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Applied to Controlled Systems

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# Experiences with Probabilistic Analysis Applied to Controlled Systems\*

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This paper presents a semi-analytic method for computing frequency dependent means, variances, and failure probabilities for arbitrarily large-order closed-loop dynamical systems possessing a single uncertain parameter or with multiple highly correlated uncertain parameters. The approach will be shown to not suffer from the same computational challenges associated with computing failure probabilities using conventional FORM/SORM techniques. The approach is demonstrated by computing the probabilistic frequency domain performance of an optimal feed-forward disturbance rejection scheme.

## Nomenclature

|         |   |   |
|---------|---|---|
| $C$     | = | cubic spline approximation of $F$ (when $U$ -space is one dimensional)    |
| $C_i$   | = | cubic polynomial that represents $C$ on interpolation interval number $i$ |
| $F$     | = | $U$ -space counterpart of $f$ : $F(U) = f(T^{-1}(U))$ so $Y = F(U)$       |
| $F_X$   | = | cumulative distribution function of $X$                                   |
| $F_Y$   | = | cumulative distribution function of $Y$                                   |
| $f$     | = | function relating $X$ and $Y$ : $Y = f(X)$                                |
| $G$     | = | counterpart of $g$ on $U$ -space: $G(u) = g(T^{-1}(u))$                   |
| $g$     | = | limit state function on $X$ -space  |
| $P_f$   | = | probability of failure  |
| $T$     | = | transformation from $X$ -space into $U$ -space                            |
| $U$     | = | scalar or vector of uncorrelated standard normal random variables         |
| $u^*$   | = | most probable point (MPP) of failure                                      |
| $X$     | = | scalar or vector of random variables                                      |
| $x$     | = | generic point in $X$ -space   |
| $Y$     | = | scalar or vector of random variables dependent on $X$ : $Y = f(X)$        |
| $Y_a$   | = | approximation of $Y$ : $Y_a = C(U) = C(T(X))$                             |
| $y$     | = | generic point in $Y$ -space   |
| $\beta$ | = | $\ u^*\ $ , the reliability index   |
| $\Phi$  | = | 1 dimensional standard normal cumulative distribution function            |

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## I. Introduction

Stability and performance robustness are two of the fundamental metrics that must be considered in control system analysis and design. The first systematic procedures for analyzing the robustness of closed-loop dynamical systems were developed by H. W. Bode<sup>1</sup> and H. Nyquist<sup>2</sup>. These researchers were instrumental in developing some of the fundamental concepts for the analysis and design of feedback control systems and their frequency domain methodologies for single-input/single-output (SISO) systems that are still in use. Today in the digital world with high performance computers and powerful numerical software, the SISO concepts for analyzing robustness have been generalized to include multi-input/multi-output (MIMO) systems using loop-gain singular value analysis<sup>3</sup>, structured singular-values<sup>4,5</sup>, and  $\mu$ -analysis<sup>6</sup>. Unfortunately, these conventional approaches in many cases result in very conservative estimates of stability and performance and rely upon a mathematical definition of the uncertainties that may not necessarily reflect the true nature of the parameter and model-form uncertainties. In an attempt to overcome these limitations, researchers are now investigating methods for analyzing the stability of feedback systems with parameter uncertainties defined in terms of probability distribution functions. These approaches have been referred to as “probability of stability”, see Refs. 7–9. Initial work in this area relied upon Monte Carlo-based methods to assess the effect of probabilistic parametric uncertainties on system stability. Although viable in many cases, simulation based approaches have computational weaknesses when analyzing very low probability events, such as those related to aerospace vehicle stability.<sup>a</sup> More recently, researchers<sup>10,11</sup> have explored the use of first-order and second-order reliability methods (FORM and SORM) for computing probabilistic performance robustness in terms of root mean square (RMS) output response and probabilistic stability analysis in terms of the system’s eigenvalues. Although very important in the analysis of performance robustness of controlled systems, RMS response is an integrated quantity and does not provide the control system designer with important response information about the spectral nature of dynamical systems.

This paper extends previous probabilistic works by considering the full spectral domain of a closed-loop dynamical system and demonstrates the computation of means, variances, and probabilistic confidence intervals across a broad range of frequencies. Furthermore, this paper will highlight some of the

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<sup>a</sup>See also the paper “Multicriteria Optimization Methods for Design of Aircraft Control Systems” by Albert A. Schy and Daniel P. Giesy in the book *Multicriteria Optimization in Engineering and in the Sciences* (Plenum Press, New York, 1988; pp. 225-262) edited by Wolfram Stadler, and references [16] and [17] from that paper. In the example Schy and Giesy present, the control effectiveness and sideslip aerodynamic derivatives of the Space Shuttle are the uncertain parameters. These appear in the equations of lateral motion which are, in turn, used to model a lateral stability augmentation system (SAS) based on state feedback. Using the feedback gains as design variables, the SAS is designed by simultaneously optimizing multiple criteria. The multiple criteria are based on the eigenstructure of the closed loop lateral equations of motion and on the roll response, peak sideslip, peak control deflection, and peak control rate values over the 6 second roll maneuver duration. In order to make the design process *Stochastic Insensitive*, not only are the nominal values of the multiple criteria optimized but also estimates (based on the assumption that the design criteria are linear functions of the uncertain parameters) of the probabilities that each criterion will fall within designer specified desirable bounds.

computational challenges associated with computing failure probabilities using conventional FORM/SORM techniques. Additionally, this paper will present a semi-analytic method for computing means, variances, and failure probabilities for arbitrarily large-order systems possessing a single uncertain parameter or with multiple highly correlated uncertain parameters. The methodology will be demonstrated by computing the probabilistic performance of an optimal feed-forward disturbance rejection scheme with a single uncertain parameter. The case of multiple highly correlated uncertain parameters is dealt with by approximating the uncertain parameters as functions of a single uncertain parameter and applying the techniques developed in this paper.

## II. Mathematical model of uncertainty

In a standard mathematical formulation used for uncertainty modeling<sup>12</sup>, a scalar or vector  $X$  of random variables is used to represent one or more uncertain parameters. A real valued function  $g$ , called the limit state function, divides the uncertain domain into success ( $g(X) > 0$ ) and failure ( $g(X) \leq 0$ ) regions. The boundary between these regions, defined by  $g(X) = 0$ , is called the limit state surface.

One statistic of interest in design and analysis problems is the probability of failure,  $P_f = P[g(X) \leq 0]$ . This is given exactly by

$$P_f = \int_{g(X) \leq 0} 1 dF_X \quad (1)$$

where  $F_X$  is the cumulative distribution function (CDF) of  $X$ . However, evaluation of this integral is usually computationally intractable and so, historically, recourse has been made to approximation techniques such as Monte Carlo, FORM, or SORM.

To approximate failure probability using FORM or SORM,  $X$  is transformed by a transformation  $T$  so that  $U = T(X)$  is a vector of uncorrelated standard normal random variables. The  $X$ -space limit state function  $g$  is transformed to a  $U$ -space limit state function  $G$  by  $G(u) = g(T^{-1}(u))$ . The closest point to the origin on the  $U$ -space limit state surface  $G(u) = 0$ , called the most probable point (MPP) of failure and denoted  $u^*$ , is determined by solving the optimization problem “minimize  $\|u\|$  subject to the constraint  $G(u) = 0$ ”. The norm of  $u^*$  is denoted by  $\beta$  and called the reliability index. FORM approximates  $P_f$  by replacing the  $U$ -space limit state surface  $G(u) = 0$  by the hyperplane tangent to that surface at  $u^*$ . Then  $P_f$  is approximated by  $1 - \Phi(\beta)$  where  $\Phi$  is the CDF of the one dimensional standard normal random variable. This  $P_f$  approximation is exactly the probability that  $U$  is in the half-space determined by the tangent hyperplane that does not contain the origin. This is appropriate if  $G(0) > 0$ . If  $G(0) < 0$ ,  $P_f$  is approximated by  $\Phi(\beta)$ , while if  $G(0) = 0$ , the FORM approximation to  $P_f$  is 0.5.

The SORM approximation makes a second order approximation to the limit state surface at the MPP and so can be expected to provide a better approximation to the probability of failure. However, a cost is incurred both in the

need to have curvature information for the limit state function and in a more complicated evaluation of the probability of a standard normal random variable falling on one side of a parabolic or quadratic surface<sup>12-14</sup>. The present study made no use of SORM.

FORM has been successfully applied in problems of real engineering interest, such as structural component reliability problems. These successes occur when the probability of failure is small and the failure set (at least in  $U$ -space) does not have too pathological a geometry. However, if there are multiple points on the limit state surface that are not in close proximity to each other and whose norm is nearly  $\beta$ , or if  $u^*$  is fairly close to 0 and the limit state surface has substantial curvature at  $u^*$ , FORM can provide a poor approximation to the failure probability.

A simple example of this problem is given in Fig. 1. In this one-dimensional example, the failure set consists of two semi-infinite intervals. The limit state "surface" is zero-dimensional, i.e., a collection of (in this case 2) discrete points: namely -1.0 and +1.1. Thus the MPP is the closer of these two to 0, namely -1.0. The FORM estimation of the failure probability is based on the probability  $P_1$  that  $U$  falls in the left component of the failure set and ignores the contribution  $P_2$  of the right component of the failure set. It thus misses the mark by about 46%.

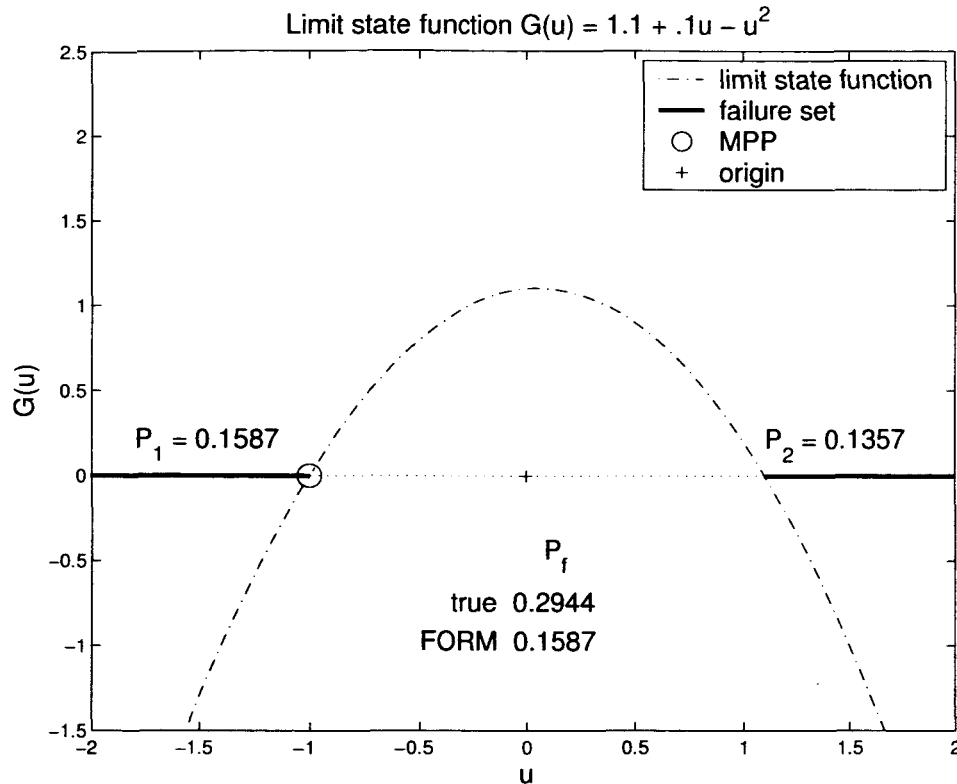
### III. Cumulative Distribution Functions of Dependent Random Parameters

Mathematically, an engineering system may be viewed as a vector-valued function of several variables. The independent variables could include design parameters such as component dimensions and controller gains and physical givens such as material properties. The dependent variables could include a wide variety of metrics of interest such as total mass, load capacity, stability margins, etc. If any of the independent variables are uncertain, and are therefore represented by random variables, the dependent variables also become uncertain. It is of interest in analyzing the system to determine what is the CDF of each dependent variable. Once this CDF is known, it can be used to compute statistics of interest such as mean, variance, confidence intervals, etc. This CDF is determined by the function that describes the system and the joint CDF of the independent variables. The next topic of interest is determining the CDF of a dependent variable; multiple dependent variables would be treated one at a time.

Consider a random parameter  $Y$  that is a function of other random parameters:  $Y = f(X)$ . The probabilistic behavior of  $Y$  is described by its CDF,  $F_Y$ , which is determined by  $f(X)$  and the CDF  $F_X$  of  $X$ :

$$F_Y(y) = P[Y \leq y] = P[f(X) \leq y] \quad (2)$$

This can be modeled in the same mathematical pattern as is used in the FORM approximation. If the limit state function  $g$  is defined as  $g(x) = f(x) - y$ , the



**Fig. 1. A Simple Limit State Function with Multiple MPPs**

“probability of failure” computed using this  $g$  approximates  $F_Y(y)$ . (It is probability a misnomer to continue to use the historical phrase “probability of failure” to describe the results of this calculation. The event  $[Y \leq y]$  is a “failure” only in that the random variable  $Y$  fails to exceed the sample value  $y$ . Notwithstanding, we will not introduce new terminology to use when traditional uncertainty analysis tools are used to approximate CDFs instead of failure probabilities.) This probability may now be approximated by FORM or SORM. Note that for each  $y$  for which the value of  $F_Y(y)$  is to be calculated, an optimization problem must be solved to find the MPP for the corresponding limit state function. The computational intensity of this task is mitigated by using the MPP from one value of  $y$  as a starting point in the calculation of the next – although a small change in  $y$  might result in a large jump in the location of the MPP, it is reasonable to expect that in most cases the MPP will change smoothly with changing  $y$ , and using the MPP from a neighboring problem will provide a good approximation to the MPP of the present problem leading to fast convergence of the optimization process.

One CDF of interest in the present study was the CDF of the Bode magnitude plot of a system with at least one random parameter. This adds a new dimension of complication – the CDF is really a family of functions, one for each frequency. If the FORM technique is to be used to approximate these CDFs, it would take advantage of information from calculations made at one frequency value to give a head start to optimization problems being solved at a near by frequency.

However, when this technique was applied in practice, validity checking with Monte Carlo simulation revealed that some of the problems that can beset FORM approximation were actually occurring. Failure probability sets of more than one component, such as is depicted in Fig. 1, were occurring in the Bode magnitude data considered in section V. This resulted in FORM based approximations to the CDF which were unacceptably inaccurate. When the uncertainties could be expressed in terms of a single random variable, an alternative to FORM was found to approximate the CDFs.

#### IV. The one-dimensional case

The case considered here is the one in which  $X$  and therefore  $U$  are scalar random parameters. Recall that  $U$  has the standard normal distribution. We wish to find the CDF of  $Y = f(X)$ . In the application of interest in this paper,  $f(X)$  will be the value of the Bode magnitude plot at a fixed value of frequency of some system in which  $X$  is a parameter.<sup>b</sup> When  $X$  is transformed into  $U$ -space,  $Y$  becomes a function of  $U$ :  $Y = F(U)$  where  $F(u) = f(T^{-1}(u))$ . For each  $y$ ,  $F_Y = P[Y \leq y] = P[F(U) \leq y]$ . Suppose that the function  $F$  is reasonably well-behaved (as could be expected in engineering applications). Then, if the set  $\{u | F(u) \leq y\}$  is not empty, it is a collection of intervals (possibly some finite, up to two semi-infinite, or just the whole real line). The boundary points of these intervals come from the solution set of the equation  $F(u) = y$ . If a complete set of solutions to this equation can be found, and if  $F$  is nice enough that the solution set contains only finitely many points, then these points divide the real line into a finite number of intervals on the interior of each of which we have that  $F$  is either strictly greater than  $y$  or strictly less than  $y$ . If  $u_1 < u_2$  are consecutive solutions of  $F(u) = y$ , then it can be determined whether  $F(u) > y$  or  $F(u) < y$  on the open interval  $(u_1, u_2)$  by evaluating  $F$  at a single interior point or evaluating  $F'$  at either endpoint. Adding up the probabilities that  $U$  lies in each interval of the set of intervals where  $F(u) \leq y$  produces the exact value of  $F_Y(y)$ ; i.e., the integral in Eq. (1) has been evaluated.

There are two problems with this.

1. For a general function  $F$ , it is difficult to be sure that a numerical procedure has found all solutions to  $F(u) = y$ . To see the problem if the

<sup>b</sup>The following discussion has been set in  $U$ -space. It could just as easily have been set in  $X$ -space. One would just use the original  $f$  instead of the transformed  $F$ . In place of using  $\Phi$  to find the probability that  $U$  falls in a given interval, one would use  $F_X$  to find the probability that  $X$  falls in the corresponding  $X$ -space interval.

solution set is incomplete, presume that a numerical procedure has calculated a collection of solutions to  $F(u) = y$ . Presume also that  $u_1 < u_2$  are adjacent members of this collection of solutions. Presume further that  $u_0$  is a solution of  $F(u) = y$  which the numerical procedure has failed to detect and which falls between  $u_1$  and  $u_2$ . The most common behavior of  $F$  in the neighborhood of  $u_0$  is that it changes signs as  $u$  passes through  $u_0$ . This means that the interval  $(u_1, u_2]$  intersects both of the events  $[F(u) > y]$  and  $[F(u) < y]$  with positive probability. Thus, either including or excluding  $P[u_1 < U \leq u_2]$  in the calculation of  $P[F(U) \leq y]$  introduces an error in that calculation.

2. Any attempt to calculate all of the solutions of  $F(u) = y$  would require many (potentially expensive) evaluations of the function  $F$ .

The solution to these problems used in this study was to use a Response Surface method:  $F$  is replaced by a cubic spline approximation  $C$  and  $Y = F(U)$  is approximated by  $Y_a = C(U)$ . An effective and efficient procedure is devised for calculating all of the solutions of  $C(u) = y$  over the support interval of  $C$ . As above, this provides the necessary information to calculate  $P[Y_a \leq y]$  which is then taken as an approximation for  $P[Y \leq y]$ . (The contribution to  $P[Y_a \leq y]$  from points  $u$  outside the support of the cubic spline  $C$  is actually ignored, but this error can be made negligible by making the support of  $C$  big enough. For example, if the support of  $C$  is  $[-5, 5]$ , so that the  $\pm 5\sigma$  range of  $U$  is covered, then the ignored probability is less than  $5.8 \times 10^{-7}$ .)

The repeated evaluations of  $F(u)$  that would be required for each value of  $y$  for which the equation  $F(u) = y$  was to be solved are now replaced by the evaluation of  $F$  at a collection of points (the so called knots of the cubic spline) that are chosen to produce a good approximation of  $F$  by  $C$ . Instead of doing many evaluations of  $F$  for each  $y$ , the calculation of  $F$  at the knots of  $C$  need be done only once, the coefficients of each of the cubic polynomials which represents  $C$  on the interpolating intervals are also calculated only once, and this defining information for  $C$  is reused for each value of  $y$ .

At any value of  $y$  for which the CDF of  $Y$  is to be approximated, the contribution to the CDF of  $Y_a = C(U)$  on the support of  $C$  can now be evaluated exactly. The solutions (if any) of  $C(u) = y$  are found on each interpolating interval. If  $C_i$  is the cubic polynomial that  $C$  uses to interpolate  $F$  over the  $i^{\text{th}}$  sub-interval, then the cubic equation  $C_i(u) - y = 0$  is solved for its three roots, and any that are real numbers actually falling within the  $i^{\text{th}}$  subinterval are retained as solutions of  $C(u) = y$ . After some alternative ways of finding these roots were studied (use of the Matlab<sup>®</sup> polynomial root finder (`roots`), use of Sturm sequences to avoid actually calculating the roots of a cubic if they did not lie in the target interval), the most efficient way found to calculate the solutions of  $C(u) - y = 0$  was to use the cubic formula<sup>15-17</sup> to calculate the solutions of  $C_i(u) - y = 0$  for every  $i$ .

Since it is possible that individual polynomials of the cubic spline degenerate to lower order than 3, the polynomial solver implemented to locate the zeros of  $C(u) - y$  must be able to solve linear, quadratic, and cubic equations. This was



done in the present study with software which provided numerical approximations to the roots of polynomial equations of degree 3 or less which had relative errors on the order of a small multiple of machine epsilon. This was adequate solution accuracy in most cases. However, numerical problems can occur at the knots and at derivative zeros (including local maxima and minima) of the cubic spline,  $C$ , and special care must be exercised at these points.

The problem which can occur if a solution to  $C_i(u) = y$  falls at one of the knots which constitute the boundary of the  $i^{\text{th}}$  subinterval is that, when the solver has approximated the solution, a small error in its computed value might move it out of that subinterval. This results in the solution logic rejecting the value as a solution of  $C(u) = y$ . If that happens on both intervals which share that knot, the solution point will be overlooked. If it happens in neither interval, then the knot is entered twice in a list of computed solutions. The problem of duplicate solutions is fixed by post processing the list of solutions to remove duplicates. This also deals with any problem the probability calculation might have with any multiple zeros of  $C(u) = y$  detected by the polynomial solver. The problem of a solution at a knot being overlooked is solved by just directly evaluating the cubic spline at each knot and adding any knots  $u_i$  for which  $C(u_i)$  is within some small error tolerance of  $y$  to the solution list. Any knot detected as a solution point is deflated out of its cubic polynomial before any remaining solutions in that subinterval are sought.

The problem which occurs at a point  $u'$  where  $C(u') = y$  and  $C'(u') = 0$  is that  $u'$  ought to appear as a multiple solution to  $C(u) = y$ . However, as a result of round-off error while evaluating the formulae which give these answers, what might come out of the solution process is 2 or 3 distinct real or complex approximations to the multiple solution of  $C(u) = y$  at  $u'$  which are closely clustered about  $u'$ . If all these approximations are real numbers, this is not a problem with regard to computing  $P[Y_a \leq y]$ . The spurious intervals introduced by partitioning the real line using these approximate solutions are so short that any contribution to  $P[Y_a \leq y]$  which comes from them is negligible. However, since only real roots are sought, any of these approximations to  $u'$  which are complex by virtue of having small, but nonzero, imaginary parts are discarded. To avoid this, the solution software is structured so that the user can pass in a collection  $\{u_j\}$  of zeros of  $C'$  together with the values  $\{y_j = C(u_j)\}$  which  $C$  takes at these derivative zeros. If the software trying to solve  $C(u) = y$  detects that any of the  $y_j$  are equal to  $y$ , the corresponding  $u_j$  is included in the solution set. The same software used to solve equations of the form  $C(u) = y$  can be used to solve  $C'(u) = 0$  and so locate the derivative zeros of  $C$ .

(The authors are aware that, normally, any test involving comparing two floating point numbers for equality, as was just suggested in comparing the value  $y$  at which the CDF of  $Y_a$  is being computed to members of the set  $\{y_j\}$  of zeros of  $C'$ , is questionable from the point of view of numerical analysis. That notwithstanding, it is done here because of experience based on a peculiar property of the CDF of a random variable dependent on another through a function which has derivative zeros. If any random variable is dependent on

another through a function which possesses points with horizontal tangent lines, then at the corresponding point in the function image space the CDF of the dependent random variable has a vertical tangent. The practice followed in this study was to refine the grid of points at which the dependent variable CDF was being calculated in the vicinity of such vertical asymptotes to better resolve the effect of this sudden rise in the CDF on such quantities as the mean and variance of the dependent random variable which are determined from the CDF. In this refinement, the point of actual vertical tangency was specifically included. At such points, the equality test between  $y$  and at least one of the  $y_j$  is met.)

Once this is done, all solutions of  $C(u) = y$  can be reliably computed, and (except for the neglected portions outside the support of  $C$ ) the CDF of  $Y_a$  is computed exactly at  $y$ .

## V. Application to Optimal Feed-forward Control

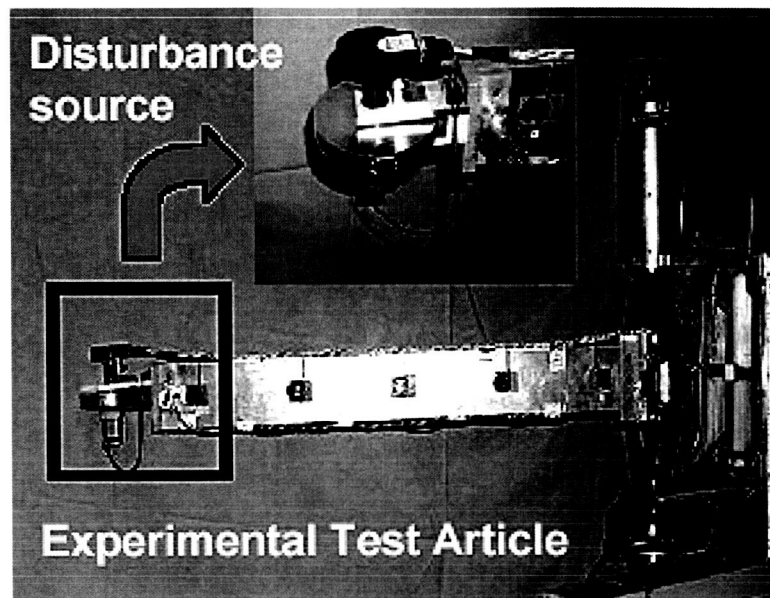
A model of the flexible beam test article, shown in Fig. 2, will be used to validate the probabilistic analysis methodology developed in this work. The system consists of a very flexible thin aluminum blade, approximately one-meter long, attached at its base to a hub motor. The hub motor is the primary control effector for the system. At the tip of the beam is a reaction wheel that serves as a disturbance generator to provide harmonic imbalance forces. The test article has nine sensors that may be used in any combination for either feedback or performance output monitoring. The finite element method was used to model this system and employed a simple implementation of Euler-Bernoulli planar beam elements.<sup>c</sup> For a complete description of the flexible beam test article as well as the optimal feed-forward disturbance rejection methodology see Ref. 18.

### A. Probabilistic Analysis

The analysis approach presented in this paper has been used to evaluate the probabilistic robustness of the system's pointing performance for two different control system architectures. Two architectures were chosen as a representative example of the design choices facing control engineers and to illustrate how probabilistic performance analysis may be used to enhance the design process. The first control system that is considered is purely a Linear Quadratic Gaussian (LQG) feedback controller using the hub actuator to mitigate the effect of reaction wheel imbalance forces on the tip acceleration. The second control system utilizes the same LQG feedback structure as the first, but adds a feed-forward path to improve the ability to reject the reaction wheel imbalance forces. Although many control system performance metrics can be evaluated using the approach presented in this paper, these architectures were evaluated solely in terms of their probabilistic Bode magnitude response.

The uncertain parameter considered in this study was a normally distributed mass density property, i.e., mass density of the beam elements used in the finite

<sup>c</sup>The finite element model was used to produce a modal model using one rigid body mode (rotation about the hub motor axis) and the first seven flexible modes. This gave a 16 state plant model. With the addition of a full state estimator, the analysis model was a 32 state closed loop system.

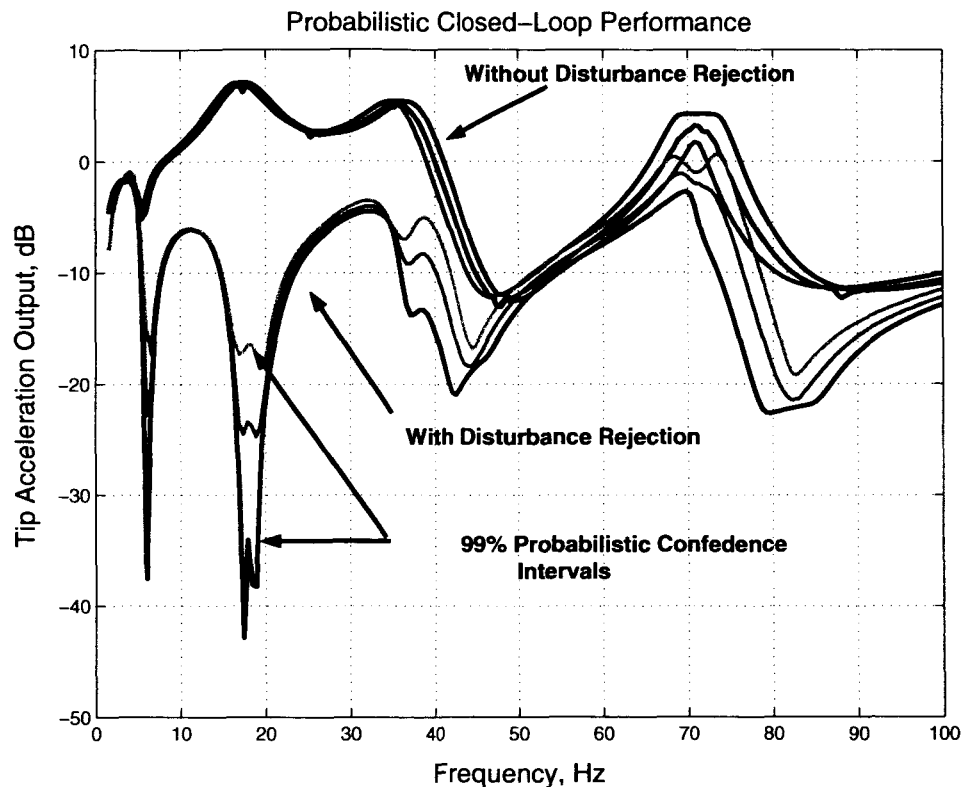


**Fig. 2. Flexible Beam Test Article**

element model.<sup>d</sup> This multi-disciplinary problem required integrating structural analysis, control system analysis and probability tools. This integration was accomplished by developing all software tools in Matlab®. The results of the probabilistic robustness analysis are shown in Figs. 3 and 4. Figure 3 presents the probabilistic performance for both control architectures in terms of their mean responses together with 99% confidence intervals. The computation of the mean as well as the confidence intervals requires the CDF of the Bode magnitude response for each frequency of interest. The results shown in this paper were generated using 200 frequency points with 40 points per CDF analysis and 50 evaluations over a  $\pm 5\sigma$  range of the uncertain mass density variable used to generate the cubic spline interpolating functions.

Once given the frequency dependent CDF, higher order statistics may also be easily computed. Fig. 4 is an example of a second-order moment (variance) of the system's acceleration response. This figure indicates that the system with feed-forward control has (in general) smaller variances than the system with feedback only. From this, we can conclude that the robustness of the feed-forward approach is not adversely affected by uncertainty in the mass density. More specific conclusions regarding the smaller variances are outside the scope of this paper, but credible variance information should be seen as a significant enhancement over the information resulting from convention robust control analysis and is a particularly valuable metric to be considered in design.

<sup>d</sup>Beam element mass density mean was  $2850 \text{ Kg/m}^3$ . Variance is defined by the requirement that  $3\sigma = 285 \text{ Kg/m}^3$ , which is 10% of the mean.



**Fig. 3. Probabilistic Analysis Results**

The results in Fig. 3 show that for the frequency range between 5–45 Hz the control system with feed-forward has substantially better mean value performance than the system with feedback only. This assessment is true not only in terms of the mean value performance, but also for all possible responses within the 99% confidence intervals. In the frequency ranges above and below this range, the probabilistic analysis shows that the two architectures have overlapping performances. This information is of great value for considering resource allocation trade studies. For example, if the system's dominant operating range is in the 50–70 Hz range, then the more expensive feed-forward architecture provides little or no performance advantage over the standard LQG controller. The real value of the approach is that the quantitative behavior of different control system architectures can be accessed in the presence of uncertainties that are truly consistent with the physical phenomenon.

As a final note, the overall central processor unit time required to compute the data presented in Figs. 3 and 4 was 113 seconds for each control architecture. Timing data was obtained on an AMD 2600XP processor system.

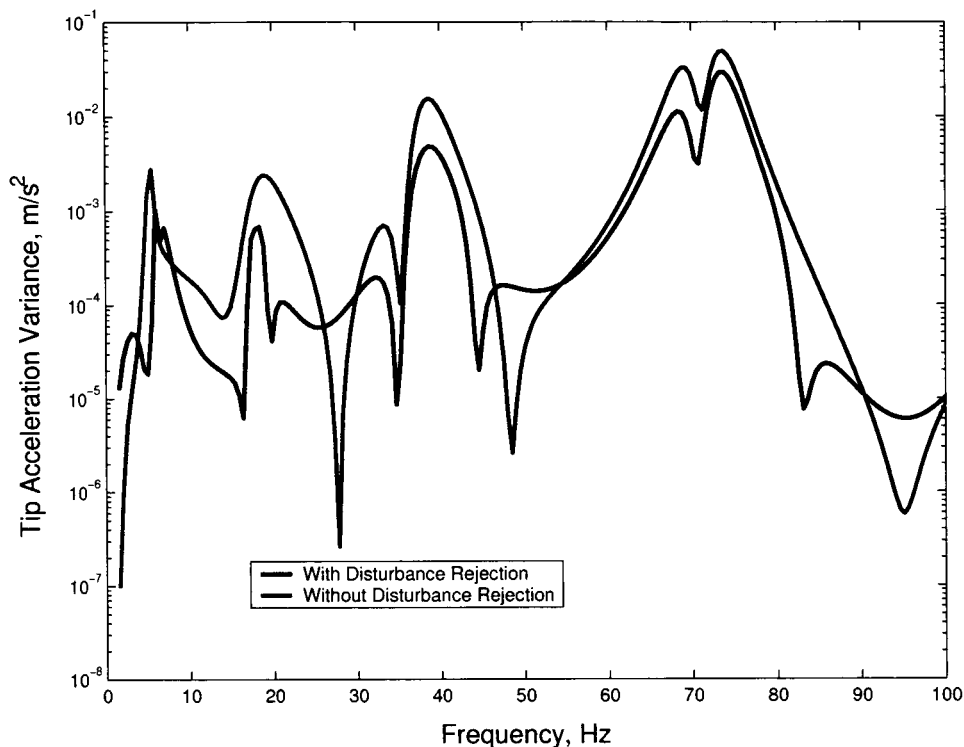
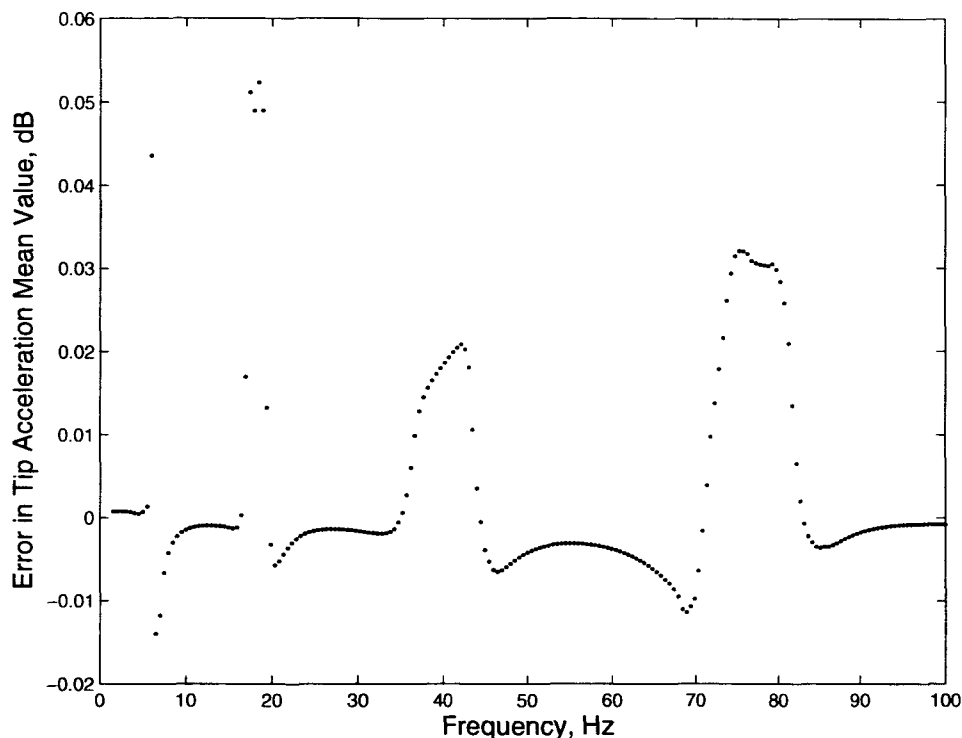


Fig. 4. Acceleration Output Variance

#### B. Monte Carlo Verification

Since some approximation has been involved in the preceding analysis (replacing, at each frequency, the actual function by which the Bode magnitude response is determined by the uncertain mass density with a cubic spline approximation; ignoring the chance that the mass density falls outside the  $\pm 5\sigma$  range), a Monte Carlo study was performed to validate the results for the controller which included feed-forward. A set of 60,000 samples of the uncertain mass density was generated and the Bode magnitude analysis was performed at each of these 60,000 sample values at each of the 200 frequency points considered previously. For each frequency point, the mean and variance of the Monte Carlo results were calculated.

The Monte Carlo results agree so well with the analytical results using the cubic spline approximation that, when they were plotted together, the difference between the two is all but lost in the limits of the resolution of the plotting device. The actual error curves are shown in Figs. 5 and 6. The discrepancy between the mean as calculated by the cubic spline approximation and as calculated by Monte Carlo is never more than 0.2% of the maximum magnitude Monte Carlo value. The discrepancy between variances is never more than 1.6%.



**Fig. 5. Difference between Monte Carlo and Semi-analytic – Mean**

This validates the technique presented here.

The Monte Carlo calculation required 40 hours of computer time. Comparing this computation time with the 113 seconds required for the cubic spline approximation method shows the advantage in efficiency of the technique presented here.

## **VI. Conclusions**

This paper has presented an approach for computing frequency dependent cumulative distribution functions for closed-loop dynamical systems. The approach has been shown to not suffer from the same computational challenges associated with computing failure probabilities using conventional FORM/SORM techniques. The computation of means, variances, and probabilistic confidence intervals has been demonstrated on a multi-disciplinary model of a laboratory test article for a broad range of frequencies. The advantages of this approach over Monte Carlo methods have been demonstrated.

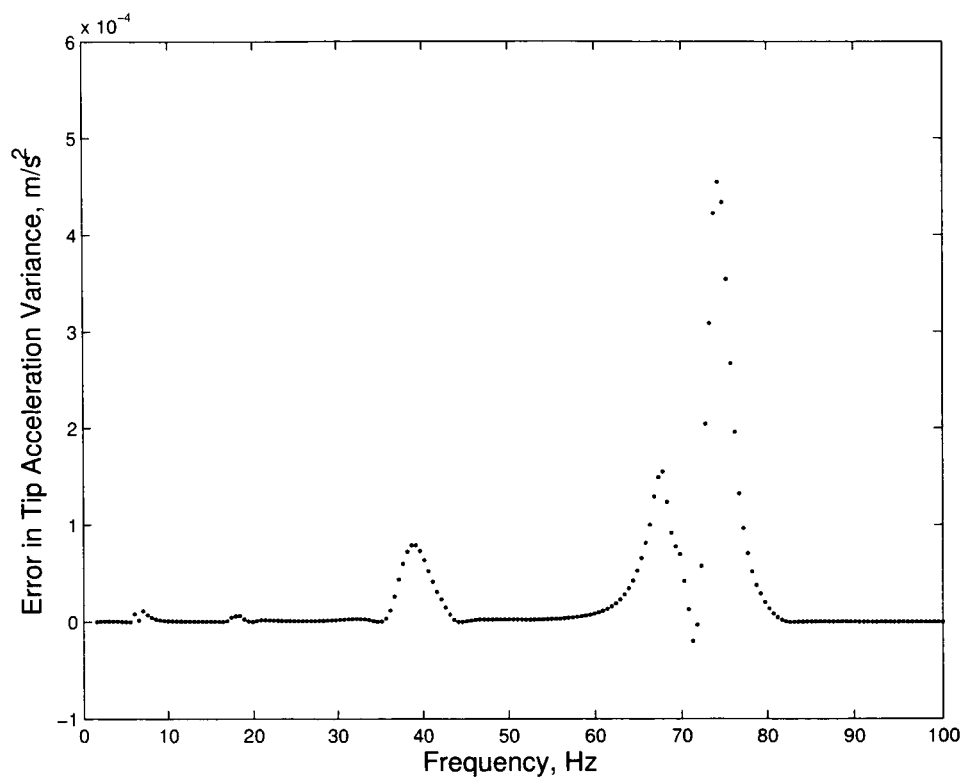


Fig. 6. Difference between Monte Carlo and Semi-analytic – Variance

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