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On the Exploitation of Sensitivity Derivatives for Improving Sampling Methods

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Many application codes, such as finite-element structural analyses and computational fluid dynamics codes, are capable of producing many sensitivity derivatives at a small fraction of the cost of the underlying analysis. This paper describes a simple variance reduction method that exploits such inexpensive sensitivity derivatives to increase the accuracy of sampling methods. Three examples, including a finite-element structural analysis of an aircraft wing, are provided that illustrate an order of magnitude improvement in accuracy for both Monte Carlo and stratified sampling schemes.

Introduction

Sampling methods for evaluating moments and distributions of random functions have been used extensively, but relatively little attention has been paid to utilizing sensitivity derivatives of the random function to improve the efficiency of sampling methods. (A sensitivity derivative is the derivative of the dependent random function with respect to one of the independent random variables.)

Recently, Cao, Hussaini and Zang[1] formulated a sampling method for stochastic optimal control problems that exploits the sensitivity derivatives. There appear to have been no previous attempts to exploit derivative information in Monte Carlo methods for uncertainty analysis. For example, this possibility is not mentioned in the recent texts by Fishman[2] and Liu.[3]

A variety of engineering analyses are capable of producing sensitivity derivatives at a small fraction of the cost of the analysis itself. This is certainly true of many applications of finite-element structural analysis. For example, data in Storaasli, Nguyen, Baddourah and Qin[4] (Table 1, p. 350) indicate that for a relatively small finite-element structural model (16,000 DoF), a single derivative can be obtaining in 7% of the time for an analysis. Since this relative time is inversely proportional to problem size, the relative cost of a derivative drops below 1% of the analysis time for 130,000 DoF. The recent development of efficient adjoint solvers for computational fluid dynamics (CFD) codes indicates that aerodynamic sensitivity derivatives can be obtained very efficiently. As one example we cite the work of Carle, Fagan and Green,[5] who reported that they have obtained 88 derivatives for Euler CFD at the cost of 10 analyses. As another, more dramatic example we refer to the work of Sundaram, Agrawal and Hager,[6] who have obtained 400 derivatives at the cost of 10 analyses for viscous, turbulent CFD. Thus, there are important applications in which derivative information, even for tens of parameters, can be obtained at less cost than an analysis. The challenge is to devise sampling methods which exploit this additional inexpensive information to reduce the overall computational cost.

On their optimal control problem application, Cao, Hussaini and Zang (CHZ) demonstrated that, compared with conventional Monte Carlo sampling, exploiting the sensitivity derivatives produced an order of magnitude increase in the efficiency of the sampling method on a model problem with one random variable. The present paper furnishes additional numerical support for the benefits of the use of sensitivity derivatives. In particular, we demonstrate (1) that this improved efficiency is even greater when the baseline sampling scheme is stratified sampling; (2) that improved efficiency is realized on a moderately complex, finite-element analysis of an aircraft wing structure; and (3) that the improved efficiency extends to problems with more than one random variable. On the other hand, whereas CHZ's work was in the context of optimal control problems, our demonstrations are confined to simply the estimation of first and second moments of random functions.

The paper is organized as follows. In Section 2, we summarize the relevant formulation of sensitiv-

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ity derivative-enhanced sampling (SDES) methods from CHZ. In Section 3 we present the verification of our Monte Carlo and stratified sampling procedures on an analytical function of several variables. In Section 4, we demonstrate SDES using stratified sampling on the Burgers equation problem studied by CHZ. Finally, in Section 5, we present results for an aircraft wing structure.

Sensitivity Derivative-Enhanced Sampling Framework

Consider a real-valued function $y(\xi)$, where ξ is a real-valued random variable with probability density function $\rho(\xi)$. We assume that the sensitivity derivatives of y with respect to ξ are available. Let $J(y)$ be a functional of y. The expected value of $J(y)$, denoted by $E(J)$, is given by

$$E(J) = \int J(y(\xi))\rho(\xi)d\xi \; . \tag{1}$$

Let $V(J)$ denote its variance

$$V(J) = \int \Big(J(y(\xi)) \; - \; E(J)\Big)^2 \rho(\xi)d\xi \; . \tag{2}$$

We use $\overline{\xi}$ to denote the mean (or expected) value of ξ. The most straightforward way to compute the expected value of J is to use a Monte Carlo method. The problem with the Monte Carlo method is its slow convergence. It can easily take hundreds or thousands of samples to obtain satisfactory approximation to the moments.

In a Monte Carlo method, the approximation of the integral (1) is given by

$$\hat{E}(J) \approx \frac{1}{N}\sum_{i=1}^{N} J(y(\chi_i)), \tag{3}$$

where $\chi_1, \chi_2, \cdots, \chi_N$ is a sequence of samples of ξ generated according to the density function $\rho(\xi)$. The convergence of (3) is, of course, guaranteed by the large number theorem. But the approximation error in (3) is proportional to $\frac{V(J)}{\sqrt{N}}$. One naturally looks for ways to reduce variance to improve convergence. The current effort exploits the information regarding the sensitivity of the function $J(y(\xi))$ with respect to the stochastic parameter ξ to achieve variance reduction.

Let $J_1(\xi)$ be the linear Taylor expansion of J at $\overline{\xi}$, i.e.,

$$J_1(\xi) = J(y(\overline{\xi})) + J_y(y(\overline{\xi}))y_\xi(\overline{\xi})(\xi - \overline{\xi}), \tag{4}$$

where y_ξ is the sensitivity of y with respect to ξ. Notice that

$$\int \big(J(y(\xi)) - J_1(\xi)\big)\rho(\xi)d\xi = \int J(y(\xi))\rho(\xi)d\xi - J(y(\overline{\xi})).$$

This suggests the following Monte Carlo approximation of $\hat{E}(J)$:

$$\hat{E}(J) \approx J(y(\overline{\xi})) + \frac{1}{N}\sum_{i=1}^{N}\Big(J(y(\chi_i)) - J_1(\chi_i)\Big). \tag{5}$$

The variance of $J(y(\xi)) - J_1(\xi)$ is given by

$$\int \Big(J(y(\xi)) - \overline{J(y(\xi))} - J_y(y(\overline{\xi})) - y_\xi(\overline{\xi})(\xi - \overline{\xi})\Big)^2 \rho(\xi)d\xi, \tag{6}$$

where $\overline{J(y(\xi))}$ is the mean of $J(y(\xi))$. In the following theorem, we use the variance of ξ to estimate the variance of $J - J_1$. Without loss of generality, we assume that ξ is a scalar random variable. CHZ proved the following result, which is repeated here for completeness.

Let $m = \max|\frac{d}{d\xi}J(y(\xi))|$ and $M = \max|\frac{d^2}{d\xi^2}J(y(\xi))|$. The following estimate holds

$$V(J) \leq 2m^2 V(\xi)$$

and

$$V(J - J_1) \leq \frac{M^2}{2}\Big(V^2(\xi) + E\big((\xi - \overline{\xi})^4\big)\Big) \; .$$

Proof: The proof of the first inequality is straightforward. We only provide a proof for the second inequality. By the Taylor remainder formula there exists ξ_1 such that

$$J(y(\phi)) - J(y(\overline{\xi})) = J_y(y(\overline{\xi}))y_\xi(\overline{\xi})(\phi - \overline{\xi}) + \frac{1}{2}\frac{d^2}{d\xi^2}J(y(\xi))|_{\xi=\xi_1}(\phi - \overline{\xi})^2 \; .$$

Since $\overline{\xi}$ is the expectation of ξ, we have that

$$\overline{J(y(\xi))} - J(y(\overline{\xi})) = \int \frac{1}{2}\frac{d^2}{d\xi^2}J(y(\xi))|_{\xi=\xi_1}(\phi - \overline{\xi})^2\rho(\phi)d\phi.$$

Thus

$$|\overline{J(y(\xi))} - J(y(\overline{\xi}))| \leq \frac{M}{2}\int(\phi - \overline{\xi})^2\rho(\phi)d\phi = \frac{M}{2}V(\xi). \tag{7}$$

Using the Tayor remainder formula, we get

$$\begin{aligned}|J(y(\xi)) - J(y(\overline{\xi})) &- J_y(y(\overline{\xi}))y_\xi(\overline{\xi})(\xi - \overline{\xi})| \\ &\le \frac{M}{2}(\xi - \overline{\xi})^2.\end{aligned}$$

Combining (6), (7) and the above inequality yields

$$\begin{aligned}V(J - J_1) &= \int \Big(J(y(\xi)) - J(y(\overline{\xi})) \\ &\quad - J_y(y(\overline{\xi}))y_\xi(\overline{\xi})(\xi - \overline{\xi}) \\ &\quad - (\overline{J(y(\xi))} - J(y(\overline{\xi}))\Big)^2 d\xi \\ &\le 2\int \Big(J(y(\xi)) - J(y(\overline{\xi})) \\ &\quad - J_y(y(\overline{\xi}))y_\xi(\overline{\xi})(\xi - \overline{\xi})\Big)^2 \rho(\xi)d\xi \\ &\quad + 2\int \Big(\overline{J(y(\xi))} - J(y(\overline{\xi}))\Big)^2 \rho(\xi)d\xi \\ &\le 2\int \frac{M^2}{4}(\xi - \overline{\xi})^4 \rho(\xi)d\xi + \frac{M^2}{2}V^2(\xi) \\ &= \frac{M^2}{2}\Big(V^2(\xi) + E\big((\xi - \overline{\xi})^4\big)\Big).\end{aligned}$$

This completes the proof. These results extend to functions of multiple random variables in obvious fashion.

The foregoing analysis indicates that the SDES method is effective when the variance of ξ is small. In these examples, we focus on the expected value and variance of y. In the former case,

$$J_1(\xi) = \overline{y} + y_\xi(\overline{\xi})(\xi - \overline{\xi}) ,$$

and in the latter case,

$$J_1(\xi) = \overline{y}^2 + 2\overline{y}y_\xi(\overline{\xi})(\xi - \overline{\xi}) .$$

Verification of Sampling Procedures

Two different sampling procedures are considered in this work. One is the vanilla Monte Carlo method, given by (3). The other is stratified sampling, which we describe in the case of one random variable. Let $\Phi(\xi)$ denote the cumulative distribution function of the random variable ξ, i.e.,

$$\Phi(\xi) = \int_{-\infty}^{\xi} \rho(\zeta)d\zeta . \tag{8}$$

The function Φ is non-decreasing with range $[0, 1]$. The interval $[0, 1]$ is divided into S strata, assumed here for simplicity to be of equal length:

$$[\eta^s, \eta^{s+1}] \quad s = 0, 1, \ldots, S - 1 , \tag{9}$$

where

$$\eta^s = s/S \quad s = 0, 1, \ldots, S . \tag{10}$$

In the standard stratified sampling method, for each s one chooses N_S random samples, $\psi_i^s, i = 1, \ldots, N_S$, uniformly distributed in $[\eta^s, \eta^{s+1}]$, and computes the corresponding random samples in the variable ξ by inverting the cumulative distribution function:

$$\chi_i^s = \Phi^{-1}(\psi_i^s) \quad i = 1, \ldots, N_S . \tag{11}$$

This procedure assures that the χ_i^s are distributed according to the density function $\rho(\xi)$. The expected value of J is then approximated by

$$\hat{J} \approx \frac{1}{S\ N_S} \sum_{s=0}^{S-1} \sum_{i=1}^{N_S} J(y(\chi_i^s)) . \tag{12}$$

For the SDES version of stratified sampling, one first computes the contribution to (12) from each stratum by an application of (5). In particular,

$$\hat{J} \approx \frac{1}{S} \tilde{J}^s(y) , \tag{13}$$

where

$$\tilde{J}^s(y) = J(y(\overline{\xi^s})) + \frac{1}{N_S} \sum_{i=1}^{N_S} \big(J(y(\chi_i)) - J_1^s(\chi_i)\big) , \tag{14}$$

with

$$J_1^s(\xi) = J(y(\overline{\xi^s})) + J_y(y(\overline{\xi^s}))y_\xi(\overline{\xi^s})(\xi - \overline{\xi^s}) , \tag{15}$$

where $\overline{\xi^s}$ is the mean value of ξ in the s-th stratum, given by

$$\overline{\xi^s} = \int_{\xi^s}^{\xi^{s+1}} \xi\rho(\xi)d\xi \Big/ \int_{\xi^s}^{\xi^{s+1}} \rho(\xi)d\xi , \tag{16}$$

where ξ^s is computed from the η^s from (11). Note that for stratified sampling the SDES method makes more use of sensitivity information than for the Monte Carlo method, i.e., S sensitivity derivatives are used in the former case and only one in the latter. For d random variables the SDES stratified sampling method uses S^d sets of sensitivity derivatives, whereas the SDES Monte Carlo method still just uses a single set of sensitivity derivatives. (A set of sensitivity derivatives consists of the d derivatives of y with respect to the d random variables.)

In order to verify our procedures for Monte Carlo sampling, stratified sampling, and their sensitivity derivative-enhanced variants, we have conducted tests on the simple function of d random variables

$$y(\boldsymbol{\xi}) = \sum_{j=1}^{d} \xi_j \tag{17}$$

We treat the d random variables as independent. In one test case their one-dimensional density function is the Gaussian

$$\rho(\boldsymbol{\xi}) = \prod_{j=1}^{d} \frac{1}{\sqrt{2\pi}\sigma} e^{-(\xi_j - \overline{\xi}_j)^2/2\sigma^2} \ . \tag{18}$$

Here,

$$\begin{aligned} \overline{y} &= d \\ \overline{y^2} &= d^2 + d\sigma^2 \ . \end{aligned} \tag{19}$$

The numerator in the formula (16) for the mean values of ξ in each strata, $\overline{\xi^s}$, are evaluated analytically, and the denominator is computed with the aid of an IMSL routine for the cumulative distribution function for the standard normal distribution. We use $\overline{\xi_i} = 1$ and $\sigma_i = 0.20$ for all variables in this model problem example.

In the other test case the density function is the uniform distribution

$$\rho(\boldsymbol{\xi}) = \begin{cases} 1 & \text{if} \quad \xi_1, \ldots, \xi_d \in [0.5, 1.5] \\ 0 & \text{otherwise} \ , \end{cases} \tag{20}$$

for which

$$\begin{aligned} \overline{y} &= d \\ \overline{y^2} &= d^2 + d/12 \ . \end{aligned} \tag{21}$$

As expected for this linear test function, the SDES results for the estimate of the first moment, $E(y)$, are exact. Indeed, the numerical results for all test cases demonstrate this to the full 14-bit precision of the computations. The numerical results for the estimates of the second moment (variance), $V(y)$, are provided in detail in Table 1 (for the Gaussian distribution) and Table 2 (for the uniform distribution). Demonstrations of the effectiveness of the SDES approach are provided for both Monte Carlo (MC) sampling and stratified sampling with 4 strata ($S = 4$). The column labeled N gives the total number of random samples. (The number of samples in each stratum is $N_S = N/S^d$.) The columns labeled "ratio" give the absolute value of the ratio between the baseline sampling scheme error, using the exact results from (19) or (21), and those of its SDES variant. The average ratios in the legend are the geometric means of the individual results. The ratio is cut off at 1,000 to avoid undue influence from peculiar cases. All computations in this paper were performed in 64-bit arithmetic on a Macintosh G4 with dual 1 GHz. processors.

For the case of 1 random variable, there are roughly 5-fold and 10-fold improvements in accuracy on the variance from the SDES approach for the

Table 1 SDES improvement ratios for second moment estimates of the model problem: Gaussian distribution

	1 variable		4 variables	
N	MC	$S=4$	MC	$S=4$
8	1.71	8.1	592.66	—
16	0.54	6.87	7.16	—
32	1	2.7	24.95	—
64	2.64	0.28	2.09	—
128	2.85	8.47	23.55	—
256	1.25	2.69	2.81	95.7
512	197.62	376.77	6.18	14.19
1024	3.07	16.24	3.84	69.72
2048	1.08	11.82	23.98	38.26
4096	2.3	0.88	24.34	30.71
8192	1000	23.01	32.6	9.15
16384	5.1	23.46	32.01	35.68
32768	8.26	10.62	35.2	618.63
65536	5.76	119.61	6.53	12.97
131072	14.57	10.44	2.1	6.99
262144	—	—	—	123.6
524288	—	—	—	38.22
mean	5.14	9.47	13.61	38.19

Table 2 SDES improvement ratios for second moment estimates of the model problem: uniform distribution

	1 variable		4 variables	
N	MC	$S=4$	MC	$S=4$
8	3.15	14.91	30.86	—
16	0.54	0.61	4.63	—
32	0.83	0.96	166.78	—
64	3.2	26.32	4.46	—
128	3.67	16.58	15.4	—
256	2.8	13.42	6.73	30.81
512	151.14	564.02	5.54	12.47
1024	7.87	50.55	1.21	239.58
2048	1.62	12.39	19.69	45.92
4096	3.34	0.75	16.42	46.29
8192	65.04	375.96	13.79	45.5
16384	33.63	204.5	166.58	87.92
32768	3.75	17.68	28.49	70.82
65536	17.23	74.33	4.42	124.16
131072	93.22	200.12	0.91	101.66
262144	—	—	—	52.38
524288	—	—	—	34.72
mean	6.91	23.04	11.30	57.37

Monte Carlo and stratified sampling schemes (with 4 strata), respectively. For the case of 4 random variables, the improvements in accuracy are more than twice as great. However, since there are 256 strata in this case, the cost of the sensitivity derivatives is somewhat greater, but for, say, the $N = 8192$ case in Table 1, there are 256 evaluations of the function plus its sensitivity derivatives and 256×32 simple function evaluations. The cost of the sensitivity derivatives is a negligible component of the total cost.

Demonstration on a Solution of Burgers Equation

CHZ's main example was based on the generalized steady-state Burgers equation,[7]

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(\nu \frac{\partial y}{\partial x}\right) \quad \text{for} \quad x \in (-\frac{1}{2}, \frac{1}{2}) \\ f(y) &= \frac{1}{2} y(1-y) \\ y(-\frac{1}{2}) &= \frac{1}{2}\big(1 + \tanh(\frac{-1}{8\nu})\big) \\ y(\frac{1}{2}) &= \frac{1}{2}\big(1 + \tanh(\frac{1}{8\nu})\big) \end{aligned} \tag{22}$$

This equation has the exact solution

$$y(x;\nu) = \frac{1}{2}\big(1 + \tanh(\frac{x}{4\nu})\big) \tag{23}$$

The parameter ν (viscosity) is treated as a random variable. As a result, the solution $y = y(x;\nu)$ of Burgers equation is also a random function. Whereas CHZ considered this problem in the context of optimal control, here we confine ourselves to just the estimation of the first and second moments.

We again consider both a Gaussian distribution and a uniform distribution. To conform to the cases from CHZ, the parameters of this Gaussian are $\overline{\nu} = 2.0$ and $\sigma = 0.1$, with the Gaussian cut-off below $\nu = 0.1$ and above $\nu = 3.9$. For this reason we use a quadrature formula (Simpson's rule with 100 intervals) to evaluate the integrals in (16). The uniform distribution case has the probability density function

$$\rho(\xi) = \begin{cases} 1 & \text{if} \quad \xi \in [0.1, 0.3] \\ 0 & \text{otherwise}\,, \end{cases} \tag{24}$$

We revisit the example here to illustrate the improvement that the SDES method provides over the baseline stratified sampling scheme. We include conventional Monte Carlo (MC) for reference. Our figure of merit is the root mean square (RMS) error

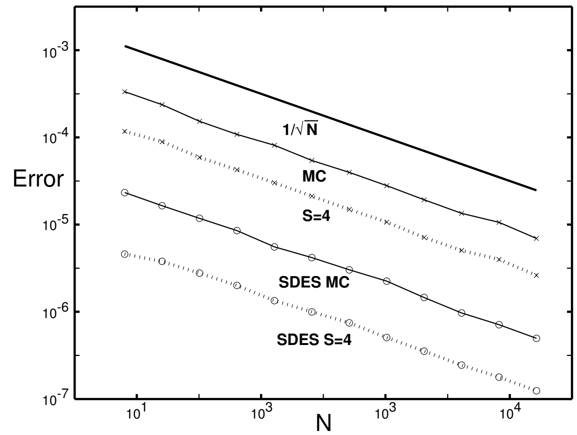

Fig. 1 Errors in the second moment estimates for the Burgers problem: Gaussian distribution

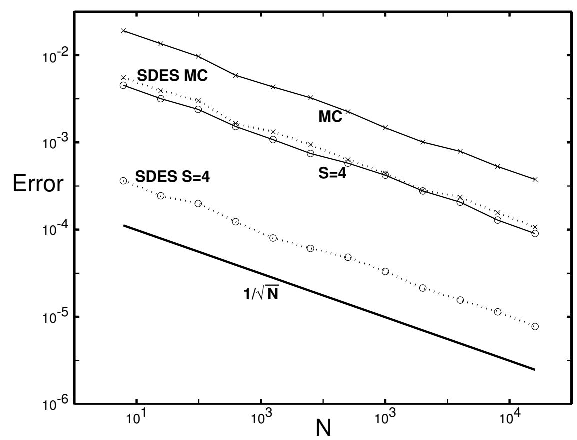

Fig. 2 Errors in the second moment estimates for the Burgers problem: uniform distribution

(in x) of the approximation of the second moment of $y(x;\nu)$:

$$E(y^2(x)) = \int y^2(x;\nu)\rho(\nu)d\nu\ .$$

The "exact" value of the second moment, as well as the root mean square errors, are evaluated by numerical quadrature. (In our numerical examples, we use Simpson's rule with 100 intervals to compute the "exact" values of the second moments at each x and 40 intervals to compute the RMS errors over x.) Moreover, in order to reduce the statistical fluctuations in this quantity, we average this error over 20 independent computations for each value of N, the number of random samples of ν. Figures 1 and 2 illustrate the errors as a function of N for Monte Carlo and for stratified sampling with $S = 4$, both with and without the SDES procedure. The heavy solid line illustrates $1/\sqrt{N}$ decay. The results in-

Table 3 SDES improvement ratios for second moment estimates of the Burgers problem: Gaussian distribution

N	MC ratio	$S = 4$ ratio	$S = 8$ ratio
8	14.33	25.53	28.72
16	14.41	23.55	28.94
32	12.95	21.24	22.34
64	12.67	21.19	23.29
128	14.63	22.53	23.68
256	12.98	21.22	22.68
512	13.09	19.98	21.91
1024	12.42	21.05	22.48
2024	13.23	20.11	22.88
4048	13.89	20.65	23.02
8192	14.88	22.36	23.59
16384	13.97	21.06	23.75
mean	13.60	21.66	23.84

Table 4 SDES improvement ratios for second moment estimates of the Burgers problem: uniform distribution

N	MC ratio	$S = 4$ ratio	$S = 8$ ratio
8	4.23	15.18	29.59
16	4.27	16.05	31.91
32	4.06	15.16	31.13
64	3.86	13.40	27.22
128	4.02	16.48	32.49
256	4.36	15.44	31.03
512	3.90	13.15	25.54
1024	3.50	13.42	25.75
2024	3.63	13.19	25.63
4048	3.82	15.18	30.42
8192	4.12	13.63	26.94
16384	4.17	13.81	26.44
mean	3.99	14.46	28.56

dicate that (1) the errors of all methods decay at the expected $1/\sqrt{N}$ rate; (2) the stratified sampling methods afford a significant improvement over conventional Monte Carlo; and (3) the SDES approach achieves an order of magnitude reduction in the error. Tables 3 and 4 document further the final point. They report the ratio of the RMS errors for the baseline schemes to the RMS error for their SDES counterparts. The mean values of the ratio in the bottom row of the tables are the geometric means of the individual values. The corresponding results for the RMS error in the first moments (not shown here) give ratios that are about 10% greater than those for the second moment.

Aircraft Wing Structure Application

The final numerical example is for a structural analysis problem using finite-element analysis. The specific problem is taken from the work of Gumbert, Hou and Newman[8] (GHN). The tapezoidal-planform, semispan wing is illustrated in Figure 3. The wing is divided into 6 zones, marked by the different colors in the figure, with zone 1 near the wing root. The airfoil sections vary linearly from a NACA 0012 section at the root to a NACA 0008 section at the tip.The finite-element model consists of 583 nodes, with 2,141 constant-strain triangle (CST) elements and 1,110 truss elements. Linear elasticity is assumed. We adopt the same grouping of structural thicknesses as used by GHN. In the case of the CST elements, which are all that are considered in the present work, there are 3 parameters for each zone. Whereas, for GHN, these parameters were design variables, for us they are the random variables. These parameters are multiplicative factors for the baseline values of the element thicknesses. For example, variable 1 is the thickness multiple for the skin elements in zone 1, variable 2 is the thickness multiple for the web elements of the ribs in zone 1, and variable 3 is the thickness multiple for the web elements of the spars in zone 1, variable 4 is the thickness multiple for the skin elements in zone 2, etc. These scaling parameters are denoted by $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d)$, where d denotes the number of parameters used in the particular case. The output functional of the analysis, $y(\xi)$, is the compliance, which is the work done by the aerodynamic pressure to deflect the structure. It is given by the integral over the wing of the aerodynamic pressures times the structural displacements. The wing leading edge has a 9.46° sweep, a root of 20 ft. and a span of 60 ft. The trailing edge is unswept. The pressures

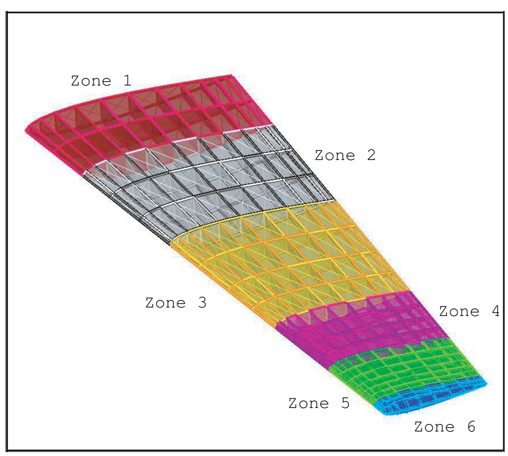

Fig. 3 Aircraft wing structural model

were based on a static aeroelastic computation computation (using Euler CFD and the finite-element structural analysis) of the flow past the rigid wing at a freestream Mach number of 0.80 and an angle-of-attack of 1°. The baseline values of the first 4 parameters are 0.188 in, 0.0375 in., 0.1200 in. and 0.125 in. The compliance is 213285 lbs.-in., and the sensitivities of the compliance with respect to these 4 parameters are -396777, -1040, -591496 and -498708 lbs.-in./in., respectively. The second variable (the thickness of the web of the rib in zone 1 contributes relatively little to the compliance. This is readily understandable on physical grounds as most of the effect of the aerodynamic load is felt by the spars and the skin.

The finite-element code is described by Hou, Arunkumar and Tiwari.[9] It was developed under contract to NASA Langley Research Center to enable basic research on simultaneous, coupled aero-structural optimization using first- and second-order sensitivity derivatives of both sizing and shape variables. (See GHN and their earlier papers for the optimization applications.) Gumbert, Newman and Hou[10] have recently used this code for uncertainty analysis applications as well.

Numerical examples for this problem are given for both a uniform distribution of the scaling parameter ξ on $[0.5, 1.5]$ and a Gaussian distribution with unit mean and a standard deviation of 0.20. Since a single structural analysis for this problem takes roughly 1/5 sec. of CPU time, the accuracy of the "exact" solution for the second moment of the compliance is limited. We used the result for a stratified sampling computation with 4 strata and a total of 65536 samples as the "exact" value. For the 1 variable cases we estimate that this result is accurate to 1 part in 10^7 for the uniform distribution results and 1 part in 10^6 for the Gaussian distribution results. The accuracy for the estimates of the "exact" second moments for the 4 variable cases are roughly 1 order of magnitude worse.

Tables 5 and 6 give the results for the Gaussian and uniform distributions, respectively, in terms of the ratio of the estimated errors from the conventional sampling computations to those of their SDES counterparts. Cases are included for both 1 and 4 random variables. Certainly, there is a very substantial gain produced by the SDES method for the 1 variable case. The improvement is not as dramatic for the 4 variable cases, although still significant. More analysis is required to understand why the SDES improvements for these 4-variable cases are not as great as those for the model problem.

Table 5 SDES improvement ratios for second moment estimates of the aircraft wing structure problem: Gaussian distribution

	1 variable		4 variables	
N	MC	$S=4$	MC	$S=4$
8	13.21	44.51	1.08	—
16	3.07	12.79	0.99	—
32	9.94	29.89	5.91	—
64	17.45	15.55	4.23	—
128	19.45	62.05	3.85	—
256	0.06	5.07	2.97	—
512	798.21	598.69	1.92	2.83
1024	8.12	38.57	12.30	0.44
2048	10.12	44.99	2.01	2.39
4096	4.88	3.98	14.04	4.36
8192	1000.00	47.83	6.51	4.23
16384	26.84	—	2.79	148.37
32768	—	—	—	0.88
mean	14.27	22.46	3.53	3.56

Table 6 SDES improvement ratios for second moment estimates of the aircraft wing structure problem: uniform distribution

	1 variable		4 variables	
N	MC	$S=4$	MC	$S=4$
8	19.63	106.38	0.90	—
16	3.05	10.51	5.50	—
32	9.20	41.48	6.52	—
64	19.08	67.63	1.92	—
128	22.79	105.96	2.37	—
256	6.88	23.20	17.47	—
512	1000.00	361.64	2.27	3.80
1024	26.10	70.98	9.54	2.37
2024	12.94	50.55	5.61	3.81
4048	8.87	51.44	12.08	13.91
8192	346.19	347.98	4.60	2.20
16384	81.83	369.27	2.15	15.16
32768	12.03	37.86	9.78	30.64
65536	81.15	—	2.72	8.13
131072	—	—	—	41.23
mean	27.44	55.30	4.39	8.18

Concluding Remarks

In conclusion, we have furnished numerical results attesting to the advantage of exploiting sensitivity derivatives in sampling schemes. The examples range from an analytic model problems to full finite-element structural analyses. For a fixed number of samples there is typically an order of magnitude reduction in the error achieved by the sensitivity derivative-enhanced sampling approach. Equivalently, SDES computations require two orders of magnitude fewer samples to achieve the same accuracy in the moments compared with baseline Monte Carlo and stratified sampling schemes. The overhead for the extra sensitivity derivative calculations is less than 5%, even for the structural analysis example.

The near-term next steps in this project include: (1) studying the 4-variable structural analysis examples more thoroughly in order to obtain definitive values for the SDES improvement; (2) evaluating the impact of SDES on Latin hypercube sampling in order to handle a larger set of random variables in the structural problem; and (3) extending the SDES to exploit the semi-analytic second-order sensitivity derivatives available from some codes. For example, the particular finite-element code used for this work was chosen because it has a second-order sensitivity capability.[9] We should note that there has been some promising work on obtaining second-order sensitivity derivatives efficiently from CFD codes.[11,12]

In the present work, we have only made very minor use of derivative information, and have obtained a significant speed-up over two conventional sampling methods. This suggests that more attention should be devoted to exploiting relatively inexpensive sensitivity derivatives in traditional sampling methods. The long-term research challenge is to make even better use of derivative information in otherwise conventional sampling methods.

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