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STOCHASTIC CONTROL SYNTHESIS OF SYSTEMS WITH STRUCTURED UNCERTAINTY

Luis G. Crespo*

ABSTRACT

This paper presents a study on the design of robust controllers by using random variables to model structured uncertainty for both SISO and MIMO feedback systems. Once the parameter uncertainty is prescribed with probability density functions, its effects are propagated through the analysis leading to stochastic metrics for the system's output. Control designs that aim for satisfactory performances while guaranteeing robust closed loop stability are attained by solving constrained non-linear optimization problems in the frequency domain. This approach permits not only to quantify the probability of having unstable and unfavorable responses for a particular control design but also to search for controls while favoring the values of the parameters with higher chance of occurrence. In this manner, robust optimality is achieved while the characteristic conservatism of conventional robust control methods is eliminated. Examples that admit closed form expressions for the probabilistic metrics of the output are used to elucidate the nature of the problem at hand and validate the proposed formulations.

Keywords: Robust control, probabilistic methods, structured uncertainty, robust optimization.

1 INTRODUCTION

The main requirement of feedback control is to achieve acceptable levels of performance in the presence of uncertainty. Fundamental trade offs and compromises between these two aspects motivate the entire body of feedback theory. While performance concerns aspects such as reference tracking, disturbance rejection, bounded control effort, etc., uncertainty appears as a result of the inevitable discrepancies between the physical problem and its deterministic mathematical model. Ignorance on the system's exact dynamics, on the actual operating conditions and the purposeful choice of a simplified representation of the physical problem exemplify this aspect. In this context, uncertainty can be classified as *structured* (or *parametric*) and *unstructured* [16]. The first kind corresponds to inaccuracies on the parameters of the model while the second one corresponds to unmodeled dynamics.

Uncertainty can be modeled in many ways depending upon the desired quality of its mathematical description. Differential sensitivity, multi-models, interval analysis, perturbations, fuzzy sets and probabilistic methods have been used [2, 7, 21]. The effects of uncertainty on the stability associated with the prescribed control solutions have been studied by both deterministic [13] and stochastic [8, 17, 18, 19] means. These analysis tools however, have not been integrated to the control design process.

The methods most commonly used for robust control design are μ-synthesis [1, 4, 9, 15] and H_∞ optimization [3, 6, 14, 20]. In these, uncertainty is modeled with norm-bounded

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complex perturbations of fixed but arbitrary structure about its nominal form. This treatment is extensively used primarily because it leads to a tractable set of sufficient conditions for robust stability. Such approach however, have the following drawbacks: (i) the crudeness of the uncertainty description usually leads to redundant and physically impossible plants, then to highly conservative designs, (ii) it is not feasible to favor scenarios with higher chance of occurrence among all the possible ones, and as a result, robust optimality is precluded, (iii) a quantitative description of the robustness of the solution is unattainable and (iv) the resulting controllers are so complex that model reduction techniques are usually required. While such perturbations account for unstructured uncertainty coarsely, an augmented plant model with structured uncertainty can be used to conciliate the uncertainty representation with the physics of the problem. While robust optimization has been studied in various disciplines using different uncertainty models [5, 10, 11], stochastic control synthesis remains, to a large extent, unexplored.

This paper studies the control design of plants with structured uncertainty for both single-input-single-output (SISO) and multiple-input-multiple-output (MIMO) systems using a probabilistic approach. The joint probability-density-function (PDF) of the parameters is prescribed *a priori*, and then propagated, leading to a probabilistic description of the metrics of the controlled response. Control design, involving decoupling, performance and stability aspects, is carried out by solving constrained non-linear optimization problems in the frequency domain. The content of this paper is organized as follows. Background on the performance requirements for feedback SISO and MIMO systems is presented in Section 2. Section 3 introduces closed loop stability considerations and the concept of *robust stability* in the probabilistic sense. Several design strategies, based on the minimization of a frequency dependent cost function, are proposed in Section 4 and evaluated in Section 5. A discussion on the developments and some conclusions, are presented in Sections 6 and 7.

2 CONTROL REQUIREMENTS

Consider a linear-time-invariant (LTI) system with transfer function matrix $G(s, \mathbf{p})$, where $\mathbf{p}$ is a vector of random variables corresponding to the uncertain parameters. Let $\mathbf{p}$ be prescribed *a priori* by the joint probability density function $f_{\mathbf{P}}(\mathbf{p})$. In this notation, $f_{\mathbf{P}}(\tilde{\mathbf{p}})$ is the PDF of $\mathbf{p}$ at $\mathbf{p} = \tilde{\mathbf{p}}$. The *support* of $f_{\mathbf{P}}(\mathbf{p})$, denoted with Δ, is the set of values the random vector $\mathbf{p}$ can take, i.e. $\tilde{\mathbf{p}} \in \Delta$. Note that the propagation of Δ leads to a family or a set of uncertain plant models in which the physical system is assumed to reside. The probability of occurrence of a plant within this set is fully determined by the structure of G and the above mentioned PDF. If the components of $\mathbf{p}$ are independent, they can be prescribed individually. Let $K(s, \mathbf{k})$ be the transfer function matrix of a controller, where $\mathbf{k}$ is a vector of gains to be determined. Control requirements for SISO and MIMO systems are presented next.

2.1 SISO Systems

Let $L(s, \mathbf{p}, \mathbf{k}) \equiv GK$ be the well-known *loop transfer function.* For the one degree-of-freedom system shown in figure 1, where r, u, z, d and n stand for reference, control, output, disturbance and noise signals; the following equations apply

$$z = Tr + Sd - Tn \tag{1}$$
$$e \equiv z - r = -Sr + Sd - Tn \tag{2}$$
$$u = KS(r - d - n) \tag{3}$$

where $S \equiv (I + L)^{-1}$ is the *sensitivity function* and $T \equiv SL = 1 - S$ is the *complementary sensitivity function*. The following analysis can now be made. For $e \approx 0$, disturbance rejection and tracking are achieved if $S \approx 0$. This implies that $|L| \gg 1$. On the other hand, noise rejection implies that $S \approx 1$, which is obtained if $L \approx 0$. This illustrates a typical conflict in the design process. Hence, the control requirements are

1. Disturbance rejection: $|L| > 1$.
2. Noise attenuation: $|L| < 1$
3. Reference tracking: $|L| > 1$
4. Control energy reduction: $|L| < 1$
5. Robust stability: $Re\{s_i(\mathbf{p}, \mathbf{k})\} < 0$ for all $s_i(\mathbf{p}, \mathbf{k})$ such that $1 + L(s_i, \mathbf{p}, \mathbf{k}) = 0$.

Fortunately, the conflicting objectives are generally in different frequency ranges, and we can meet most of them by having large loop gains at low frequencies (usual range of d and r) and small gains at high frequencies (usual range of n).

Due to the probabilistic nature of $\mathbf{p}$, L is a random complex variable parameterized by the vector $\mathbf{k}$ and the frequency ω. This makes the closed loop poles random variables and the Bode and Nyquist diagrams of G and L random processes continuously parameterized. In other words, at a fixed $\mathbf{k}$ and ω, L is a random variable utterly described by a PDF, in contrast to the single complex function it is when $\mathbf{p}$ is deterministic. In this context, stochastic control design intends to shape the random process of the system's output by manipulating $\mathbf{k}$ such that the control requirements are satisfied for all the plants generated by the propagation of Δ.

2.2 MIMO Systems

The system dynamics are now described by transfer function matrices. Let $C(s, \mathbf{c})$ be the transfer function matrix of a compensator. For the system shown in figure 2, where the open *loop transfer matrix* is $L(s, \mathbf{p}, \mathbf{k}) = GCK$, Equations (2)-(1) hold once K is replaced by KC. Denoting with $\bar{\sigma}(L)$ and $\underline{\sigma}(L)$ the maximum and minimum singular values of L, the control requirements are

1. Disturbance rejection: $\underline{\sigma}(L) > 1$
2. Noise attenuation: $\bar{\sigma}(L) < 1$
3. Reference tracking: $\underline{\sigma}(L) > 1$
4. Control energy reduction: $\bar{\sigma}(L) < 1$

5. Robust stability: $Re\{s_i(\mathbf{p},\mathbf{c},\mathbf{k})\} < 0$ for all s_i such that $Det\{I + L(s_i,\mathbf{p},\mathbf{c},\mathbf{k})\} = 0$.

As before, the open loop requirements (1) and (3) are valid at low frequencies while requirements (2) and (4) are valid at high frequencies. Then, at frequencies where high gains are required the 'worst-case' direction is related to $\underline{\sigma}(L)$, whereas at frequencies where low gains are required the 'worst-case' direction is related to $\bar{\sigma}(L)$. Both 'worst-case' directions can be unified by the function

$$\Gamma = \rho(\omega)\underline{\sigma}(L) + (1 - \rho(\omega))\bar{\sigma}(L) \tag{4}$$

where $\rho(\omega) \in [0,1]$ is a weighting function whose value approaches one at low frequencies and zero at high frequencies, e.g. $\rho(\omega) = 1/(1+\omega)^m$ for $m > 1$. In this context, stochastic control design intends to shape the random process Γ (or equivalently the random process of the singular values of L) by manipulating $\mathbf{k}$ such that the control requirements are satisfied for all the plants generated by the propagation of Δ.

Compensator design is another area where a probabilistic treatment of uncertainty is valuable. The objective of the compensator is to eliminate the effects of undesired cross-couplings among inputs and outputs such that the MIMO system can be treated as a set of independent SISO systems. This can be achieved if CG is diagonal, e.g. if G is square and and D is the desired uncoupled behavior, $C \approx DG^{-1}$. Since the exact form of G is always unknown, compensators based on a nominal deterministic plant will not perform as planned. With this in mind, we let $\mathbf{c}$ be an additional design variable to be used for decoupling only. Optimal decoupling can be pursued by finding $\mathbf{c}$ in $C(s,\mathbf{c})$ such that a stochastic norm of the off-diagonal terms of $C(s,\mathbf{c})G(s,\mathbf{p})$ is minimized in the frequency range of interest. This problem will be considered in Section 4.4.

3 ROBUST STABILITY

In this framework, we say that a LTI system with structured uncertainty is *robust stable* if all its poles reside in the left hand side of the complex plane for all possible values of the uncertain parameters, i.e the system is asymptotically stable $\forall \mathbf{p} \in \Delta$. The requirement (5) listed above implies the robust stability of the closed loop response.

Denote with $\mathbf{s}$ a vector formed by the poles of a random transfer function. Stability conditions can be written as

$$P\{Re\{\mathbf{s}(\mathbf{p},\mathbf{k})\} < \mathbf{0}\} = \boldsymbol{\epsilon} \tag{5}$$

where $P\{A\}$ is the probability of occurrence of the event A and $\boldsymbol{\epsilon} = \{\epsilon_i\}$ satisfies $\epsilon_i \in [0,1]$ for $i = 1,\ldots,n$. Defining $\mathbf{q} \equiv Re\{\mathbf{s}\}$, these conditions can be alternatively written as

$$F_{\mathbf{Q}}(\mathbf{0}) > \mathbf{1} - \boldsymbol{\epsilon} \tag{6}$$

where $F_{\mathbf{M}}(\tilde{\mathbf{m}})$ is the cumulative distribution function (CDF) of $\mathbf{m}$ at $\mathbf{m} = \tilde{\mathbf{m}}$. The Routh-Hurwitz (R-H) test applied to the corresponding characteristic equation leads to the vector inequality $\mathbf{r} \equiv \{r_i\} > \mathbf{0}$, where r_i is a R-H determinant. In this context, stability conditions can be written as

$$F_{\mathbf{R}}(\mathbf{0}) < \tilde{\boldsymbol{\epsilon}} \tag{7}$$

where $\tilde{\boldsymbol{\epsilon}} = \{\tilde{\epsilon}_i\}$ satisfies $\tilde{\epsilon}_i \in [0,1]$ for $i = 1,\ldots n$. Robust stability is attained iff $\boldsymbol{\epsilon} = \tilde{\boldsymbol{\epsilon}} = \mathbf{0}$. In this paper robust stability will be enforced via Equation (7).

4 CONTROL DESIGN

Formulations that involve the solution to optimization problems in the frequency domain are proposed below. In each problem, a metric of the controlled response, denoted with y, is used to build the cost function. Let $T(s, \mathbf{p}, \mathbf{k})$ and $\hat{T}(s)$ be complex variables associated to the current and desired designs respectively and $\Theta(s, \mathbf{p}, \mathbf{k}) = 0$ the closed loop stability equation. For SISO systems, take $T = L$, $\hat{T} = \hat{L}$, and $\Theta = 1 + L$. For MIMO systems take $T = \Gamma$, $\hat{T} = \hat{\Gamma}$ and $\Theta = det\{I + L\}$.

4.1 Minimizing the Likelihood of Instability

Let y_i be the ith inequality constraint for closed loop stability, i.e. the ith component of Equation (6) or (7), applied to the closed loop characteristic equation. Hence, $f_{Y_i}(y_i)$ is the PDF of the corresponding random variable. Designs that maximize robust stability can be found by solving the following unconstrained optimization problem:

$$\mathbf{k}^* = argmin \left\{ \sum_{i=1}^{n} F_{Y_i}(y_i) \right\} \tag{8}$$

This problem will be referred to as P1 and its cost as J_1. The minimization of J_1 shapes the PDFs of the R-H determinants such that their excursion into the negative part of the r-axes is minimized. Robust stability is achieved when $inf_{\mathbf{k}^*}\{J_1\} = 0$. Notice that robust stability might not be possible for the chosen control structure $K(s, \mathbf{k})$.

4.2 Minimizing the Variability About a Target Response

Assume that $\hat{T}$ is a deterministic target transfer function whose magnitude $\hat{y}$ satisfies the control requirements (1-4). Let $y \equiv |T(s, \mathbf{p}, \mathbf{k})|$. Hence, $f_Y(y, \omega)$ is the PDF of the corresponding random process. Designs that improve the control performance by reducing the variability of the response can be pursued by solving the following constrained optimization problem

$$\mathbf{k}^* = argmin \left\{ \int_0^\infty \int_{\Phi(\omega)} [\hat{y}(\omega) - y(\omega, \mathbf{p}, \mathbf{k})]^2 \; f_Y(y, \omega) \; g(\omega) \; dy \; d\omega \right\} \tag{9}$$

subject to the set of stability constraints, i.e. either Equation (6) or (7), for a given $\boldsymbol{\epsilon}$ or $\tilde{\boldsymbol{\epsilon}}$, prescribed in advance. In this equation, $\Phi(\omega)$ is the support of y and $g(w)$ is a weighting function used to favor particular frequency ranges. This optimization problem will be referred to as P2 and its cost as J_2.

The minimization of J_2 concentrates the entire family of PDFs of the output about the target. Not only the expected value of y is moved towards $\hat{y}$ but also the variability around it is reduced. Notice that this treatment involves shaping the whole PDF, including all its moments and corresponding support. While the open loop requirements (1-4) are pursued by manipulating the cost directly, the closed loop requirement (5) is enforced via inequality constraints.

The main advantage of this formulation is the need for only the first two order moments of y to calculate J_2. Notice however that this approach is over-restrictive because it penalizes solutions that do not necessarily violate the control requirements.

4.3 Minimizing the Likelihood of Unfavorable Performances

Let $y \equiv |T(s, \mathbf{p}, \mathbf{k})|$. Hence, $F_Y(y, \omega)$ is the CDF of the corresponding random process. Define the non-negative functions $h_1(\omega)$ in $\omega \in [0, \omega_1]$ and $h_2(\omega)$ in $\omega \in [\omega_2, \infty)$ such that a design satisfying $y < h_1(\omega)$ violates the requirements (1) and (3) and a design satisfying $y > h_2(\omega)$ violates the requirements (2) and (4). Clearly, h_1 and h_2 are envelopes for the undesired low and high frequency behavior respectively. Excursions of y below h_1 or above h_2 are unfavorable from the performance point of view. In this context, designs that improve the control performance can be found by solving the following constrained optimization problem

$$\mathbf{k}^* = argmin \left\{ \int_0^{\omega_1} F_Y(h_1(\omega), \omega) \; g(\omega) \; d\omega + \int_{\omega_2}^{\infty} [1 - F_Y(h_2(\omega), \omega)] \; g(\omega) \; d\omega \right\} \tag{10}$$

subject to the set of closed loop stability constraints for a a given probability of instability. This problem will be referred as P3 and its cost as J_3.

The minimization of J_3 reduces the probability of violating requirements (1-4) directly. In contrast to J_2, J_3 does not penalize satisfactory solutions. Notice however, that the calculation of the some of the first order moments of y_i is insufficient to properly estimate J_3. At this point, let's us define $P_{ex}(\omega)$ as $F_Y(h_1(\omega))$ for $\omega \in [0, \omega_1]$, as $1 - F_Y(h_2(\omega))$ for $\omega \in [\omega_2, \infty)$ and as zero otherwise. Hence, J_3 can be obtained by integrating $P_{ex}(\omega)g(\omega)$.

4.4 Maximizing the Decoupling Effect of the Compensator

The probabilistic description of $\mathbf{p}$ makes the components of the $n_1 \times n_2$ matrix $D \equiv CG = \{d_{jk}\}$, random variables. Perfect decoupling is achieved if the random processes of the off-diagonal terms are Dirac deltas at zero. This is clearly impossible unless G is known exactly. Let $y_{jk} \equiv |d_{jk}(s, \mathbf{p}, \mathbf{c})|$. Hence, $F_Y(y, \omega)$ is the CDF of the corresponding random process. Providing that the decoupler is causal and stable, the compensator can be designed by solving:

$$\mathbf{c}^* = argmin \left\{ \sum_{j \neq k}^{n_1} \sum_{k}^{n_2} \int_{\Omega} \int_{\Phi_{jk}(\omega)} y_{jk}(\omega, \mathbf{p}, \mathbf{c})^2 f_{Y_{jk}}(y_{jk}, \omega) g(\omega) \; dy_{jk} \; d\omega \right\} \tag{11}$$

where Ω is the frequency range of interest and $\Phi_{jk}(\omega)$ is the support of $f_{Y_{jk}}(y_{jk}, \omega)$ at ω. This problem will be referred as P4 and the corresponding cost as J_4. Constraints on the robust stability of C can also be imposed.

5 EXAMPLES

In this section, the formulations proposed above are evaluated using problems that admit closed form expressions for the PDFs of y and r_i. This allows us to have a good understanding of the problem at hand by avoiding numerical errors caused by sampling and asymptotic approximations.

Notice that the set of solutions of the P1 problem defines the domain where the solutions to the P2 and P3 problems exist. Then, a solution to the P1 problem can be used as initial condition in the numerical search for the P2 and P3 solutions. In all the examples, $\mathbf{p} = \{a\}$ and $a \in \Delta = [a^-, a^+]$, $f_A(a)$ are given.

5.1 SISO System

Consider the plant and the PID controller

$$G(s, \mathbf{p}) = \frac{b(1+ds)}{(1+cs)(e+as)} \tag{12}$$

$$K(s, \mathbf{k}) = \frac{k_1 + k_2 s + k_3 s^2}{s} \tag{13}$$

where $\mathbf{k} = \{k_1, k_2, k_3\}$. The magnitude of the resulting loop transfer function is given by

$$y \equiv |L| = \frac{h}{\sqrt{e^2 + (\omega a)^2}} \tag{14}$$

$$h = \frac{|b|}{\omega}\sqrt{\frac{(1+(d\omega)^2)[(k_2 - k_3\omega^2)^2 + (k_1\omega)^2]}{(c\omega)^2 + 1}}$$

Notice that the effect of the uncertainty is not noticeable at low frequencies. The random process of the output is given by

$$f_Y(y, \omega) = \frac{h^2}{\omega^2 y^3 \hat{a}}\left[f_A(\hat{a}) + f_A(-\hat{a})\right] \tag{15}$$

where $\hat{a} = (1/\omega)\sqrt{(h/y)^2 - e^2}$. The support of y, i.e. $y \in [y^-, y^+]$, is bounded by

$$y^-(\omega) = min\{y(a^+), y(a^-)\}, \qquad y^+(\omega) = \begin{cases} h/|e| & \text{if } 0 \in \Delta \\ max\{y(a^+), y(a^-)\} & \text{otherwise} \end{cases}$$

The characteristic equation for closed loop stability is $\Theta(s, \mathbf{p}, \mathbf{k}) = \alpha s^3 + \beta s^2 + \gamma s + \delta = 0$, where $\alpha = ac + bdk_3$, $\beta = a + ce + b(dk_1 + k_3)$, $\gamma = e + b(k_1 + dk_2)$ and $\delta = bk_2$. The R-H test leads to the inequalities $r_i > 0$ for $i = 1, 2, 3$ where $r_1 = \beta/\alpha$, $r_2 = \beta\gamma - \alpha\delta$ and $r_3 = \delta/\alpha$. In general, the constraints can be written as $r_i = (a\zeta_i + \eta_i)/(a\xi_i + \kappa_i) > 0$ for $i = 1, 2, 3$. The expressions for ζ_i, η_i, ξ_i and κ_i, which are non-linear functions of the components of $\mathbf{k}$, are omitted due to their extension. The corresponding CDFs are given by

$$F_{R_j}(r_j) = \begin{cases} 1 - sign\{m_j\}[F_A(\bar{a}_j) - F_A(\tilde{a}_j)] & \text{if } r_j > \zeta_j/\xi_j \text{ and } m_j \neq 0 \\ sign\{m_j\}[F_A(\tilde{a}_j) - F_A(\bar{a}_j)] & \text{if } r_j \leq \zeta_j/\xi_j \text{ and } m_j \neq 0 \\ H\{r_j - (a\zeta_i + \eta_i)/(a\xi_i + \kappa_i)\} & \text{otherwise} \end{cases} \tag{16}$$

$$F_{R_2}(r_2) = \begin{cases} F_A(\tilde{a_2}) & \text{if } \zeta_2 > 0 \\ 1 - F_A(\tilde{a_2}) & \text{if } \zeta_2 < 0 \\ H\{r_2 - \eta_2\} & \text{otherwise} \end{cases} \tag{17}$$

where $m_j = \zeta_j\kappa_j - \eta_j\xi_j$, $\bar{a}_j = -\kappa_j/\xi_j$ for $j = 1, 3$; $\tilde{a}_i = (r_i\kappa_i - \eta_i)/(\zeta_i - r_i\xi_i)$ for $i = 1, 2, 3$ and $H\{\cdot\}$ is the Heaviside function. Differentiation leads to

$$f_{R_j}(r_j) = \begin{cases} |m_j| f_A(\tilde{a_j})/(\zeta_j - r_j\xi_j)^2 & \text{if } m_j \neq 0 \\ \delta\{r_j - (a\zeta_i + \eta_i)/(a\xi_i + \kappa_i)\} & \text{otherwise} \end{cases} \tag{18}$$

$$f_{R_2}(r_2) = \begin{cases} f_A(\tilde{a_2})/|\zeta_2| & \text{if } \zeta_2 \neq 0 \\ \delta\{r_2 - \eta_2\} & \text{otherwise} \end{cases} \tag{19}$$

where $\delta\{\cdot\}$ is the Dirac function. Notice that a combination of the gains can eliminate the randomness of the constraint making it purely deterministic. This has also been observed in [2]. The following observations are made: (i) when $\bar{a_j} \in \Delta$, some designs lead to $f_{R_i}(r_i)$ with unbounded support, (ii) unbounded $f_{R_i}(r_i)$ are obtained where $m_j(\mathbf{k}, a) = 0$, i.e. $\exists \tilde{r_i}$ such that $lim_{r_i \to \tilde{r_i}} f_{R_i}(r_i) = \infty$.

The problems P1, P2 and P3 can now be solved numerically using Equations (15), (16) and (17). In the results that follow, the values $b = 3$, $c = 10$, $d = -2$, $e = 2/5$, $\tilde{\boldsymbol{\epsilon}} = \mathbf{0}$ and the functions $\hat{y} = |3(1-2s)/(20s(2s+1)(s/3+1))|$, $h_1(\omega) = 1$ for $\omega \in [10^{-4}, 10^{-2}]$ and $h_2(\omega) = 3/(4\omega)$ for $\omega \in [0.5, 10^2]$ are assumed. The improper integrals used to build the cost functions will be approximated in accordance with these intervals, i.e. $\omega \in [10^{-4}, 10^2]$.

First, assume that a is a beta random variable with parameters $\{2, 10\}$ and support $\Delta = [0, 2]$. In this setup, the uncontrolled system is robust stable. The corresponding bode diagram is shown in figure 3. The system's response at high frequencies is clearly inadequate, then some control action is needed. The P2-solution leads to the diagram shown in figure 4. Here, $g(\omega)$ has been used to favor the performance at low frequencies. The corresponding PDFs for the R-H determinants are shown in figure 5. Notice that three supports extend towards positive infinity. The same information is shown in figures 6 and 7 for a cost function that favors the performance at high frequencies.

Figure 8 shows the stochastic bode diagram for a robust stable design. Notice the undesired performance regions created by $h_1(\omega)$ and $h_2(\omega)$ and the corresponding P_{ex}. This design is used as initial condition for solving the P3-problem. The corresponding solution is shown in figure 9, where $\tilde{\epsilon} = 0.005\{1, 1, 1\}$. The PDFs of the R-H determinants and the stochastic Nyquist diagram are shown in figures 10 and 11. Due to $\boldsymbol{\epsilon} \neq \mathbf{0}$ $s = -1$ is encircled by a portion of the PDF of L and a tail of $f_{R_2}(r_2)$ is in the negative part of the r_2-axis. In all three optimal solutions the control is strictly proper, i.e. $k_3^* = 0$.

Now, assume that a is beta distributed with parameters $\{2, 2\}$ and support $\Delta = [-0.5, 1]$. The corresponding uncontrolled plant is nominally stable, i.e. $G(s, E[\mathbf{p}])$, but robust unstable. This can be seen from figure 12 where the PDFs of the R-H determinants are shown. It is interesting to notice that the support of those distributions is given by sets with the form $S = (-\infty, \Psi] \cup [\Phi, \infty)$, where Ψ and Φ are finite values that depend on a^- and a^+. This occurs due to singularities in r_i and λ caused by vanishing denominators. This can be explained by thinking of a as a moving parameter within Δ. When a approaches zero from the right, a pole moves towards positive infinity and the system is stable . At $a = 0$, such a pole disappears at infinity, i.e. the relative degree of the plant decreases by one, and the system stability is now given by a degenerated characteristic equation. When a is decreased further, such a pole moves from minus infinity towards zero making the response unstable. This analysis explains the non-connected support of the bi-modal PDF.

The first task is to find $\mathbf{k}$ for robust stabilization. The stochastic bode diagram and the PDFs for the R-H determinants corresponding to a solution of the P1 problem are shown in figure 13 and 14. It is observed that: (i) there is a substantial difference between the nominal and the expected response at high frequencies in both magnitude and phase, (ii) the uncertainty on the sign of a makes the phase plot to spread over a range 180 degrees wide at high frequencies, (iii) robust stability cannot be achieved without the derivative action and (iv) r_1 behaves as a deterministic constraint at the optimum, i.e. $f_{R_1}(r_1) \approx \delta(r_1 - 10^{-3})$. Notice that observation (iii) precludes the search for designs on the PID control structure that lead to satisfactory performances at high frequency. The P1-solution leads to figures 15, 16 and 17.

5.2 MIMO System

The following model is derived by studying the angular velocity control of a satellite spinning about one of its principal axes [12]

$$G(s, \mathbf{p}) = \frac{1}{s^2 + a^2} \begin{bmatrix} s - a^2 & a(s+1) \\ -a(s+1) & s - a^2 \end{bmatrix} \tag{20}$$

The plant has a pair of poles at $s = \pm a\mathbf{j}$ so it needs to be stabilized. A diagonal PID control structure without compensator is considered, i.e. $\mathbf{k} = \{k_1, k_2, k_3\}$, $K(s, \mathbf{k}) = (k_1 + k_2/s + k_3 s)I$, $C(s, \mathbf{c}) = I$. Equation (4) leads to

$$y \equiv \Gamma = \rho \frac{\mu\sqrt{1 + a^2}}{|a + \omega|} + (1 - \rho) \frac{\mu\sqrt{1 + a^2}}{|a - \omega|} \tag{21}$$

where $\mu = \sqrt{(\omega k_2)^2 + (k_1 - k_3\omega^2)^2}/\omega$. For the sake of simplicity in the notation we will use $\underline{\sigma}$ and $\bar{\sigma}$ to denote the minimum and maximum singular values of L respectively. The corresponding random processes are given by

$$f_{\underline{\sigma}}(\underline{\sigma}, \omega) = sign\{\mu - \underline{\sigma}\} \left[f_A(a_2) \frac{da_2}{d\underline{\sigma}} + f_A(a_1) \frac{da_1}{d\underline{\sigma}} \right] \tag{22}$$

$$f_{\bar{\sigma}}(\bar{\sigma}, \omega) = sign\{\mu - \bar{\sigma}\} \left[f_A(a_3) \frac{da_3}{d\bar{\sigma}} + f_A(a_4) \frac{da_4}{d\bar{\sigma}} \right] \tag{23}$$

$$a_{1,2} = \frac{-\omega\underline{\sigma}^2 \mp \mu\sqrt{\underline{\sigma}^2(1 + \omega^2) - \mu^2}}{\mu^2 - \underline{\sigma}^2}$$

$$a_{3,4} = \frac{\omega\bar{\sigma}^2 \mp \mu\sqrt{\bar{\sigma}^2(1 + \omega^2) - \mu^2}}{\mu^2 - \bar{\sigma}^2}$$

Denote with $\tilde{a}_i$ the real roots of Equation (21) such that $\tilde{a}_i \le \tilde{a}_j$ for $j > i$. Denote with $y_1^* = y(a_1^*)$ and $y_2^* = y(a_2^*)$ the extrema of Equation (21) such that $y_1^* \le y_2^*$. If $\dot{a} \equiv da/dy$, the random process of y is given by

$$f_Y(y, \omega) = \begin{cases} f_A(\tilde{a}_2)\dot{\tilde{a}}_2 - f_A(\tilde{a}_1)\dot{\tilde{a}}_1 & \text{if } (y_1^* \le y \le \mu < y_2^*) \text{ or} \\ & (y_1^* \le y \le y_2^* \le \mu) \\ f_A(\tilde{a}_1)\dot{\tilde{a}}_1 - f_A(\tilde{a}_2)\dot{\tilde{a}}_2 & \text{if } y_1^* \le \mu \le y \le y_2^* \\ f_A(\tilde{a}_4)\dot{\tilde{a}}_4 + f_A(\tilde{a}_2)\dot{\tilde{a}}_2 - f_A(\tilde{a}_1)\dot{\tilde{a}}_1 - f_A(\tilde{a}_3)\dot{\tilde{a}}_3 & \text{if } y_1^* \le y_2^* \le y \le \mu \\ f_A(\tilde{a}_1)\dot{\tilde{a}}_1 + f_A(\tilde{a}_3)\dot{\tilde{a}}_3 - f_A(\tilde{a}_2)\dot{\tilde{a}}_2 - f_A(\tilde{a}_4)\dot{\tilde{a}}_4 & \text{otherwise} \end{cases} \tag{24}$$

Analytical expressions for the $\tilde{a}$s and the y^*s were derived but omitted due to their length. The support of $y \in [y^-, y^+]$ is bounded by

$$y^-(\omega) = \begin{cases} y_i^* & \text{if } a_i^* \in \Delta \text{ and } \mathrm{a_j^*} \notin \Delta \text{ for } \mathrm{i} \neq \mathrm{j} \\ y_1^* & \text{if } a_1^* \in \Delta \text{ and } \mathrm{a_2^*} \in \Delta \\ min\{y(a^+), y(a^-)\} & \text{otherwise} \end{cases}$$

$$y^+(\omega) = \begin{cases} max\{y\} = \infty & \text{if } |\omega| \in \Delta \\ max\{y(a^+), y(a^-)\} & \text{otherwise} \end{cases}$$

Notice that $f_Y(y, \omega)$ is unbounded at y_1^* and y_2^* and its support is unbounded from above for all possible resonant frequencies, i.e. $\forall \omega \in \Delta$.

The closed loop stability equation is $\Theta(s, \mathbf{p}, \mathbf{k}) = \alpha s^4 + \beta s^3 + \gamma s^2 + \psi s + \theta = 0$, where $\alpha = 1 + 2k_3 + (1+b)k_3^2$, $\beta = 2[-bk_3 + k_2(1 + k_3 + bk_3)]$, $\gamma = k_2^2 + 2k_1(1 + k_3) + b(1 - 2k_2 + k2^2 + 2k_1k_3)$, $\psi = 2k_1[b(k_2 - 1) + k_2]$ and $\theta = (1+b)k_1^2$ where $b \equiv a^2$. The R-H test leads to the non-trivial inequalities $r_1 = \beta > 0$, $r_2 = \beta\gamma - \alpha\psi > 0$ and $r_3 = \beta\gamma\psi - \alpha\delta^2 - \theta\beta^2 > 0$. These constraints can be written as $r_i = \kappa_i b^3 + \xi_i b^2 + \zeta_i b + \eta_i$ for $i = 1, 2, 3$. The corresponding CDFs are given by

$$F_{R_i}(r_i) = \begin{cases} F_B(b_1) & \text{if } (\kappa_i > 0,\ \mathrm{n} = 1) \text{ or } (\kappa_\mathrm{i} = 0,\ \xi_\mathrm{i} = 0,\ \zeta_\mathrm{i} > 0) \\ 1 - F_B(b_1) & \text{if } (\kappa_i < 0,\ \mathrm{n} = 1) \text{ or } (\kappa_\mathrm{i} = 0,\ \xi_\mathrm{i} = 0,\ \zeta_\mathrm{i} < 0) \\ F_B(b_3) - F_B(b_2) - F_B(b_1) & \text{if } \kappa_i > 0 \text{ and } \mathrm{n} = 3 \\ 1 + F_B(b_2) - F_B(b_1) - F_B(b_3) & \text{if } \kappa_i < 0 \text{ and } \mathrm{n} = 3 \\ F_B(b_2) - F_B(b_1) & \text{if } \kappa_i = 0 \text{ and } \mathrm{n} = 2 \text{ and } \xi_\mathrm{i} > 0 \\ 1 - F_B(b_2) + F_B(b_1) & \text{if } \kappa_i = 0 \text{ and } \mathrm{n} = 2 \text{ and } \xi_\mathrm{i} < 0 \\ H\{r_i - \eta_i\} & \text{otherwise} \end{cases} \tag{25}$$

where $i = 1, 2, 3$, $F_B(b) = F_A(\sqrt{b}) - F_A(-\sqrt{b})$ and n is the number of real roots of $Q(b) = \kappa_i b^3 + \xi_i b^2 + \zeta_i b + \eta_i - r_i = 0$, such that $Q(b_i) = 0$ and $b_i \leq b_j$ for $j > i$. Notice that the coefficients of this polynomial are non-linear functions of the components of $\mathbf{k}$. Analytical expressions for the bs were derived but omitted here due to their length. Expressions for the PDFs of the constraints can be derived by differentiating Equation (25).

The problems P1, P2 and P3 can now be solved numerically using Equations (24) and (25). In the results that follow, a is a beta random variable with parameters $\{6, 6\}$, $\Delta = [0.5, 1.5]$, $\tilde{\boldsymbol{\epsilon}} = \mathbf{0}$ and $\hat{y}$, $h_1(\omega)$ and $h_2(\omega)$ are as before. Figures 18 and 19 correspond to the P2-solution. In this case, a strictly proper control leads to a robust stable system in which r_2 is purely deterministic. The uncertainty on the natural frequency is clearly reflected in the support of $f_Y(y, \omega)$. Notice that in spite of this, $E[y]$ is bounded. Figure 20 shows the section of the $\mathbf{k}$-domain where $k_3 = 0$ for $\Delta = [0.5, 10]$. The support of the corresponding constraints is filled with dotted lines. Notice that the bounds of such supports are not always given by the deterministic constraints evaluated at the extreme values of Δ, i.e. $r_i(b^-) \neq r_i^-$ and $r_i(b^+) \neq r_i^+$. By *constraint envelopes* we mean a deterministic set of inequalities satisfying the stability conditions for $\boldsymbol{\epsilon} = \hat{\boldsymbol{\epsilon}} = \tilde{\boldsymbol{\epsilon}} = \mathbf{0}$. Having this information at hand will considerably reduce the computational demands of searching for the optimum. The admissible domain, where robust solution exists, is colored in gray. Such region is bounded by the tightest set of constraint envelopes. Figure 21 corresponds to the P3-solution, whose features are very similar to figure 19.

5.3 Decoupling

Below, a compensator for the plant given in Equation (20) is designed by solving the P4-problem given the configuration shown in figure 2. The compensator will be first designed via generalized decoupling. Define $G(s, E[\mathbf{p}])$ as the nominal plant. If the desired uncoupled behavior is given by $Diag\{G(s, E[\mathbf{p}])\}$, the resulting compensator is

$$C(s, \mathbf{c}) = \frac{1}{(s^2 + E[a]^2)(1 + E[a]^2)} \begin{bmatrix} (E[a] - s)^2 & E[a](1+s)(E[a]^2 - s) \\ -E[a](1+s)(E[a]^2 - s) & (E[a] - s)^2 \end{bmatrix}$$

Notice that the compensator is Bounded-Input-Bounded-Output (BIBO) stable and that $E[a]$ acts as a design variable, i.e. $\mathbf{c} = \{c\} \equiv \{E[a]\}$. Few manipulations lead to

$$D = \alpha \begin{bmatrix} -a^2(c^2 - s) - ac(1+s)^2 - s(s - c^2) & -(a-c)(1+s)(ac+s) \\ (a-c)(1+s)(ac+s) & -a^2(c^2 - s) - ac(1+s)^2 - s(s - c^2) \end{bmatrix} \tag{26}$$

where $D(s, \mathbf{c}, \mathbf{p}) = CG$ and $\alpha = (c^2 - s)/[(s^2 + c^2)(1 + c^2)(a^2 + s^2)]$. Notice that if a is deterministic, i.e. $f_A(a) = \delta(a - E[a])$, and $c = E[a]$ perfect decoupling is achieved, i.e. $D(s, \mathbf{c}, \mathbf{p}) = Diag\{G(s, E[\mathbf{p}])\}$. The presence of uncertainty prevents this to happen. In this example the compensator is based on the inverse of the plant, what give us as many design variables as uncertain parameters. This, however, does not have to be the case.

The structure of the matrix (26) allows us to define an equivalent metric for the off-diagonal error as follows

$$y \equiv y_{12} = y_{21} = \frac{\mu |a - c|}{|a^2 - \omega^2|} \sqrt{a^2 c^2 + \omega^2} \tag{27}$$

Denote with $\tilde{a}_i$ the real roots of Equation (27) provided that $\tilde{a}_i \leq \tilde{a}_j$ for $j > i$. Let y_i^* be one of the n extrema of y, i.e. $y_i^* = y(a_i^*)$ such that $dy/da = 0$ at $a = a_i^*$, and $y_i^* \leq y_j^*$ for $i < j$. If $\dot{a} \equiv da/dy$, the random process of y is given by

$$f_Y(y, \omega) = \begin{cases} f_A(\tilde{a}_2)\dot{\tilde{a}}_2 - f_A(\tilde{a}_1)\dot{\tilde{a}}_1 & \text{if } C_1 \text{ holds} \\ f_A(\tilde{a}_1)\dot{\tilde{a}}_1 - f_A(\tilde{a}_2)\dot{\tilde{a}}_2 & \text{if } C_2 \text{ holds} \\ f_A(\tilde{a}_4)\dot{\tilde{a}}_4 + f_A(\tilde{a}_2)\dot{\tilde{a}}_2 - f_A(\tilde{a}_1)\dot{\tilde{a}}_1 - f_A(\tilde{a}_3)\dot{\tilde{a}}_3 & \text{if } C_3 \text{ holds} \\ f_A(\tilde{a}_1)\dot{\tilde{a}}_1 + f_A(\tilde{a}_3)\dot{\tilde{a}}_3 - f_A(\tilde{a}_2)\dot{\tilde{a}}_2 - f_A(\tilde{a}_4)\dot{\tilde{a}}_4 & \text{otherwise} \end{cases} \tag{28}$$

where the C_1, C_2 and C_3 are

$$C_1 = \begin{cases} \{n = 1 \text{ and } [(|c| \geq \omega,\ y_1^* \geq \hat{y} \geq y) \text{ or } (|c| \geq \omega,\ \hat{y} \geq y_1^* \geq y) \text{ or } (|c| \leq \omega,\ y_1^* \geq y)]\} \text{ or} \\ \{n = 3 \text{ and } [(y_2^* \geq \hat{y} \geq y_1^* \geq y) \text{ or } (y_3^* \geq \hat{y} \geq y \geq y_2^*) \text{ or } (y_3^* \geq \hat{y} \geq y_2^* \geq y_1^* \geq y) \text{ or} \\ (y_1^* \geq \hat{y} \geq y) \text{ or } (\hat{y} \geq y_3^* \geq y \geq y_2^*) \text{ or } (\hat{y} \geq y_3^* \geq y_2^* \geq y_1^* > y)]\} \end{cases}$$

$$C_2 = \begin{cases} \{n = 1 \text{ and } (y_1^* \geq y \geq \hat{y})\} \text{ or} \\ \{n = 3 \text{ and } [(y_3^* \geq y \geq y_2^* \geq \hat{y} \geq y_1^*) \text{ or } (y_3^* \geq y \geq \hat{y} \geq y_2^*) \text{ or} \\ (y_3^* \geq y \geq y_2^* \geq y_1^* \geq \hat{y}) \text{ or } (y_1^* \geq y \geq \hat{y})]\} \end{cases}$$

$$C_3 = \begin{cases} \{n=1 \text{ and } [(|c| \geq \omega,\ y \geq y_1^* \geq \hat{y}) \text{ or } (|c| \geq \omega,\ y \geq \hat{y} \geq y_1^*) \text{ or } (|c| \leq \omega,\ y \geq \hat{y})]\} \text{ or} \\ \{n=3 \text{ and } [(y \geq y_3^* \geq y_2^* \geq \hat{y} \geq y_1^*) \text{ or } (y_3^* \geq y \geq y_2^* \geq y \geq \hat{y} \geq y_1^*) \text{ or } (y \geq \hat{y} \geq y_3^*) \\ \text{or } (y \geq y_3^* \geq \hat{y} \geq y \geq y_2^*) \text{ or } (y \geq y_3^* \geq y_2^* \geq y_1^* \geq \hat{y}) \text{ or } (y_3^* \geq y_2^* \geq y \geq y_1^* \geq \hat{y})]\} \end{cases}$$

and $\hat{y} = |c|\mu$. Analytical expressions for a_i, a_i^* and y_i^* can be found. The support of y is bounded by

$$y^-(\omega) = \begin{cases} min\{y\} = 0 & \text{if } \ c \in \Delta \\ y_1^* & \text{if } \ a_1^* \in \Delta \text{ and } c \notin \Delta \\ y_3^* & \text{if } \ a_3^* \in \Delta \text{ and} \{\bar{a},\ a_1^*\} \notin \Delta \text{ and } n=3 \\ min\{y(a^+), y(a^-)\} & \text{otherwise} \end{cases}$$

$$y^+(\omega) = \begin{cases} max\{y\} = \infty & \text{if } \ |\omega| \in \Delta \\ y_2^* & \text{if } \ a_2^* \in \Delta \ \text{ and } |\omega| \notin \Delta \text{ and } n=3 \\ max\{y(a^+), y(a^-)\} & \text{otherwise} \end{cases}$$

Problem P4 can now be solved numerically using Equation (28). In the results that follow the same data used in Example 5.2 is assumed and $\Omega \in [0, \infty)$. Figure 22 shows the magnitude of the off-diagonal terms of D versus frequency for $c = E[a] = 1$. This is the resulting behavior of a conventional decoupling design, where perfect decoupling is achieved if the plant is known exactly. Notice that y^- and $y(E[a])$ coincide. This figure indicates significant discrepancies between deterministic and stochastic results.

The P4-solution leads to $c^* \neq E[a]$. Figure 23 shows a slight improvement in the decoupling at low-frequencies. Compensators with more design variables and/or different structures can achieve better results. Note that: (i) the compensator's structure does not have to be based on the inverse of G, (ii) the decoupling of non-square plants can be studied by the same means.

6 DISCUSSION

In the cases where the uncertain parameters represent a set of changing operating conditions this methodology leads to valuable solutions that otherwise are very difficult to find. In problems where the parameters are physical quantities modeled as uncertain variables as a result of our ignorance on their real value, the solutions can be improved further by refining and even eliminating their probabilistic description using additional information. Adaptive control and model predictive techniques can be used in this respect.

The ignorance on the actual implemented values of the solution can be studied in advance by modeling physical design quantities as additional random variables. We refer to this type of uncertainty as *input uncertainty*. This can be achieved by using variables that parameterize prescribed PDFs as design variables in the optimization problem. Notice that while the physical quantity is deterministic, its modeling is stochastic and its manipulation is done via deterministic variables. For example, if a gain is modeled as a Gaussian random variable with a fixed relation between its first two order moments -say, due to limitations in the precision of the solution to be implemented- optimal solutions can be found by using a single deterministic variable in the optimization problem. Designs that carry out this practice will lead to satisfactory performances for the entire range of gain values within the support of the corresponding PDFs. Notice that in spite of using the same modeling for both parameter and input uncertainty, their physical significance is very different.

The behavior of the probabilistic constraints and their envelopes in the **k**-space must be explored. Such envelopes cannot, in general, be easily built (providing they exist). This fact prevents us from relaxing the computational demands of the optimization problem. The tightest constraint envelopes, given by the set that maximizes the size of the robust stable region in the **k**-space, can be obtained by propagating Δ at a high computational expense. On the other hand, over-constraining envelopes might be easier to build at the expense of introducing conservatism into the problem. This is especially valid in the cases where at least one of the constraints is active (in the probabilistic sense) at the optimum. Circumventing the probabilistic nature of the constraints leads to the loss of the sensitivity of the probability of instability to changes in **k**. This information is very valuable from the design point of view. This feature posts a formidable trade-off between numerical convenience and mathematical significance.

The control design of systems with multiple uncertain parameters lead to additional challenges. In such problems, the calculation of moments and probabilities have to be done, almost exclusively, by means of sampling techniques and asymptotic approximations. While sampling based techniques are suitable for calculating first order moments and large probabilities; asymptotic approximations are reliable for small probabilities. Because the probability of occurrence of the metrics used in the optimization problem can fluctuate considerably, algorithms that dynamically discriminate between these tools are desirable. In addition, the repeated evaluation of the cost function suggest the need for adjusting the trade off between accuracy and computational cost dynamically. These issues are currently being investigated.

7 CONCLUSIONS

In this paper, a stochastic approach to the control synthesis of systems with structured uncertainty is introduced. This strategy allows us to quantify the probability of instability and/or unfavorable performance such that robust optimality can be pursued. Formulations for the stochastic optimization of SISO and MIMO control systems are proposed taking into account decoupling, performance and stability considerations. Examples that admit an exhaustive analysis of the formulation and its solution are used to validate the approach. The application of this methodology to large scale problems, where the use of reliable and inexpensive numerical tools is crucial, is currently being addressed.

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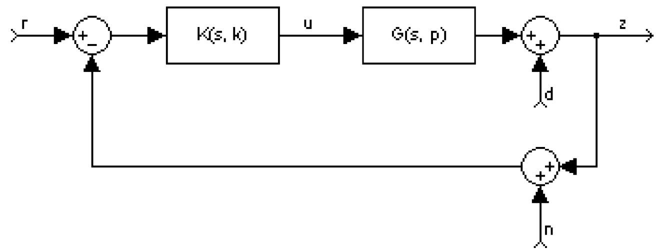

Figure 1: *Block diagram of one degree-of-freedom SISO control system*

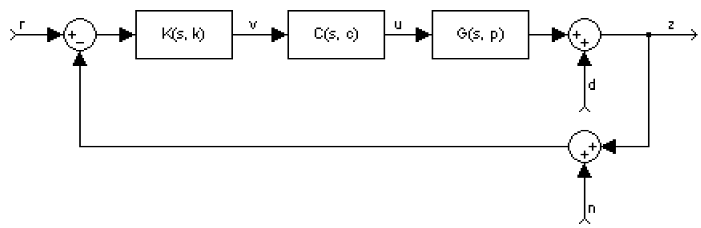

Figure 2: *Block diagram of one degree-of-freedom MIMO control system*

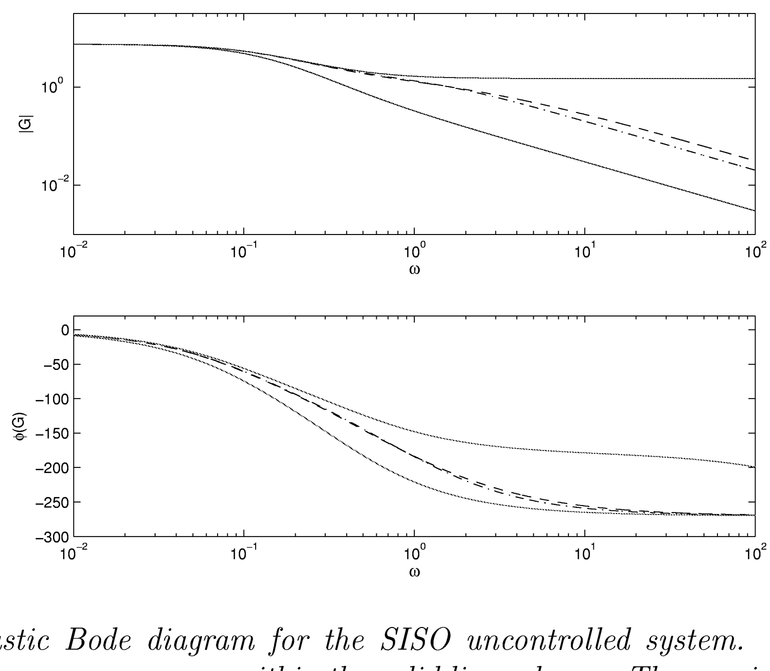

Figure 3: *Stochastic Bode diagram for the SISO uncontrolled system. The PDFs of the magnitude and phase are non-zero within the solid lines shown. The nominal plant* $|G(E[\mathbf{p}])|$ *and its expected value* $E[|G(\mathbf{p})|]$ *are indicated with dash-dotted and dashed lines respectively.*

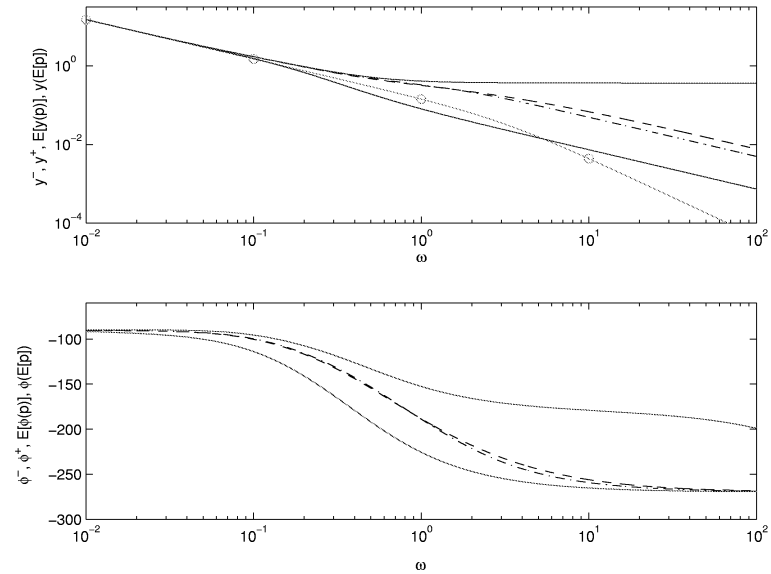

Figure 4: *Stochastic Bode diagram of a design that favors the performance at low frequencies. The PDFs are non-zero within the solid lines shown, i.e.* $y^- < y < y^+$. *The nominal plant* $y(E[\mathbf{p}])$ *and its expected value* $E[y(\mathbf{p})]$ *are indicated with dash-dotted and dashed lines respectively. The curve of the target function* $\hat{y}$ *is shown using a line-circle pattern. The same line conventions are used in the bottom for* $\phi(L)$.

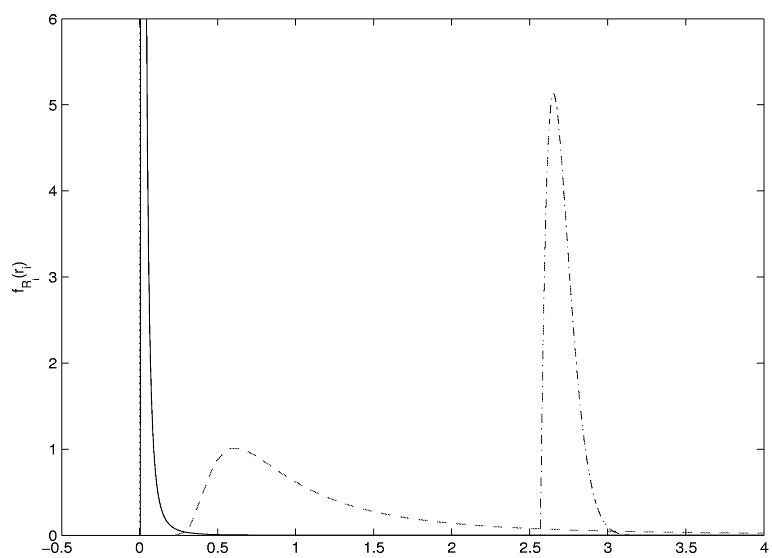

Figure 5: *PDFs of the R-H determinants for the optimal solution.* $f_{R_1}(r_1)$, $f_{R_2}(r_2)$ *and* $f_{R_3}(r_3)$ *are show using dashed, dash-dotted and solid lines respectively.*

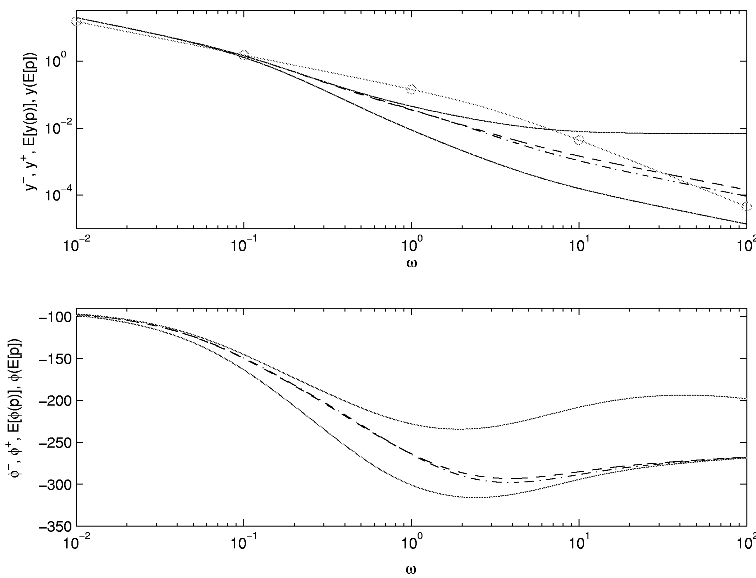

Figure 6: *Stochastic Bode diagram for a design that favors the performance at high frequencies. Conventions used previously apply.*

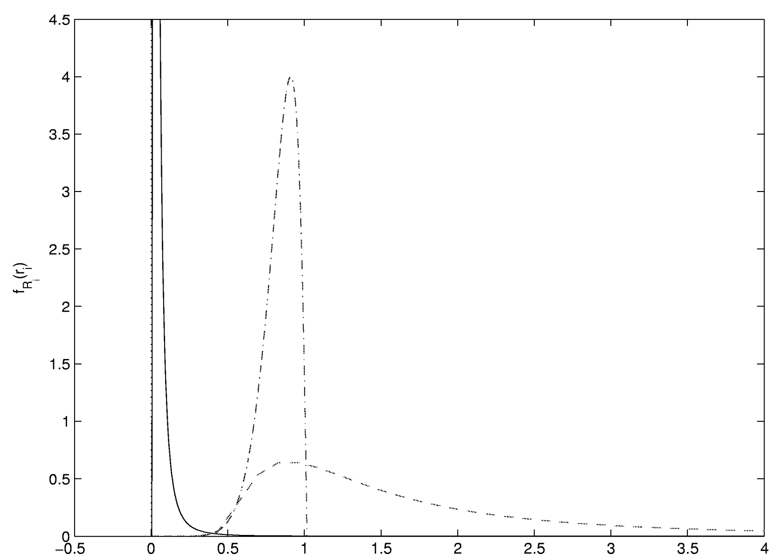

Figure 7: *PDFs of the R-H determinants for the optimal solution. Conventions used previously apply.*

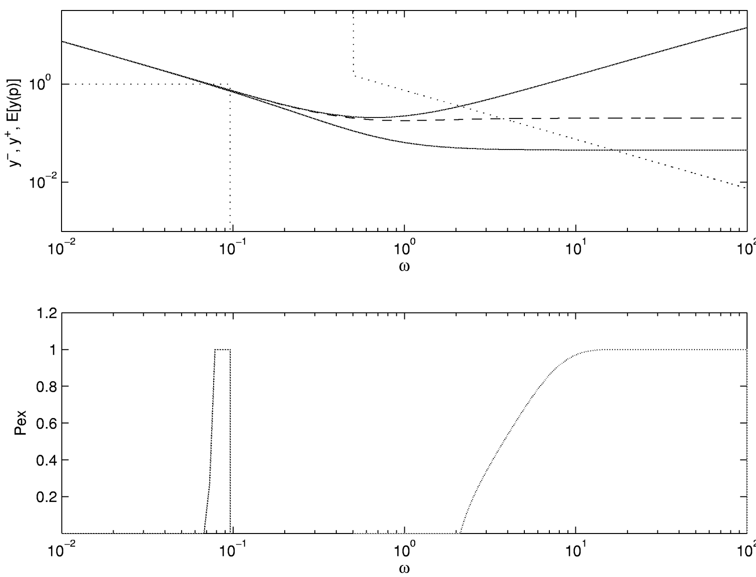

Figure 8: *Top: upper view of $f_Y(y, \omega)$ for a robust stable design. Conventions used previously apply. Dotted lines are used to show the boundary of the sets defined by $y \leq h_1(\omega)$ and $y \geq h_2(\omega)$. Bottom: corresponding P_{ex}.*

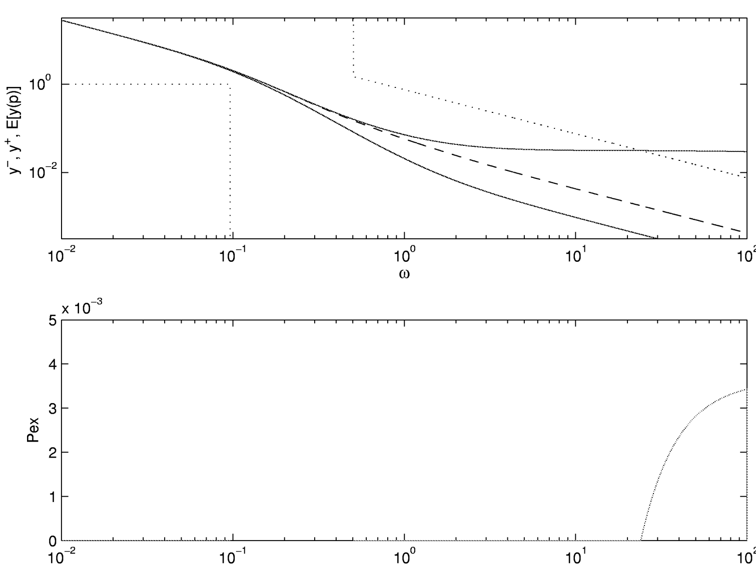

Figure 9: *Top: upper view of $f_Y(y, \omega)$ corresponding to the optimal solution. Conventions used previously apply.*

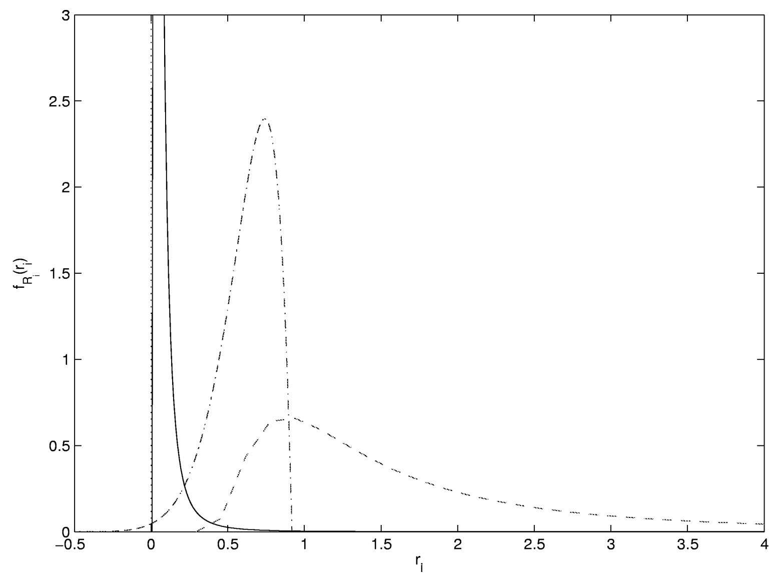

Figure 10: *PDFs of the R-H determinants for the optimal solution. Conventions used previously apply.*

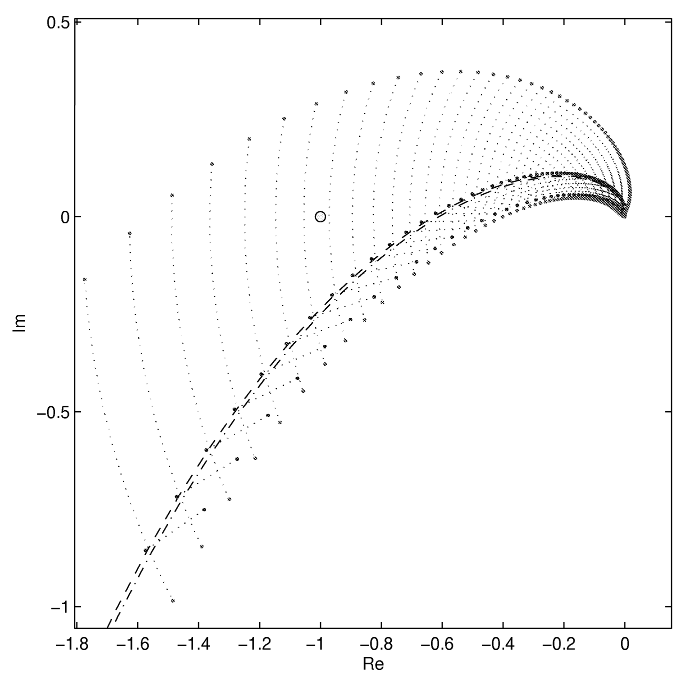

Figure 11: *Stochastic Nyquist plot corresponding to the optimal solution. Crosses centered at the expected value are shown for discrete frequency values. Dotted lines are used to indicate the support of* y *in the radial and tangential directions. A dash-dotted line and a dashed line are used to show the nominal and expected Nyquist plots respectively.*

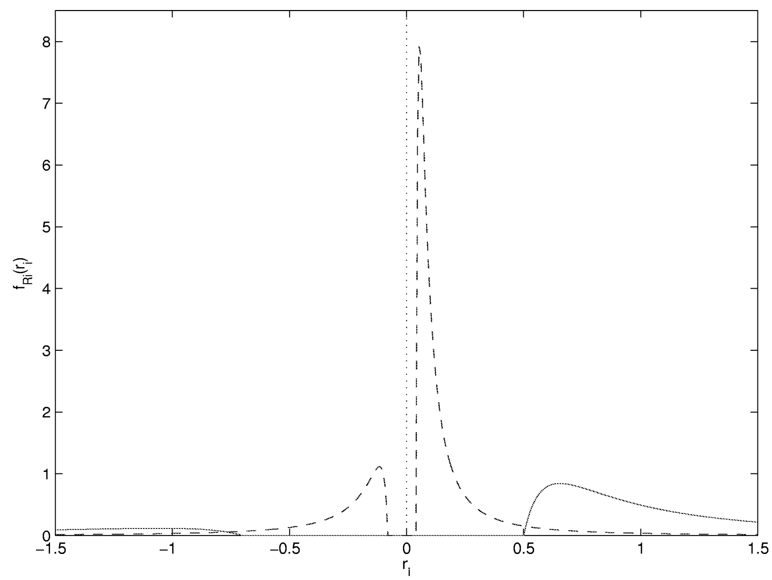

Figure 12: *PDFs of the R-H determinants for the uncontrolled plant. Conventions used previously apply.*

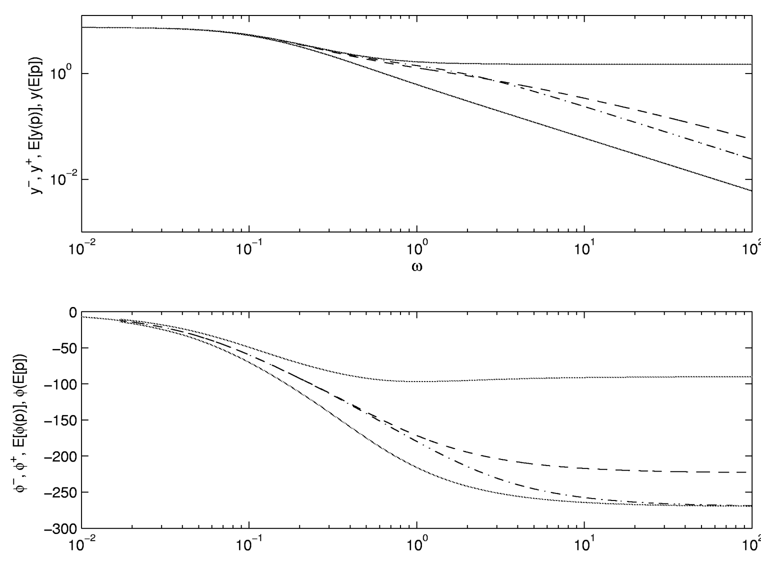

Figure 13: *Stochastic Bode diagram for a robust stable plant. Conventions used previously apply.*

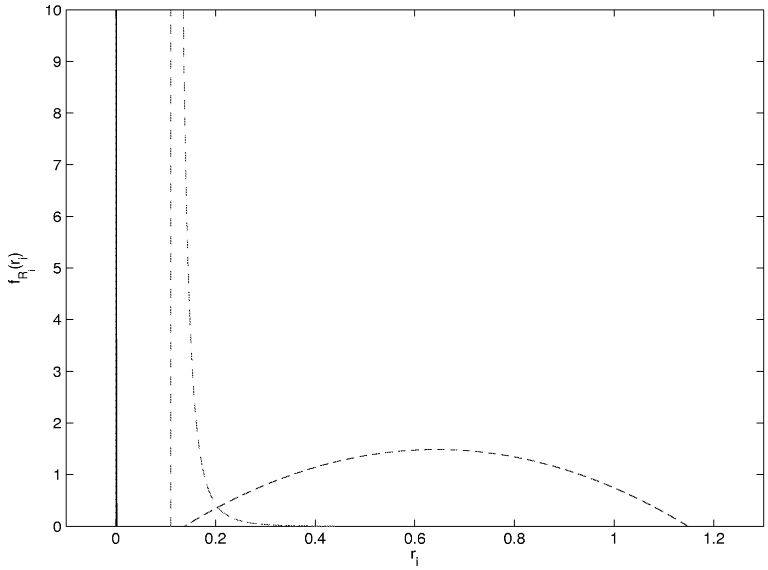

Figure 14: *PDFs of the R-H determinants for a robust stable plant. Conventions used previously apply.*

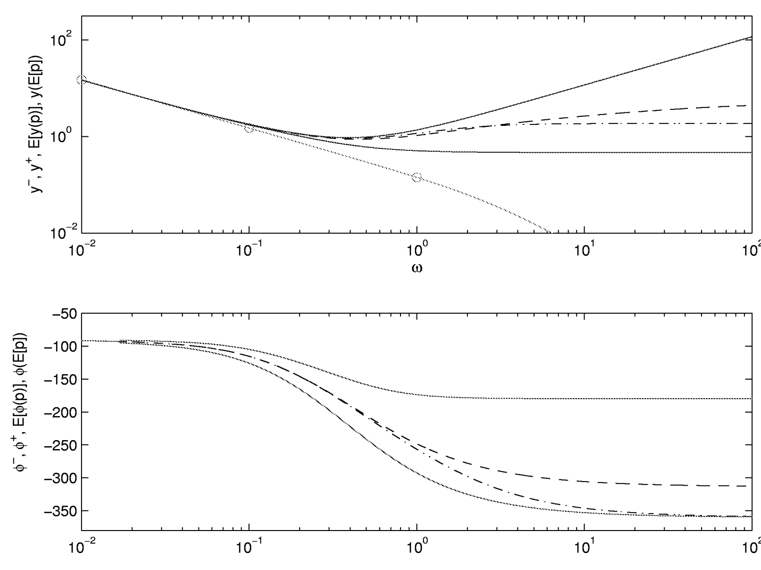

Figure 15: *Stochastic Bode diagram. Conventions used previously apply.*

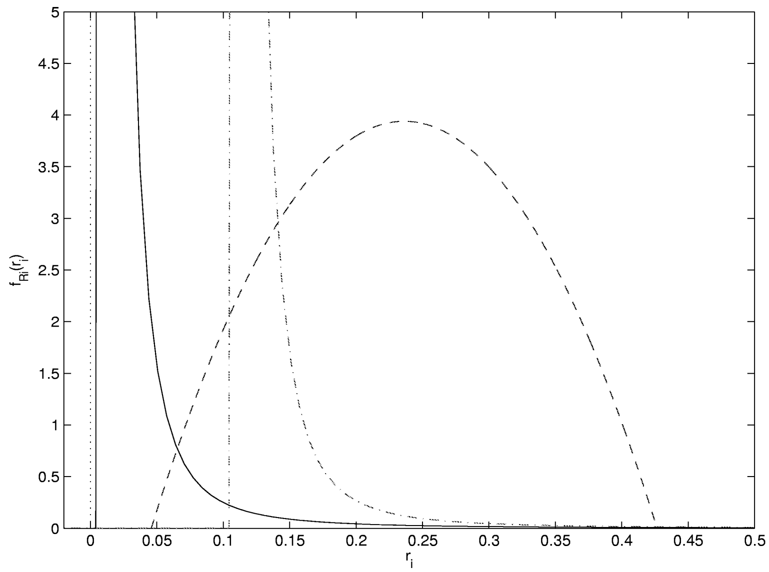

Figure 16: *PDFs of the R-H determinants. Conventions used previously apply.*

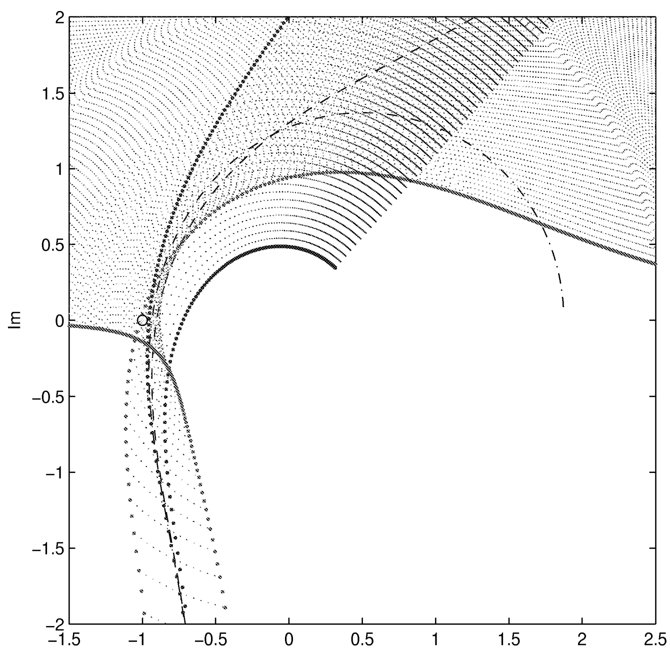

Figure 17: *Stochastic Nyquist plot corresponding to the optimal solution. Conventions used previously apply*

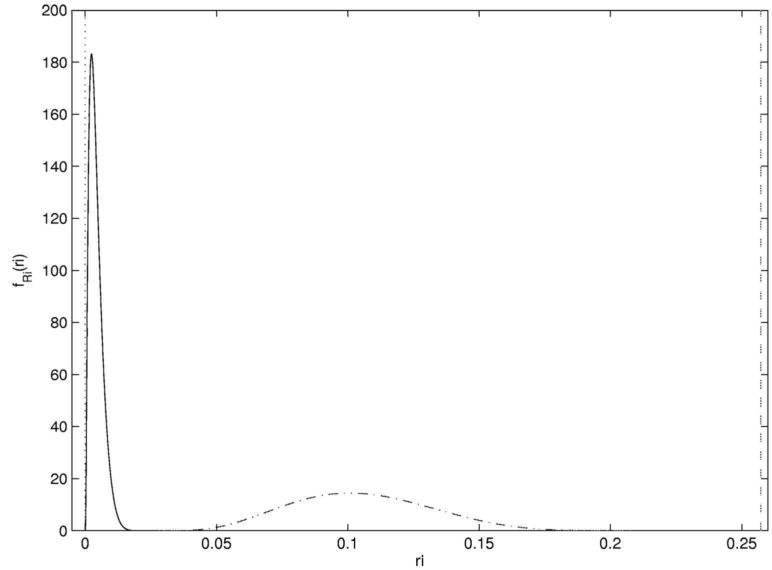

Figure 18: *PDFs of the R-H determinants. Conventions used previously apply.*

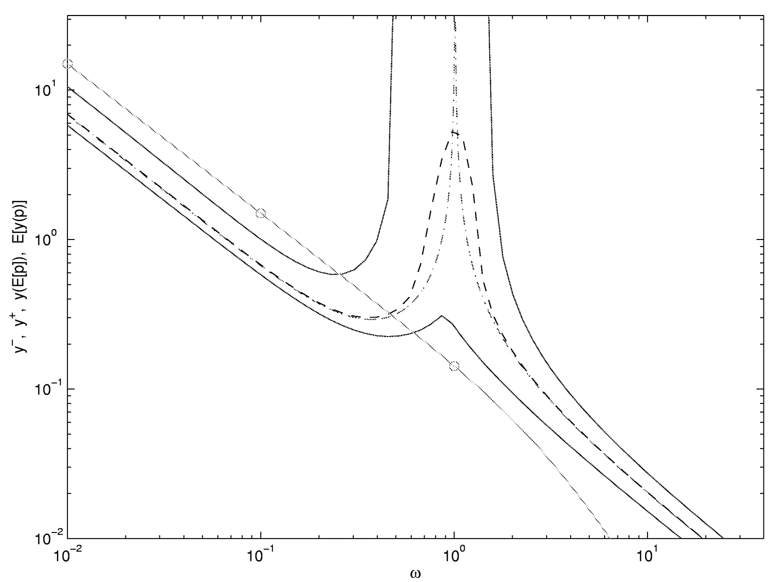

Figure 19: *Optimal solution to the P2 problem.* $y(E[\mathbf{p}])$, $E[y(\mathbf{p})]$, y^{-} *and* y^{+} *are indicated with dash-dotted, dashed and solid lines respectively. The curve of the target function* $\hat{y}$ *is shown using a line-circle pattern.*

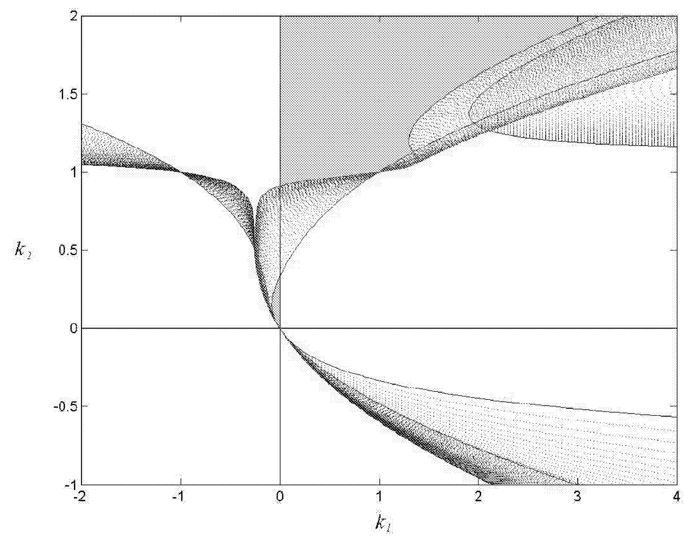

Figure 20: *Section of the k-domain for $k_3 = 0$. The support of the constraints is filled with dotted lines and the functions $r_i(a^+)$ and $r_i(a^-)$ are shown using solid lines. Robust stable regions, where the optimal resides, are colored in gray.*

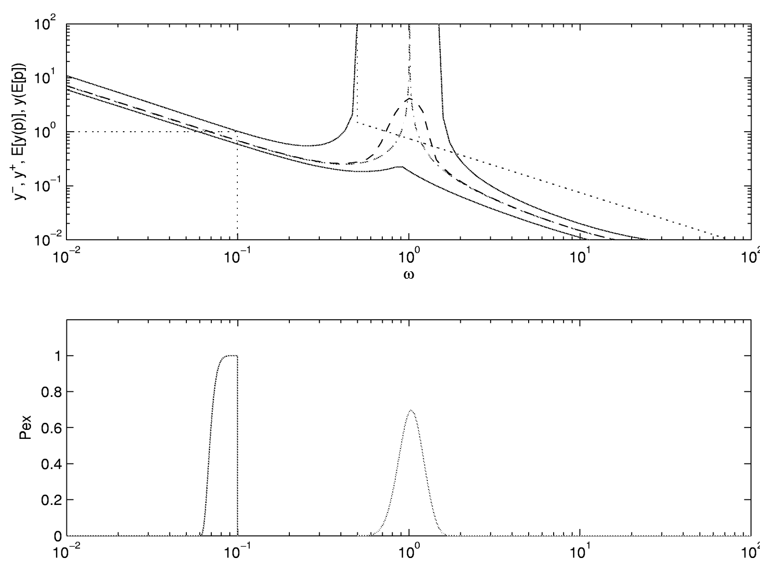

Figure 21: *Top: upper view of $f_Y(y)$ for a robust stable design. Conventions used previously apply. Bottom: corresponding P_{ex}.*

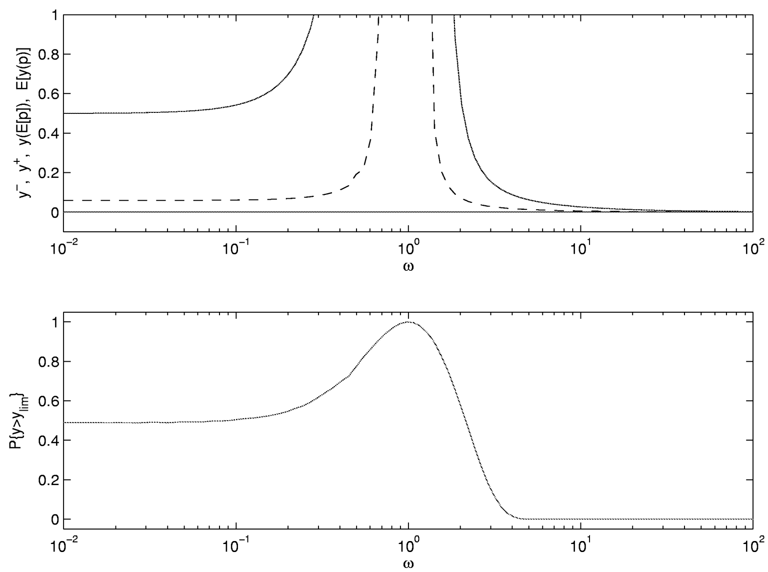

Figure 22: *Results for $c = E[a]$. Top: magnitude of the off-diagonal term of B. The PDFs are non-zero within the solid lines, i.e. $y^{-} < y < y^{+}$. The nominal and expected behaviors, namely $y(E[\mathbf{p}])$, $E[y(\mathbf{p})]$ are marked with dash-dotted and dashed lines respectively. In this case, y^{-} and $y(E[\mathbf{p}])$ coincide. Bottom: probability of the norm of the off-diagonal term to exceed $y_{lim} = 0.05$.*

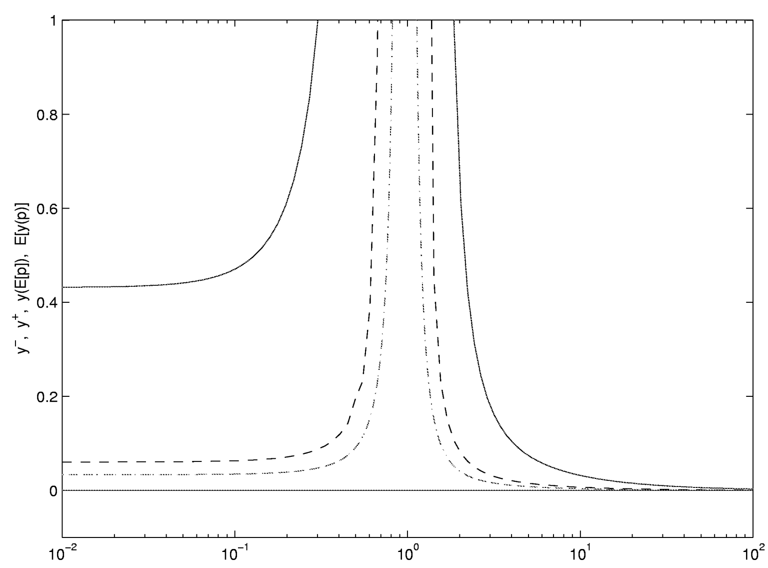

Figure 23: *Results to the P4-problem. Conventions and values used in the previous figure apply.*

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14. ABSTRACT

This paper presents a study on the design of robust controllers by using random variables to model structured uncertainty for both SISO and MIMO feedback systems. Once the parameter uncertainty is prescribed with probability density functions, its effects are propagated through the analysis leading to stochastic metrics for the system's output. Control designs that aim for satisfactory performances while guaranteeing robust closed loop stability are attained by solving constrained non-linear optimization problems in the frequency domain. This approach permits not only to quantify the probability of having unstable and unfavorable responses for a particular control design but also to search for controls while favoring the values of the parameters with higher chance of occurrence. In this manner, robust optimality is achieved while the characteristic conservatism of conventional robust control methods is eliminated. Examples that admit closed form expressions for the probabilistic metrics of the output are used to elucidate the nature of the problem at hand and validate the proposed formulations.

15. SUBJECT TERMS

robust control, probabilistic methods, structured uncertainty, robust optimization

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