NATIONAL ADVISORY COMMITTEE
FOR AERONAUTICS

TECHNICAL MEMORANDUM 1413

IMPACT ON A COMPRESSIBLE FLUID

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Translation of "Udar o szhimaemuiu zhidkost."
Prikladnaia Matematika i Mekhanika.
Vol. 20, No. 1, 1956

NACA

Washington

February 1958

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IMPACT ON A COMPRESSIBLE FLUID*

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Upon impact of a solid body on the plane surface of a fluid, there occurs on the wetted surface of the body an abrupt pressure rise which propagates into both media with the speed of sound.

Below, we assume the case where the speed of propagation of sound in the body which falls on the surface of the fluid may be regarded as infinitely large in comparison with the speed of propagation of sound in the fluid; that is, we shall assume that the falling body is absolutely rigid. In this case, the entire relative speed of the motion which takes place at the beginning of the impact is absorbed by the fluid. The hydrodynamic pressures arising thereby are propagated from the contact surface within the fluid with the speed of sound in the form of compression and expansion waves and are gradually damped. After this, they are dispersed like impact pressures, reach ever larger regions of the fluid remote from the body and become equal to zero; in the fluid there remain hydrodynamic pressures corresponding to the motion of the body after the impact (ref. 1).

Neglecting the forces of viscosity and taking into account, furthermore, that the motion of the fluid begins from a state of rest, according to Thomson's theorem, we may consider the motion of an ideal compressible fluid in the process of impact to be potential.

We examine the case of impact upon the surface of a compressible fluid of a flat plate of infinite extent or of a body, the immersed part of the surface of which may be called approximately flat. In this report we discuss the first phase of the impact pressure on the surface of a fluid, prior to the appearance of a cavity, since at this stage the hydrodynamic pressures reach their maximum values. Observations, after the fall of the bodies on the surface of the fluid, show that the free surface of the fluid at this stage is almost completely at rest if one does not take into account the small rise in the neighborhood of the boundaries of the impact surface.

1. Let us consider the motion of a fluid in the coordinate system Oxy, rigidly connected with the solid body (fig. 1). In the selected coordinate system, we have as the potential of the velocity of

the motion φ -in the case of the two-dimensional problem - the following linearized equation (ref. 2):

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} \qquad \left(c = \sqrt{\frac{dp}{d\rho}}\right) \tag{1.1}$$

Here, c is the speed of sound. The boundary conditions in the case considered will be, on the free surface of the fluid,

$$\varphi = 0 \qquad\qquad \text{for } |x| > a, \quad y = 0 \tag{1.2}$$

on the surface of the plate,

$$\frac{\partial \varphi}{\partial y} = y' \qquad\qquad \text{for } |x| < a, \quad y = 0 \tag{1.3}$$

To the conditions required for a unique determination of the solution, we add yet another, the condition at infinity. Let us use the principle of radiation and express the condition that it is impossible to propagate disturbances from infinity inside the flow; in other words, the waves arising from the impact dissipate at infinity (ref. 2):

$$\lim_{r \to \infty} \sqrt{r}\left(\frac{\partial \varphi}{\partial r} + iv\varphi\right) = 0 \qquad\qquad \lim_{r \to \infty}\left|\sqrt{r}\varphi\right| = \text{Constant} \tag{1.4}$$

We shall also introduce the stationary system of coordinates $O_1 x_1 y_1$. We shall place the axis $O_1 x_1$ on the free surface; the axis $O_1 y_1$ inside the fluid.

At the initial instant of the impact, the motion of the plate is determined by the conditions

$$y_1 = 0 \qquad\qquad y_1' = v \qquad\qquad \text{for } t = 0 \tag{1.5}$$

2. We may continue the function $\varphi(x,y,t)$ to the upper half plane and obtain the function $\varphi(x,y,t)$, which is analytic in the entire

plane with the exception of the cut $|x| \leq a$, $y = 0$ whereby $\varphi(x,y,t) = -\varphi(x,-y,t)$.

We shall seek the partial integral of equation (1) in the form

$$\varphi(x,y,t) = \psi(x,y)e^{-\beta t} \qquad (2.1)$$

Substituting expression (2.1) into equation (1.1), we obtain

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - \frac{\beta^2}{c^2} \psi = 0 \qquad (2.2)$$

Assuming in this equation $x = ax_1$, $y = ay_1$, we can bring it into the form

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \nu^2 \psi = 0 \qquad \left(\nu^2 = -\frac{a^2 \beta^2}{c^2}\right) \qquad (2.3)$$

Here, and in what follows, the subscript 1 for the variables x_1 and y_1 is omitted.

The function $\psi(x,y)$ must also satisfy the boundary conditions

$$\psi = 0 \quad \text{for } |x| > 1, y = 0 \qquad \frac{\partial \psi}{\partial y} = a\nu \quad \text{for } |x| < 1, y = 0$$

$$(2.4)$$

and also the conditions of the principle of radiation

$$\lim \sqrt{r}\left(\frac{\partial \psi}{\partial r} + i\nu\psi\right) = 0 \qquad \lim |\sqrt{r}\psi| < \text{Constant} \quad \text{for } r = \sqrt{x^2 + y^2} \to \infty$$

$$(2.5)$$

We introduce the elliptical coordinates ξ and η (fig. 2) with the aid of the equalities

$$x = \cosh \xi \cos \eta \qquad y = \sinh \xi \sin \eta \qquad (2.6)$$

This corresponds to the conformal transformation $z = \cosh \zeta$ of the plane $z = x + iy$ into the plane $\zeta = \xi + i\eta$. From the relationships in equations (2.6), there follows

$$\frac{x^2}{\cosh^2\xi} + \frac{y^2}{\sinh^2\xi} = \cos^2\eta + \sin^2\eta = 1$$

$$(2.7)$$

$$\frac{x^2}{\cos^2\eta} - \frac{y^2}{\sin^2\eta} = \cosh^2\xi - \sinh^2\xi = 1$$

To the lines ξ = Constant in the ζ-plane, there corresponds a family of confocal ellipses in the z-plane and to the lines η = Constant there correspond a family of confocal hyperbolas, orthogonal to the family of ellipses. It is easy to see that the region of interest to us, $|x| < 1$, represents the degenerated ellipse $\xi = 0$, $|\eta| < \pi$. To the region of the coordinate axis $x > 1$, there corresponds the degenerated hyperbola $\eta = 0$, $\xi > 0$ and to $x < -1$, the degenerated hyperbola $\eta = \pm\pi$, $\xi > 0$.

We may transform equation (2.3) after substitution of equations (2.6), with the aid of Lamé's relationship (ref. 5)

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \left(\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2}\right)\left|\frac{d\zeta}{dz}\right|^2 \qquad (2.8)$$

into the form

$$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} + \nu^2(\cosh^2\xi - \cos^2\eta)\psi = 0 \qquad (2.9)$$

Assuming its solution in the form $\psi(\xi,\eta) = F(\xi)G(\eta)$, we obtain, after substitution of equation (2.9) for the functions F and G, the equations of Mathieu

$$\frac{d^2G}{d\eta^2} + (\alpha - 2\theta_* \cos 2\eta)G = 0 \qquad \frac{d^2F}{d\xi^2} - (\alpha - 2\theta_* \cosh 2\xi)F = 0$$

$$(2.10)$$

There

$$\theta_* = \frac{1}{4}\nu^2 \qquad \theta_* = -\theta \qquad \theta = \frac{a^2\beta^2}{4c^2} \qquad (\alpha \text{ is a Constant}) \quad (2.11)$$

As a result of the fact that α may assume any value, it is necessary - in order to made the solution unique - that the function $G(\eta)$ in the z-plane should be periodic, with the period 2π. This condition defines the multiplicity of eigenvalues $\alpha_{2n+1}(\theta_*)$ where $n = 0, 1, 2, \ldots$. The corresponding family of fundamental functions forms a complete orthogonal system.

The function $F(\xi)$ must be expressed through the modified Mathieu functions which satisfy the principle of radiation. In accordance with what has been said above, we shall seek the potential $\psi(\xi,\eta)$ in the form of an expansion in odd Mathieu functions of odd index

$$\psi(\xi,\eta) = \sum_{n=0}^{\infty} a_{2n+1} Ne^{(1)}_{2n+1}(\xi,-\theta) se_{2n+1}(\eta,-\theta) \qquad (2.12)$$

Here, $se_{2n+1}(\eta,-\theta)$ is an odd periodic Mathieu function which is the solution of the first kind of the first equation of (2.10) and is expressed through trigonometric functions in the form of the series

$$se_{2n+1}(\eta,-\theta) = (-1)^n \sum_{r=0}^{\infty} (-1)^r A^{2n+1}_{2r+1} \sin(2r+1)\eta \qquad (2.13)$$

where A^{2n+1}_{2r+1}, the expansion coefficients, are functions of θ. The function $Ne^{(1)}_{2n+1}(\xi,-\theta)$ constitutes the combined Mathieu function, expressed through the product of the Bessel functions of imaginary argument in the form of the following series:

$$Ne^{(1)}_{2n+1}(\xi,-\theta) = \frac{2iP_{2n+1}}{\pi A_1^{2n+1}} \sum_{r=0}^{\infty} A^{2n+1}_{2r+1}\left[I_r(v_1)K_{r+1}(v_2) + I_{r+1}(v_1)K_r(v_2)\right]$$

$$(2.14)$$

Here,

$$P'_{2n+1} = \frac{(-1)^{n+1}}{kA_1^{2n+1}} ce_{2n+1}(0,\theta)ce'_{2n+1}\left(\frac{1}{2}\pi,\theta\right)$$

$$v_1 = ke^{-\xi}$$

$$v_2 = ke^\xi$$

$$k^2 = \theta$$

$ce_{2n+1}(\xi, \theta)$ is the odd periodic Mathieu function, $I_m(v_1)$ is the Bessel function of imaginary argument, and $K_m(v_2)$ is a Macdonald function.

The function $Ne_{2n+1}^{(1)}(\xi, -\theta)$ has the following asymptotic representation:

$$Ne_{2n+1}^{(1)} \sim P'_{2n+1} \frac{e^{-v}}{\sqrt{2\pi v}} \qquad (2.15)$$

where $v = 2k \cosh \xi$, whence it is evident that for $\xi \to \infty$ the function (2.15) tends monotonically toward zero. Hence, it follows immediately that the expression we derived for the potential (2.12) satisfies the principle of radiation.

The function of the potential (2.12) satisfies the first boundary condition (2.4) since, for $\eta = 0$ and $\eta = \pm\pi$, the function $se_{2n+1}(\eta, -\theta)$ is transformed into zero.

For determining the arbitrary integration constant, a_{2n+1}, we use the second boundary condition (eq. (2.4)) which, in the elliptical coordinate system, is written in the form

$$\frac{\partial \psi}{\partial \xi} = av \sin \eta \quad \text{for} \quad |\eta| < \pi, \ \xi = 0 \qquad (2.16)$$

Subjecting the expression of the potential (2.12) to the condition (2.16), we obtain

$$av \sin \eta = \sum_{n=0}^{\infty} a_{2n+1} Ne_{2n+1}^{(1)'}(0, -\theta) se_{2n+1}(\eta, -\theta) \qquad (|\eta| < \pi) \quad (2.17)$$

Here

$$Ne_{2n+1}^{(1)'}(0, -\theta) = \left[\frac{\partial Ne_{2n+1}^{(1)}(\xi, -\theta)}{\partial \xi} \right]_{\xi=0} \qquad (2.18)$$

We multiply the equation obtained by $se_{2n+1}(\eta, -\theta)d\eta$, integrate from $-\pi$ to $+\pi$ and find

$$av \int_{-\pi}^{\pi} se_{2n+1}(\eta, -\theta)\sin \eta \, d\eta = \sum_{n=0}^{\infty} a_{2n+1} Ne_{2n+1}^{(1)'}(0, -\theta) \int_{-\pi}^{\pi} se_{2n+1}^{2}(\eta, -\theta)d\eta$$

$$(2.19)$$

On the basis of the orthogonality of the Mathieu functions

$$\int_{-\pi}^{\pi} se_{2n+1}(\eta, -\theta)\sin \eta \, d\eta = (-1)^{n} A_{1}^{2n+1} \pi$$

In conformity with the second condition of normalization of the Mathieu functions (ref. 4),

$$\int_{-\pi}^{\pi} se_{2n+1}^{2}(\eta, -\theta)d\eta = \pi$$

Substituting the values of integrals found into equation (2.12) and solving it with respect to a_{2n+1}, we obtain

$$a_{2n+1} = (-1)^{n} \frac{A_{1}^{2n+1}}{Ne_{2n+1}^{(1)'}(0, -\theta)} av \qquad (2.20)$$

Substituting, in turn, equation (2.20) into equation (2.12), we find

$$\psi(\xi, \eta) = av \sum_{n=0}^{\infty} (-1)^{n} A_{1}^{2n+1} \frac{Ne_{2n+1}^{(1)}(\xi, -\theta)}{Ne_{2n+1}^{(1)'}(0, -\theta)} se_{2n+1}(\eta, -\theta) \qquad (2.21)$$

Taking into account formula (2.1), we introduce the velocity potential of the motion of the fluid considered in the form

$$\varphi(\xi, \eta, t) = a \sum_{n=0}^{\infty} (-1)^{n} A_{1}^{2n+1} \frac{Ne_{2n+1}^{(1)}(\xi, -\theta)}{Ne_{2n+1}^{(1)'}(0, -\theta)} se_{2n+1}(\eta, -\theta)ve^{-\beta t} \qquad (2.22)$$

The expression (2.22) is a partial integral of the wave equation (1.1) which satisfies the initial and boundary conditions that we set up.

3. The data obtained regarding the flow of the fluid enable us to turn to the determination of the hydrodynamic forces. For this purpose, we use Lagrange's integral. Neglecting the weight and the squares of the magnitudes of the absolute velocity of the fluid, we can find the overall value of the hydrodynamic forces per unit width of the plate. We integrate the pressure $p - p_0 = -\rho_0 \partial\varphi/\partial t$ along the length of the plate and obtain

$$R^* = -\rho_0 \int_{-a}^{a} \frac{\partial\varphi}{\partial t}\, dx \bigg|_{y=0} \quad \text{or} \quad R^* = \rho_0 a \int_{-\pi}^{\pi} \frac{\partial\varphi}{\partial t} \sin \eta\, d\eta \bigg|_{\xi=0} \qquad (3.1)$$

$$R^* = \rho_0 \pi a^2 \sum_{n=0}^{\infty} \frac{Ne_{2n+1}^{(1)}(0,-\theta)}{Ne_{2n+1}^{(1)\,'}(0,-\theta)} \left(A_1^{2n+1}\right)^2 \frac{d}{dt}\left(ve^{-\beta t}\right) \qquad (3.2)$$

Formula (3.2) is derived for the case of a plate moving in an infinite fluid. But according to the principle of symmetry, the values φ and $\partial\varphi/\partial t$ above and below the axis of the plate are equal in magnitude and opposite in sign. Hence, it follows that the force acting on the plate upon impact on the free surface of the fluid is equal to half the force acting on the plate in unbounded flow of the fluid:

$$R = \frac{\rho_0 \pi a^2}{2} \sum_{n=0}^{\infty} \frac{Ne_{2n+1}^{(1)}(0,-\theta)}{Ne_{2n+1}^{(1)\,'}(0,-\theta)} \left(A_1^{2n+1}\right)^2 \frac{d}{dt}\left(ve^{-\beta t}\right) \qquad (3.3)$$

For simplification of the further notation we shall write

$$Le(\theta) = \sum_{n=0}^{\infty} \frac{Ne_{2n+1}^{(1)}(0,-\theta)}{Ne_{2n+1}^{(1)\,'}(0,-\theta)} \left(A_1^{2n+1}\right)^2 \qquad (3.4)$$

The diagram of the function $Le(\theta)$ is presented in figure 3. Let us note that

$$\lim Le(\theta) = -\infty \quad \text{for} \quad \theta \to 0 \qquad \lim Le(\theta) = 0 \quad \text{for} \quad \theta \to \infty$$

Thereby, in the first limiting case for $\theta \to 0$ (which corresponds to a transition to an incompressible fluid $c = \infty$), the function $Le(\theta) \to \infty$ like k^{-2}.

Carrying out the differentiation in formula (3.3) and substituting equation (3.4), we obtain the following expression for determining the force of the impact:

$$R = -\frac{1}{2}\rho_0 \pi a^2 \beta \; Le(\theta)y' \qquad \left(y' = ve^{-\beta t}\right) \tag{3.5}$$

Here, y' is the speed of motion of the plate during the process of impact.

In order to determine the parameter β, we substitute equations (3.5) into the equation of motion of the plate which, in the case of weightless impact, has the form $my'' = -R$. As a result, we obtain the transcendental equation

$$m = -\frac{1}{2}\rho_0 \pi a^2 Le(\theta) \tag{3.6}$$

It is not difficult to determine from this equation, for the given quantities m, a, and ρ_0, making use of the diagram of the function $Le(\theta)$ (fig. 3), the value of θ and, consequently, the value of the parameter β. (See (2.11).) Taking into account equation (3.6), we obtain the expression for the force R in the form: $R = m\beta v e^{-\beta t}$; we integrate this expression and obtain

$$J = \int_0^t R \; dt = mv\left(1 - e^{-\beta t}\right) \tag{3.7}$$

This relationship permits us to determine the momentum of the fluid for any instant of time during the first phase of the impact. Using formula (3.5), we can also determine the kinetic energy of the plate.

Figure 4 shows $T = T(t)$, the variation with time of the kinetic energy of the plate ($m = 100$ kg sec^2 m^{-1}, $a = 1.10$ m, $v = 5.0$ m sec^{-1}) upon impact on water ($c = 1{,}485$ m sec^{-1}, $\rho_0 = 102$ kg sec^2 m^{-4}); on ethyl alcohol ($c = 1{,}170$ m sec^{-1}, $\rho_0 = 79$ kg sec^2 m^{-4}); and on pentane ($c = 800$ m sec^{-1}, $\rho_0 = 64$ kg sec^2 m^{-4}). Figure 5 shows for these same cases the diagram of the variation of the momentum of the plate during the process of impact.

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REFERENCES

1. Prandtl, L.: Gidroaeromekhanika. [Hydro and Aero Mechanics.] Izd. I,
     M., 1949.

2. Sedov, L. I.: Ploskie zadachi gidrodinamiki i aerodinamiki. [Two-
     Dimensional Problems in Hydrodynamics and Aerodynamics.] Gos. izd.
     tekhn.-teor. literaturi, M., 1950.

3. Khaskind, M. D.: Akusticheskie izlucheniia kolebliushchikhcia tel v
     szhimaemoi zhidkosti. [Acoustic Radiation of Oscillating Bodies
     in a Compressible Fluid.] Zhurnal eksperimentalnoi i teoreticheskoi
     fiziki, vyp. 7, 1946.

4. Mac-Lachlan, N.: Teoriia i prilozhenie funktsii Mate. [Theory and
     Application of the Mathieu Functions.] Izd. IL, M., 1953.

5. Whittaker, E. T., and Watson, G. N.: Kurs sovrennogo analiza, ch. II.
     [Course of Contemporary Analysis, Part II.] Gos. izd. tekhn.-teor.
     literaturi, 1934.
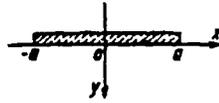
Figure 1

Figure 2

Figure 3

$$T = \frac{m(y')^2}{2}$$

1200

c=800 m sec⁻¹

800

c=1170 m sec⁻¹

c=1485 m sec⁻¹

400

c=∞

0   0.0004   0.0008   t

Figure 4

J(Kg sec)

500   mv

250

c=800 m sec⁻¹

c=1170 m sec⁻¹

c=1485 m sec⁻¹

c=∞

0   0.001 t sec

Figure 5