Discontinuous Galerkin Finite Element Method for Parabolic Problems

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H. Kaneko, K. S. Bey and G. J. W. Hou

Abstract

In this paper, we develop a time and its corresponding spatial discretization scheme, based upon the assumption of a certain weak singularity of $\|u_t(t)\|_{L_2(\Omega)} = \|u_t\|_2$, for the discontinuous Galerkin finite element method for one-dimensional parabolic problems. Optimal convergence rates in both time and spatial variables are obtained. A discussion of automatic time-step control method is also included.

Key words: Discontinuous Galerkin Method, Parabolic Equations.

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1 Introduction

In this paper, the following standard model problem of parabolic type is considered:

Find u such that

$$\begin{aligned} &u_t(x,t) - u_{xx}(x,t) = f(x,t), && x \in \Omega, t \in R^+, \\ &u(x,t) = 0, && x \in \partial\Omega, t \in R^+, \\ &u(x,0) = u_0(x), && x \in \Omega, \end{aligned} \tag{1.1}$$

where Ω is a closed and bounded set in R with boundary $\partial\Omega$, $R^+ = (0,\infty)$, $u_{xx} = \partial^2 u/\partial x^2$, $u_t = \partial u/\partial t$, and the functions f and u_0 are given data.

In this paper, we take advantage of a certain regularity of $\|u_t\|_2$, to discretize the time variable in a way that the optimal convergence rate of the discontinuous Galerkin (DG) method is obtained. In a series of papers [1], [2], [3] and [4], Eriksson and Johnson presented adaptive DG finite element methods in which time and space variables are adjusted using *a posteriori* estimates of the error. The current method is concerned with establishing *a priori* error estimate for the DG method which is optimal.

The following conditions will be assumed. Let (h, T, S) denote a finite element discretization satisfying

1. h is a positive function in $C^1(\bar{\Omega})$ such that

$$|\nabla h(x)| \leq M, \qquad \text{for all } x \in \bar{\Omega} \text{ and for some } M > 0.$$

2. $T = \{K\}$ is a set of triangular subdomain of Ω with each triangular element having diameter h_K such that

$$c_1 h_K^2 \leq \int_K dK \qquad \text{for all } K \in T,$$

   and associated with the function h through

$$c_2 h_K \leq h(x) \leq h_K, \qquad \text{for all } x \in K, K \in T,$$

   where $c_1 > 0$, $c_2 > 0$.

3. S is the set of all continuous functions on $\bar{\Omega}$ which are polynomials of order r in $x = (x_1, x_2)$ on each $K \in T$ and vanish on $\partial\Omega$.

For the DG method for (1.1), we partition R^+ as $0 = t_0 < t_1 < \cdots < t_n < \cdots$ where we let $I_n \equiv (t_{n-1}, t_n]$ with $k_n \equiv t_n - t_{n-1}$. For each time interval, with q a nonnegative integer, define

$$W_{hk} \equiv \{v: R^+ \to V_m: v|_{I_n} \in P_q(I_n), n = 1, \ldots N\},$$

where

$$V_m = \left\{ \begin{array}{l} \text{the space of piecewise linear splines} \\ \text{defined over } \Omega = [0, L] \text{ with breakpoints} \\ 0 = x_0 < x_1 < \cdots < x_m = L \\ \text{and } h_m = \max_{0 \leq i \leq m-1} x_{i+1} - x_i. \end{array} \right\}$$

and

$$P_q(I_n) = \{v(t) = \sum_{i=0}^{q} v_i t^i : v_i \in V_m\}.$$

Even though linear splines are used in the definition of V_m, splines of any order can be used to form V_m.

The DG method is defined as follows:

Find U such that for $n = 1, 2, \cdots$, $U|_{\Omega \times I_n} \in W_{hk}$ and

$$\int_{I_n} \{(U_t, v) + a(U, v)\} dt + ([U]_{n-1}, v_{n-1}^+) = \int_{I_n} (f, v) dt \qquad \text{for all } v \in W_{hk}, \tag{1.2}$$

where $[w]_n = w_n^+ - w_n^-$, $w_n^{+(-)} = \lim_{s\to 0^{+(-)}} w(t_n+s)$, $U_0^- = u_0$, $(u,v) = \int_\Omega u(x)v(x)dx$ and $a(u,v) = (\nabla U, \nabla v)$.

As stated earlier, the approach here is to first estimate $\|u_t\|_2$. For example, if $u_0(x) = \pi - x$ in (1.1) with $f(x,t) \equiv 0$ and $\Omega = (0,\pi)$, then the actual solution is given by

$$u(x,t) = \sum_{j=1}^{\infty} u_j^0 e^{-j^2 t} \sin(jx), \tag{1.3}$$

where

$$\begin{aligned} u_j^0 &= \tfrac{2}{\pi}\textstyle\int_0^\pi (\pi - x)\sin(jx)dx \\ &= \tfrac{2}{\pi}\{\tfrac{\pi}{j} - \tfrac{1}{j^2}\sin^2 j\pi\} \\ &= O(\tfrac{1}{j}). \end{aligned}$$

In the following, C's denote generic constants whose values change as they appear. From (1.3)

$$\|u_t(t)\|_2^2 = \|u_t(t)\|_{L_2(\Omega)}^2 = \sum_{j=1}^{\infty} Cj^2 e^{-2j^2 t} = \frac{d}{dt}\sum_{j=1}^{\infty} Ce^{-2j^2 t}. \tag{1.4}$$

The last equality in (1.4) is justified because of the uniform convergence in t of $\sum_{j=1}^{\infty} Ce^{-2j^2 t}$. Now using the fact that $\int_0^\infty e^{-x^2}dx < \infty$, a simple change of variables (say, $y = j\sqrt{2t}$) will show that the last expression in (1.4) is $\frac{d}{dt}Ct^{-1/2}$, which leads to

$$\|u_t(t)\|_2 = O(t^{-3/4}).$$

A similar argument shows that if $u_j^0 = O(\frac{1}{j^2})$ for some initial value function $u_0(x)$, then $\|u_t(t)\|_2 = O(t^{-1/4})$. This case arises when $u_0(x) = \min(x, \pi - x)$ for $x \in (0,\pi)$. If u_j^0 decays faster than $j^{-2.5}$ as $j \to \infty$, then $\|u_t(t)\|_2$ will be bounded as $t \to 0$, and in this case, time interval can be partitioned uniformly. Before we present the current method in the next section, one of the main theorems (theorem 2.3) from [2], which is pertinent to this paper, is recalled below.

Theorem 1.1 *(Eriksson and Johnson [2]) Let u be the solution of (1.1) and U that of (1.2). Assume that $V_m \subseteq V_{m-1}$ and $k_n \leq \gamma k_{n+1}$ for all n and for some $\gamma > 0$. Then there exists constants C only depending on c_1 and c_2 (see condition 2 above) such that for $q = 0, 1$, and $N = 1, 2, \ldots$,*

$$\|u - U\|_{I_n} \leq CL_N \max_{1\leq n\leq N} E_{qn}(u),$$

and for $q=1, N=1,2,\ldots,$

$$\|u(t_N)-U_N^-\|_2 \le CL_N \max_{1\le n\le N} E_{2n}(u),$$

where $L_N = (\log(t_N/k_N)+1)^{1/2}$,

$$E_{qn} = \min_{j\le q+1} k_n^j \|u_t^{(j)}\|_{I_n} + \|h_n^2 D^2 u\|_{I_n}, \qquad q=0,1,2 \tag{1.5}$$

with $u_t^{(1)} = u_t$, $u_t^{(2)} = u_{tt}$, $u_t^{(3)} = \Delta u_{tt}$ *and* $\|w\|_{I_n} = \max_{t\in I_n} \|w(t)\|_2$.

The term $\min_{j\le q+1} k_n^j \|u_t^{(j)}\|_{I_n}$ in (1.5) describes the error associated with discretization in time. Thus, if $\|u_t^{(j)}\|_{I_n}$ is bounded for each n and $j=1,2,3$, then the DG method is of the jth order accuracy in time. In many cases where the initial condition is incompatible with the boundary conditions, $\|u_t\|_{I_1}$ is unbounded as $t\to 0^+$. We demonstrate in Section 2 that the graded time partitions can be established to restore the optimal rates of convergence even in the presence of non-smooth data. It should be pointed out that similar non-uniform graded discretization schemes were used in capturing the solutions of a class of weakly singular integral equations, -e.g., [8], in capturing the solution of parabolic integro-differential equations with memory term, -e.g., [9], in capturing the solution of a class of initial value problems, -e.g.,[13] as well as in capturing the solution of linear parabolic equations in the h-version of the DG finite element method, -e.g., [12]. Some other notable papers in this topic are, -e.g., [5], [10] and references cited therein. The recent paper by Schötzau and Schwab [12] is of particular interest here, since it also deals with the h-version of the DG method for parabolic problems with non-smooth data, particularly the topic covered in section 5.2 of [12]. Schötzau and Schwab derive a graded time partition scheme by analyzing a function defining the initial condition. More specifically, u_0 in (1.1) is examined by the K-method of interpolation as a function which belongs to an intermediate space between H^2, the Sobolev space, and L_2 and the graded time partitions are chosen accordingly. In this paper, graded time partitions are derived in Section 2 by analyzing the regularity of $\|u_t\|_2$. The difference between the current approach and that of Schötzau and Schwab in [12] manifests itself in an example (Section 3) in which we demonstrate that the current method gives rise to more relaxed time partitions. The paper [12] by Schötzau and Schwab is a must-read for anyone who is interested in the DG finite element method for parabolic problems. The paper goes far beyond what are explored in this paper, which deals exclusively with the h-version of the DG method. In Section 3, a numerical example is given to demonstrate the validity of our theory. Also included in Section 3 is the discussion of automatic time step control algorithm which is based upon the results presented in Section 2. The term $\|h_n^2 D^2 u\|_{I_n}$ in (1.5) describes the spatial discretization error and it is of the second order due to the use of linear splines in defining the space V_m as long as $\|D^2 u(t)\|_{I_n}$ is well behaved. Unfortunately, for solutions of parabolic problems, this is not always the case. We investigate the behaviour of $D^2 u$ and establish in Section 2 the spatial discretization scheme that achieves the quadratic convergence in space variable.

2 Discontinuous Galerkin Method for Parabolic Problems

Consider one dimensional parabolic problem (1.3). Even though the method will be outlined in one dimensional setting, extensions to higher dimensions are possible and straightforward.

For $0 < \alpha < 1$ and q a nonnegative integer, define an index of singularity $Q \equiv \frac{q+1}{1-\alpha}$. For a positive integer N and $T > 0$, let

$$t_n^* = (\frac{n}{N})^Q, \qquad n = 0, 1, \ldots, N$$

and

$$t_n = t_n^* T. \tag{2.1}$$

Define $I_n = (t_{n-1}, t_n]$, $\quad n = 1, 2, \ldots, N$, and let k_n denote the length of I_n so that

$$k_n = [(\frac{n}{N})^Q - (\frac{n-1}{N})^Q]T, \qquad n = 1, 2, \ldots N.$$

Note that

$$k_n \leq Q[\frac{n}{N}]^{Q-1}\frac{1}{N}T \tag{2.2}$$

by the mean value theorem.

The solution $u(x,t)$ of (1.1) is then approximated in t over each I_n by a polynomial of degree q. For instance, with $q = 1$, let $P_1 w$ denote the linear interpolatory projection of $w \in H_0^2 = \{v \colon D^{(j)} v \in L_2, j = 0, 1, 2; v \equiv 0 \,\text{on}\, \partial\Omega\}$ in time onto W_{hk}, viz,

$$P_1 w(x,t) = \frac{t_n - t}{k_n} w(x, t_{n-1}) + \frac{t - t_{n-1}}{k_n} w(x, t_n), \qquad \text{for each } t \in I_n.$$

Note that P_1 is bounded with respect to the norm $\|\cdot\|_{\infty, I_n}$ where

$$\|w(t)\|_{\infty, I_n} \equiv \max_{t \in I_n} \|w(t)\|_{L_\infty(\Omega)}.$$

Since Ω is assumed to be bounded, P_1 is bounded with respect to $\|\cdot\|_{I_n}$ also. Since P_1 equals the identity on polynomials of degree ≤ 1, examining the error terms in the Taylor expansion of order 0 and 1 respectively, we obtain, for each $n = 1, 2, \ldots, N$,

$$\|u - P_1 u\|_{I_n} \leq \int_{I_n} \|u_t(t)\|_2 dt, \tag{2.3}$$

and

$$\|u - P_1 u\|_{I_n} \leq C k_n^2 \|u_{tt}\|_{I_n}. \tag{2.4}$$

For higher values of q, carrying out the Taylor expansion to the order q, the best possible estimate that can be achieved for the projection $P_q : H_0^2 \to W_{hk}$ is given by

$$\|u - P_q u\|_{I_n} \leq \min\{\int_{I_n} \|u_t(t)\|_2 dt, C k_n^{q+1} \|\frac{\partial^{q+1}}{\partial t^{q+1}} u\|_{I_n}\}. \tag{2.5}$$

The type of estimate obtained in (2.5) plays a useful role in bounding the error in time variable for the DG finite element method for parabolic equations. The first term on the right of (2.5) is used to capture the error during the transient phase of the solution, while the second term is used for the solution later in time, upon which $\|u_t\|_2$ is expected to be smooth. The following lemma examines the first term on the right hand of (2.5). In [9], for integro-differential parabolic equations with memory, a similar singular behaviour in solution is treated. However, the analysis used in [9] restricts the construction of graded time partitions to constant and linear cases only.

Lemma 2.1 *Let $0 < \alpha < 1$, q a nonnegative integer and $T > 0$, we assume that t_n, $n = 1, \ldots, N$ are defined by (2.1). Then*

$$\int_{I_1} s^{-\alpha} ds \leq C \frac{1}{N^{q+1}},$$

where C is a constant independent on N and for $n > 1$,

$$\max_{1 < n \leq N} \int_{I_n} s^{-\alpha} ds \leq C \frac{1}{N},$$

where C is a constant independent on N.

Proof: For $n = 1$,

$$\int_{I_1} s^{-\alpha} ds = \int_0^{(\frac{1}{N})^Q T} s^{-\alpha} ds = \frac{1}{1-\alpha} [(\frac{1}{N})^Q T]^{1-\alpha} = O((\frac{1}{N})^{q+1}).$$

For $1 < n \le N$, we obtain

$$\begin{aligned}
\int_{I_n} s^{-\alpha} ds &\le \int_{I_n} [(\tfrac{n-1}{N})^Q T]^{-\alpha} ds && \text{as } s^{-\alpha} \text{ is decreasing over } I_n, \\
&= T^{-\alpha} (\tfrac{n-1}{N})^{-\alpha Q} [(\tfrac{n}{N})^Q - (\tfrac{n-1}{N})^Q] \\
&\le C (\tfrac{n-1}{N})^{-\alpha Q} (\tfrac{n}{N})^{Q-1} (\tfrac{n}{N} - \tfrac{n-1}{N}) && \text{by the mean value theorem} \\
&= C (\tfrac{1}{N})^{Q-\alpha Q} \tfrac{n^{Q-1}}{(n-1)^{\alpha Q}} = C \tfrac{1}{N^{q+1}} (\tfrac{n}{n-1})^{\alpha Q} n^{Q-1-\alpha Q} \\
&= C \tfrac{n^q}{N^{q+1}} \le C \tfrac{N^q}{N^{q+1}} = C \tfrac{1}{N}.
\end{aligned}$$

Q.E.D.

Using Lemma 2.1 with $n = 1$ and $q = 1$, the first term on the right of inequality (2.5) is of order $O(\frac{1}{N^2})$, provided that $\|u_t(t)\|_2 = O(t^{-\alpha})$ for some $\alpha \in (0, 1)$ and for $t \in I_1$. As stated earlier, it is reasonable to expect that $\|u_{tt}\|_{I_n} < \infty$ for $n > 1$, so that (2.5) yields

$$\|u - P_1 u\|_{I_n} = O(\frac{1}{N^2}), \qquad \text{for each } n = 1, \ldots N. \tag{2.6}$$

Lemma 2.2 *Let t_n and k_n be defined by (2.1). Then, with any positive integer N,*

$$(1 + \log \frac{t_n}{k_n})^{1/2} \le \sqrt{2}, \qquad \textit{for each } n = 0, 1, \ldots, N.$$

Proof:

$$\begin{aligned}
(1 + \log \tfrac{t_n}{k_n})^{1/2} &= (1 + \log \tfrac{(n/N)^Q}{(n/N)^Q - (n-1/N)^Q})^{1/2} \\
&= (1 + \log \tfrac{1}{1-(\frac{n-1}{n})^Q})^{1/2} \\
&= (1 - \log(1 - (\tfrac{n-1}{n})^Q))^{1/2} \\
&\le (1 - \log(1 - (\tfrac{N-1}{N})^Q)^{1/2} \\
&\le (1 + (\tfrac{N-1}{N})^Q)^{1/2} \le \sqrt{2}.
\end{aligned}$$

The second to the last inequality is obtained from $\log(1 - x) < -x$ for $x < 1$.
Q.E.D.

Lemma 2.2 will be used in Theorem 1.1 to guarantee the stability of the DG method currently proposed.

For spatial discretization, let M be a positive integer and let x_k for $k = 0, 1, \ldots, 2^M$ denote the spatial knots which will be defined precisely later in this section. Denote by $\varphi_k(x)$ the linear spline over $\Omega_k = [x_{k-1}, x_{k+1}]$ for $k = 1, 2, \ldots, 2^M - 1$ and $\varphi_0(x)$ and $\varphi_{2^M}(x)$ are linear splines on $[x_0, x_1]$ and $[x_{2^M-1}, x_{2^M}]$ respectively. Extensions to higher order splines are straightforward. At each time level t_n, we approximate $u(t_n, x)$ by

$$U^n = U^n(x) = U(t_n, x) = \sum_{i=0}^{2^M} \xi_i^n \varphi_i(x), \qquad n = 0, 1, \ldots, N,$$

where $\xi_i^0 = u_0(x_i)$. Now, equations (1.2) can be restated as follows: For $n = 1, 2, \ldots, N$, given $U^{n-1,-}$, find $U \equiv U|_{I_n} \in P_q(I_n)$ such that

$$\int_{I_n} [(U_t, v) + a(U, v)]dt + (U^{n-1,+}, v^{n-1,+}) = \int_{I_n} (f, v)dt + (U^{n-1,-}, v^{n-1,+}) \tag{2.7}$$

for all $v \in P_q(I_n)$ where $U^{0,-} = u_0$.

For demonstration, let us take a detour to see how equations in (2.7) take specific forms for $q = 0$ and $q = 1$. First, consider the case $q = 0$, -i.e., constant in time. As $U^n = U^{n,-} = U^{n-1,+}$ in this case, with $U^n = \sum_{i=0}^{2^M} \xi_i^n \varphi_i(x)$, (2.7) reduces to

$$\sum_{i=0}^{2^M} \xi_i^n [(\varphi_i, \varphi_j) - k_n a(\varphi_i, \varphi_j)] = \sum_{i=0}^{2^M} \xi_i^{n-1} (\varphi_i, \varphi_j) + (f, \varphi_j), \quad j = 0, 1, \ldots, 2^M. \tag{2.8}$$

For $q = 1$, we let $U|_{I_n} = \bar{\varphi}_n(x) + \frac{t-t_{n-1}}{k_n} \bar{\psi}_n(x)$ where $\bar{\varphi}_n = \sum_{i=0}^{2^M} \xi_i^{\bar{\varphi},n} \varphi_i(x)$ and $\bar{\psi}_n = \sum_{i=0}^{2^M} \xi_i^{\bar{\psi},n} \varphi_i(x)$. As $U^{n-1,+} = \bar{\varphi}_n$ and $U^{n-1,-} = \bar{\varphi}_{n-1} + \bar{\psi}_{n-1}$, (2.7) becomes

$$\begin{aligned} &\textstyle\sum_{i=0}^{2^M} \xi_i^{\bar{\varphi},n} \{(\varphi_i, \varphi_j) + k_n a(\varphi_i, \varphi_j)\} + \sum_{i=1}^{2^M} \xi_i^{\bar{\psi},n} \{(\varphi_i, \varphi_j) + \frac{k_n}{2} a(\varphi_i, \varphi_j)\} \\ &\quad = \textstyle\int_{I_n} (f(t), \varphi_j) + \sum_{i=0}^{2^M} [\xi_i^{\bar{\varphi},n-1} + \xi_i^{\bar{\psi},n-1}](\varphi_i, \varphi_j), \qquad j = 0, 1, \ldots, 2^M \\ &\textstyle\sum_{i=0}^{2^M} \xi_i^{\bar{\varphi},n} \frac{k_n}{2} a(\varphi_i, \varphi_j) + \sum_{i=1}^{2^M} \xi_i^{\bar{\psi},n} \{\frac{1}{2}(\varphi_i, \varphi_j) + \frac{k_n}{3} a(\varphi_i, \varphi_j)\} \\ &\quad = \textstyle\frac{1}{k_n} \int_{I_n} (t - t_{n-1})(f(t), \varphi_j)dt, \qquad j = 0, 1, \ldots, 2^M \end{aligned} \tag{2.9}$$

The following theorem is a generalization of Theorem 1.1. A proof can be obtained by applying minor changes in the proof of theorem 2.3, [2].

Theorem 2.3 *Suppose that there is a constant γ such that time steps k_n satisfy $k_n \leq \gamma k_{n+1}$, $n = 1, \ldots, N-1$ and let U_n denote the solution of (1.2) approximating u at t_n. Here u is approximated by a polynomial of degree $q \geq 0$ over each I_n for $n = 1, \ldots, N-1$. Then there is a constant C depending only on γ and a constant β, where $\rho_K \geq \beta h_K$ and ρ_K is the diameter of the circle inscribed in K for all $K \in T$, such that for $n = 1, 2, \ldots, N$,*

$$\|u(t_n) - U_n\|_2 \leq C(1 + \log \frac{t_n}{k_n})^{1/2} \{\max_{m \leq n} \|u - P_q u\|_{I_m} + \|h_n^2 D^2 u\|_{I_n}\}. \tag{2.10}$$

We proceed to examine the assumptions of Theorem 2.3 relative to the current time and space discretization scheme. First of all, the condition $k_n \le \gamma k_{n+1}$ is satisfied with $\gamma = 1$ as k_n is increasing. Second, for one-dimensional problem (1.1), $h_K = \rho_K = |\Omega_K|$ so that $\frac{\rho_K}{h_K} = 1$ for all Ω_K. Hence the assumptions are fulfilled under the current time discretization scheme. Lemma 2.2 guarantees the uniformly bounded property of $(1 + \log \frac{t_n}{k_n})^{1/2}$, establishing the stability of the current DG method.

By (2.5) along with Lemma 2.1, with any nonnegative integer q, we obtain, provided that $\|u_t(t)\|_2 = O(t^{-\alpha})$ for some $0 < \alpha < 1$,

$$\|u - P_q u\|_{I_n} = O(\frac{1}{N^{q+1}}). \tag{2.11}$$

It remains to examine the spatial discretization error term $\max_{t \le t_n} \|h_n^2 D^2 u(t)\|_{I_n}$ in (2.10) for the current problem. If $\|D^2 u(t)\|_{I_n}$ is bounded for each n and for each t, then using the graded time partitions described in Lemma 2.1, Theorem 2.3 transforms to the following:

Theorem 2.4 *Consider the parabolic problem (1.1) and assume that the initial value $u_0(x)$ is defined in such a way that $\|u_t(t)\|_2 = O(t^{-\alpha})$, for $0 < \alpha < 1$. Denote by U_n the solution of (1.2) approximating u at t_n. Let time discretization $\{t_n\}$ be defined by (2.1) and $(0, \pi)$ is divided into 2^M subintervals each of equal length. Also assume $\|D^2 u(t)\|_{I_n}$ is bounded for each n. If q denotes the degree of polynomials used in approximating u in time variable, then for each $n = 1, 2, \ldots, N$,*

$$\|u(t_n) - U_n\|_2 = O(\frac{1}{N^{q+1}} + \frac{1}{2^{2M}}).$$

In many practical problems, the assumption in the previous theorem that $\|D^2 u(t)\|_{I_n}$ is bounded for each n and t may not be possible, -e.g., examine the solution of one dimensional problem in (1.3). This topic is now considered.

For elliptic problem,

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma, \end{aligned}$$

it is well known (cf. see [6], p.92) that, for a smooth $\partial\Omega$,

$$\|u\|_{H^2(\Omega)} \le C\|f\|_2. \tag{2.12}$$

Now for convenience, let us assume that $f = 0$ in (1.1). Then using the definitions of the respective norms and from (1.1) and (2.12), we obtain

$$\|D^2 u(t)\|_{I_n} \le \|u(t)\|_{H^2(\Omega)} \le C\|u_t(t)\|_2. \tag{2.13}$$

If $\|u_t(t)\|_2$ is bounded as $t \to 0^+$, then spatial discretization can be made arbitrarily with step size h and attain

$$\max_{t \le t_n} h^2 \|u_t(t)\|_2 = O(h^2), \qquad \text{where } h = 2^{-M} \text{ at each time level } t_n.$$

In the case, $\|u_t(t)\|_2 = O(t^{-\alpha})$, (2.13) implies that we must select a set of knots $\{x_k(t)\}$ that depends upon t. The approach here is similar to the adaptive strategy employed by Eriksson and Johnson [2] in which the spatial increment h_n is selected according to the size of $\|u_t(t)\|_{\infty, I_n}$. Let $h(t) = \max_{1 \le k \le 2^M}(x_k(t) - x_{k-1}(t))$ for each $t \in (0, T]$. Then $h(t)$ is determined from the condition that $h(t)^2 t^{-\alpha} = O(t^2)$ as $t \to 0+$. In terms of N, we require that $h(t)^2 t^{-\alpha} = O(\frac{1}{N^2})$. For $t = t_j = (\frac{j}{N})^{\frac{q+1}{1-\alpha}} T$, $j = 1, \ldots, N$, we have, for some $C > 0$,

$$h^2(t_j)[(\frac{j}{N})^{\frac{q+1}{1-\alpha}} T]^{-\alpha} = C \frac{1}{N^2}. \tag{2.14}$$

Solving (2.14) for $h(t_j)$ with $C = 1$ for convenience,

$$h(t_j) = [\frac{j^{\alpha Q}}{N^{2+\alpha Q}} \cdot T^{\alpha}]^{1/2}. \tag{2.15}$$

Finally, select M so that

$$2^M > \frac{L}{h(t_j)} \tag{2.16}$$

for each time level t_j, $j = 1, \ldots, N$, and $x_k(t_j) = \frac{kL}{2^M}$ (note that M depends upon j) , for $k = 1, \ldots, 2^M - 1$ to form the spatial partition points. Reflecting on what was just discussed, we finally have the following theorem which characterizes the convergence of the DG method both in time and in space variables in terms of N.

Theorem 2.5 *Consider the parabolic problem (1.1) and assume that the initial value $u_0(x)$ is defined in such a way that $\|u_t(t)\|_2 = O(t^{-\alpha})$, for $0 < \alpha < 1$. Denote by U_n the solution of (1.2) approximating u at t_n. Let time discretization $\{t_n\}$ be defined by (2.1) and $(0, \pi)$ is divided into 2^M subintervals each of equal length, where M is defined by (2.16). If q denotes the degree of polynomials used in approximating u in time variable, then for each $n = 1, 2, \ldots, N$,*

$$\|u(t_n) - U_n\|_2 = O(\frac{1}{N^{q+1}} + \frac{1}{N^2}). \tag{2.17}$$

Note that equality (2.15) implies that less partition points in space would be required as $j \to N$. This characterizes the situation described previously typical of parabolic problems in which solutions become smoother after certain transient periods. Also note that, at each time level t_j particularly for t near 0, it is possible to reduce computational cost by relaxing the size of spatial elements over the regions where $u(\cdot, t_j)$ is well behaved. This will be made more evident when a numerical example is discussed in the next section.

3 A Numerical Example and Automatic Time Step Control

In this section, the graded partitions described in the previous section is tested in the following standard one dimensional problem. Find u such that

$$\begin{aligned} &u_t(x,t) - u_{xx}(x,t) = 0, && 0 < x < \pi, t \in R^+, \\ &u(0,t) = u(\pi,t) = 0, && t \in R^+, \\ &u(x,0) = \pi - x, && 0 < x < \pi \end{aligned} \tag{3.1}$$

The initial function is incompatible with the zero boundary conditions in (3.1). The exact solution is given in (1.3) and $\|u_t(t)\|_{L_2(\Omega)} = O(t^{-\frac{3}{4}})$. Thus, the indices of singularity $Q = \frac{q+1}{1-\alpha}$ are 4 and 8 respectively for constant and linear approximations in time variable. Schötzau and Schwab (example 7.1 [12]) discuss a similar situation in which the initial function is taken to be $u(x,0) = 1$. For this case, the grading function h is selected as $h(t) = t^{3(2q+3)}$ in [12]. With N denoting, as before, the number of partitions in time variable, Schötzau and Schwab's approach results in $t_1 = (\frac{1}{N})^9$ and $t_1 = (\frac{1}{N})^{15}$ for the constant and linear approximations respectively in [12]. In the present method, as $\|u_t(t)\|_{L_2(\Omega)} = O(t^{-\frac{3}{4}})$ with $u(x,0) = 1$, $t_1 = (\frac{1}{N})^4$ and $t_1 = (\frac{1}{N})^8$ for the constant and linear approximations. This shows that the present method provides more relaxed time increments at the beginning, which is critically important in order for numerical computation to proceed with a reasonable cost.

We select $T = 5$ and $N = 10$ so that the solution is sampled at each time level $t_j = (\frac{j}{10})^Q T$, $j = 1, \ldots, 10$. First, the case for constant approximation in time, $q = 0$, is considered. The exact solution and the approximate solutions are plotted in Appendix for t_n, $t_n = 1, \ldots, 9$. For $q = 0$, the errors are dominated from the time discretization term which is of order $O(\frac{1}{N})$. In Fig. 1.1 with t_1, from (2.15), $h(t_1) \simeq .0058$. Hence we take $M \sim 9$ in (2.16) to guarantee the convergence rates in Theorem 2.5. It is interesting to note that, for the example under consideration, we can relax M significantly while maintaining an overall quality of the numerical solution. Also recall that the size of spatial discretization can be relaxed over the region away from $x = 0$ in this case where $u(x, t_1)$ is well-behaved, but, of course, this requires *a priori* knowledge of $u(x, t_1)$. For $q = 1$, $h(t_1) \simeq .0002$ from (2.15) so that choose $M = 14$. In this case, (2.17) gives

$$\|u(t_1) - U_1\|_2 = O(\frac{1}{N^2}).$$

Table 1 below lists L_2 errors at each time level t_n, $n = 1, 2, \ldots, 10$.

	Table 1	
n	$q = 0$	$q = 1$
1	.0133	.0037
2	.0571	.0020
3	.0673	.0089
4	.0729	.0087
5	.0763	.0086
6	.0780	.0081
7	.0878	.0076
8	.0933	.0072
9	.0702	.0059
10	.0850	.0073

We close this paper by considering an automatic time step control scheme for the DG method for parabolic problems having non-smooth data. Johnson considered a similar automatic time step control method for a class of stiff ordinary differential equations in [7]. The result was extended to an automatic time control method for a nonlinear parabolic problem in [1]. Let us recall the following algorithm in [1] for choosing the time step size k_n:

Algorithm:[1] Given a tolerance $\delta > 0$,

1. Choose $k_n = k_{n-1}$,

2. Given k_n, compute the corresponding approximate solution $U(t_n)$.

3. If

   $$\frac{\delta}{\gamma C} \leq \|U(t_n) - U(t_{n-1})\|_2 \leq \frac{\delta}{C},$$

   where γ is a suitable constant, $\gamma \approx 2$ or 3, then stop and accept the time step k_n. Otherwise, decrease or increase k_n by a factor of, say, 2 and return to step 2.

To demonstrate that the algorithm above may become expensive in the presence of a singular transient phase, consider the following: Suppose that $\delta = 10^{-5}$ and $\|u_t(t)\|_2 = O(t^{-1/2})$. Approximating $\|U(t_1) - U(t_0)\|_2$ by $t_1^{1/2}$, and decreasing the time step size by a factor of 2 beginning with $k_1 \equiv \frac{1}{10}$, it would take 8 steps to find k_1 which produces the numerical solution within the prescribed tolerance. The current approach is to estimate the parameter α in $\|u_t(t)\|_2 = O(t^{-\alpha})$. The following algorithm contains a process of adjusting each time step (step 4), if necessary, as computation progresses, but, in principle, if α is estimated accurately in step 1, the amount of adjustments would be minimal.

Algorithm: Given a tolerance $\delta > 0$,

1. Choose $t_1 = k_1$ small. Compute $\|U(t_1) - U(0)\|_2$. As
$$\|u_t(t_1)\|_2 \simeq \|\frac{U(t_1) - U(0)}{k_1}\|_2 \simeq Ck_1^{-\alpha},$$
so α is estimated by
$$\alpha^* = 1 - \frac{\ln \|U(t_1) - U(0)\|_2}{\ln Ck_1}. \tag{3.2}$$

2. Time discretization error is controlled by requiring
$$\int_{I_1} \|u_t(t)\|_2 dt < \delta.$$
t_1 is adjusted by
$$\frac{C}{1-\alpha^*} t_1^{1-\alpha^*} < \delta,$$
and subsequently choose the number N of time partitions from
$$t_1 \simeq (\frac{1}{N})^{\frac{1+q}{1-\alpha^*}} T,$$
where q is the degree of polynomials used in time approximation.

3. With $Q^* \equiv \frac{1+q}{1-\alpha^*}$, define
$$t_i = (\frac{i}{N})^{Q^*} T, \qquad i = 2, 3, \ldots N,$$
to complete the partitions $0 < t_1 < t_2 < \cdots t_N = T$.

4. For $i \geq 2$, compute $U(t_i)$ and check an accuracy requirement
$$\|U(t_i) - U(t_{i-1})\|_2 < \delta.$$
If this inequality is satisfied, then proceed to compute $U(t_{i+1})$. If not, adjust t_{i+1} by decreasing $k_{i+1} = t_{i+1} - t_i$ by the factor of, say, 2.

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Appendix

Fig 1.1 Discontinuous Galerkin Finite Element Method for Parabolic Problem – constant time

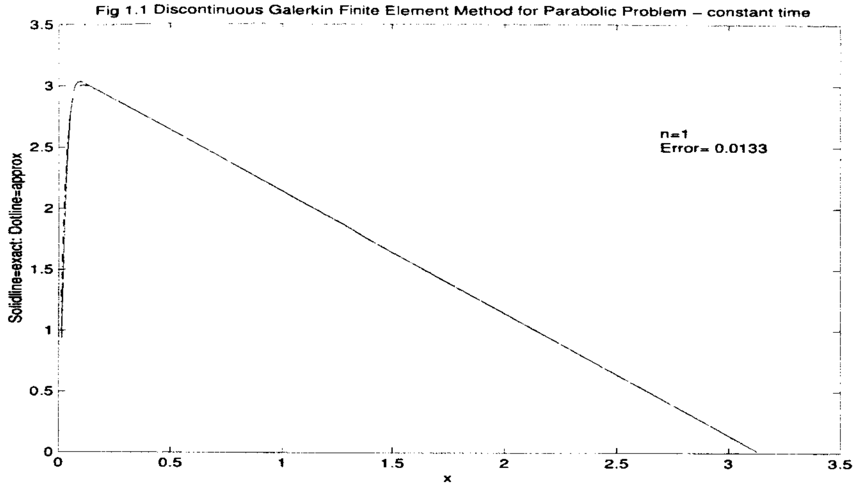

Fig 1.2 Discontinuous Galerkin Finite Element Method for Parabolic Problem – constant time

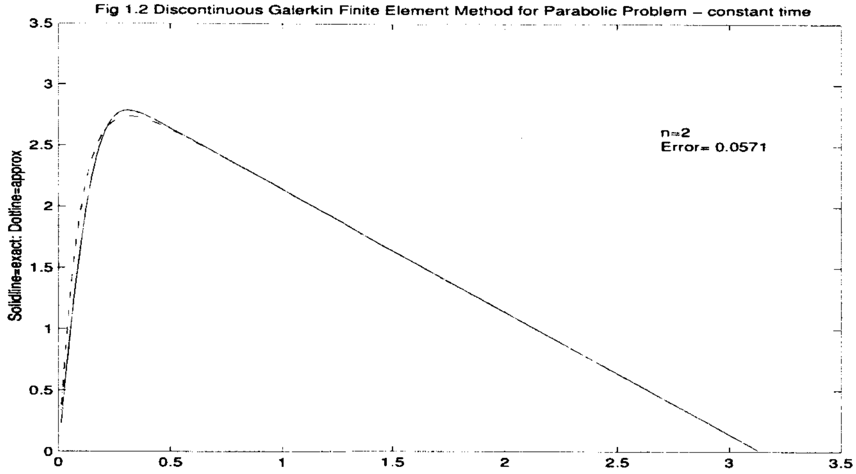

Fig 1.3 Discontinuous Galerkin Finite Element Method for Parabolic Problem – constant time

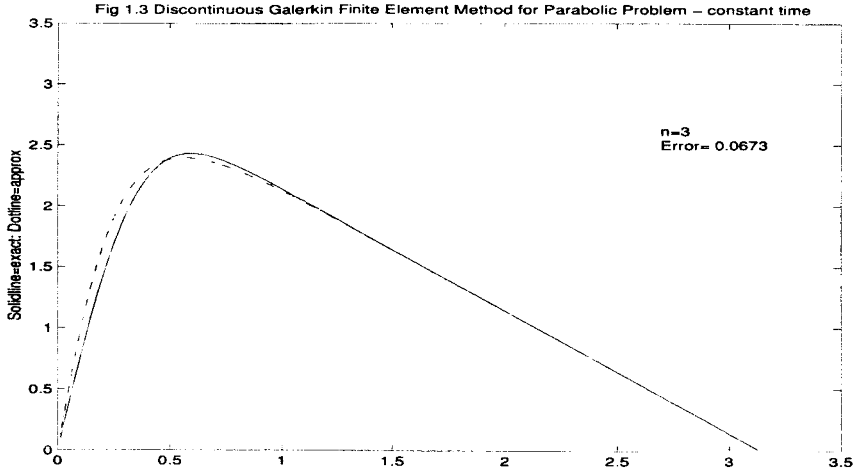

Fig 1.4 Discontinuous Galerkin Finite Element Method for Parabolic Problem – constant time

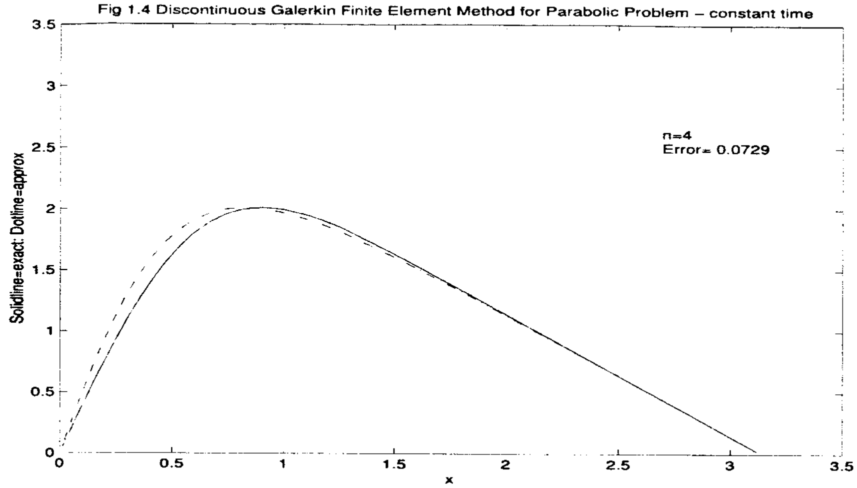

Fig 1.5 Discontinuous Galerkin Finite Element Method for Parabolic Problem – constant time

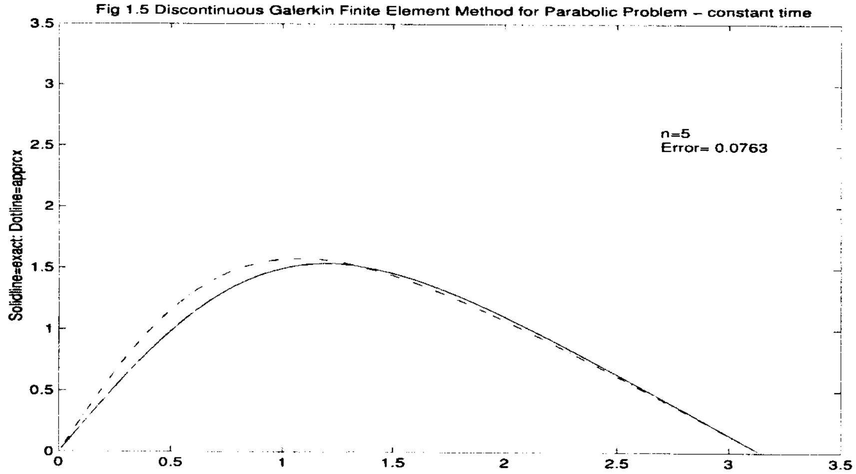

Fig 1.6 Discontinuous Galerkin Finite Element Method for Parabolic Problem – constant time

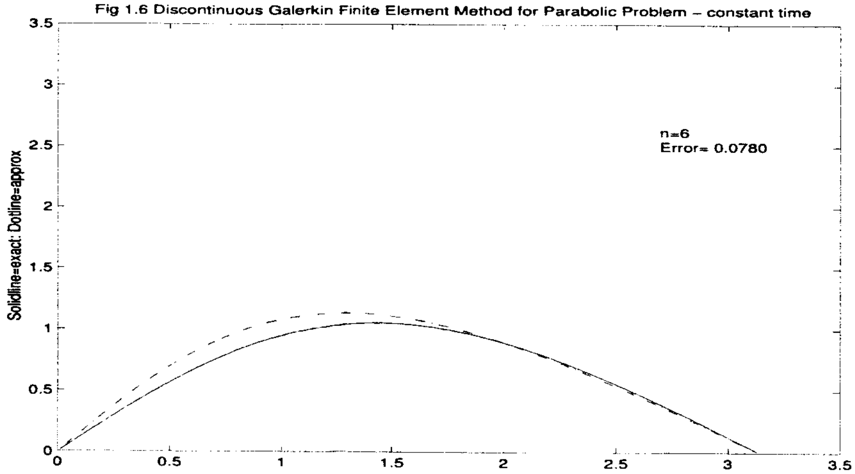

Fig 1.7 Discontinuous Galerkin Finite Element Method for Parabolic Problem – constant time

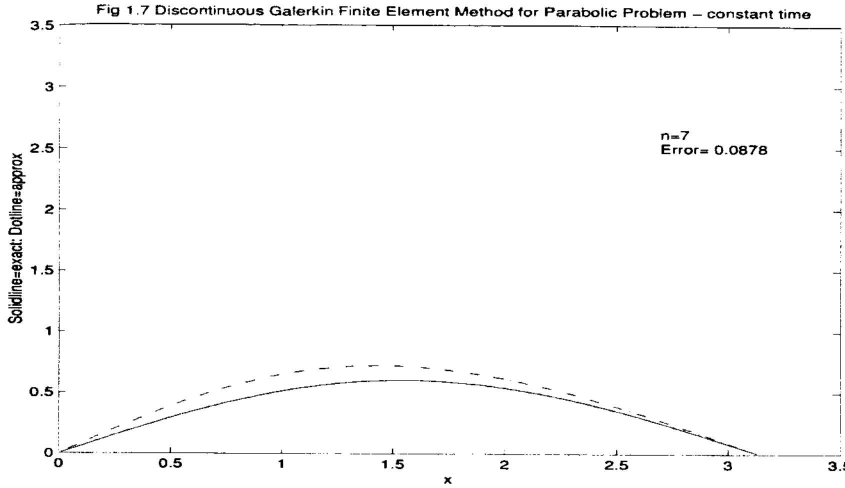

Fig 1.8 Discontinuous Galerkin Finite Element Method for Parabolic Problem – constant time

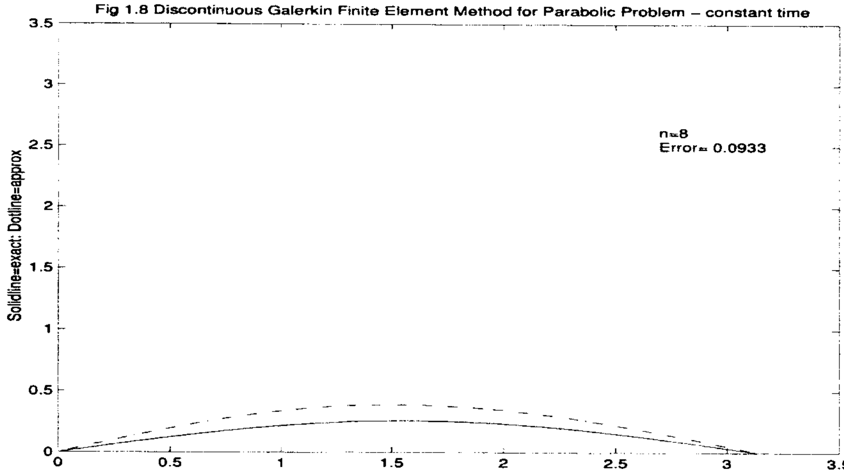

Fig 1.9 Discontinuous Galerkin Finite Element Method for Parabolic Problem – constant time