpha s} \int_{1}^{\infty} \frac{e^{-i\omega\xi}}{\sqrt{\xi_1^2 - 1}} \, d\xi_1 = -\frac{l\pi}{2\alpha s} B_0(\omega, \tau)$$

and we obtain, because of (16)

$$-\gamma_0 \frac{2}{\pi} \int_{-1}^{1} \frac{w_1(v_1)}{\sqrt{1 - v_1^2}} \, dv_1 = \frac{1}{2}\left[1 - T(\omega, \tau)\right] \frac{2}{\pi} \int_{-1}^{1} g_1(x_1) \sqrt{\frac{1 + x_1}{1 - x_1}} \, dx_1$$

where T is the expression defined by (23). There follows finally the ultimate representation for the lift distribution

$$P(x) = \frac{2}{\pi}(sx_1 + c)\sqrt{\frac{1 - \xi_1}{1 + x_1}} \oint_{-1}^{1} \sqrt{\frac{1 + \xi_1}{1 - \xi_1}} \frac{g_1(\xi_1)}{x_1 - \xi_1} d\xi_1 +$$

$$\frac{i\omega s}{\tau} \frac{2}{\pi} \sqrt{1 - x_1^2} \oint_{-1}^{1} \frac{g_2(\xi_1)}{(x_1 - \xi_1)\sqrt{1 - \xi_1^2}} d\xi_1 +$$

$$\frac{1}{2}\left[1 - T(\omega,\tau)\right] \left\{(sx_1 + c)\sqrt{\frac{1 - x_1}{1 + x_1}} - s\sqrt{1 - x_1^2}\right\} \qquad (29)$$

which for $\tau \to 0$ is transformed into the representation known for the single wing (refs. 4, 5). For the lift we obtain from (29)

$$A = \int_{-l/2}^{l/2} \Delta p \, dx' = \frac{1}{2} \rho V l \frac{s}{\tau} \int_{-1}^{1} \frac{P(x)}{sx_1 + c} dx_1 \times e^{i\nu t}$$

$$= \rho V l \frac{s}{\tau} \left\{ \left[(1 - T(\omega,\tau)) \frac{1}{1 + e^{\tau}} - 1 \right] \int_{-1}^{1} g_1(x_1) \sqrt{\frac{1 + x_1}{1 - x_1}} dx_1 + \right.$$

$$\left. \frac{i\omega}{\tau} \int_{-1}^{1} \frac{g_2(x_1)(1 - c - sx_1)}{(c + sx_1)\sqrt{1 - x_1^2}} dx_1 \right\} e^{i\nu t} \qquad (30)$$

BENDING VIBRATIONS

As an example of application, we shall consider the case where the blades of the ring perform bending vibrations. If Δ is the amplitude of the deflection, $i\omega 2V \frac{\Delta}{l} e^{i\nu t}$ is the velocity with which each blade point moves in the direction of the y' axis. Therefore,

$$g(x') = i\omega 2V \frac{\Delta}{l} = \delta = \text{const}$$

and hence there follows from the definition given above:

$$g_1(x_1) = \frac{\delta}{sx_1 + c} \quad \text{and} \quad g_2(x_1) = \frac{\delta}{s} \ln(sx_1 + c)$$

If we enter these into (30), we obtain for the lift the expression

$$A = \pi\rho V^2 \Delta \frac{s}{\tau} \left\{ i\omega \left[(1 - T) \frac{1}{1 + e^{\tau}} - 1 \right] \frac{4}{1 + e^{\tau}} + \omega^2 \frac{8}{\tau s} \ln \operatorname{Cosh} \frac{\tau}{2} \right\} \quad (31)$$

which for $\tau \to 0$ is transformed into

$$\pi\rho V^2 \Delta \left\{ \omega^2 - i\omega(1 + T(\omega)) \right\}$$

That is the value known for the single wing (ref. 4). If we assume, on the other hand that τ is large, and limit ourselves to a linear theory in α/l, we obtain, in agreement with reference 1

$$A = \rho V^2 \Delta \frac{\alpha}{l} \left[4\dot{\omega}^2 - 2i\omega \right] \quad (32)$$

If we use for spacings $\alpha/l \leqq 1$ the approximation (24') for $1 - T$, the lift may be written in the form

$$A = \rho V^2 \Delta \frac{\alpha}{l} \left\{ 4\omega^2 f_1(\omega', \tau) - 2i\omega f_2(\omega', \tau) \right\} \quad (33)$$

with

$$f_1(\omega', \tau) = \frac{2}{\tau} \ln \operatorname{Cosh} \frac{\tau}{2} - \frac{e^{\tau} - e^{-\tau}}{\tau\left(1 + 4\omega'^2\right)\left(1 + e^{\tau}\right)^2}$$

$$f_2(\omega', \tau) = \frac{e^{\tau} - e^{-\tau}}{e^{\tau} + 1} \left[1 - \frac{4\omega'^2}{\left(1 + 4\omega'^2\right)\left(1 + e^{\tau}\right)} \right]$$

where these two functions assume the value 1 for a linear theory in α/l. Hence we can see that for spacings $\alpha/l \leqq 0.5$, that is, for τ values $\geqq 2\pi$, the linear theory yields the left term which is proportional to the velocity; thus the damping, with an error of not quite 1 percent. In contrast, for the term which is proportional to the deflection, that is, behaves like a spring force, conditions are considerably more unfavorable. This term is very inadequately included by the linear theory. We must note, however, that in vibration calculations this term is opposed by a mechanical spring force which is generally very large compared to the corresponding term of an aerodynamic force so that the deficiency has only a slight effect.

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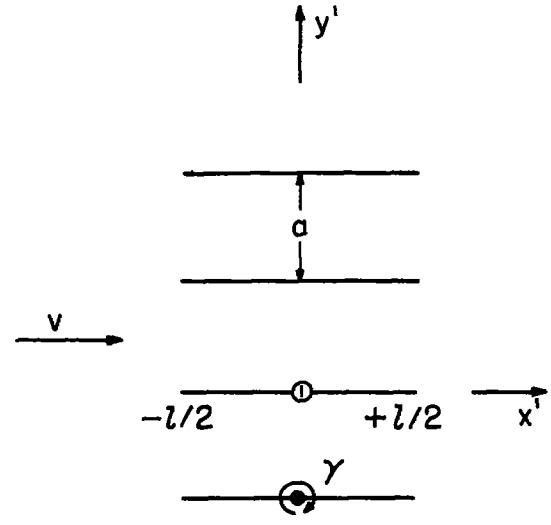

Figure 1. - Cascade parameters.

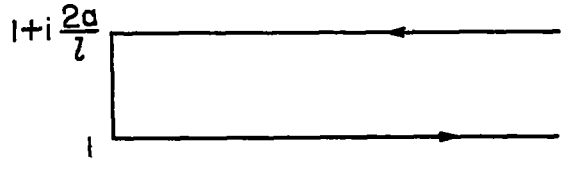

Figure 2. - Path of integration C.