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An Introduction to Logic for Students of Physics and Engineering

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An Introduction to Logic for Students of Physics and Engineering

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Summary

A physicist with an engineering background, the author presents a brief tutorial on logic. In his work at NASA and in his encounters with students, he has often found that a firm grounding in basic logic is lacking—perhaps because there are so many other demands on people that time simply cannot be taken to really examine the roots of human reasoning. This report provides an overview of this all-too-important subject with the dual hope that it will suffice insofar as it goes and that it will spur at least some to further study.

Introduction

When I began college in the 1970s to pursue a major in engineering science with a physics option, I began calculus in my first quarter but ran aground on the issue of proofs. At this stage, I had not yet studied logic and found that the wording of the theorems I was attempting to prove could be very confusing. Needless to say, I was often far off track in constructing my proof because I lacked the knowledge of their logical structure.

I later studied logic on my own and became fascinated with the subject of proofs. I kept notes on the various topics and began to realize that if I had had such notes in college (and had understood them), I would have advanced more quickly in my study of mathematics. Also, since mathematics is the underpinning of science and engineering, such an advancement would have greatly contributed to my progress in those courses and to my study of the logic itself, which forms the foundation of all modern thought.

Logic is applied daily in almost every situation in our lives. Although our thinking is seldom as formal as that which characterizes mathematics or the sciences, logic of some sort is used. In fact, the word “logic,” derived from Greek and Latin roots, means “to reason.” When we, “Reason it out,” we are explicitly or implicitly using logic.

Logic, a formalization of language, takes basic terms used everyday and sharpens their definitions to mathematical precision. The types of statements that mathematical logic most often deals with are hypothetical and categorical propositions. Other types of statements involving copulas such as *should* or *ought* are also dealt with in certain branches of logic but are not discussed in this presentation, which is written for the technical student pursuing a science or engineering degree.

Sometimes the engineering or science student is put off by the unfamiliar language of rigorous mathematical texts. This statement carries through to logic as well. In this presentation, I have tried to steer a middle ground between the strictness of mathematical rigor and the somewhat more intuitive language of engineering and science. I believe that it is important for students to master the basics of thinking, especially in the current academic and work environment where computer literacy is often stressed to the exclusion of language or mathematical literacy. After all, we will still have to think even when the power goes out.

No single volume can ever contain all that is to be said about logic. As a discipline, logic is as ancient as the human race. It is also as modern as the 21st century. Millennia ago, the Greeks developed a system of logic that is still actively used today. In the 19th century, European mathematicians developed—or one could say, formalized—another system of logic more suited to the needs of their rapidly developing discipline. These logical systems are related, but they are also individual with each containing its unique elements.

There is no one system of logic. Even within the formal logic of modern mathematics, there exist fascinating differences of meaning and method. Of all the systems of logic extant today, this overview deals with the two I deem most useful to students of science and engineering: hypothetical, the logic formalized by 19th century mathematicians, and Aristotelian (syllogistic), a system developed by the ancient Greeks and used throughout most of Western history. Hypothetical logic is directly applicable to mathematics, especially in the development of proofs; Aristotelian logic is useful just about everywhere else and deserves equally serious study.

This introductory work defines significant terms, lists important theorems, and suggests an algebraic method of reasoning that is useful in developing proofs. The student should not rely heavily upon method alone, for it does not replace genuine thought. However, method can often provide the beacon by which a sound reasoning mind can steer through rough waters. The student is challenged to master the topics presented herein and then go on to a more formal text.

I hope to share insights obtained mostly after graduation when I had the time to begin the long process of gathering up all the loose ends that college had left me and to start making real sense out of what I had been taught. As in most of my writing for students, I focus on basics because one needs to know the grammar and syntax before becoming a fluent writer. Is this too basic a piece to be bothered with? Is it going to be too simple to warrant serious time or study? The answer is best given by the composer Beethoven, who once retorted in a very different context, "It is only simple when you know how."

This presentation is not a substitute for a formal text but is an introduction to a vast subject area and presents a simple approach for constructing proofs of logical propositions. Although the algebraic approach is not new, it should be very useful to the student as a springboard for further study.

Basics of Hypothetical Logic

Simple Propositions

A simple proposition is a statement with a simple subject and a simple predicate. The statement "3 is a numeral" is a simple proposition; "3" is the subject and "is a numeral" is the predicate. The function of the predicate is to say, or predicate, something about the subject. For example,

3 is a numeral.
3 is between 2 and 4.
3 is half of 6.

Each predicate adds to the knowledge of the subject "3."

In logic, propositions have properties independent of their specific content. The propositions "3 is a numeral" and "I am a man" both have the general form "*S* is a *P*," where *S* is the subject term and *P* is the predicate term. We can represent any specific proposition of the form "*S* is a *P*" by the letter *a* or the letter *b* and so on.

In the language of logic, we assign a single letter or variable such as *a* the task of representing a simple proposition. Thus, if we assign *a* the role of representing the proposition "3 is a numeral," this assignment is often called a definition:

$$a = 3 \text{ is a numeral} \tag{1}$$

If there were only simple propositions, then all would be inordinately simple, and logic would be boring and useless. In reality, there are combinations of simple propositions, and in the combinations are found the beauty, complexity, and versatility of logic as a system for representing human thought.

Compound Propositions

Any structure such as *a* and *b*, where *a* and *b* are both simple propositions, is a compound proposition with two variables. The term (in this case, "and") linking the variables in a compound proposition is called a copula. If *b* represents the simple proposition "*x* is a variable," then

$$a \text{ and } b = 3 \text{ is a numeral and } x \text{ is a variable} \tag{2}$$

There are several types of compound propositions:

1. Conjunction: "and" statement (*a* and *b*)
2. Disjunction: "or" statement (*a* or *b*)
3. Implication: *a* implies *b*
4. Equivalence: *a* is equivalent to *b*
5. Existential, strong: (For all *a*, *b*)
6. Existential, weak: (There exists *a* such that *b*)

These compound propositions in turn may be combined in various ways to form other compound propositions, such as "For all *a* and *b*,…," or "For all *a* and *b*, *c*," "If there exists *c* such that *d* and *e*, then *f*." These will be discussed later in the section Other Forms of Compound Propositions.

Negations

If one statement is true, then there must be another related statement that is false. For example, if it is raining today, then it is false to say that it is not raining today. There are two statements here:

It is raining today.
It is not raining today.

Both cannot be true at the same time because both deal with a state of being, a rainy day, in exactly opposite senses. Each statement is related to the other. The relation between the two statements is called negation. The truth of one of the statements negates the truth of the other.

Let a single letter preceded by a tilde, such as ~*a*, represent the negation of the simple proposition represented by the letter *a*. If *a* is true, ~*a* is false; if *a* is false, ~*a* is true. If *a* = "3 is a numeral," then ~*a* = "3 is a non-numeral." Obviously in this case, *a* is true and ~*a* is false. Any compound proposition also has a negation; that is, the negation of (*a* and *b*) is ~(*a* and *b*). Using the expressions given above yields

$$a \text{ and } b = \left(3 \text{ is a numeral and } x \text{ is a variable}\right) \quad (3)$$

$$\sim\left(a \text{ and } b\right) = (3 \text{ is a non- numeral or } x \text{ is a nonvariable (or both)}) \quad (4)$$

Note that in the negation, the copula *and* has been replaced with the copula *or*. The reason is clear if one considers the simple grammar involved in saying that the proposition "*a* and *b*" is true. To say that "*a* and *b*" as a composite statement is true is to say that both "*a*" and "*b*" as individual statements are both true. Tell a child, "I will give you a toy and take you for ice cream." Then give the toy but forget about the ice cream and watch what happens. Hence, the falsity of either statement individually negates the original composite statement.

For the discussions that follow, note that the written variable *a* is assumed to be equivalent to "*a* is true," and the written variable ~*a* (with a tilde) is assumed to be equivalent to "*a* is false." This provision does not compromise the generality[1] and is introduced merely as a convenience.

[1]Define another variable, say, *s*, and let *s* = ~*a*; then one has the case that *s* is false and ~*s* (= ~~*a* = *a*) is true.

Defining Compound Propositions

The basis of a logical system must be a set of agreed-upon definitions that all adhere to. These definitions are not absolute in that they do not apply to all times and places; however, they are the best and broadest statements that can be agreed upon at the time of their establishment. A set of definitions is considered a formal structure; a formal structure carries weight but is not forever unchangeable.

To formally define compound propositions, it must be stated specifically when a compound proposition will be considered true and when false. Since compound propositions consist of simple propositions, the truth or falsity of any compound proposition must depend on the truth or falsity of its component simple propositions.

In a compound proposition of two variables *a* and *b*, there are only four possibilities for the truth and falsity of the component simple propositions; one of the following four possibilities must be true:

a is true, *b* is true
a is true, *b* is false
a is false, *b* is true
a is false, *b* is false

When defining a compound proposition and its negation, each of these possibilities must appear exactly once. In the example just given,

$$\left(a \text{ and } b\right) = \left(3 \text{ is a numeral and } x \text{ is a variable}\right) \quad (5)$$

$$\sim\left(a \text{ and } b\right) = (3 \text{ is a non- numeral or } x \text{ is a nonvariable (or both)}) \quad (6)$$

The right-hand side (RHS) for ~(*a* and *b*) may also be written:

$$(3 \text{ is a non- numeral, } x \text{ is a variable}) \quad (7)$$

or

$$(3 \text{ is a numeral, } x \text{ is a nonvariable}) \quad (8)$$

or

$$(3 \text{ is a non- numeral, } x \text{ is a nonvariable}) \quad (9)$$

Note that when the definitions for (a and b) and ~(a and b) are taken together, each possibility appears exactly one time:

3 is a numeral, x is a variable
3 is a numeral, x is a nonvariable
3 is a non-numeral, x is a variable
3 is a non-numeral, x is a nonvariable

Strict Forms of Compound Propositions

The writing of definitions may now begin with the types of compound propositions considered to be strict forms. These include the following: "and" statements, "or" statements, implications, equivalences, and their negations. In the following sections, definitions for each of these types of propositions will be written.

***"And" statements*: *conjunctions*.**— The compound proposition (a and b), in which the simple propositions a and b are connected with the copula *and*, will be represented as ($a \wedge b$), where $\wedge$ is the usual logic symbol for *and*. The proposition is called a conjunction, and the variables a and b are called conjuncts.

Note: Using the logic notation just introduced, the four truth conditions given above may be written:

1. $a \wedge b$
2. $a \wedge {\sim}b$
3. ${\sim}a \wedge b$
4. ${\sim}a \wedge {\sim}b$

Since ($a \wedge b$) is equivalent to ($a \wedge b$), it follows that the negation of ($a \wedge b$) must be ~($a \wedge b$) is equivalent to the compound "or" statement:

$$(a \wedge {\sim} b) \text{ or } ({\sim}\mathrm{a} \wedge \mathrm{b}) \text{ or } ({\sim}\mathrm{a} \wedge {\sim}\mathrm{b}) \qquad (10)$$

Again, note that each of the four truth possibilities appears just once in the definition of ($a \wedge b$) and its negation.

***"Or" statements*:[2] *disjunctions*.—**The compound proposition (a or b), in which the simple propositions a and b are connected with the copula *or*, will be represented as ($a \vee b$), where $\vee$ is the usual logic symbol for *or*. The proposition is called a disjunction, and the variables a and b are called disjuncts. The two types of "or" statements are inclusive

$$(a \vee b) \text{ is equivalent to } (a \wedge {\sim}b) \text{ or } ({\sim}\mathrm{a} \wedge \mathrm{b}) \text{ or } (\mathrm{a} \wedge \mathrm{b}) \qquad (11)$$

and exclusive

$$(a \vee b) \text{ is equivalent to } (a \wedge {\sim}b) \text{ or } ({\sim}\mathrm{a} \wedge \mathrm{b}) \qquad (12)$$

Note that the exclusive *or* does not include the possibility ($a \wedge b$) whereas the inclusive *or* does.

Example of inclusive "or:" The proposition "I was born on the 21st of September *or* she was born on the 4th of February" is an inclusive "or" statement since either one or both of the simple propositions it contains may be true.

Example of exclusive "or:" The proposition "Either the Sun will shine today *or* it will not" is an exclusive "or" statement since either one alone may be true, but both cannot be true together.

Note: By convention, when ($a \vee b$) is written, it will always mean the inclusive *or* unless otherwise specified.

Since the disjunction ($a \vee b$) is equivalent to ($a \wedge {\sim}b$) or (${\sim}a \wedge b$) or ($a \wedge b$), its negation must be

$${\sim}(a \vee b) \text{ is equivalent to } ({\sim}\mathrm{a} \wedge {\sim}\mathrm{b}) \qquad (13)$$

Using the definition of ($a \vee b$), note that (${\sim}a \vee {\sim}b$) is equivalent to ($a \wedge {\sim}b$) *or* (${\sim}a \wedge b$) *or* (${\sim}a \wedge {\sim}b$). But the RHS of this expression is identical to the definition of ~(a ∧ b). This identity was previously used in equation (10). The expression

$${\sim}(a \wedge b) \text{ is equivalent to } ({\sim}\mathrm{a} \vee {\sim}\mathrm{b}) \qquad (14)$$

is one of two important laws of logic (and set theory) that bear the name of Augustus De Morgan (1806–71). These laws may have first been encountered in set theory. They apply to operations relating to the union and intersection of sets. Using the usual set notation in which $\cup$ is set union, $\cap$ is set intersection, and superscript C is complement, De Morgan's laws state that

[2]"Or" statements are equivalent to "Either…or…" statements. The statement "a or b" is equivalent to "Either a or b."

$$(A \cup B)^C = A^C \cap B^C$$

$$(A \cap B)^C = A^C \cup B^C$$

The equivalent forms in hypothetical logic are

$$\sim(a \vee b) \Leftrightarrow (\sim a \wedge \sim b)$$

$$\sim(a \wedge b) \Leftrightarrow (\sim a \vee \sim b)$$

De Morgan's laws are an important part of each of these respective disciplines.

Implications.—The implication is defined in terms of "and" and "or" statements: The implication "*a* implies *b*" is written $a \Rightarrow b$ and is defined by

$(a \Rightarrow b)$ is equivalent to

$$(a \wedge b) \vee (\sim a \wedge b) \vee (\sim a \wedge \sim b) \qquad (15)$$

The proposition *a* is called the hypothesis (antecedent), and the proposition *b* is called the conclusion (consequent). Note that the possibility in which *a* is true but *b* is false is the only possibility missing. Logicians have argued over this definition but despite its difficulties, it remains the best available and is the definition used in all mathematics texts. Equivalently, it may be written

$$(a \Rightarrow b) \text{ is equivalent to } \sim(a \wedge \sim b) \qquad (16)$$

Using expression (16), the negation of the implication may be seen as

$$\sim(a \Rightarrow b) \text{ is equivalent to } (a \wedge \sim b) \qquad (17)$$

The English language forms of the implication include

1. *a* implies *b*
2. *a* only if *b*
3. If *a*, then *b*
4. If *a*, *b*
5. *a* is a sufficient condition for *b*
6. *a* is sufficient for *b*
7. *a* only on the condition that *b*
8. Whenever *a*, *b*
9. Given that *a*, *b*
10. In case *a*, *b*
11. *b* is implied by *a*
12. *b* if *a*
13. *b* is a necessary condition for *a*
14. *b* is necessary for *a*
15. *b* on the condition that *a*
16. *b* provided that *a*

In mathematics and logic texts, these English forms are the most commonly encountered and all have the same meaning: $a \Rightarrow b$. For example, the algebraic quantity a/b, where *a* and *b* are integers, is a rational number only if $b \neq 0$ is equivalent to the phrase "the quantity a/b, where *a* and *b* are integers, is a rational number $\Rightarrow b \neq 0$" is equivalent to the statement "For *a* and *b* integers, a/b rational implies $b \neq 0$" is equivalent to the statement "For *a* and *b* integers, if a/b is rational, then $b \neq 0$," and so on.

Equivalences.—At this point, equivalence will be defined in terms of implication. The equivalence "*a* equivalent *b*" is written $a \Leftrightarrow b$ and is equivalent to $(a \Rightarrow b) \wedge (b \Rightarrow a)$. Since $(a \Rightarrow b)$ is equivalent to $(a \wedge b) \vee (\sim a \wedge b) \vee (\sim a \wedge \sim b)$, and $(b \Rightarrow a)$ is equivalent to $(a \wedge b) \vee (a \wedge \sim b) \vee (\sim a \wedge \sim b)$, the compound statement $(a \Rightarrow b) \wedge (b \Rightarrow a)$ must be equivalent to $(a \wedge b)$ or $(\sim a \wedge \sim b)$, the parts that the two individual implications have in common. Thus,

$$(a \Leftrightarrow b) \text{ is equivalent to } (a \wedge b) \vee (\sim a \wedge \sim b) \quad (18)$$

The English language forms of the equivalence include

1. *a* equivalent *b*
2. *a* is equivalent to *b*
3. *a* and *b* are equivalent
4. *a* implies and is implied by *b*
5. *a* if and only if *b* (sometimes abbreviated *a* iff *b*)
6. *a* is a necessary and sufficient condition for *b*
7. *a* is necessary and sufficient for *b*
8. *a* just in case *b*

The negation of an equivalence is $\sim(a \Leftrightarrow b)$ is equivalent to $(a \wedge \sim b) \vee (\sim a \wedge b)$.

Other Forms of Compound Propositions

This section discusses the compound proposition "*a* therefore *b*" and the existential forms "There exists *a* such that *b*" and "For all *a*, *b*." (The forms "*a* because b," "a but b," "a ought b," and others are not dealt with because they belong to areas of logic not of immediate concern to mathematics and science.)

"***Therefore" propositions***.—The copula *therefore* usually designates the conclusion of an argument. For example, if it has already been proven that some compound proposition $a \wedge b$ is true, then it may also be concluded that a is true and b is true individually. The logic shorthand for *therefore* is written with the symbol $\therefore$. For example, $(a \wedge b) \therefore a$ or $(a \wedge b) \therefore b$. Any statement, "$a \therefore b$" is true when the conclusion b is true; otherwise it is false and is written $\sim(a \therefore b) \Leftrightarrow (a \therefore \sim b) \vee (\sim a \therefore \sim b)$.

"***Existential" propositions***.—The copula *There exists…such that…* is called an existential quantifier. The existential proposition using this copula is usually abbreviated in logic notation as $\exists a \ni b$, where $\exists \ldots \ni \ldots$ is the logic shorthand for "There exists…such that…" Any proposition $\exists a \ni b$ is true whenever there is at least one actual instance of the variable a for which the variable b is true. Only when there is demonstrably no such instance is $\exists a \ni b$ false. In such a case, it is written $\sim(\exists a \ni b) \Leftrightarrow (\forall a, \sim b)$.

The copula *For all…, …* is another (the other) existential quantifier. The existential proposition using this copula is usually abbreviated in logic notation as $\forall a, b$, where $\forall \ldots, \ldots$ is the logic shorthand for "For all…,… ." Any proposition $\forall a, b$ is true provided that b is true for every actual instance of a. If there is even one demonstrable instance of a for which b is false, then $\forall a, b$ is false and is written $\sim(\forall a, b) \Leftrightarrow \exists a \ni \sim b$. Such a single instance is called a counterexample.

Table 1 summarizes the logic propositions and their negations presented thus far.

TABLE 1.—SUMMARY OF LOGIC PROPOSITIONS AND THEIR NEGATIONS

Proposition	Logic notation
a	a
a and b	$a \wedge b$
not (a and b)	$\sim(a \wedge b) \Leftrightarrow (\sim a \wedge b) \vee (a \wedge \sim b) \vee (\sim a \wedge \sim b)$
a or b (inclusive)	$a \vee b \Leftrightarrow (a \wedge \sim b) \vee (\sim a \wedge b) \vee (a \wedge b)$
not [a or b (inclusive)]	$\sim(a \vee b) \Leftrightarrow (\sim a \wedge \sim b)$
a implies b	$(a \Rightarrow b) \Leftrightarrow (a \wedge b) \vee (\sim a \wedge b) \vee (\sim a \wedge \sim b)$
not (a implies b)	$\sim(a \Rightarrow b) \Leftrightarrow (a \wedge \sim b)$
a equivalent b	$(a \Leftrightarrow b) \Leftrightarrow (a \wedge b) \vee (\sim a \wedge \sim b)$
not (a equivalent b)	$\sim(a \Leftrightarrow b) \Leftrightarrow (a \wedge \sim b) \vee (\sim a \wedge b)$
a therefore b	$a \therefore b$
not (a therefore b)	$(a \therefore \sim b) \vee (\sim a \therefore \sim b)$
there exists a such that b	$\exists a \ni b$
not (there exists a such that b)	$\forall a, \sim b$
for all a, b	$\forall a, b$
not (for all a, b)	$\exists a \ni \sim b$

Equivalence Laws and Arguments

Equivalence laws and arguments differ with respect to form. We have already defined the forms "and," "or," "implication, "equivalence," "therefore," "there exists…such that…," and "for all…,… ." Using these forms, it is possible to write innumerable other logic statements that have the general form "P is equivalent to P^*," or $P \Leftrightarrow P^*$. Such forms are called equivalence laws or tautologies. Many of these laws are useful in constructing higher order arguments and/or proofs, such as those in mathematics, science, or philosophy.

Note that laws use the copula $\Leftrightarrow$. Arguments, on the other hand, use the copula *therefore* and have the form (Some proposition P) therefore (some other proposition P^*), or $P \therefore P^*$. Arguments are generally proven directly from the laws using a method of direct or indirect proof. Such methods will be examined in the section Techniques of Proof in Mathematics, Science, and Philosophy. Arguments can reach any level of complexity. The art of mathematical proof is really the art of constructing consistent arguments (consistent at every step with the laws of logic).

The following section presents without proof a sampling of the basic equivalence laws of logic along with some of the classical argument forms that appear or are used in more extended proofs. The laws and arguments are introduced first as text and then are restated in table 2 using the logic notation already introduced. Sample proofs are provided in the section Proofs of Selected Laws.

Examples of Equivalence Laws

Laws from the ancient Greeks.—Three classic laws of logic are derived from the ancient Greeks:

Law of identity: A proposition is equivalent to itself: "It will rain" if and only if "it will rain."

Law of the excluded middle: Either a proposition or its negation (but not both) must be true; there can be no

TABLE 2.—SUMMARY OF EQUIVALENCE LAWS AND ARGUMENTS

Law	Logic form
Equivalence laws	
Derived from the ancient Greeks	
Identity	$a \Leftrightarrow a$
Excluded middle	$a \vee \sim a \Leftrightarrow U$ (universal truth)
Contradiction	$a \wedge \sim a \Leftrightarrow \varphi$ (universal falsehood)
Idempotence	
Copula $\wedge$ (and)	$a \Leftrightarrow a \wedge a$
Copula $\vee$ (or)	$a \Leftrightarrow a \vee a$
Commutative, associative, and distributive	
Commutative: $\vee$ (or)	$(a \vee b) \Leftrightarrow (b \vee a)$
Commutative: $\wedge$ (and)	$(a \wedge b) \Leftrightarrow (b \wedge a)$
Commutative: $\Leftrightarrow$ (is equivalent to)	$(a \Leftrightarrow b) \Leftrightarrow (b \Leftrightarrow a)$
Associative: $\vee$	$(a \vee b) \vee c \Leftrightarrow a \vee (b \vee c)$
Associative: $\wedge$	$(a \wedge b) \wedge c \Leftrightarrow a \wedge (b \wedge c)$
Distributive: $\wedge$ over $\vee$	$a \wedge (b \vee c) \Leftrightarrow (a \wedge b) \vee (a \wedge c)$
Distributive: $\vee$ over $\wedge$	$a \vee (b \wedge c) \Leftrightarrow (a \vee b) \wedge (a \vee c)$
De Morgan's	
Copulas $\sim, \vee$	$\sim(a \vee b) \Leftrightarrow \sim a \wedge \sim b$
Copulas $\sim, \wedge$	$\sim(a \wedge b) \Leftrightarrow \sim a \vee \sim b$
Double negation	$a \Leftrightarrow \sim\sim a$
Implication and equivalence, and their negations	
Implication: $\Rightarrow$	$(a \Rightarrow b) \Leftrightarrow [(a \wedge b) \vee (\sim a \wedge b) \vee (\sim a \wedge \sim b)]$
Negation of an implication	$\sim(a \Rightarrow b) \Leftrightarrow (a \wedge \sim b)$
Equivalence: $\Leftrightarrow$	$(a \Leftrightarrow b) \Leftrightarrow [(a \wedge b) \vee (\sim a \wedge \sim b)]$
Negation of an equivalence	$\sim(a \Leftrightarrow b) \Leftrightarrow [(a \wedge \sim b) \vee (\sim a \wedge b)]$
Contraposition, exportation, importation, and absorption	
Contraposition in implication	$(a \Rightarrow b) \Leftrightarrow (\sim b \Rightarrow \sim a)$
Contraposition in equivalence	$(a \Leftrightarrow b) \Leftrightarrow (\sim b \Leftrightarrow \sim a)$
Exportation/importation: $\Rightarrow/\Leftarrow$	$[(a \wedge b) \Rightarrow c] \Leftrightarrow [a \Rightarrow (b \Rightarrow c)]$
Absorption	$(a \Rightarrow b) \Leftrightarrow [\, a \Rightarrow (a \wedge b)]$
Arguments	
Pierce's law	$[(a \Rightarrow b) \Rightarrow a] \therefore a$
Syllogisms and other forms	
Hypothetical syllogism	$(a \Rightarrow b) \wedge (b \Rightarrow c) \therefore (a \Rightarrow c)$
Modus tollendo ponens (disjunctive syllogism)	$(a \vee b) \wedge \sim a \therefore b$
Modus ponendo ponens (affirming the antecedent or rule of detachment)	$(a \Rightarrow b) \wedge a \therefore b$
Modus tollendo tollens (denying the consequent)	$(a \Rightarrow b) \wedge \sim b \therefore \sim a$
Modus ponendo tollens	$\sim(a \wedge b) \wedge a \therefore \sim b$
Equivalence ponens	$(a \Leftrightarrow b) \wedge a \therefore b$ (or $(a \Leftrightarrow b) \wedge b \therefore a$)
Equivalence tollens	$(a \Leftrightarrow b) \wedge \sim a \therefore \sim b$ (or $(a \Leftrightarrow b) \wedge \sim b \therefore \sim a$)
Dilemmas	
Simple constructive	$[(a \Rightarrow b) \wedge (c \Rightarrow b)] \wedge (a \vee c) \therefore b$
Complex constructive	$[(a \Rightarrow b) \wedge (c \Rightarrow d)] \wedge (a \vee c) \therefore b \vee d$
Simple destructive	$[(a \Rightarrow b) \wedge (a \Rightarrow c)] \wedge (\sim b \vee \sim c) \therefore \sim a$
Complex destructive	$[(a \Rightarrow b) \wedge (c \Rightarrow d)] \wedge (\sim b \vee \sim d) \therefore \sim a \vee \sim c$
Special	$(a \Rightarrow b) \wedge (\sim a \Rightarrow b) \therefore b$
Other miscellaneous arguments	
Law of simplification	$(a \wedge b \therefore a)$ or $(a \wedge b \therefore b)$
Law of absurdity (also impossible antecedent)	$[a \Rightarrow (b \wedge \sim b)] \therefore \sim a$
Law of addition	$a \therefore a \vee b$
Law of the true consequent	$b \therefore (a \Rightarrow b)$
Law of the false antecedent (Law of Duns Scotus)	$a \therefore (\sim a \Rightarrow b)$
Conjunction introduction	$a, b \therefore (a \wedge b)$
Disjunction introduction	$a \therefore (a \vee b)$

middle ground:[3] Either "It will rain" or "It will not rain."

Law of contradiction: Both a proposition and its negation cannot be true together: "It will rain" and "It will not rain" at the same time at the same spot is absurd.

These laws are fundamental to all logic and are used in most arguments and proofs.

***Idempotence laws*.**—The truth of a proposition *a* is equivalent to the truth of *a* and *a* or of *a* or *a* or both: "It will rain." Therefore, "It will rain and/or it will rain."

***Commutative, associative, and distributive laws*.**—Variables in logic behave somewhat analogously to variables in algebra and have similar basic laws:

Commutative: The order of variables in a conjunction, a disjunction, or an equivalence does not affect the truth of the respective propositions: (a) "It will be cloudy and it will rain" if and only if "it will rain and it will be cloudy." (b) "It will be cloudy or it will rain" if and only if "it will rain or it will be cloudy."

Associative: In a proposition of more than two variables, the variables may be grouped in pairs without reference to position or order and without affecting the truth of the proposition: (a) "It will rain, Jack will get wet, and Jill will stay in" if and only if "Jack will get wet, Jill will stay in, and it will rain." (b) "It will rain or Jack will stay dry or Jill will come out" if and only if "Jack will stay dry or Jill will come out or it will rain."

Distributive: The variables in a compound proposition are permitted to be distributed over the copulas *and* and *or* without affecting the truth of the proposition: (a) "It will rain and Jack will get wet or Jill will stay in" if and only if "it will rain and Jack will get wet" or "It will rain and Jill will stay in." (b) "It will rain or Jack will stay dry and Jill will come out" if and only if "it will rain or Jack will stay dry" and "It will rain or Jill will come out."

***De Morgan's laws*.**—These are two of the most important laws of logic and of set theory. In logic, they take the form of negations of conjunctions and disjunctions and might be regarded as distributive laws for the negator $\sim$. When the negator is distributed over the individual variables, the copula is changed either from $\wedge \rightarrow \vee$ or from $\vee \rightarrow \wedge$, depending on whether the original proposition was conjunctive or disjunctive: (a) Not "to run and get wet" is equivalent to "not to run" or "not to get wet" or both. (b) Not "to run or get wet" is equivalent to "not to run" and "not to get wet."

***Double negation law*.**—The negation of a negated proposition results in the original proposition before negation. This law is akin to the law of double multiplication by –1 in basic algebra ($1 = -(-1)$): Not "not to run" is equivalent to "to run."

***Implication and equivalence laws and their negations*.**—These laws have already been discussed in the section Strict Forms of Compound Propositions.

***Contraposition, exportation, importation, and absorption laws*.**—These laws involve statements such as $(a \Rightarrow b)$, $\sim(a \Rightarrow b)$, $(a \Leftrightarrow b)$, $\sim(a \Leftrightarrow b)$, which we have already examined.

Law of contraposition for implications: A given implication is equivalent to another implication in which the variables are both negated and their order is reversed: "If the front comes through, it will rain" is equivalent to "If it does not rain, the front has not come through."

Law of contraposition for equivalences: A given equivalence is equivalent to another equivalence in which the variables are both negated: "It will rain if and only if the front comes through" is equivalent to "It will not rain if and only if the front does not come through."

Law of exportation/importation: An implication whose antecedent is a conjunction is equivalent to another implication whose antecedent is one of the conjuncts and whose consequent is another implication in which the antecedent is the other conjunct and the consequent is the original consequent: When read from left to right, the law is known as "exportation" and when read from right to left, "importation." "Rain and thunder imply a storm" is equivalent to "Rain implies that thunder implies a storm" is also equivalent to "Thunder implies that rain implies a storm."

Law of absorption: Any implication is equivalent to another implication whose antecedent is the same

[3]In the early 20th century, the mathematician Kurt von Gödel (1906–78) demonstrated the existence of a third class of propositions, typically self-referential, the truth or falsity of which implied their own falsity or truth. He called such statements undecidable. Thus, if one were to write, "This statement is false," one would first note that the statement refers to itself and that it is a negative statement. Finally, one would note that its truth would imply its falsity and its falsity would imply its truth. By showing that such statements *must always occur* in logical systems, Gödel thereby shattered the hope of ever finding a logically consistent scheme of mathematics (i.e., one in which such statements could never occur). The finding of such a scheme was exactly the programme undertaken by Russell, Whitehead, and others in the late 19th and early 20th centuries.

and whose consequent is the conjunction of the antecedent and the consequent: “Rain implies a storm” if and only if “rain implies rain and a storm.”

Arguments

***Pierce’s law**.*—If the truth of an implication implies the truth of its own antecedent, the antecedent must be true: “Rain implies a ruined picnic” implies “rain” therefore “rain.” (A cynic’s logic)

***Syllogisms and other forms**.*—Syllogisms are ancient forms dating back to the Greek philosopher Aristotle. Two types are considered:

(1) Hypothetical: Two implications have a common (or middle) term, which is the consequent of one implication (the first) and the antecedent of the other (the second). The conclusion is that the antecedent of the first implication implies the consequent of the second: “If thunder implies rain.” and “Rain implies no picnic.” then “Thunder implies no picnic.”

(2) Disjunctive: This form consists of a disjunction and the denial (negation) of one of the disjuncts. The conclusion is the other disjunct: “It will rain” or “we will have a picnic.” “It will not rain,” therefore “we will have a picnic.”

This syllogism is also referred to as modus tollendo ponens, the “mode of taking and putting.” The remaining arguments use these same Latin words as their descriptors.

Affirming the antecedent or rule of detachment (modus ponendo ponens): This form consists of an implication and the confirmation of the antecedent. The conclusion is the consequent: “If it rains, the picnic will be cancelled.” “It is raining,” therefore “the picnic will be cancelled.”

Denying the consequent (modus tollendo tollens): This form consists of an implication and the negation of the consequent. The conclusion is the negation of the antecedent: “If it rains, the picnic will be cancelled.” “The picnic is not cancelled,” therefore “it is not raining.”

Negation of a conjunction and affirmation of one of the conjuncts (modus ponendo tollens): The conclusion is the negation of the other conjunct: “It is not true that it will rain and the picnic will be held anyway.” “The picnic will be held,” therefore “it will not rain.” (or “It will rain,” therefore “the picnic will not be held.”)

Equivalence ponens: This form consists of an equivalence of two propositions and the affirmation of one of the propositions. The conclusion is the other proposition: “The picnic will be held” if and only if “it does not rain.” “It does not rain,” therefore “the picnic will be held.”

Equivalence tollens: This form consists of an equivalence of two propositions and the negation of one of the propositions. The conclusion is the negation of other proposition: “The picnic will be held” if and only if “it does not rain.” “It rains,” therefore “the picnic will not be held.”

***Dilemmas**.*—Dilemmas consist of two compound propositions (lemmas) and a conclusion:

Simple constructive: Two implications have different antecedents but have a common consequent. If either antecedent (or both) is true, then the consequent must be true: “Clouds imply rain.” and “Thunder implies rain.” “There are clouds.” or “There is thunder,” therefore “it will rain.”

Complex constructive: Two implications have different antecedents and different consequents. If either antecedent (or both) is true, then one (or both) of the consequents must be true: “Clouds imply rain.” and “High winds imply damage.” “There are clouds.” or “There is high wind,” therefore “it will rain” or “there will be damage.”

Simple destructive: Two implications have the same antecedent and different consequents. If either consequent (or both) is false, then the antecedent must be false: “Clouds imply rain.” and “Clouds imply high wind.” “There is no rain.” or “There is no high wind,” therefore “there are no clouds.”

Complex destructive: Two implications have different antecedents and different consequents. If either consequent (or both) is false, then one (or both) of the antecedents must be false: “Clouds imply rain.” and “High winds imply damage.” “There is no rain.” or “There is no damage,” therefore “there are no clouds” or “there are no high winds.”

Special: Two implications have the same consequent and antecedents that are negations, one of the other. The conclusion is the common consequent: “We will have a party whether it rains or shines.” Therefore, “we will have a party.”

Question for the reader: Can you write out examples for the category “Other miscellaneous arguments” in table 2?

Algebraic Representation and Proofs of Laws

Several methods of proof for the laws of hypothetical logic are available in any standard text. Many laws are amenable to proof using a modified type of algebra, a “logic algebra,” which will be introduced herein as it provides a direct and intuitive method for approaching the laws and their proofs. However, the student is encouraged to learn about other and more advanced techniques.

All the equivalence laws given in table 2 may also be represented as algebraic forms when the following substitutions are made:

$$\wedge \leftarrow \cdot \tag{19}$$

$$\vee \leftarrow + \tag{20}$$

$$\Leftrightarrow \leftarrow = \tag{21}$$

These substitutions, along with some hypotheses for a “logic algebra” (as opposed to a number algebra), enable ready proofs of the logic laws using means easily accessible to anyone who has learned basic algebra. Since all the equivalence laws involve propositions that can be reduced to compound “and” and “or” statements, one need only focus on the behavior of $\cdot$ and $+$.

(NB: Substitute the symbol = whenever the symbol $\Leftrightarrow$ occurs; the symbol = has the same meaning that it has in basic arithmetic.)

The logic algebra for compound “and” and “or” propositions is defined in table 3 by the nine hypotheses:

Using the definition of implication $(a \Rightarrow b) \Leftrightarrow (a \wedge b) \vee (\sim a \wedge b) \vee (\sim a \wedge \sim b)$ and substituting, we may write its algebraic form as

$$(a \Rightarrow b) = (a \cdot b) + (\sim a \cdot b) + (\sim a \cdot \sim b) \tag{22}$$

Similarly, the negation of the implication is

$$\sim (a \Rightarrow b) = (a \cdot \sim b) \tag{23}$$

Proofs of Selected Laws

Algebraic representation is now used to prove some basic laws of hypothetical logic presented in the previous section and in table 2.

Example 1: identity.—Begin with the law of identity $a \Leftrightarrow a$, which appears almost self-evident. Its algebraic form is $a = a$. Q.E.D.

Example 2: distributive: $\vee$ over $\wedge$: $a \vee (b \wedge c) \Leftrightarrow (a \vee b) \wedge (a \vee c)$.—Begin with the algebraic form for the left-hand side (LHS) of this law $(a \vee (b \wedge c))$ and rewrite it as an algebraic form:

$$a \vee (b \wedge c) = a + b \cdot c \tag{24}$$

Now, invoke the disjunction introduction on b and c to write $(b \therefore b + a)$ and $(c \therefore c + a)$. Also, invoke the idempotent for $\cdot$ on a to write $a = a \cdot a$. Substitute for a, b, and c on the RHS and find

$$\begin{aligned} a \vee (b \wedge c) &= a \cdot a + (b + a) \cdot (c + a) \\ &= a \cdot a + b \cdot c + b \cdot a + a \cdot c + a \cdot a \\ &= a \cdot a + b \cdot a + c \cdot a + c \cdot b \\ &= a \cdot (a + b) + c \cdot (a + b) \\ &= (a + c) \cdot (a + b) \end{aligned} \tag{25}$$

By reverse substitution, conclude that $a \vee (b \wedge c) \Leftrightarrow (a + c) \cdot (a + b)$. Q.E.D**.**

(NB: If variables in basic algebra were being treated, what has just been shown would make no sense because it would have proven that $a + b \cdot c = (a + c) \cdot (a + b)$, which is certainly not true (in basic algebra). Remember that logic algebra is being used and that the initial hypotheses and treatment of variables are different from those in basic algebra.)

Example 3: extended De Morgan's: $\sim(a + b + c) \Leftrightarrow \sim a \cdot \sim b \cdot \sim c$.—Begin by letting $u = a + b$ so that

$$\sim (a + b + c) = \sim (u + c) \tag{26}$$

Apply De Morgan's law to the LHS to obtain

$$\sim (u + c) = \sim u \cdot \sim c \tag{27}$$

Finally, in the LHS, replace u with $a + b$, apply De Morgan's law again, and the desired result is obtained. Q.E.D.

Similarly, it may be shown that $\sim(a \cdot b \cdot c) \Leftrightarrow \sim a + \sim b + \sim c$.

Example 4: equivalence $(a \Leftrightarrow b) = (a \cdot b) + (\sim a \cdot \sim b)$.—The algebraic form for implication may be used to show that $(a \Leftrightarrow b)$ is equivalent to $(a \wedge b) \vee (\sim a \wedge \sim b)$. Recall that originally by definition $(a \Leftrightarrow b)$ is

TABLE 3.—NINE HYPOTHESES FOR COMPOUND PROPOSITIONS

Rule	Logic form	Algebraic form
Idempotent law for •	$a \wedge a \Leftrightarrow a$	$a \bullet a = a$
Idempotent law for +	$a \vee a \Leftrightarrow a$	$a + a = a$
Commutative law for •	$a \wedge b \Leftrightarrow b \wedge a$	$a \bullet b = b \bullet a$
Commutative law for +	$a \vee b \Leftrightarrow b \vee a$	$a + b = b + a$
Associative law for •	$(a \wedge b) \wedge c \Leftrightarrow a \wedge (b \wedge c)$	$(a \bullet b) \bullet c = a \bullet (b \bullet c)$
Associative law for +	$(a \vee b) \vee c \Leftrightarrow a \vee (b \vee c)$	$(a + b) + c = a + (b + c)$
Distributive law: • over +	$a \wedge (b \vee c) \Leftrightarrow (a \wedge b) \vee (a \wedge c)$	$a \bullet (b + c) = (a \bullet b) + (a \bullet c)$
Universal falsehood	$a \wedge \sim a \Leftrightarrow$ [a]φ	$a \bullet \sim a = 0$
Universal truth	$a \vee \sim a \Leftrightarrow$ [b]U	$a + \sim a = 1$

[a]φ, null or empty set.
[b]U, universe of discourse.

equivalent to $(a \Rightarrow b) \wedge (b \Rightarrow a)$. By writing out the algebraic forms for each of the implications and using rules 1 to 9, we are able to arrive at the law of equivalence: $(a \Leftrightarrow b) = (a \bullet b) + (\sim a \bullet \sim b)$.

Begin by writing

$$(a \Rightarrow b) \wedge (b \Rightarrow a) = \left[(a \bullet b) + (\sim a \bullet b) + (\sim a \bullet \sim b)\right] \times \left[(b \bullet a) + (\sim b \bullet a) + (\sim b \bullet \sim a)\right] \qquad (28)$$

Next, manipulate the RHS using the methods of algebra:

$$\begin{aligned}\text{RHS} = {} & (a \bullet b) \bullet (b \bullet a) + (a \bullet b) \bullet (\sim b \bullet a) \\ & + (a \bullet b) \bullet (\sim b \bullet \sim a) + (\sim a \bullet b) \bullet (b \bullet a) \\ & + (\sim a \bullet b) \bullet (\sim b \bullet a) + (\sim a \bullet b) \bullet (\sim b \bullet \sim a) \\ & + (\sim a \bullet \sim b) \bullet (b \bullet a) + (\sim a \bullet \sim b) \bullet (\sim b \bullet a) \\ & + (\sim a \bullet \sim b) \bullet (\sim b \bullet \sim a)\end{aligned} \qquad (29)$$

Then, rearrange terms and use the ninth hypothesis from table 3:

$$\begin{aligned}\text{RHS} = {} & (a \bullet a \bullet b \bullet b) + (a \bullet a \bullet b \bullet \sim b) \\ & + (a \bullet \sim a \bullet b \bullet \sim b) + (\sim a \bullet a \bullet b \bullet b) + (\sim a \bullet a \bullet b \bullet \sim b) \\ & + (\sim a \bullet \sim b \bullet b) + (\sim a \bullet a \bullet \sim b \bullet b) + (\sim a \bullet a \bullet \sim b) \\ & + (\sim a \bullet \sim a \bullet \sim b \bullet \sim b)\end{aligned} \qquad (30)$$

$$\begin{aligned}\text{RHS} = {} & (a \bullet a \bullet b \bullet b) + 0 + 0 + 0 + 0 + 0 \\ & + 0 + 0 + (\sim a \bullet \sim a \bullet \sim b \bullet \sim b)\end{aligned} \qquad (31)$$

$$\text{RHS} = (a \bullet b) + (\sim a \bullet \sim b) \qquad (32)$$

By reverse substitution, we can conclude that $(a \Leftrightarrow b) = (a \bullet b) + (\sim a \bullet \sim b)$. Q.E.D.

Example 5: contraposition: $\boldsymbol{\Rightarrow}$***:*** $\boldsymbol{(a \Rightarrow b) \Leftrightarrow (\sim b \Rightarrow \sim a)}$.—Begin by writing the algebraic form of the implication $a \Rightarrow b$:

$$(a \Rightarrow b) = (a \bullet b) + (\sim a \bullet b) + (\sim a \bullet \sim b) \qquad (33)$$

Then use the law of double negation to replace b with $\sim\sim b$ wherever b occurs:

$$(a \Rightarrow b) = (a \bullet \sim\sim b) + (\sim a \bullet \sim\sim b) + (\sim a \bullet \sim\sim\sim b) \qquad (34)$$

Now, observe that $\sim\sim\sim b = \sim b$:

$$(a \Rightarrow b) = (a \bullet \sim\sim b) + (\sim a \bullet \sim\sim b) + (\sim a \bullet \sim b) \qquad (35)$$

An examination of the RHS of this last expression reveals that it is identical to the algebraic statement of the implication $\sim b \Rightarrow \sim a$, so that

$$(a \Rightarrow b) = (\sim b \Rightarrow \ \sim a) \qquad (36)$$

from which we obtain the law of the contrapositive: $(a \Rightarrow b) \Leftrightarrow (\sim b \Rightarrow \sim a)$. Q.E.D.

Example 6: Pierce's: $\boldsymbol{(a \Rightarrow b) \Rightarrow a \ \therefore \ a}$.—Begin by writing

$$(a \Rightarrow b) = (a \bullet b + \sim a \bullet b + \sim a \bullet \sim b) \qquad (37)$$

so that

$$\left[(a \Rightarrow b) \Rightarrow a\right] = \left[(a \cdot b + \sim a \cdot b + \sim a \cdot \sim b) \cdot a\right]$$
$$+ \left[\sim (a \cdot b + \sim a \cdot b + \sim a \cdot \sim b) \cdot a\right]$$
$$+ \left[\sim (a \cdot b + \sim a \cdot b + \sim a \cdot \sim b) \cdot \sim a\right] \quad (38)$$

Now, observe that $\sim(a \bullet b + \sim a \bullet b + \sim a \bullet \sim b) = a \bullet \sim b$. There are three means of demonstrating this equivalence (any one is sufficient):

1. Cite either the rule for negating an implication or the rule for a compound proposition having two variables and its negation, which states that each of the four truth possibilities must appear exactly once.
2. Note that the expression $(a \bullet b + \sim a \bullet b + \sim a \bullet \sim b)$ is also the definition for $a \Rightarrow b$. The negation of $a \Rightarrow b$ is equivalent to $a \bullet \sim b$. Cite this equivalence directly.
3. Use algebraic means,[4] an exercise the reader should try before checking the footnote. Thus,

$$\left[(a \Rightarrow b) \Rightarrow a\right] = \left[(a \cdot b + \sim a \cdot b + \sim a \cdot \sim b) \cdot a\right]$$
$$+ \left[(a \cdot \sim b) \cdot a\right] + \left[(a \cdot \sim b) \cdot \sim a\right]$$
$$= a \cdot b + a \cdot \sim b$$
$$= a \cdot (b + \sim b)$$
$$= a \quad (39)$$

from which we conclude a. Q.E.D.

Techniques of Proof in Mathematics, Science, and Philosophy

In this section, it is necessary to suspend the earlier proviso that a variable a (without a tilde) is true and a variable $\sim a$ (with a tilde) is false.[5]

There are three major approaches to proving a proposition in mathematics, science, or philosophy: direct proofs, indirect proofs, and proofs by induction. For any given proposition, the method must be carefully chosen. The wrong choice can lead to frustration, whereas the correct choice may lead to a quick solution.

Direct.—Suppose that we are given the truth of a proposition a and are asked to prove the truth of a proposition b. In the method of direct proof, the truth of a is assumed (taken without proof or argument) and the truth of b must be demonstrated. If the proposition b is a conjunct or a disjunct, proceed as follows:

Given $S, \therefore a \wedge b$: Assume the truth of S and reason from this assumption to the truth of $a \wedge b$. We might reason individually first to the truth of a and then to the truth of b, but once they have been shown to be true individually, invoke the law of conjunction introduction to conclude the truth of the compound proposition $a \wedge b$.

Given $S, \therefore a \vee b$: It is sufficient to assume the truth of S and to reason from this assumption the truth of either a or b but not necessarily both.

To prove propositions of the form

$a \Rightarrow b$: Assume the truth of a and reason to the truth of b. Or, invoke the law of the contrapositive and write the equivalent proposition $\sim b \Rightarrow \sim a$, assume the truth of $\sim b$ and reason to the truth of $\sim a$.

$a \Leftrightarrow b$: Prove in two separate steps the implications $a \Rightarrow b$ and $b \Rightarrow a$.

$a \therefore b$: Assume the truth of a and reason to the truth of b.

$\exists a \ni b$: It is sufficient to find one true instance of a for which b is also true.

$\forall a, b$: It is necessary to show that the truth of b is ensured regardless of the choice of any true instance of a. This method is often very difficult, and the indirect method of proof by counterexample is used instead (see the next section).

Indirect (reductio ad absurdum).—The argument reductio ad absurdum (reduction to absurdity) is based on the Greek law of the excluded middle; that is, either a proposition or its negation must be true. Suppose that one is given the truth of a proposition a and is asked to prove the truth of a proposition b. It is known that b is either true or false; that is, either b is true or $\sim b$ is true, but it is not known which. In the method of indirect proof, assume the truth of a and of $\sim b$ and reason to a contradiction of the form $p \wedge \sim p$. Recall that $p \wedge \sim p$ is a universal falsehood. Then, since the choice of $\sim b$

[4]Using the extended De Morgan's law, $\sim(a \bullet b + \sim a \bullet b + \sim a \bullet \sim b) = [\sim(a \bullet b) \bullet \sim(\sim a \bullet b) \bullet \sim(\sim a \bullet \sim b)] = [(\sim a + \sim b) \bullet (a + \sim b) \bullet (a + b)] = [(\sim a \bullet a + \sim a \bullet \sim b + \sim b \bullet a + \sim b) \bullet (a + b)] = [(\sim a \bullet a) + (\sim a \bullet a \bullet \sim b) + (\sim b \bullet a) + (a \bullet \sim b) + (\sim a \bullet a \bullet b) + (\sim a \bullet \sim b \bullet b) + (\sim b \bullet b \bullet a) + (b \bullet \sim b)] = (a \bullet \sim b)$. Q.E.D.

[5]This suspension is necessary because the variables a and/or $\sim a$ often must be considered as being individually true or false, so that herein the more general provision cannot strictly hold.

leads to a universal falsehood, $\sim b$ must be false, and b must be true. Thus, to prove propositions of the form

Given S, $\therefore a \wedge b$: Assume the truth of S and the truth of $\sim(a \wedge b)$. Since $\sim(a \wedge b) \Leftrightarrow \sim a \vee \sim b$ by De Morgan's law for $\wedge$, we may also assume the truth of S and either $\sim a$ or $\sim b$ individually and show that a contradiction follows. Since $\sim(a \wedge b)$ leads to a contradiction, it follows that $a \wedge b$ must be true.

Given $S \therefore a \vee b$: Assume the truth of S and the truth of $\sim(a \vee b)$. Since $\sim(a \vee b) \Leftrightarrow \sim a \wedge \sim b$ by De Morgan's law for $\vee$, we may assume the truth of S and the truth of the compound proposition $\sim a \wedge \sim b$ and show that a contradiction follows. Since $\sim(a \vee b)$ leads to a contradiction, it follows that $a \vee b$ must be true.

Given: $a \Rightarrow b$: Assume the truth of $\sim(a \Rightarrow b) \Leftrightarrow (a \Rightarrow \sim b) \Leftrightarrow (a \wedge \sim b)$ and show that a contradiction follows. Since $\sim(a \Rightarrow b)$ leads to a contradiction, it follows that $a \Rightarrow b$ must be true.

Given: $a \Leftrightarrow b$: Assume the truth of

$$\sim(a \Leftrightarrow b) \Leftrightarrow \left[(a \Leftrightarrow \sim b) \vee (\sim a \Leftrightarrow b)\right] \Leftrightarrow \left[(a \wedge \sim b) \vee (\sim a \wedge b)\right] \qquad (40)$$

and show that a contradiction follows. Since $\sim(a \Leftrightarrow b)$ leads to a contradiction, it follows that $a \Leftrightarrow b$ must be true.

Given: $a \therefore b$: Assume the truth of a and of $\sim b$ and show that a contradiction follows. Since $\sim(a \therefore b)$ leads to a contradiction, it follows that $a \therefore b$ must be true.

Given: $\exists a \ni b$: Assume the truth of $\forall a \sim b$ and show that a contradiction follows. Since $\sim(\exists a \ni b)$ leads to a contradiction, it follows that $\exists a \ni b$ must be true.

Given: $\forall a, b$: Assume that $\exists a \ni \sim b$ (proof by counterexample) and show that a contradiction follows. Since $\sim(\forall a, b)$ leads to a contradiction, it follows that $\forall a, b$ must be true.

Inductive.—Suppose that we wish to prove an infinite sequence of related propositions, $a_0, a_1, a_2, a_3, \ldots, a_\infty$. Each proposition cannot be proven individually because there is an infinite number of them. So we must resort to proof by induction. The two steps to complete an inductive proof are

1. Prove the truth of a_0 and a_1.
2. Show that the truth of any a_n implies the truth of a_{n+1}; that is, $a_n \Rightarrow a_{n+1}$.

Basics of Aristotelian Logic

Mediate Versus Immediate Inference

Aristotelian logic uses arguments called syllogisms. This section introduces syllogisms and the concepts involving their use and construction. Hypothetical logic, discussed in the first section of this report, was mainly developed during the 19th century for use in mathematics because during this time developments in mathematics increased more than they had in previous centuries. Therefore, the logic of mathematics had to be formalized in an agreed-upon system.

Aristotelian logic is used in the news, the courtroom, and other nonmathematical disciplines. Because Aristotle was concerned with a different set of problems than those of the 19th century mathematicians, he developed his logic accordingly. Thus, we will discover that many of the rules developed for hypothetical logic were also developed for Aristotelian logic.

Let us begin with concept of *inference* as it relates to argument. To infer implies that we have a subject we wish to speak about and a predicate we wish to apply to the subject. When we infer something, we draw a conclusion based on given statements that we take to be true. For example, given the statements

All Greeks are philosophers.
Aristotle is a Greek.

we can infer that

Aristotle is a philosopher.

In the first section, Basics of Hypothetical Logic, simple propositions were presented. Note that in the foregoing statements, there are three simple propositions, two given and one inferred. Each proposition has a subject and a predicate. The subject is what we are talking about; the predicate is what we have to say about the subject.

The proposition "Rover is a dog" has the subject "Rover" and the predicate "is a dog." Changing the predicate to "is a giraffe" would completely change the meaning of the proposition. However, changing the subject to "Spot" would not alter the original proposition. That Spot is a dog does not affect the status of Rover at all.

The three propositions in the previous example, "All Greeks are philosophers," "Aristotle is a Greek," and

TABLE 4.—THE FOUR FIGURES

Proposition[a]	Figure			
	First	Second	Third	Fourth
Premises	*MP* and *SM*	*PM* and *SM*	*MP* and *MS*	*PM* and *MS*
Conclusion	Therefore *SP*	Therefore *SP*	Therefore *SP*	Therefore *SP*

[a]*S*, subject term; *P*, predicate term; *M*, middle term.

"[Therefore] Aristotle is a philosopher," taken together form a type of argument called a syllogism. Note that the terms "Aristotle" and "philosopher" appear in the conclusion, "[Therefore] Aristotle is a philosopher," but that the term "Greek" does not. The term "Greek" is part of the argument but acts only as a mediator and has no other role.

We may restate our argument in the notation of hypothetical logic as

$$\begin{aligned}&\text{Greek} \Rightarrow \text{philosopher}\\&\text{Aristotle} \Rightarrow \text{Greek}\\&\therefore \text{Aristotle} \Rightarrow \text{philosopher}\end{aligned}$$

Each of the three propositions is shown as an implication, and each implication makes an immediate statement (i.e., a statement without a term that acts as a mediator). Only when the three propositions are taken together does the mediatory role of the term "Greek" appear.

To summarize based on the foregoing discussion, we may define two types of inference:

1. Mediate—inference with a middle step
2. Immediate—inference without a middle step

Note that the Latin prefix *im* means "without." In immediate inference, we reason directly from a hypothesis to a conclusion. If, instead of "Greek ⇒ philosopher," we write "$x + 1 = 0 \Rightarrow x = -1$," the role of hypothetical implication as inference becomes clear. The proof of "$x + 1 = 0 \Rightarrow x = -1$" is obvious to one who has had first-year algebra.

In mediate inference, we reason from antecedent propositions using a middle term to a concluding proposition in which the middle term is absent.

Syllogistic logic is mediate inference. There are three terms: the subject *S*, predicate *P*, and the mediate or middle *M*. The point of the syllogism is to prove that the predicate term applies to the subject term; that is, the predicate tells (or predicates) something about the subject.

The conclusion of a syllogism always has the form *SP*, the subject followed by the predicate. For clarity, here is an illustration of the symbol *SP*. Let us say that "Aristotle" is the subject and "is a philosopher" is the predicate. Then, the symbol *SP* means "Aristotle is a philosopher." In shorthand,

$$\begin{aligned}&S = \text{Aristotle}\\&P = \text{is a philosopher}\\&SP = \text{Aristotle is a philosopher}\end{aligned}$$

In a syllogism, the middle term occurs once in each of the antecedent arguments. In this case, the middle term occurs once as a subject ("All Greeks are philosophers") and once as a predicate ("Aristotle is a Greek"). The four ways the middle term can occur or "be distributed" in a syllogism are as

1. Subject and subject
2. Subject and predicate (example just given)
3. Predicate and subject
4. Predicate and predicate

Each of these distributions is called a figure and there are four. A syllogism is characterized by the type of figure it employs. For easy reference, table 4 presents the four figures.

Definition by Genus and Species

When hypothetical logic was introduced in the first section of this report, we spoke of making definitions. Recall that we had to define the terms that would be used to build the logical system. Aristotle was very concerned with definition and wrote a great deal about it. The following summarizes what he had to say.

Definitions are the starting point of most philosophical and mathematical arguments. To construct a definition, we first have to select a "universe of discourse" in which the argument can take place. Suppose that we are talking about your new golden retriever. The universe of discourse might be the set of all dogs. Within the set of all dogs we may then speak of the set of all retrievers, and within the set of all retrievers, the set of all golden retrievers, and within this set, your particular retriever.

Each paring down requires recognition of both general and specific characteristics. Aristotle called these general and specific characteristics *genus* and *species*, respectively. An Aristotelian definition is a proposition (or set of propositions) that involves the concepts of genus and species. In defining the term "tree," we recognize the various species of trees as oak, maple, pine, and so on. In defining a maple tree, on the other hand, we recognize the various species as red, silver, or Japanese.

A genus is a class of a certain kind that can be partitioned into smaller equivalence subclasses of *species*. A class is a set of objects having some element or elements in common, and it may generally be partitioned into species by further dividing among these common elements. The set union of all the species in a class comprises the class itself. The set intersection of any two species is the null set.

Example of null set.—All trees have trunks, roots, and foliage. The set of all trees is a class (genus). Foliage may be subdivided into leaves and needles; hence, the class *trees* may be partitioned into the subclasses (species): trees with leaves and trees with needles. The set union of these subclasses is the original class. Their intersection is the null set.

Aristotle's definition of a term by genus and species first specifies a characteristic belonging to the genus of the term being defined.

A second characteristic, the differentia, is then added to indicate the species. The characteristic supplied by the differentia distinguishes the term being defined from all other terms belonging to the same genus.

Example of differentia.—The genus is again trees. The differentia are needles and leaves.

The genus is predicated of the species but not conversely. For example, we say, "A tree has leaves," not "Leaves has a tree." The species term contains more information than the genus term.

Propositions

In the first section, Basics of Hypothetical Logic, we encountered the concept of *proposition*. Aristotelian logic is a logic of propositions. In general, a proposition is a statement that proposes an idea. More formally, a proposition

1. Is a specific type of statement that is capable of being judged true or false. There are many types of statements that do not concern us in the study of logic. The explicative "Hey you!" is one such statement. Note that the genus at this point is the term "statement" and the species is the term "proposition."
2. Predicates something by asserting or denying it. Thus, the statement "Hay is for horses" is a proposition, but the statement, "Hey you!" is not.
3. Is capable of being true or false, thereby allowing us to judge or assert that truth or falsity. Thus, the statement "Pink elephants sing pretty songs" is not a proposition since its truth or falsity cannot be strictly judged.

Three types of propositions are used in logic:

1. Categorical—All Greeks are philosophers.
2. Hypothetical—If he/she is a Greek, then he/she is a philosopher.
3. Disjunctive—Either Aristotle is a Greek or Aristotle is not a philosopher.

Note: In continuing the process of definition, the genus has now become "propositions" and the species, "categorical," "hypothetical," and "disjunctive."

The propositions of classical Aristotelian logic are categorical and consist of a subject term and a predicate term connected by the copula *is*. The predicate affirms or denies something about the subject. If S stands for subject and P stands for predicate, the general categorical schema is

$$\left[\text{Some or all}\right] S \text{ is } \left[\text{or is not}\right] P \tag{41}$$

We have already seen that a hypothetical proposition consists of an antecedent (hypothesis) and a consequent (conclusion) connected by the copula *If…then…* . If these terms are represented respectively as H and C, the general hypothetical schema is

$$\text{If } H\text{, then } C. \tag{42}$$

The disjunctive proposition consists of two disjuncts connected by the copula *Either…or…* . If P and Q are the disjuncts, the general disjunctive schema is

$$\text{Either } P \text{ or } Q. \tag{43}$$

Categorical Propositions

Now that we have some of the definitions and concepts in hand, we can advance farther in our study of Aristotle's logic. Before doing so, we must state the obvious: To talk about something, we must have something to talk about. (This type of statement is called a tautology. Another example of a tautology is a mathematical theorem.) In this section, we will explore the relation of language in general and propositions in particular to the world at large. Recall that a categorical proposition is a proposition that consists of a subject term and a predicate term connected by the copula *is*.

Subject and substance.—In Aristotelian[6] logic, a subject is the term about which something is to be predicated. Whatever is subject in a categorical proposition is also substance, that which exists in the world. The term "subject" refers to language; the term "substance" refers to the stuff of the real world. Thus, a categorical proposition is a statement (proposition) about the world at large.

We may read in Aristotle and in Aquinas of "substance" and "form." For Aristotle, substance is the highest reality and is composed of formed matter; conversely, matter is the raw material which, when formed, becomes substance.

Form cannot exist without matter, and matter cannot exist without form. Prime matter (prima materia or formless matter) is an abstraction, a passive, undifferentiated potentiality, capable of becoming anything, awaiting determination. Similarly, matterless form by itself (c.f., Plato's ideal forms) is a similar abstraction.

Although these concepts may seem antiquated, they are still in modern usage as reflected by terms such as quintessential, potential, and actualization. For example, consider the use of the word "potential" in such terms as "potential energy," "vector potential," and "scalar potential." Potential in physics is a formless something that underlies an important aspect of the physical world. Potential energy is capable of becoming kinetic energy in a mechanical system. The vector and scalar potentials in electromagnetic theory are the driving functions in d'Alembert's general equations of the electromagnetic field and give rise to the actual field quantities.

Essence and existence.—We speak of the substance of an argument or a situation. In a common cliché, "the sum and substance of …," substance is the "bird in hand" of a given argument. Perhaps it would be better to say "The something in hand." However, with something in hand, we are then compelled to identify exactly what it is that we have. After all, bug substance is different from rock substance or vegetable substance. Whatever the something in hand, it is referred to by the general term "essence." We often speak of the essence of an argument; that is, what the speaker is really trying to say. According to Aristotle, the essence of a thing is that set of qualities by which the thing is what it is. Essence comprises a unique set of properties that every member of a species must possess to belong to that species and to no other.

Being or "existence" is also a quality of the something in hand. The concept of being has always challenged philosophers and continues to do so today. H.G. Wells asked in his novel *The Time Machine*, "Can an instantaneous cube exist? I mean, a cube that lasts for no time at all?" In other words, can nothing be imagined? The answer is "hardly," since nothing itself is represented by something; it is represented by the term "nothing," which has a very real existence.

Aristotle attempted to comprehend existence. He associated existence with substance. He argued that neither form by itself nor matter by itself has existence except as an abstraction or an idea; only when form and matter combine in substance does something come into being. So potential is actualized in substance (formed matter) which represents something specific—essence—and has being; that is, it exists in the world.

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[6]The material in these sections represents no more than an introduction to Aristotelian thought. The author hopes that students will further investigate the subject on their own. Although Aristotle is often denigrated in modern classrooms, it must be remembered that he has much to teach us and we own him a great deal of respect.

Propositions deal with the things of the world. Therefore, they deal with substance, form, essence, and existence; propositions are statements about these qualities. When studying categorical propositions, recall the existential quantifier ∃, which means "there exists." Modern logic is also concerned with existence, although in a slightly different way.

***Predication*.—**When something exists, we are able to say something about it—to "predicate" something about it. Every subject has qualities of interest. The grammatical form that relates to these qualities is called the predicate. In categorical propositions, we use the simple grammatical form "*a* is *b*" with some variations. Recall that a proposition is a specific type of statement that asserts or denies (predicates) something and is capable of being judged true or false. The five types of predicates are

1. Genus—a universal class that may be partitioned into various equivalence subclasses called species (e.g., tree).
2. Differentia—a characteristic or set of characteristics that identify the members of a particular species as belonging to that species and to no other (e.g., oak).
3. Species—differentia and genus taken together (e.g., oak tree).
4. Property—a necessary attribute of any member of a species (e.g., ability to produce acorns).
5. Accident—a contingent attribute of any member of a species (e.g., a large oak tree).

Table 5 summarizes the five predicate types.

TABLE 5.—TYPES OF PREDICATES

Type	Example
Genus	Tree
Differentia	Oak
Species	Oak tree
Property	Ability to produce acorns
Accident	Size (large or small)

Let us continue to analyze the concept of predication to learn more about its characteristics. Each type of predicate is either necessary or contingent. Necessary refers to that which stands alone and may not be otherwise, whereas contingent refers to that which is dependent on something else and may be otherwise.

The opposite of any contingent is another contingent. If we were to say, "This leaf is not red; this leaf is green," the reference would be to the color of the leaf, a contingent quality. Leaf color is contingent on the time of the year the leaf is observed, on the color of the light under which it is observed, on the type of tree or plant that produces the leaf, even on whether or not the observer is color blind (think about the definition of color). Contingent qualities are sometimes referred to as "accidental qualities" or "accidents."

The opposite of the necessary is the impossible. If one were to say, "This oak tree produces acorns; this oak tree produces no acorns," the reference would be to the acorn-producing propensity of oak trees. For an oak tree, the production of acorns is a necessary quality, intrinsic to the tree and to its being an oak. There is nothing contingent or accidental about it. The oak tree that produces no acorns is either not an oak at all or is dead, in which case it is not strictly an oak tree either (assuming that when we name a tree, we are naming a living entity and not some hollowed out hull).

Another way to analyze predication is to consider the internal qualities of the predicate: what it is versus what its external qualities are (i.e., how it pertains to the world at large). In this type of analysis, one speaks of the characteristics of intension and extension.

The intension (also called the comprehension or connotation) of a term entails those characteristics present in the definition of the term itself. Consider the categorical proposition "A vector is a line segment with length and direction." The intension of "vector" is "line segment with length and direction."

The extension (also called the denotation) of a term entails that set of objects to which the term refers. Consider the categorical proposition "Field quantities in physics are vectors." The particular extension of "vector" used here is "field quantities in physics."

Intension and extension are related in that the intension of a term determines its extension, and the extension of a term determines its intension.

***Words: univocal, equivocal, analogical*.—**Categorical propositions consist of terms that are usually words—language. Because words play an integral role in propositions, let us examine how words function in other language structures, such as puns.

Words may be classified as univocal, equivocal, and analogical. A word is univocal when it has the same meaning in all its various uses. For example, the word "bird" is univocal as are the words "dog" and "man."

A word is equivocal when it has more than one meaning depending on its use. For example, the word "bark" is equivocal as in the bark of a tree, the bark of a dog, or a bark (type of boat). Equivocation is involved when one observes the play on words in the simile "Time flies like an arrow" or in the statement "Fruit flies like a banana."

A word is analogical when it is used for something it is not but suggests something to which it bears some similarity. For example, the word "lady" is analogical when it is used to refer to a boat.

Immediate inference.—Now that the building blocks of Aristotelian logic have been surveyed, we are ready to put the blocks together and build the structure. In the section Basics of Hypothetical Logic, the three basic laws of thought were introduced. These laws are so fundamental and apparently self-evident that they are taken as universally true. Therefore, they are restated for reference:

1. Law of identity—Any statement is equivalent to itself.
2. Law of the excluded middle—Either a statement is true, or it is false. There is no middle ground.
3. Law of contradiction—A statement and its negation cannot both be true at the same time and place.

We now consider immediate inference in its classical or Aristotelian form. In the section Basics of Hypothetical Logic, the existential quantifiers $\forall$ and $\exists$ were introduced: "For all" and "There exists…such that..." These quantifiers also appear here with only small modification.

Categorical propositions involving the quantifier $\forall$ have two expressions or moods. In hypothetical logic, we wrote $\forall p$, q, which states that "For all p, q is true." The two moods of this quantifier in Aristotelian logic are designated by the Roman letters A and E:

A: All S are P.
E: No S are P.

where S is the subject and P is the predicate.

The mood A is called the universal affirmative and the mood E, the universal negative. Again, in hypothetical logic, we wrote, $\exists p \ni q$, which means "There exists at least one p such that q is true." The two moods of this quantifier in Aristotelian logic are designated by the Roman letters I and O:

I: Some S are P
O: Some S are not P

The mood I is called the particular affirmation and the mood O, the particular negation. The four moods are summarized in table 6:

TABLE 6.—MOODS[a] OF CATEGORICAL PROPOSITIONS

Proposition	Mood
A: All S are P.	Universal affirmation
E: No S are P.	Universal negative
I: Some S are P.	Particular affirmation
O: Some S are not P.	Particular negative

[a]In some texts, "type" is the same as "mood."

The relationships between the four moods is illustrated and illuminated in diagrammatic form using a device called the square of opposition. The square of opposition introduces new terminology: contraries, contradictories, subcontraries, and subalternates and superalternates. The notation of hypothetical logic is used to define these terms:

1. Contraries—$\forall S$, P and $\forall S$, $\sim P$
2. Contradictories—$\forall S$, P and $\exists S \ni \sim P$ or $\forall S$, $\sim P$ and $\exists S \ni P$
3. Subcontraries—$\exists S \ni P$ and $\exists S \ni \sim P$
4. Subalternates or superalternates—$\forall S$, P and $\exists S \ni P$ or $\forall S$, $\sim P$ and $\exists S \ni \sim P$

These same relationships are more easily comprehended when represented in the square of opposition (fig. 1):

This diagram is worth committing to memory because it provides an easy and quick way to compare statements made during a discussion or an argument. Specifically, if one of the four moods arises and is asserted to be true or false, the truth or falsity of the other three moods can be readily determined.

In the square of opposition, if the truth or falsity of each of the four moods of categorical propositions is assumed individually, then the remaining moods are either true, false, or undetermined. For example, if I is true, then E must be false, but A and O are undetermined. The truth of I tells us that some S are P. Thus, we know that the proposition "No S are P" must be false. Whether all S are P or some S are not P cannot be determined from I taken alone.

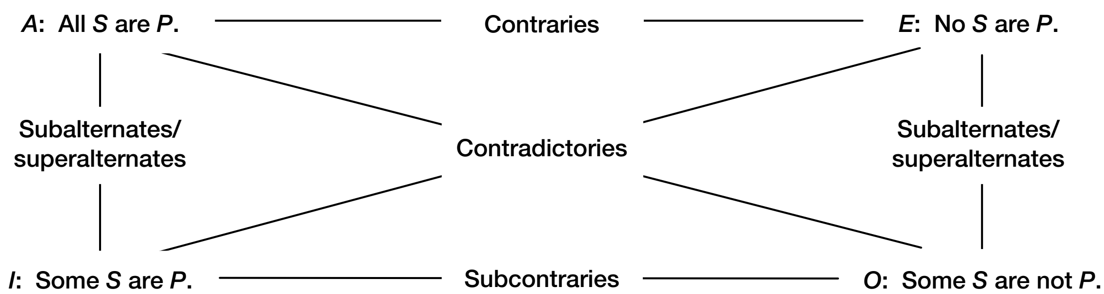

Figure 1.—Square of opposition.

TABLE 7.—INFERENCES DERIVED FROM ASSERTING THE TRUTH OR FALSITY OF A PARTICULAR CATEGORICAL PROPOSITION

Proposition[a]	Inference		
If *A* is true	*E* is false	*I* is true	*O* is false
If *E* is true	*A* is false	*I* is false	*O* is true
If *I* is true	*A* is undetermined	*E* is false	*O* is undetermined
If *O* is true	*A* is false	*E* is undetermined	*I* is undetermined
If *A* is false	*E* is undetermined	*I* is undetermined	*O* is true
If *E* is false	*A* is undetermined	*I* is true	*O* is undetermined
If *I* is false	*A* is false	*E* is true	*O* is true
If *O* is false	*A* is true	*E* is false	*I* is true

[a]*A*, universal affirmation; *E*, universal negative; *I*, particular affirmation; *O*, particular negative.

Table 7 summarizes all the possibilities for the four moods when *A*, *E*, *I*, and *O* are taken individually to be true or false. A mood and its truth or falsity is selected from the left-hand column and the remaining moods are determined by reading across.

***Converse, obverse, contrapositive, and inverse of categorical propositions*.**—As with predication, the concept of immediate categorical propositions can be analyzed in different ways to illuminate their different characteristics. In this section, we will examine what happens when a given proposition is interchanged and/or its terms are negated in various ways.

In the section Basics of Hypothetical Logic, we encountered the concept of a statement and its contrapositive. Recall that given the proposition

$$A \text{ implies } B$$

the contrapositive proposition is

$$\sim B \text{ implies } \sim A.$$

The truth of one of these propositions is equivalent to the truth of the other.

Note that in this particular case, the terms *A* and *B* were interchanged (their roles as hypothesis and conclusion were switched) while at the same time each was negated. What would happen if these changes were made in different ways? Logicians have examined this question and have determined that given a proposition, there are four variants that are worth mentioning:

TABLE 8.—CONVERSE, OBVERSE, CONTRAPOSITIVE, AND INVERSE OF CATEGORICAL PROPOSITIONS

Proposition[a]	Converse	Obverse	Contrapositive	Inverse
A: All *S* are *P*.	*I*: Some *P* are *S*.	*E*: No *S* are non-*P*.	*A*: All non-*P* are non-*S*.	Full: Some non-*S* are non-*P*. Partial: Some non-*S* are not *P*.
E: No *S* are *P*.	*E*: No *P* are *S*.	*A*: All *S* are non-*P*.	*O*: All non-*P* are not non-*S*.	Full: Some non-*S* are not non-*P*. Partial: Some non-*S* are *P*.
I: Some *S* are *P*.	*I*: Some *P* are *S*.	*O*: Some *S* are not non-*P*.	Undetermined	Undetermined
O: Some *S* are not *P*.	Undetermined	*I*: Some *S* are non-*P*.	*O*: Some non-*P* are not non-*S*.	Undetermined

[a]*A*, universal affirmation; *E*, universal negative; *I*, particular affirmation; *O*, particular negative.

1. The original proposition: $A \Rightarrow B$
2. The converse: $B \Rightarrow A$ (the terms are simply interchanged)
3. The obverse: $\sim(A \Rightarrow \sim B)$ (The conclusion is first negated then the result is negated)
4. The contrapositive: $\sim B \Rightarrow \sim A$
5. The inverse: $\sim A \Rightarrow \sim B$ (the terms are negated without changing places)

Evidently, given that $A \Rightarrow B$ is true, then the obverse and the contrapositive are also true, but the converse and the inverse are undetermined. In Aristotelian logic, these same relationships are defined for the four moods in table 8:

Categorical Syllogisms

Mediate inference.—Now let us examine categorical propositions in mediate inference. Recall that syllogisms are arguments that consist of three categorical propositions. Two of these propositions are called premises; the third is called the conclusion. Each premise may be an *A*, *E, I*, or *O* categorical proposition. Syllogisms also contain three terms: a subject, a middle, and a predicate. The middle term occurs in both premises. The subject occurs in one premise and in the conclusion. The predicate occurs in the other premise and the conclusion.

In the syllogism, “All dogs have four legs; Fluff is a dog; therefore, Fluff has four legs,” “dog” is the middle term, “having four legs” is the predicate, and “Fluff” is the subject. The first two statements are premises (the first is an *A* categorical proposition, and the second is an *I* categorical proposition), and the third is the conclusion, also an *I* categorical proposition.

The premises in a syllogism are distinguished by the terms “major” and “minor.” The major premise contains the predicate and the middle terms. The minor premise contains the subject and the middle terms. The conclusion contains the subject and the predicate terms. Table 9 illustrates the foregoing example.

TABLE 9.—PREMISES OF A SYLLOGISM

Premise	Categorical proposition
Major	All dogs have four legs.
Minor	Fluff is a dog.
Conclusion	Therefore, Fluff has four legs

Four figures of a syllogism.—Obviously, the subject, middle term, and predicate can occur in different orders in different propositions. When these variations are taken into account, four distinct figures appear for categorical syllogisms. These figures are most easily seen when presented in table 4, which is repeated here for this discussion. These figures are the columns labeled first through fourth. The first row in each column shows the premise and the placement of the middle term (*M*), the predicate (*P*), and the subject (*S*), respectively. The second row in each column gives the conclusion, which is the same in all four figures as it must be by definition.

TABLE 4.—THE FOUR FIGURES

Proposition[a]	Figure			
	First	Second	Third	Fourth
Premises	*MP* and *SM*	*PM* and *SM*	*MP* and *MS*	*PM* and *MS*
Conclusion	Therefore *SP*	Therefore *SP*	Therefore *SP*	Therefore *SP*

[a]*S*, subject term; *P*, predicate term; *M*, middle term.

Each categorical syllogism can be classified into one of these four figures.

Four categorical moods.—Since each proposition in a categorical syllogism can be of mood *A*, *E*, *I*, or *O*, it follows that there are many variants of categorical propositions. To classify these variants, we must know the role of the various terms in the propositions. For example, in the proposition "All Greeks are philosophers," it is clear that the set of all Greeks is a subset of all philosophers. A non-Greek philosopher is possible in light of this premise, but *not* a nonphilosophical Greek. In the proposition, "Some Greeks are philosophers," it is at least implied that some Greeks are nonphilosophers. This second proposition is somewhat weaker than the first. Thus, the "inclusive" role played by the various terms in a proposition is classified under the name "distribution." In the proposition "All Greeks are philosophers," we say that the term "Greeks" is distributed since a nonphilosophical Greek is not permitted. In other words, to be Greek is to be philosophical. The term "philosopher," on the other hand, is not distributed since to be philosophical is not necessarily to be Greek. This may seem confusing at first, but thinking about it for a while will make it clearer.

Distribution of terms in the four categorical moods.—Venn diagrams are a useful way to think about distribution. In fact, since a distributed term is any term for which the proposition conveys information about every member of a class, Venn diagrams provide a direct method for determining the distribution of a term. The following steps describe the construction of a Venn diagram:

1. Use only the terms given in the proposition.
2. When the diagram is complete, the term that is *contained* is distributed; the term that *contains* is undistributed. (If neither contains nor is contained, then both are undistributed.)
3. If a given term is distributed, its negation is undistributed, or if a given term is undistributed, its negation is distributed.

Table 10 shows the distribution of terms in the four moods. Figure 2 shows the Venn diagrams corresponding to these four moods. The distributions in the table should become clear once these diagrams are understood.

TABLE 10.—DISTRIBUTION OF TERMS[a] IN FOUR CATEGORICAL MOODS

Categorical mood[a b]	Distribution
A: All *S* are *P*.	The subject term is distributed; the predicate term is undistributed.
E: No *S* are *P*.	The subject and predicate terms are both distributed.
I: Some *S* are *P*.	The subject and predicate terms are both undistributed.
O: Some *S* are not *P*.	The subject term is undistributed; the predicate term is distributed.

[a]*A*, universal affirmation; *E*, universal negative; *I*, particular affirmation; *O*, particular negative.
[b]*S*, subject term; *P*, predicate term.

Rules for constructing a valid categorical syllogism.—The five classical rules for constructing a valid categorical syllogism follow:

1. The middle term must be distributed at least once.
2. No term may be distributed in the conclusion if it is not distributed in the premises.
3. If both premises are negative, there is no conclusion.
4. If one premise is negative, the conclusion must be negative.
5. If both premises are affirmative, the conclusion must be affirmative.

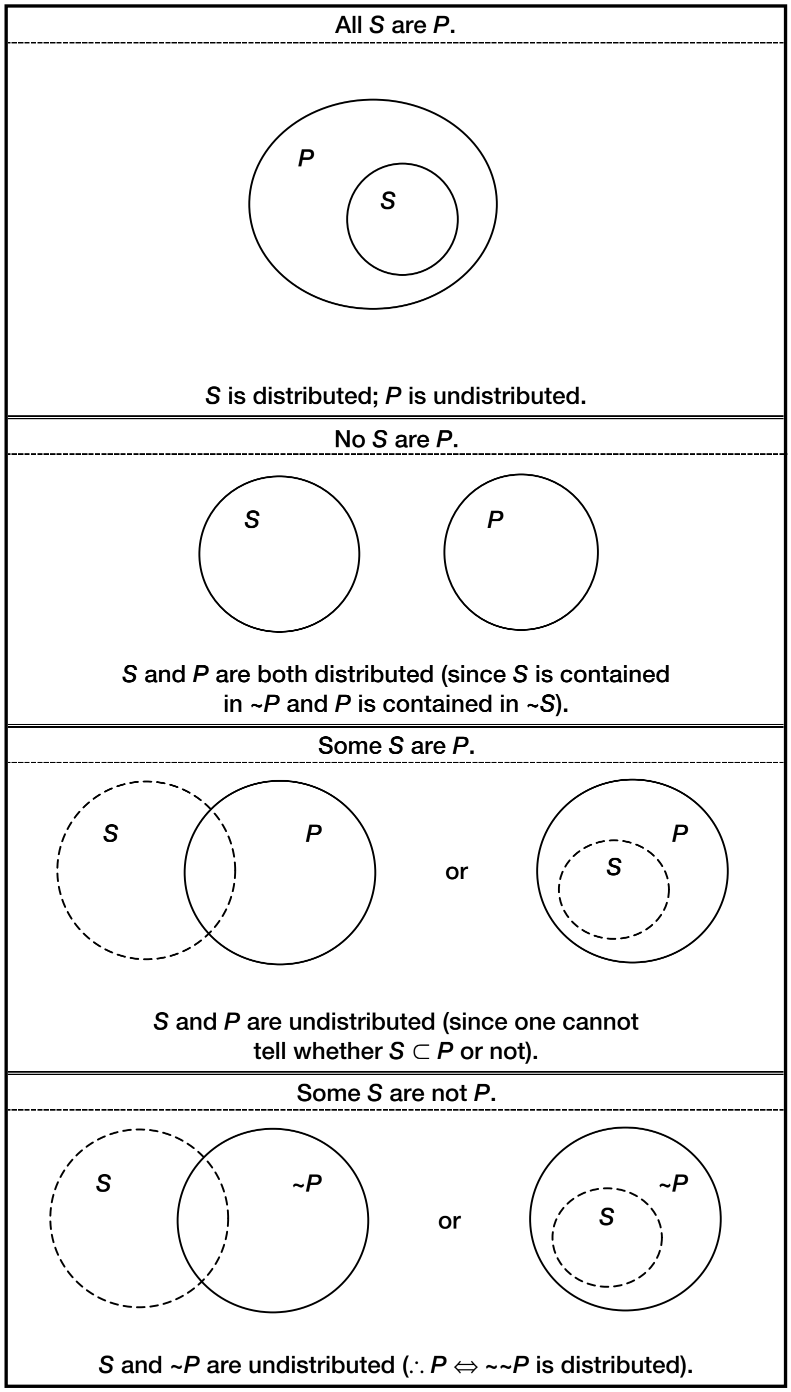

Figure 2.—Venn diagrams showing distribution of terms in four categorical moods.

***Twenty-four valid syllogisms*.**—The rules for constructing a valid syllogism can be used to form 24 valid combinations of moods for premises and conclusions. These 24 possibilities are subdivided into the 4 figures already described (see table 4). In table 11, read the notation as follows: *AAA* ⇔ premise 1 is an *A* categorical proposition; premise 1 is an *A* categorical proposition; and premise 3 is an *A* categorical proposition.

TABLE 11.—TWENTY-FOUR VALID SYLLOGISMS
[Four moods of categorical propositions: *A*, universal affirmation; *E*, universal negative; *I*, particular affirmation; *O*, particular negative.]

Figure			
First	Second	Third	Fourth
AAA	*AEE*	*AAI*	*AAI*
AAI	*AEO*	*AII*	*AEE*
AII	*AOO*	*EAO*	*AEO*
EAE	*EAE*	*EIO*	*IAI*
EAO	*EAO*	*IAI*	*EAO*
EIO	*EIO*	*OAO*	*EIO*

The following are some selected syllogisms for the student to ponder:

First figure (*SM*, *MP*, therefore *SP*), *AAA*:

All fowl are birds.
All birds are vertebrates.
Therefore all fowl are vertebrates.

First figure, *EAE*:

No birds are dogs.
All dogs are canine.
Therefore no birds are canine.

Second figure (*PM*, *SM*, therefore *SP*), *AOO*:

All dogs bark, "woof, woof!"
Some mammals do not bark, "woof, woof!"
Some mammals are not dogs.

Second figure, *EIO*:

No cats are pink.
Some fish are pink.
Some fishes are not cats.

Third figure (*MP*, *MS*, therefore *SP*), *AAI*:

All people are bipedal.
All people eat breakfast.
Therefore, some breakfast eaters are people.

Third figure, *OAO*:

Some butterflies are not green.
All butterflies have wings.
Therefore, some winged things are not green.

Fourth figure (*PM*, *MS*, therefore, *SP*), *AAI*:

All snakes are crawling things.
All crawling things slither along the ground.
Some slithering things are snakes.

Fourth figure, *IAI*:

Some whales are blue.
All blue things reflect blue light.
Therefore, some blue-light reflectors are whales.

Remember, practice makes perfect.

Concluding Remarks

The basics of logic have been introduced, but the challenge for the student is to become proficient in the various areas presented and then to move beyond. The only advice that I can offer is practice. Practice can help one to meet this challenge. In addition to the material presented in the present work, the student can use other sources. Of the many fine workbooks, some offer difficult problems and sometimes give their solutions. Also, many textbooks are available, but the student should be aware that they may specialize in small areas, such as mathematical proof, modal logic, and Boolean logic. A general text is probably best for the beginner. The student may also choose to evaluate news articles, or stories, or other writings of interest to determine how well the material encountered daily adheres to the laws of logic.

Glenn Research Center
National Aeronautics and Space Administration
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A physicist with an engineering background, the author presents a brief tutorial on logic. In his work at NASA and in his encounters with students, he has often found that a firm grounding in basic logic is lacking—perhaps because there are so many other demands on people that time simply cannot be taken to really examine the roots of human reasoning. This report provides an overview of this all-too-important subject with the dual hope that it will suffice insofar as it goes and that it will spur at least some to further study.

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