A NEW STOCHASTIC EQUIVALENT LINEARIZATION IMPLEMENTATION FOR PREDICTION OF GEOMETRICALLY NONLINEAR VIBRATIONS

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Abstract

In this paper, the problem of random vibration of geometrically nonlinear MDOF structures is considered. The solutions obtained by application of two different versions of a stochastic linearization method are compared with exact (F-P-K) solutions.

The formulation of a relatively new version of the stochastic linearization method (energy-based version) is generalized to the MDOF system case. Also, a new method for determination of nonlinear stiffness coefficients for MDOF structures is demonstrated. This method in combination with the equivalent linearization technique is implemented in a new computer program.

Results in terms of root-mean-square (RMS) displacements obtained by using the new program and an existing in-house code are compared for two examples of beam-like structures.

Introduction

Resurgent interest in high speed fight vehicles and the daily operation of the aging commercial and military aircraft fleets necessitate the further development of sonic fatigue technology to understand the fatigue mechanisms and to estimate the service life of aerospace structures subjected to intense acoustic and thermal loads. Efforts to extend the performance and flight envelope of high speed aerospace vehicles have resulted in structures which may behave in a geometrically nonlinear fashion to the imposed loads. Such behavior can have a significant effect on fatigue life. Further improvements in vehicle performance and system design are hampered by the limited understanding of the physical nature of geometrically nonlinear

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structural response. Conventional (linear) prediction techniques can lead to grossly conservative designs and provide little understanding of the nonlinear behavior. A large body of work exists on the prediction of geometrically nonlinear dynamic response of structures. All methods currently in use are typically limited by their range of applicability or excessive computational expense.

Methods currently in use to predict geometrically nonlinear dynamic structural response include perturbation, Fokker-Plank-Kolmogorov (F-P-K), Monte Carlo simulation and stochastic linearization techniques. Perturbation techniques are limited to weak geometric nonlinearities. The F-P-K approach[1,2] yields exact solutions, but can only be applied to simple mechanical systems. Monte Carlo simulation is the most general method, but computational expense limits its applicability to rather simple structures. Finally, stochastic linearization methods (e.g. equivalent linearization, see[2-6]) have seen the most broad application for prediction of geometrically nonlinear dynamic response because of their ability to accurately capture the response statistics over a wide range of response levels while maintaining a relatively light computational burden.

Implementations of stochastic linearization have been limited to special purpose computer codes until recently when the method of equivalent linearization was introduced into MSC/NASTRAN as a Direct Matrix Abstraction Program (DMAP) Alter[7]. In this study an alternative approach to the solution of nonlinear vibration problems is developed and an independent in-house code based on this approach is implemented.

Equivalent Linearization Techniques

Two versions of the equivalent linearization technique are considered. One is based on minimization of the error in the force-vector, and the other minimizes the error in potential energy.

Force Error Minimization Version

Consider a MDOF, viscously damped linear system. The equations of motion governing such a system can be written in the form

$$M\ddot{X} + C\dot{X} + KX = F$$

where M is the mass matrix, C is the damping matrix, K is the stiffness matrix, X is the displacement response vector and F is the force excitation vector.

For geometrically nonlinear problems of deformation, e.g. large deflection flexural vibration of thin plate structures, the governing equation(s) of motion will include a nonlinear force term, (X), i.e.

$$M\ddot{X} + C\dot{X} + KX + , (X) = F \qquad (1)$$

where the vector function $, (X)$ generally includes 2nd and 3rd order terms in X. There exist mathematical difficulties in the derivation of a general solution to equation (1) for the case of random excitation. An approximate solution can be achieved by formation of an equivalent linear system:

$$M\ddot{X} + C\dot{X} + (K + K_e)X = F \qquad (2)$$

where K_e is the equivalent linear stiffness matrix.

The method of equivalent linearization seeks to minimize the difference between the nonlinear force and the product of the equivalent linear stiffness and displacement response vector. The equivalent linear stiffness satisfying this requirement can be determined from the following condition:

$$error = E\left[(, (X) - K_eX)^T (, (X) - K_eX)\right] \rightarrow min$$

where $E[...]$ represents the expectation operator. The latter equation will be satisfied if

$$\frac{\partial(error)}{\partial K_{eij}} = 0 \qquad i,j = 1,2,...,N$$

In this paper, consideration is limited to the case of Gaussian, zero-mean excitation and response to simplify the solution. Omitting intermediate derivations, the final form for the equivalent linear stiffness matrix becomes (see for example Roberts et al.[3], Atalik et al.[4]):

$$K_e = E[\frac{\partial,}{\partial X}] \qquad (3)$$

Potential Energy Error Minimization Version

Elishakoff et al.[5,6] proposed another stochastic linearization approach, based on potential energy error minimization and numerical results were demonstrated for the case of SDOF systems. In this paper, that approach is generalized for the case of MDOF systems. One can begin with an expression for the error in potential energy ϵ

$$\epsilon = E[(U(X) - \frac{1}{2}X^T K_e X)^2]$$

where $U(X)$ is the potential energy of the original (nonlinear system) and K_e is the stiffness matrix of the equivalent linear system.

A condition of minimized error $\epsilon \rightarrow min$ requires the following

$$E[\frac{\partial}{\partial K_{eij}}((U(X) - \frac{1}{2}X^T K_e X)^2)] = 0 \qquad i,j = 1,N$$

where K_{eij} are elements of matrix K_e. Omiting intermediate derivations, one obtains the following system of N^2 linear equations with respect to unknown elements of matrix K_e:

$$\sum_{i,j=1}^{N} K_{eij} E[x_i x_j x_k x_l] = 2E[x_k x_l U(X)] \qquad (4)$$

where $k,l = 1,N$.

For example, equation (4) would have the following form for a two-degree-of-freedom system

$$\begin{bmatrix} E[x_1^4] & E[x_1^3 x_2] & E[x_1^3 x_2] & E[x_1^2 x_2^2] \\ E[x_1^3 x_2] & E[x_1^2 x_2^2] & E[x_1^2 x_2^2] & E[x_1 x_2^3] \\ E[x_1^3 x_2] & E[x_1^2 x_2^2] & E[x_1^2 x_2^2] & E[x_1 x_2^3] \\ E[x_1^2 x_2^2] & E[x_1 x_2^3] & E[x_1 x_2^3] & E[x_2^4] \end{bmatrix} \begin{bmatrix} K_{e11} \\ K_{e12} \\ K_{e21} \\ K_{e22} \end{bmatrix}$$

$$= 2 \begin{bmatrix} E[x_1^2 U(X)] \\ E[x_1 x_2 U(X)] \\ E[x_1 x_2 U(X)] \\ E[x_2^2 U(X)] \end{bmatrix} \qquad (5)$$

Note that the 2nd and 3rd rows are identical, thus an additional equation is required to solve this system. The additional equation(s) can be provided by the imposition of a condition of symmetry of the matrix K_e:

$$K_{eij} = K_{eji} \qquad (6)$$

The matrix of the system in equation (5) involves 4th order moments of displacements and the right-hand side (assuming that the potential energy is a function of the 2nd, 3rd and 4th order terms) involves moments of 4th, 5th and 6th order. Using the Gaussian distributed, zero-mean response assumption means that the odd order moments are zero and the higher even order statistical moments can be expressed in terms of the 2nd order moments, e.g.,

$$E[x_i x_j x_k x_l] = E[x_i x_j]E[x_k x_l] + E[x_i x_k]E[x_j x_l]+$$

$$E[x_i x_l]E[x_j x_k]$$

Therefore the matrix and right-hand side of (5) can be determined solely by the response covariance matrix. So the equivalent stiffness matrix at each iteration is determined through the use of response covariance terms from the previous iteration by solving equations (5).

Iterative scheme of equivalent linearization

Having defined the equivalent linear stiffness matrix through either the force error or potential energy error minimization techniques, one can proceed with the solution of the equivalent linear system. Assuming stationary excitation, a stationary response is sought precluding the need for initial conditions. As the equivalent stiffness matrix K_e is a function of the unknown displacement response vector, the solution to the system of equations of motion takes an iterative form, i.e.

$$M\ddot{X}_{n+1} + C\dot{X}_{n+1} + (K + K_{en})X_{n+1} = F \qquad (7)$$

where new displacement response estimates are calculated from a system based upon the previous estimate and iterations are continued until a convergence criterion is satisfied.

The solution to the equivalent system in equation (7) for each iteration can be obtained in the frequency domain using the well known relation between the spectral density matrices for a linear system:

$$S_x(\omega)_{n+1} = H_n(\omega)S_f(\omega)\overline{H}_n^T(\omega)$$

where the over-bar indicates the complex-conjugate, S_f is the spectral density matrix of the random excitation and the frequency response matrix is given by

$$H_n(\omega) = [-\omega^2 M + i\omega C + K + K_{en}]^{-1}$$

The zero-time-lag covariance matrix components participating in the matrix K_{en} are calculated from the response spectral density matrix using the Wiener-Khinchine formula

$$E[x_i x_j]_n = \int_{-\infty}^{\infty} [S_{xij}(\omega)]_n d\omega$$

An implementation of the equivalent linearization approach outlined above was recently implemented in a special purpose in-house code to provide a tool for expedient study.

Comparison with F-P-K Solutions

The two equivalent linearization methods presented above will be compared with F-P-K solutions for SDOF and 2DOF systems.

SDOF system

Consider a SDOF system (Duffing oscillator):

$$\ddot{q}(t) + 2\xi\omega_0\dot{q}(t) + \omega_0^2 q(t) + \beta q^3(t) = f(t) \qquad (8)$$

where q is a nondimensional coordinate/displacement. The addition to the potential energy originated from the nonlinear term is characterized by

$$U(q) = \frac{1}{4}\beta q^4 \qquad (9)$$

For this case, solution of the system (5) (energy-based technique) provides the following equivalent stiffness

$$k_e = 2.5\beta E[q^2]$$

and equation (3) (force-base technique) yields

$$k_e = 3\beta E[q^2]$$

Comparison of response variances for this system versus the nonlinearity parameter $\frac{\beta}{\omega_0^2}$ is illustrated in Figure 1. A white noise excitation was taken as the input spectral density function, i.e. $S_f(\omega)$ was constant and equal to $1.0e+05$. The results correspond to an oscillator with a natural frequency of $57.4\ Hz$ ($\omega_0^2 = 1.301e+05\ s^{-2}$) and damping coefficient $\xi = 0.005$. The three curves in Figure 1 correspond to the F-P-K solution, force error minimization and energy error minimization versions.

Comparison of response variances for this oscillator versus the spectral density function value S_f is illustrated in Figure 2, where the nonlinearity parameter $\frac{\beta}{\omega_0^2}$ was fixed and equal to 10. One can see that the energy error minimization version results are closest to the exact (F-P-K) solution results.

2DOF system

As a next example, consider the model of a 2DOF system in Figure 3. The equations of motion for this model have the form

$$m_1\ddot{q}_1 + c_1\dot{q}_1 + k_1 q_1 + k_2(q_1 - q_2) + \beta_1 q_1^3 + \beta_2(q_1 - q_2)^3 = f_1(t)$$

$$m_2\ddot{q}_2 + c_2\dot{q}_2 + k_2(q_2 - q_1) + \beta_2(q_2 - q_1)^3 = f_2(t) \quad (10)$$

The potential energy contribution from the nonlinear part is

$$U(q_1, q_2) = \frac{1}{4}\beta_1 q_1^4 + \frac{1}{4}\beta_2(q_2 - q_1)^4$$

White noise was taken again as the input excitation with the spectral density matrix components: $S_{f11}(\omega) = S_{f22}(\omega) = 1$, and $S_{f12}(\omega) = S_{f21}(\omega) = 0$. The rest of the parameters of this model were as follows $m_1 = m_2 = 1$, $k_1 = k_2 = 1$, $c_1 = c_2 = 0.1$ and $\beta_1 = \beta_2 = \beta$. A comparison of response variances versus the nonlinearity parameter β for this 2DOF system (10) is shown in Figure 4. Again the energy-based version results are closer to the exact F-P-K solution, than the force-based version results.

Note that in the case of general MDOF nonlinear systems, the determination of the expression for the potential energy can be complicated. This problem will be addressed in the section below.

Determination of Nonlinear Stiffness Coefficients

So far, examples were considered where the nonlinear stiffness coefficients were prescribed. In a general case of a MDOF system, these coefficients have to be determined. One method of determining the nonlinear stiffness coefficients is through the use of a finite element approach. Existing finite element commercial programs are unable to provide these nonlinear stiffness coefficients directly. It is desirable to achieve a solution within a commercial finite element code to take advantage of the comprehensive element library, etc. necessary for modeling complex structures. This section describes a method of determining the nonlinear stiffness coefficients through the use of the nonlinear static solution capability that exists in many commercial finite element codes.

For MDOF structures, it is expedient to seek a solution in modal coordinate space

$$X = \Phi q \qquad (11)$$

where Φ is generally a subset ($L \leq N$) of the linear eigenvectors (normal modes). Such a representation allows the size of the problem to be significantly reduced without a noticeable loss of accuracy in many cases.

One can obtain the following set of differential equations in terms of modal coordinates q_i ($i = 1, L$):

$$\ddot{q}_i(t) + \sum_{j=1}^{L} c_{ij} \dot{q}_j(t) + k_i q_i(t) + \gamma_i(q_1, q_2, ..., q_L) = f_i(t) \qquad (12)$$

where the nonlinear terms will be represented in the following form

$$\gamma_i(q_1, q_2, ..., q_L) = \sum_{j,k=j}^{L} a_{jk}^i q_j q_k + \sum_{j,k=j,l=k}^{L} b_{jkl}^i q_j q_k q_l \qquad (13)$$

where the first index j takes values 1,2,...,L, the index k takes values from j (the current first index value) and up $j + 1, j + 2$ ... to L and the third index l takes values from k (the current second index value) and up $k + 1$, $k + 2$ to L.

The analytical form of the nonlinear terms facilitates the solution of equations (12) when the forces and displacements are random functions of time.

A procedure for determination of the coefficients a_{jk}^i and b_{jkl}^i is described briefly. This procedure requires the application of a finite element program with a nonlinear static solution capability. In this study, the MSC/PATRAN and MSC/NASTRAN programs[9,10] are utilized.

The suggested technique is based on the restoration of nodal applied forces from enforced nodal displacements prescribed to the whole structure in a static solution (linear and nonlinear). Namely, by prescribing the physical nodal displacements (vector X_c) to the structure, one can restore the nodal forces F_T and the corresponding nonlinear contribution F_c:

$$F_c = , (X_c) = F_T - K X_c \qquad (14)$$

The displacements X_c can be prescribed by creating a displacement constraint set for the model in PATRAN, then the nodal applied forces F_T will arise as single-point-constraint forces in a NASTRAN nonlinear static solution.

To illustrate the technique, one can begin with the prescription of displacements for the whole structure in the following form

$$X_c = \phi_1 q_1 \qquad (15)$$

The nodal force vectors F_T (nonlinear static solution) and $K X_c$ (linear static solution) are provided by NASTRAN. The nonlinear term F_c can then be evaluated by equation (14). The vector of modal forces $\tilde{F}_c = \Phi^T F_c$ is calculated and it is represented as

$$\tilde{F}_c = \Phi^T F_c = \Phi^T , (X_c) = \Phi^T , (\phi_1 q_1) =$$
$$[a_{11}^i] q_1 q_1 + [b_{111}^i] q_1 q_1 q_1 \qquad (16)$$

where the sought stiffness coefficients $[a_{11}^i]$, $[b_{111}^i]$ are column-vectors $L \times 1$ ($i = 1, L$). Note that all other nonlinear terms in (16) do not appear since $q_j = 0$ for $j \neq 1$.

Prescribing a displacement field with opposite sign $X_c = -\phi_1 q_1$ results in a modal force vector (denoted by $\tilde{F}_{-c}$):

$$\tilde{F}_{-c} = \Phi^T F_{-c} = \Phi^T , (X_c) =$$
$$\Phi^T , (-\phi_1 q_1) = [a_{11}^i] q_1 q_1 - [b_{111}^i] q_1 q_1 q_1 \qquad (17)$$

where the quadratic (even) term will be the same as in (16) and the cubic (odd) term takes on a sign change.

Note that in the system of equations (16) and (17), the value of q_1 is given. The coefficients $[a_{11}^i]$, $[b_{111}^i]$ ($i = 1, L$) can be determined from this system of $2 \times L$ linear equations. In an analogous manner, i.e. prescribing $X_c = \phi_j q_j$, all other coefficients $[a_{jj}^i]$, $[b_{jjj}^i]$ can be determined.

A similar technique can be employed to determine coefficients with two or three inequal lower indices, e.g., $[a_{12}^i],[b_{112}^i]$, $[b_{122}^i]$ or $[b_{123}^i]$. Note that coefficients of the latter type appear only if the number of retained eigenvectors L in (11) is greater than or equal to 3. Determination of coefficients $[a_{12}^i],[b_{112}^i]$ and $[b_{122}^i]$ will be considered as an example. Prescribe the displacement field to the model in the following form

$$X_c = \phi_1 q_1 + \phi_2 q_2$$

then the calculated (using NASTRAN) corresponding modal force vector $\tilde{F}_c$ is represented as follows

$$\tilde{F}_c = \Phi^T F_c = \Phi^T , (\phi_1 q_1 + \phi_2 q_2) =$$

$$[a^i_{11}]q_1q_1 + [b^i_{111}]q_1q_1q_1 + [a^i_{22}]q_2q_2 + [b^i_{222}]q_2q_2q_2 +$$
$$[a^i_{12}]q_1q_2 + [b^i_{112}]q_1q_1q_2 + [b^i_{122}]q_2q_2q_1 \qquad (18)$$

Prescribing the opposite sign displacement field

$$X_c = -\phi_1q_1 - \phi_2q_2$$

one obtains a second set of equations

$$\tilde{F}_{-c} = \Phi^T F_{-c} = \Phi^T, \, (-\phi_1q_1 - \phi_2q_2) =$$

$$[a^i_{11}]q_1q_1 - [b^i_{111}]q_1q_1q_1 + [a^i_{22}]q_2q_2 - [b^i_{222}]q_2q_2q_2 +$$
$$[a^i_{12}]q_1q_2 - [b^i_{112}]q_1q_1q_2 - [b^i_{122}]q_2q_2q_1 \qquad (19)$$

Summing (18) and (19), one obtains

$$\tilde{F}_c + \tilde{F}_{-c} = 2[a^i_{11}]q_1q_1 + 2[a^i_{22}]q_2q_2 + 2[a^i_{12}]q_1q_2$$

From this equation, the coefficients $[a^i_{12}]$ are determined (note that the coefficients $[a^i_{11}]$, $[a^i_{22}]$ were already determined above).

Now we have two sets of L equations (18) and (19), but to determine cubic coefficients $[b^i_{112}]$ and $[b^i_{122}]$ from them is not possible since the system matrix has linearly dependent rows. Therefore, an additional type of displacement field is required. One can prescribe the following type

$$X_a = \phi_1q_1 - \phi_2q_2$$

Then the modal force vector is equal to

$$\tilde{F}_a = \Phi^T F_a = \Phi^T, \, (\phi_1q_1 - \phi_2q_2) =$$

$$[a^i_{11}]q_1q_1 + [b^i_{111}]q_1q_1q_1 + [a^i_{22}]q_2q_2 - [b^i_{222}]q_2q_2q_2 -$$
$$[a^i_{12}]q_1q_2 - [b^i_{112}]q_1q_1q_2 + [b^i_{122}]q_2q_2q_1 \qquad (20)$$

From the system of $2*L$ linear equations (18) and (20), the coefficients $[b^i_{112}]$ and $[b^i_{122}]$ can be determined. In a similar manner, all coefficients of the type $[b^i_{jjk}]$ and $[b^i_{kkj}]$ can be determined. A technique has been developed to determine all the coefficients $[a^i_{jk}]$, $[b^i_{jkl}]$ using a similar approach as above.

Solution of modal equations

Having the modal equations of motion (12) formulated, solution to these equations can now be undertaken through a variety of techniques. For the case of random loading, the application of the equivalent stochastic linearization was implemented in this study. Within the framework of the force-based technique, the equivalent stiffness matrix (according to the formula (3)) will have the following form

$$K_e = E[\frac{\partial(\gamma_1, \gamma_2, ..., \gamma_L)}{\partial(q_1, q_2, ..., q_L)}] \qquad (21)$$

Note that the derivatives and expectations in (21) can be easily evaluated due to the analytical representation

of the nonlinear terms in (13). A program producing the calculations described above has been developed and numerical results will be demonstrated in the next section.

Based upon the expressions derived in equation (13), one can proceed with the determination of potential energy U in terms of modal coordinates. It is known that elastic force terms (linear + nonlinear) satisfy the following

$$k_iq_i + \gamma_i(q_1, q_2, ..., q_L) = \frac{\partial U}{\partial q_i} \qquad i = 1, L \qquad (22)$$

Since all nonlinear coefficients in $\gamma_i(q_1, q_2, ..., q_L)$ have been determined, the potential energy function $U(q_1, q_2, ..., q_L)$ can be derived and it can be used in the energy-based stochastic linearization technique. An implementation of the energy-based version for application to MDOF systems is considered as future work.

Numerical Results for MDOF Structures

It is important to note that the analysis of a vibrating structure in the nonlinear setting is necessary only if the comparison of two static solutions (linear and nonlinear ones for the highest deformation level) shows a noticeable difference in the displacement fields. To illustrate this, one can consider the three beam structures shown in Fig.5a-c. It was found that for a cantilevered beam model (Fig. 5c), the difference in terms of static flexural displacements is negligible. Two curves (Fig.6) corresponding to the linear and nonlinear models are indistinguishable, where the flexural displacement of the tip node versus the applied static base acceleration (inertial loading) is plotted. However for a beam in Fig. 5b and clamped-clamped beam (Fig.5a) the difference in displacements is quite noticeable (see Fig. 6,7), indicating that the vibration analysis should be conducted in the nonlinear setting.

The numerical results presented in this section correspond to models of the structures in Figures 5a and b. The results obtained with the SEMELRR DMAP[7] are compared with the new method which employs the technique described above, i.e. the determination of nonlinear stiffness coefficients plus the conventional (force-based) stochastic linearization technique.

SEMELRR was implemented in MSC/NASTRAN using equivalent linear modal degrees of freedom. This requires an eigensolution at each iteration, but affords the most simple and versatile procedure readily adapted within the framework of the existing MSC/NASTRAN solution sequences. The original implementation was limited to spatially uniform mechanical loads, but has since been generalized to include spatial non-uniformity. The solution is also formulated to include the effects of static deflection due to mechanical or thermal loads, material nonlinearity and follower forces. Some work has been done to validate the prediction capability of SEMELRR, see Robinson et al.[7],

material	Young's modulus	Poisson's ratio	density
aluminum	0.73e+11	0.325	0.2763e+04
length beam a)	length beam b)	width	thickness
0.4572	0.2286	0.0254	0.002261

Table 1: Parameters of beams a) and b)

b_{111}^1	b_{222}^1	b_{112}^1	b_{122}^1
0.899e+12	0.977e+13	0.191e+13	0.139e+14
b_{111}^2	b_{222}^2	b_{112}^2	b_{122}^2
0.638e+12	0.608e+14	0.139e+14	0.293e+14

Table 2: Nonlinear stiffness coefficients for beam a)

b_{111}^1	b_{222}^1	b_{112}^1	b_{122}^1
0.359e+13	-0.713e+14	-0.233e+14	0.722e+14
b_{111}^2	b_{222}^2	b_{112}^2	b_{122}^2
-0.779e+13	0.247e+15	0.722e+14	-0.213e+15

Table 3: Nonlinear stiffness coefficients for beam b)

but rigorous validation of all of its features and bounds of its applicability have not been fully established.

The parameters of the models in Fig.5a and b are shown in Table 1 (system of units is SI, $[m]$, $[N/m^2]$, $[kg]$ etc.), where width and thickness are dimensions of the cross-section of the beams. The first two natural frequencies (associated with flexural modes in the excitation plane) for the beam in Fig.5a are 57.4 Hz and 310.1 Hz and the first two natural frequencies for the beam in Fig.5b are 35.6 Hz and 220 Hz. In all cases the first two (symmetric for clamped-clamped beam) flexural mass-normalized modes were chosen to approximate the motion of beams according to formula (11).

The nonlinear stiffness coefficients determined with application of the procedure described above are summarized in Tables 2 and 3. The quadratic terms were negligible, so only the 3rd order terms are shown. Since the modal coordinates q_1, q_2 are nondimensional, the units of these nonlinear coefficients are in $[N*m]$.

Note that from (22) would follow that

$$\frac{\partial \gamma_j}{\partial q_k} = \frac{\partial \gamma_k}{\partial q_j} = \frac{\partial^2 U}{\partial q_k q_j}$$

and comparing the terms with like powers in q_j and q_k leads to a certain relation between the nonlinear coefficients, for example, for the cubic coefficients b_{122}^1 and b_{112}^2 it is

$$b_{122}^1 = b_{112}^2$$

and for other types, it is

$$3b_{222}^1 = b_{122}^2 \qquad 3b_{111}^2 = b_{112}^1$$

It turned out that the computed nonlinear stiffness coefficients (see Tables 2 and 3) are in an excellent agreement with these relations.

The results in terms of the RMS displacement of the middle and tip nodes (Fig. 5a and b) are shown in Fig. 8 and 9. A vertical base white noise excitation (acceleration a_b) provided inertial loading which was spread over a 20–320 Hz range. One can see that numerical results produced with the SEMELRR code and the new method differ by about 20 % for the case of clamped-clamped beam. The difference is about 30 % for beam b). In each case, the SEMELRR's RMS displacements are less than the RMS displacements from the new approach.

Unfortunately, there are no exact solutions available for these structures, so comparisons are not possible. However, recent experimental measurements (not presented here) indicate that the new method predicts RMS responses more in agreement with their physical counterparts than the SEMELRR solution sequence. This will be quantified with further numerical and experimental work.

Summary

The energy-based version of stochastic linearization technique has been extended to MDOF systems and the numerical results have shown superior performance of this technique in comparison with the conventional linearization version.

A new method for determination of nonlinear stiffness coefficients has been suggested and applied to several examples of beam-like structures. This method has been incorporated into a program which calculates a steady-state response of a MDOF structure to a Gaussian zero-mean excitation. Efforts are presently underway to implement this capability into MSC/NASTRAN through a DMAP Alter.

Some difference (about 20–30 % range) has been found between the two independent results in terms of prediction of nonlinear response. Further numerical studies and experimental work will be devoted to this problem.

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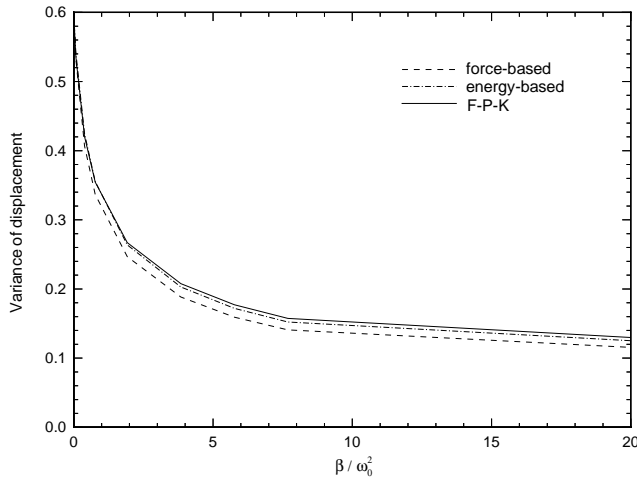[3] Roberts J.B., Spanos P.D., *Random vibration and stochastic linearization*, John Wiley and Sons, 1990.

Figure 1: Variance of displacement vs. nonlinearity parameter for the SDOF system with natural frequency $\omega_0/2\pi = 57.4\ Hz$ and damping coefficient $\xi = 0.005$

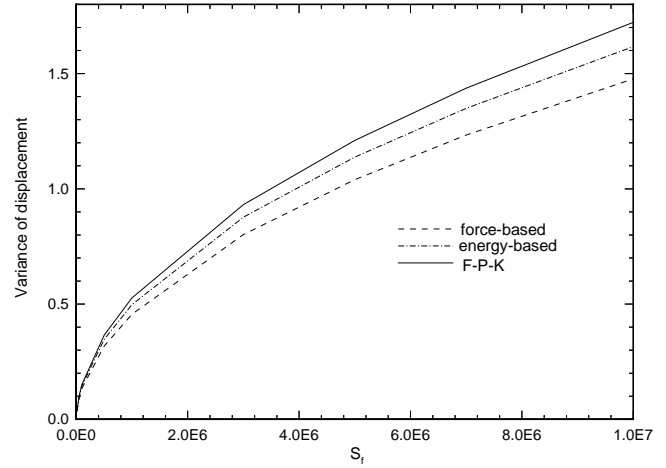Figure 2: Variance of displacement vs. spectral density of force for the SDOF system with natural frequency $\omega_0/2\pi = 57.4\ Hz$, damping coefficient $\xi = 0.005$ and $\beta/\omega_0^2 = 10$

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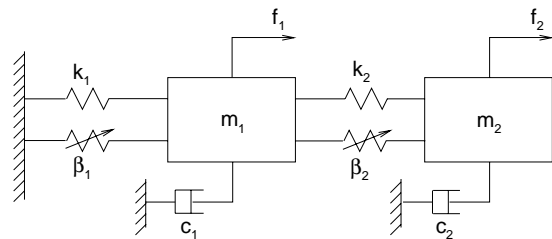[9] *MSC/NASTRAN QUICK REFERENCE GUIDE, version 70.5*, The MacNeal-Schwendler Corporation, 1998.
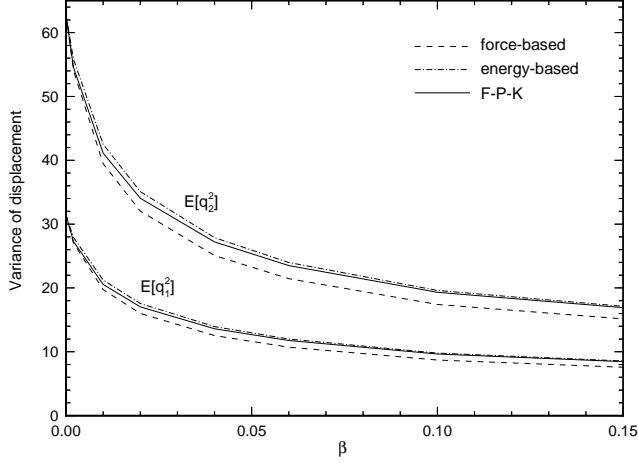
Figure 3: Example of a two-degree-of-freedom system

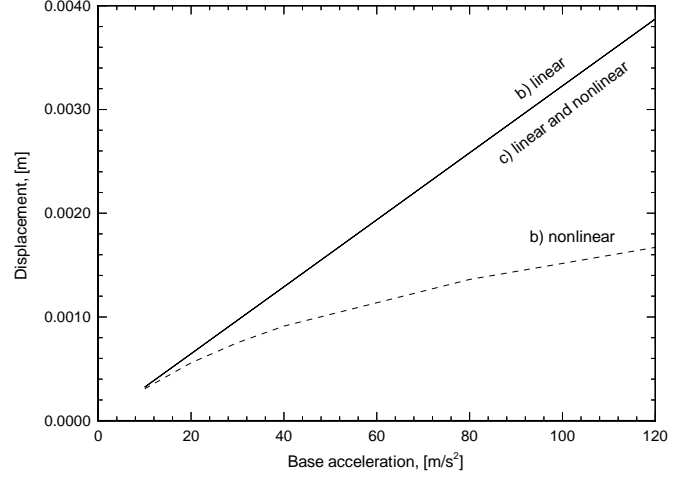Figure 4: Variancies of displacement vs. nonlinearity parameter for the 2DOF system

Figure 6: Displacement as a function of inertial load, beams in Fig.5b,c
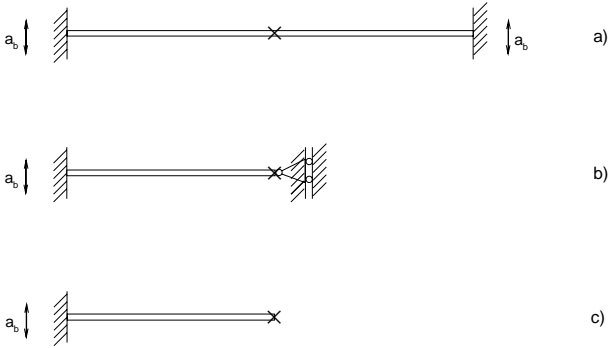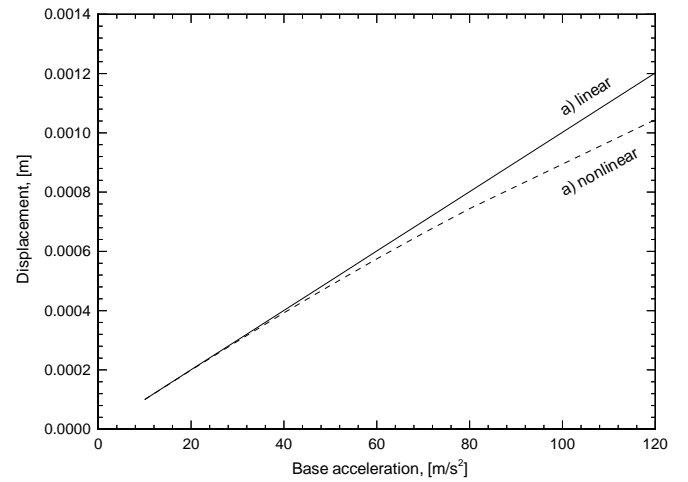

Figure 5: Examples of beam-like structures

Figure 7: Displacement as a function of inertial load, beam in Fig.5a
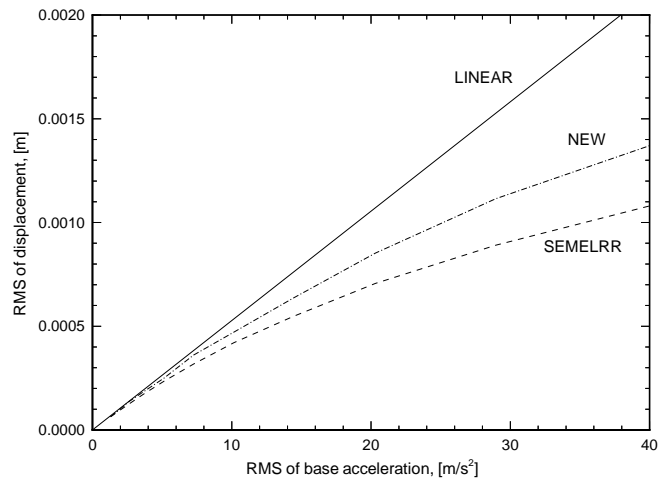
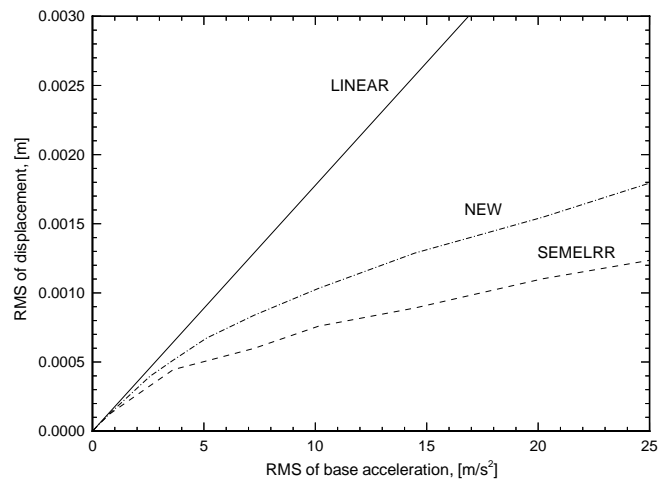Figure 8: RMS displacement for beam in Fig. 5a as a function of inertial random loading

Figure 9: RMS displacement for beam in Fig. 5b as a function of inertial random loading