# SCHULER PERIOD IN LEO SATELLITES

J. Russell Carpenter† and I.Y. Bar-Itzhack‡

## ABSTRACT

This paper generalizes and extends the concept of the Schuler oscillation that occurs in the theory of inertial navigation systems, allowing one to see how the Schuler phenomenon affects inertial navigation systems operating in space. We show why a low earth orbit satellite's orbital period is identical to the period of the Schuler pendulum, which is the period of the errors for terrestrial inertial navigation systems. We also show that the generalized form of the Schuler oscillation takes the same form as the Hill-Clohessy-Wiltshire equations for satellite relative motion, and that the period of the out-of-plane motion in neighboring satellite relative trajectories is the same as the Schuler period. Finally, we describe how INS gyro drift manifests itself in different coordinate systems for the orbital case. These results may assist orbital flight dynamics and attitude control systems engineers in the design and analysis of INS-equipped spacecraft.

## INTRODUCTION

People working in the field of inertial navigation systems (INS) are familiar with the phenomenon of Schuler oscillations [1] in INS. The majority of them are familiar with the concept of the Schuler pendulum. However, the fact that the period of both is identical to the period of low earth orbit (LEO) satellites and to the period of the oscillatory change of the relative location of two adjacent LEO satellites is not widely known in the INS community, if at all. It is also not widely known that when the INS is installed on a LEO satellite traversing a near-circular orbit, the INS error equations are the same as the Hill-Clohessy-Wiltshire equations that describe the change of the relative location of two adjacent LEO satellites. On the other hand, orbital mechanics professionals are well aware of the orbital period of LEO satellites and the equations of their relative motion, but in general they are unfamiliar with the Schuler pendulum and Schuler oscillations in INS, and thus are unaware of the identity between all four periods. In this paper we examine the four phenomena; namely, the Schuler pendulum, Schuler oscillations, the LEO satellite orbital period, and the period of the relative motion between two adjacent LEO satellites. In particular we show why the LEO satellite's orbital period is identical to the well-known INS Schuler period. We also show that if the INS error equations are generalized to a non-terrestrial case, a generalized form of the Schuler oscillation exists, which takes the form of the Hill-Clohessy-Wiltshire equations for satellite relative motion.

## SCHULER PENDULUM

In a 1923 paper [1], while claiming that INS could not be realized because "differentiation between the force of gravity and acceleration is impossible", the German Physicist, Maximilian Schuler, explained that if one can build a pendulum that has the length of the earth's radius then one could move at will the pendulum point of support and still have the pendulum point in the direction of the vertical. (Schuler explained that this stemmed from the fact that the center of mass of the apparatus remained at the center of

† Senior Aerospace Engineer, Flight Dynamics Analysis Branch, Mission Engineering and System Analysis Division, Code 595, NASA Goddard Space Flight Center, Greenbelt, MD 20771. AIAA Senior Member. Tel: (301) 286-7526, Fax: (301) 286-0369. Email: James.R.Carpenter@nasa.gov

‡ Sophie and William Shamban Professor. Faculty of Aerospace Engineering, Technion-Israel Institute of Technology, Haifa 32000, Israel. Member Technion Asher Space Research Institute. AIAA Fellow. Tel: +972-4-829-3196, Fax: +972-4-823-1848. Email: ibaritz@tx.technion.ac.il

the earth and thus at rest.) It is very well known that, $T_p$, the period of such mathematical pendulum, is given by

$$T_p = 2\pi\sqrt{\frac{R}{g}} \tag{1}$$

where R is the radius of the earth, and g is its gravity.

## SCHULER OSCILLATIONS IN INS ERROR PROPAGATION

The simplest way to describe the generation of oscillations in INS is to examine the behavior of a stable-platform INS in response to tilt error.

For simplicity, consider a stable-platform INS where the platform control system is programmed to keep the platform horizontal with one of the platform axes, say the $x$-axis, pointing north, the $y$-axis pointing east and the $z$-axis pointing in the direction of the local nadir. Consequently the preferred platform coordinate system is the North, East, and Down system. Suppose that there is a platform tilt error, denoted by $\varphi$, about the platform North-axis (see Fig. 1). As a result the East accelerometer reads gravity as acceleration component $f = -g \cdot \sin\varphi$ (the sign is negative because the accelerometer reads gravity as acceleration in the opposite direction). The INS computer is set to integrate this measured f to obtain velocity. Thus the computed East velocity is the integral of f; in other words,

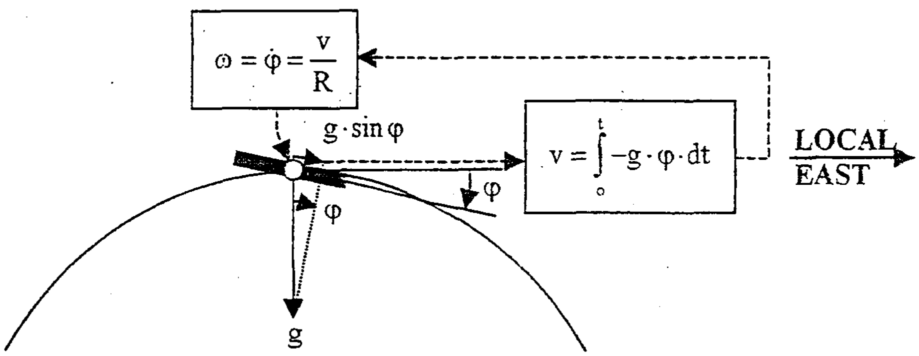


Fig. 1: Generation of Miss-level Error in a Stable Platform INS.

$$\dot{v} = -g \cdot \sin\varphi \tag{2.a}$$

Usually $\varphi \approx 0$, therefore we can write

$$\dot{v} = -g\varphi \tag{2.b}$$

In order to keep the platform level, the platform control system rotates the platform about the North axis by the angular rate $v/R$, thus

$$\dot{\varphi} - \frac{v}{R} \tag{3}$$

Differentiation of the last equation and substitution of $\dot{V}$ from Eq. (2.b) into the resultant equation yields

$$\ddot{\varphi}+\frac{g}{R}\varphi=0 \tag{4}$$

The last equation is that of a harmonic oscillator whose period, $T_{INS}$, is given by

$$T_{INS}=2\pi\sqrt{\frac{R}{g}} \tag{5}$$

This period, known as Schuler period, is well known in Inertial Navigation technology. A comparison between Eqs. (5) and (1) reveals that

$$T_{INS}=T_p \tag{6}$$

It is only ironical that this period, a characteristic feature of INS, is named after Schuler who maintained that the construction of INS was "an entirely impossible undertaking" [1, 2].

## LEO SATELLITE ORBITAL PERIOD

Consider a LEO satellite on a circular orbit at the surface of a hypothetical perfectly smooth airless earth. In order to be in orbit the satellite has to travel at the orbital velocity where the centrifugal acceleration balances gravity. That is, the orbital velocity, $V$, has to satisfy the relation

$$\frac{V^2}{R}=g \tag{7}$$

thus

$$V=\sqrt{gR} \tag{8}$$

The distance covered after one orbit is the circumference of earth, which is $2\pi R$. Therefore write:

$$VT_o=2\pi R \tag{9}$$

$T_o$ being the duration of one revolution. Substituting V from Eq. (8) into the last expression yields

$$\sqrt{gR}\cdot T_o=2\pi R \tag{10}$$

which gives

$$T_o=2\pi\sqrt{\frac{R}{g}} \tag{11}$$

It means that the orbital period also equals the Schuler period!

This can also be shown in the following way. It is well known that the Keplerian parameter, $n$, also known as Mean Motion, is given by

$$n=\sqrt{\frac{\mu}{R^3}} \tag{12}$$

but

$$g = -\frac{\mu \mathbf{R}}{R^3} = -\frac{\mu}{R^2}\frac{\mathbf{R}}{R} = -\frac{\mu}{R^2}\mathbf{1}_R \tag{13}$$

where $\mathbf{R}$ is the vector of earth radius from the center of earth to the satellite, and $\mathbf{1}_R$ is a unit vector in the direction of this vector, therefore

$$g = |g| = \frac{\mu}{R^2} \tag{14}$$

From Eqs. (12) and (14) we obtain

$$n = \sqrt{\frac{g}{R}} \tag{15}$$

and since $n = 2\pi / T_o$, we obtain

$$T_o = 2\pi\sqrt{\frac{R}{g}} \tag{16}$$

which is the Schuler period.

## GENERALIZATION OF THE SCHULER OSCILLATION IN INS ERROR PROPAGATION

The above derivation uses a simplified scalar INS error model to describe the Schuler oscillation. This section generalizes that explanation to the vector case. It then shows how the generalized form of the INS error equations results in Hill's equations.

The INS senses a specific force, $\mathbf{f}$, consisting of the true acceleration, and the acceleration of gravity, $\mathbf{g}$, where the latter is interpreted as a negative acceleration, thus:

$$\mathbf{f} = \mathbf{a} - \mathbf{g}(\mathbf{R}_t) \tag{17}$$

where gravity is a vector function of the true position, $\mathbf{R}_t$. The INS accelerometer measurement, $\mathbf{f}_m$, is corrupted by bias (among other errors neglected here), $\delta\mathbf{a}$, thus:

$$\mathbf{f}_m = \mathbf{a} - \mathbf{g}(\mathbf{R}_t) + \delta\mathbf{a} \tag{18}$$

As a result, the INS reconstruction of the true acceleration is as follows:

$$\mathbf{a}_c = \mathbf{f}_m + \mathbf{g}(\mathbf{R}_c) \tag{19.a}$$

where $\mathbf{R}_c$ is the computed position vector. Using Eq. (18) the last equation becomes

$$\mathbf{a}_c = \mathbf{a} - \mathbf{g}(\mathbf{R}_t) + \mathbf{g}(\mathbf{R}_c) + \delta\mathbf{a} \tag{19.b}$$

$$\therefore \mathbf{a}_c = \mathbf{a} + \delta\mathbf{a} + \left[\mathbf{g}(\mathbf{R}_t + \delta\mathbf{R}) - \mathbf{g}(\mathbf{R}_t)\right] \tag{19.c}$$

where $\delta\mathbf{R} = \mathbf{R}_c - \mathbf{R}_t$. Then using a truncated Taylor series expansion yields

$$\mathbf{a}_c = \mathbf{a} + \delta\mathbf{a} + \left[ \mathbf{g}(\mathbf{R}_t) + \left.\frac{\partial \mathbf{g}}{\partial \mathbf{R}}\right|_{\mathbf{R}_t} \delta\mathbf{R} - \mathbf{g}(\mathbf{R}_t) \right] + \ldots \tag{19.d}$$

Consequently

$$\mathbf{a}_c = \mathbf{a} + \delta\mathbf{a} + \left.\frac{\partial \mathbf{g}}{\partial \mathbf{R}}\right|_{\mathbf{R}_t} \delta\mathbf{R} + \ldots \tag{19.e}$$

Therefore, to compute its position, the INS performs the following integration:

$$\mathbf{R}_c = \iint \mathbf{a}_c \, dt\, dt = \iint \left( \mathbf{a} + \delta\mathbf{a} + G(\mathbf{R}_t)\delta\mathbf{R} + \ldots \right) dt\, dt \tag{20}$$

where $G(\mathbf{R}_t) = \partial \mathbf{g} / \partial \mathbf{R}|_{\mathbf{R}_t}$. Since (see Eq. 13)

$$\mathbf{g} = -\frac{\mu \mathbf{R}}{R^3} = -\frac{\mu}{R^2}\frac{\mathbf{R}}{R} = -g\mathbf{1}_R \tag{21}$$

where $\mathbf{1}_R = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$, it can be shown in a straightforward manner that

$$G(\mathbf{r}) = -\frac{g}{R}\left( I - 3\mathbf{1}_R \mathbf{1}_R^T \right) \tag{22}$$

In reality, the true position is

$$\mathbf{R}_t = \iint \mathbf{a} dt dt \tag{23}$$

To get a differential equation for the INS errors, insert Eq. (23) into Eq. (20) to get

$$\mathbf{R}_c = \mathbf{R}_t + \iint \left( \delta\mathbf{a} + G(\mathbf{R}_t)\delta\mathbf{R} + \ldots \right) dt dt \tag{24}$$

and differentiate Eq. (24) with respect to an inertial frame. This yields

$$\ddot{\mathbf{R}}_c = \ddot{\mathbf{R}}_t + \delta\mathbf{a} + G(\mathbf{R}_t)\delta\mathbf{R} + \ldots \tag{25}$$

which, using the definition of $\delta\mathbf{R}$, can be written as:

$$\delta\ddot{\mathbf{R}} - G(\mathbf{R}_t)\delta\mathbf{R} = \delta\mathbf{a} + \ldots \tag{26}$$

Substitution of $G(\mathbf{R}_t)$ from Eq. (22) into the last equation yields:

$$\delta\ddot{\mathbf{R}} + \frac{g}{R}\left( I - 3\mathbf{1}_R \mathbf{1}_R^T \right)\delta\mathbf{R} = \delta\mathbf{a} + \ldots \tag{27}$$

Normally in terrestrial INS equations the z-axis of the reference coordinates points along the vertical, therefore

$$\left(I - 3\mathbf{1}_R\mathbf{1}_R^T\right) \approx \mathrm{diag}\{1 \quad 1 \quad -2\} \tag{28}$$

and Eq. (27) becomes

$$\delta\ddot{R} + \frac{g}{R}[1 \quad 1 \quad -2]\delta R \approx \delta a \tag{29}$$

The horizontal; that is, the $x$ and $y$ components of Eq. (29) form harmonic oscillators, with a natural frequency given by the square root of the scalar coefficient of the gravity gradient, and a corresponding period $T_{INS} = 2\pi\sqrt{g/R}$ , which is identical to the expression given in Eq. (5).

## THE PERIOD OF HILL'S EQUATION

Consider a LEO satellite in the vicinity of another LEO satellite. We name the first satellite *Primary Satellite (PS)* and the second *Secondary Satellite (SS)*. Define an orbital coordinate system whose origin is in the *PS*. Its $x$-axis is in the direction of the satellite zenith, the $y$-axis is in the direction of the satellite velocity vector, and the $z$-axis is normal to the orbital plane on the side that completes the set to a right-hand coordinate system (see Fig. 2). The differential equations that describe the relative displacement of the *SS* with respect to the *PS* in this coordinate system are known as Hill's equations or Clohessy-Wiltshire equations [3]. In the absence of external forces these equations are:

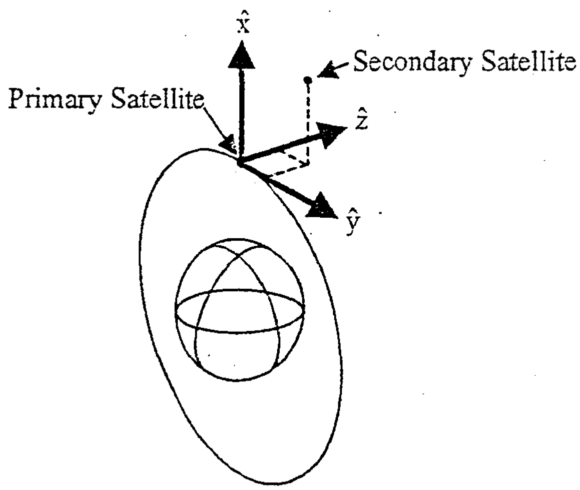


**Fig. 2:** The Orbital Coordinate System for Relative LEO. Motion

$$\ddot{x} - 3\omega^2 x - 2\omega\dot{y} = 0 \tag{30.a}$$

$$\ddot{y} + 2\omega\dot{x} = 0 \tag{30.b}$$

$$\ddot{z} + \omega^2 z = 0 \tag{30.c}$$

All three components have an oscillatory mode at the angular frequency $\omega$ (for the $z$-component this is obvious). It has been shown [3] that

$$\omega = \sqrt{\frac{\mu}{R^3}} \tag{31}$$

(we approximate here the radius of the circular orbit by the earth radius) and as we saw before, T, the period that corresponds to such angular frequency is

$$T = 2\pi\sqrt{\frac{R}{g}} \tag{32}$$

In fact it is easy to see without mathematical developments that this is the period of the *z* component. For simplicity assume that the orbit of the *PS* is circular and that the initial conditions on the *x*, *y*, and *z* components, and their derivatives are as follows $x_o = \dot{x}_o = y_o = \dot{y}_o = 0$, $z_o = 0$, $\dot{z}_o \neq 0$. Then the orbit of the *SS* is also a circular orbit that intersects the orbit of the *PS* twice in an orbit as shown in Fig. 3. Obviously the distance

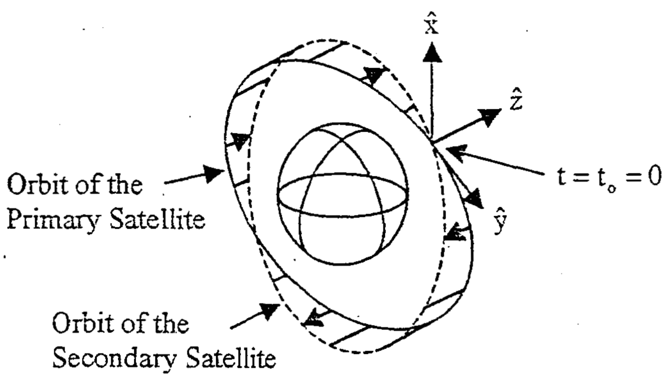


**Fig. 3:** The Relative Location of the Secondary Satellite with Respect to the Primary Satellite along the *z*-axis.

between the two satellites changes direction once on every orbit and the frequency at which this happens is the orbital frequency which, in the preceding section, was found to be the Schuler frequency.

## EXTENSION OF THE INS ERROR EQUATIONS TO NON-TERRESTRIAL MECHANIZATION

If we consider an INS installed on a satellite circling the earth in a near-circular orbit, and if Eq. (26) is written in a local-vertical local-horizontal frame, Hill's Equations result. To see this, consider the following general INS position error equation [4]:

$$\frac{{}^r d^2}{dt^2}\delta R + 2\left(\omega^r \times \frac{{}^r d}{dt}\delta R\right) + \frac{{}^r d}{dt}\omega^r \times \delta R + \omega^r \times \left(\omega^r \times \delta R\right) - G(R)\delta R = \Delta \mathbf{a} - \psi \times f \tag{33.a}$$

where $\omega^r$ is the angular velocity of frame r with respect to an inertial frame, and the pre-superscripts of the $d/dt$ operator indicate the frame in which the derivative is taken. The vectors $\Delta\mathbf{a}$ and $\psi$ denote, respectively, the accelerometer errors and the angular error caused by the gyro errors which are denoted by the vector $\boldsymbol{\varepsilon}$. (The latter errors are known as 'psi' errors). The 'psi' angular error develop according to [4]

$$\frac{{}^{r}d}{dt}\psi + \omega^{r} \times \psi = \varepsilon \tag{33.b}$$

At rest on earth $\boldsymbol{\omega}^{r}$ contributes a 24 hour oscillatory mode in the propagation of $\boldsymbol{\psi}$, which through the term $-\boldsymbol{\psi} \times \mathbf{f}$ on the right side of Eq. (33.a) contributes this mode to the propagation of $\delta\mathbf{R}$. In our case of a LEO satellite $\boldsymbol{\omega}^{r}$ is the orbital angular velocity at which the satellite orbits the earth, and according to Eq. (11) the orbital period corresponding to $\boldsymbol{\omega}^{r}$ is exactly the Schuler period. As a result, the oscillatory mode in the 'psi' angular error has a Schuler period rather than a 24 hour period. Because $\boldsymbol{\omega}^{r}$ is constant then the third term on the left of Eq. (33.a) drops, and so is the term $-\boldsymbol{\psi} \times \mathbf{f}$ on the right side of Eq. (33.a) because in orbit $\mathbf{f} = \mathbf{0}$. Therefore, if we consider only accelerometer bias, Eq. (33.a) becomes:

$$\frac{{}^{r}d^{2}}{dt^{2}}\delta R + 2\left(\omega^{r} \times \frac{{}^{r}d}{dt}\delta R\right) + \omega^{r} \times \left(\omega^{r} \times \delta R\right) - G(R)\delta R = \delta a \tag{33.c}$$

where, as before, $\delta\mathbf{a}$, is a vector whose three components are the components of the three corresponding accelerometer biases. Let us denote the magnitude of $\boldsymbol{\omega}^{r}$ by n, that is $n = |\omega^{r}|$. In a circular orbit (see Eq. 15) $n^{2} = g/R$, so Eq. (22) can be written as

$$G(R)\delta R = -n^{2}\delta R + 3n^{2}\mathbf{1}_{R}\mathbf{1}_{R}^{T}\delta R \tag{34}$$

Choose the axes of the rotating frame such that $\omega^{r} = n\hat{\mathbf{k}}_{r}$, and the *x*- axis is pointing in the direction of the zenith; that is, $\mathbf{1}_{R} = \hat{\mathbf{i}}_{r}$. Also, let

$$\delta R = x\hat{\mathbf{i}}_{r} + y\hat{\mathbf{j}}_{r} + z\hat{\mathbf{k}}_{r} \tag{35}$$

Inserting Eqs. (33.c), (34), and (35) into Eq. (26) results in

$$\ddot{x} - 2n\dot{y} - 3n^{2}x = \delta a_{x} + \ldots \tag{36.a}$$

$$\ddot{y} + 2n\dot{x} = \delta a_{y} + \ldots \tag{36.b}$$

$$\ddot{z} + n^{2}z = \delta a_{z} + \ldots \tag{36.c}$$

in which we recognize the homogenous part as the Hill-Clohessy-Wiltshire equations of relative motion, forced by the accelerometer bias and higher-order terms. We conclude that when on a circular orbit, the INS position error due to accelerometer bias behaves like the relative position between two adjacent satellites on low earth orbit. In particular, the relative position along the normal to the *PS* orbital plane oscillates at the Schuler frequency. Moreover, the 'psi' angular error too has a Schuler period, which, unlike in terrestrial INS, is not coupled into the INS position error.

*Remark:* In some spacecraft the INS reference coordinate system is an inertial one. In such cases equation (33.a) reduces to ${}^{I}d\psi / dt = \varepsilon$ where ${}^{I}$ designates the inertial frame. In such cases all three of the INS channels exhibit unbounded error growth due to gyro drift, and the spaceflight INS must use external aids such as star trackers to periodically re-align its attitude.

## CONCLUSIONS

In this paper we have presented results that may assist orbital flight dynamics and attitude control systems engineers in the design and analysis of INS-equipped spacecraft, by showing how phenomenon that occur in terrestrial INS generalizes to spacecraft applications. We showed that the period of the Schuler pendulum, the Schuler period in INS error propagation, the LEO orbital period, and the period of the oscillatory part of the well-known Hill-Clohessy-Wiltshire equations of LEO satellites' relative motion are identical. We have also shown that generalizing the INS error equations results in Hill's equations. The satellite relative position in this case corresponds to the INS position error; thus the Schuler frequency that is present in the errors of the level channels of terrestrial INS generalizes to an out-of-plane relative position oscillation at the orbit frequency. Although these oscillations do not directly enter into the attitude errors of a spacecraft-mounted INS, they do affect pointing of the spacecraft relative to the earth, since their effect on the navigation states corrupts the spacecraft's on-board computation of the local level reference frame. We also described how INS gyro drift produces an additional Schuler oscillation in the attitude error when the INS reference coordinate system is the orbital local-level system. However, when the INS reference system is inertial, the attitude exhibits unbounded error growth due to gyro drift.

## REFERENCES


[1] Schuler, M., "Die Störung von Pendel und Kreiselapparaten durch die Beschleunigung des Fahrzeuges (The Disturbance of Pendulum and Gyroscope by the Acceleration of a Vehicle)," Physikalische Zeitschrift, Vol. 24, July 1923, pp. 334-350.

[2] MacKenzie, D., *Inventing Accuracy: A Historical Sociology of Nuclear Missile Guidance (Inside Technology)*, MIT Press, Cambridge, MA, 1990.

[3] Vallado, D.A., *Fundamentals of Astrodynamics and Applications*, McGraw-Hill, New York, 1997. pp. 343-347.

[4] Pinson, J.C., "Inertial Guidance for Cruise Vehicles," *Guidance and Control of Aerospace Vehicles*, edited by C.T. Leondes, Mc Graw-Hill, New York, 1963, Chap. 4.