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Enhancement of Buckling Load with the Use of Active Materials

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Abstract

In this paper, active buckling control of a beam using piezoelectric materials is investigated. Under small deformation, mathematical models are developed to describe the behavior of the beams subjected to an axial compressive load with geometric imperfections and load eccentricities under piezoelectric force. Two types of supports, simply supported and clamped, of the beam with a partially bonded piezoelectric actuator are used to illustrate the concept. For the beam with load eccentricities and initial geometric imperfections, the load-carrying capacity can be significantly enhanced by counteracting moments from the piezoelectric actuator. For the single piezoelectric actuator, using static feedback closed-loop control, the first buckling load can be eliminated. In the case of initially straight beams, analytical solutions of the enhanced first critical buckling load due to the increase of bending stiffness by piezoelectric actuators are derived based on linearized buckling analysis.

Introduction

The stability of large space structures is of critical importance in connection with the deployment of large or precision structures in outer space for various missions. The space structures are very flexible in many cases and may be very long up to 100m, thus, initial imperfection is not avoidable; especially the materials will be rigidized in the outer space. Therefore it is necessary to control the elastic deformation and stability for proper performance. One approach to controlling structural stability is to incorporate active materials into the structural elements in which local strains can be actuated and regulated. Actuated strain is that component of the strain that is due to stimuli other than mechanical stress. Piezoelectric materials, which exhibit mechanical deformation when an electric field is applied, have recently received attention because of their potential application to the control of the flexible structure. These materials, bonded to the surface of a structural member, transfer forces to the structural member according to the magnitude of excitation voltage applied to them. These forces exerted by the piezoelectric materials may be employed to actively control the deformation and enhance the buckling strength of the structure. Active control allows members to be loaded beyond their critical buckling load by using sensors to monitor and detect the onset of buckling and applying actuation forces by piezoelectric actuators to restore the member towards the undeflected position.

Utilizing shape memory alloys, Ro and Baz investigated NiTinol reinforced plates and showed that the NiTinol fibers which are pre-tensioned and activiated can increase the critical buckling load. For simply supported plate with $a/b = 1$ the ratios of the critical load to that of the plain plate are 2.60 – 7.04. The corresponding ratios are 1.06 – 2.72 for clamped plate.

Meressi and Paden [1] analyzed numerically the vibration of a simply supported straight beam with piezoelectric actuators subjected to an axial compressive load. The design of a feedback control system is used to increase the bending stiffness of the first buckling mode. The numerical result indicated that buckling of the simply supported beam could be postponed beyond the first critical buckling load, and the beam can support up to the buckling load of the second mode. Chanrashekhara and Bhatia [2] numerically simulated active control of an ideal laminated composite plates utilizing piezoelectric material. The plate was subjected to a linearly increasing axial compressive load. The piezoelectric sensor output is used to determine the input to the actuators using a proportional control algorithm. Employing active control, full activation is generated after the lateral deflection exceeds a threshold value of one tenth of the plate thickness, and remains activated thereafter. The finite element solutions demonstrated that active control increased critical load about 4%. Berlin [3] has shown experimentally that the active control can increase the load-bearing strength of a compressively loaded member by employing a prototype actively-controlled column.

Within the above literatures, all the structures are assumed to be uniform with no imperfections and loading eccentricities. Due to these effects, the beams are deflected at the onset of compressive loading. Utilizing piezoceramic actuators, experiments of active control of column were conducted by Thompson and Loughlan [4]. The results shown that the active column can counteract the effects of imperfections enhance the buckling loads, and increases in load carrying capability are of the order of 19.8%-37.1%. A study of the piezoelectric effects on the behavior of initially imperfect composite slender (Euler-Bernoulli) beams under compression was conducted [5]. The finite element results showed that, for a simply supported beam, suitable voltages can be applied to the piezoelectric actuators attached to the imperfect beam in order to render its equilibrium path as close as possible to of the ideal perfect structure, and effectively reduce the deflections due to the initial imperfections. Berlin [6] presented experimental results showing that a modal controller stabilized column with axial loads up to 5.6 times the critical buckling load using piezoactuators.

In this paper, using piezoactuators, a technique for active buckling control of a beam with imperfections and eccentricity of loading has been developed. The sensor is used to measure the deformation, which provides the input to the actuators. The actuators are activated and controlled voltage is applied after the lateral deflection exceeds deflection allowable. The theoretical results obtained show that the active control dramatically increases the load-carrying capability of a beam under compressive load. The critical buckling load for a simply supported beam can reach the second buckling load, while the critical buckling load is increased significantly for a cantilever beam, and it can be several times of the first buckling load.

Mathematical Formulation

Consider a beam with a symmetric piezoelectric actuator pair bonded to the top and bottom of its surface. Assuming the cross section of the beam is symmetric with respect to the neutral axis. The beam is subjected to an axial compressive load and a moment is applied to the beam by the piezoactuators. The bending moment is excited by the two actuators being driven out of phase.

From the linear piezoelectric constitutive equations, the axial stress-strain relation for the piezoactuator can be expressed as

$$\varepsilon_a = \frac{\sigma_a}{E_a} + \Lambda \tag{1}$$

where

$$\Lambda = V d_{31} / t_a$$

Here, the subscript a refers to the actuator, t_a is the thickness of a piezoelectric, Λ is the free piezoelectric strain which depends on the applied voltage V, and the piezoelectric strain constant d_{31}.

For the beam the stress-strain expression can be written as

$$\varepsilon_b = \sigma_b / E_b \tag{2}$$

Here, the subscript b refers to the beam.

For the combined structure under small deflections, a linear strain distribution can be assumed across the structure thickness,

$$\varepsilon = -z\, w'' \tag{3}$$

where w is the transverse deflection.

Employing symmetry about the neutral axis, the moment equilibrium condition for any location which is covered by the piezoactuators can be expressed as follows:

$$\int_{A_b} \sigma_b z\, dA + \int_{A_a} \sigma_a z\, dA = M \tag{4}$$

where A is the cross sectional area and M is the externally applied moment.

Substituting eqs. (1), (2), and (3), into eq. (4), the governing equation for the combined structure yields

$$(E_a I_a + E_b I_b) w'' = -M - M_a \tag{5}$$

where $M_a = 2E_a \Lambda A_a d$ is the actuator induced bending moment; A_a is cross sectional area of a piezoelectric actuator and d is the distance from the neural axis to the centroid of the piezoelectric actuator.

In practice, the beam is not perfectly straight and the applied load does not necessarily pass through the centroid of the cross section. It is therefore necessary to study the behavior of beams of imperfect geometries and of beams for which the load is applied eccentrically. Note that M in eq. (5) is due to the axial compressive load only. The bending moment M can also includes the contribution due to the imperfection of the beam and eccentricities of the load P. The general governing equations of a simply supported beam under axial compressive load can be derived from the principle of virtual work given below.

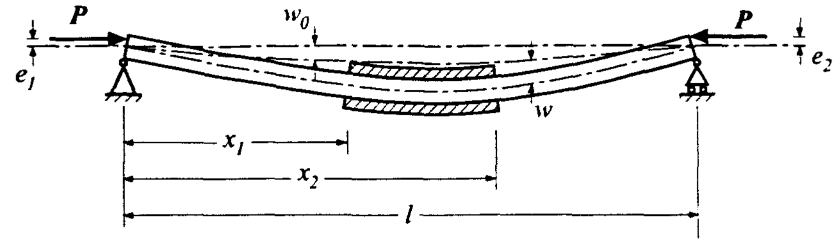

Fig. 1 A simple supported beam with a piezoelectric actuator under geometric imperfections and load eccentricities

A simply supported beam with small initial geometric imperfections is shown in Fig. 1. The beam is under an axial compressive load P with eccentricity e_A *and* e_B at the left and the right ends respectively. The beam is covered by a pair of piezoelectrics from $x_1 < x < x_2$. Voltages are applied to the piezoelectric actuators bonded to the beam symmetrically. We may write the principle of virtual work for the present problem as

$$\int_V \sigma_x \delta\varepsilon_x \, dV - P\delta\left\{\frac{1}{2}\int_0^\ell [(w'+w_o')^2 - w_0'^2]dx\right\} - M_A \delta w'(0) + M_B \delta w'(\ell) = 0 \tag{6}$$

where w_0 is the initial shape of the axis of the beam, w the deflections due to deformation, $M_A = Pe_A$, $M_B = Pe_B$.

In eq. (6), the arbitrary infinitesimal virtual displacements δw satisfy the prescribed geometrical conditions

$$w(0) = w(\ell) = 0 \tag{7}$$

After some calculation, eq. (6) via eq. (1), (2), and (3) can be simplified to

$$\begin{aligned} &-\int_0^\ell [M'' - P(w+w_0)'']\delta w \, dx + [M(0) - M_A]\delta w'(0) - [M(\ell) - M_B]\delta w'(\ell) \\ &+ M(x_1)\delta[w']_{x=x_1} - M'(x_1)\delta[w]_{x=x_1} + M(x_2)\delta[w']_{x=x_2} - M'(x_2)\delta[w]_{x=x_2} = 0 \end{aligned} \tag{8}$$

where

$$M = \int_A \sigma_x z dA, \quad M_a = \int_{A_a} E\Lambda z dA$$

$$M = -E_b I_b w'', \qquad \text{for } 0 < x < x_1 \text{ or } x_2 < x < \ell$$

$$M = -(E_a I_a + E_b I_b)w'' - M_a, \qquad \text{for } x_1 < x < x_2$$

$[w]_{x=x_c}$ and $[w']_{x=x_c}$ represent the difference in w and w' values across the location $x = x_c$ respectively

From eq. (8), we obtain the governing differential equations

$$M'' - [P(w+w_0)]'' = 0 \tag{9}$$

and the corresponding boundary conditions

$$M(0) = M_A, \text{ and } M(\ell) = M_B \tag{10}$$

with continuity conditions $[w]=[w']=0$ at $x=x_1$ or $x=x_2$.

Integrating eq. (9) twice with eq. (10), the equilibrium equation of moment for the beam is

$$E_bI_bw''=-P[w+w_o+e_A+(e_B-e_A)x/\ell],\quad \text{for } 0<x<x_1 \text{ or } x_2<x<\ell$$
$$(E_aI_a+E_bI_b)w''=-M_a-P[w+w_o+e_A+(e_B-e_A)x/\ell],\quad \text{for } x_1<x<x_2 \tag{11}$$

Due to these effects the beam is deflected at the onset of compressive loading. Therefore, for practical purposes the critical load represents the maximum load-carrying capacity of an elastic beam, because excessive deflections are not acceptable in most applications. In the following, elimination of the first buckling load of a simply supported beam by using the piezoelectric actuators will be investigated. First, the bending deformation due to the piezoelectric actuators, initial imperfections, load eccentricity under the axial compressive load will be separately derived. Then the active buckling control by a closed-loop feedback control will be considered. The buckling control of a clamped beam will be briefly derived in Appendix B.

Active Buckling Control of Simply Supported Beam

(a) Bending deformation of the beam with active piezoelectric actuators

A straight beam is subjected to an axial compressive load P without eccentricity, and moments are applied by the piezoactuators on a portion of the beam with no imperfections. The differential equations for the three parts of the deflection curve are

$$\begin{aligned}
E_bI_bw'' &= -Pw, && 0<x<(\ell-a)/2\\
(E_aI_a+E_bI_b)w'' &= -Pw-M_a, && (\ell-a)/2<x<(\ell+a)/2\\
E_bI_bw'' &= -Pw, && (\ell+a)/2<x<\ell
\end{aligned} \tag{6}$$

where V is the applied voltage on the piezoelectric with $z>0$, $-V$ on the piezoelectric with $z<0$.

To satisfy the end conditions

$$w=0 \qquad \text{at } x=0 \text{ and } \ell \tag{7}$$

solutions of the equations can be written in the form

$$\begin{aligned}
w &= B\sin k_1x, && \textit{for}\ \ 0<x<\ell_1\\
w &= C\cos k_2x+D\sin k_2x-\beta d, && \textit{for}\ \ \ell_1<x<\ell_2\\
w &= F(-\tan k_1\ell\cos k_1x+\sin k_1x), && \textit{for}\ \ \ell_2<x<\ell
\end{aligned} \tag{8}$$

where

$$k_1 = \sqrt{P/(E_b I_b)} = \sqrt{P/(EI)_1}\ , \qquad k_2 = \sqrt{P/(E_a I_a + E_b I_b)} = \sqrt{P/(EI)_2}\ ,$$

$$\ell_1 = (\ell - a)/2\ , \qquad \ell_2 = (\ell + a)/2\ ,$$

$$\beta = 2E_a A_a \Lambda / P\ .$$

Physically, β represents the ...The integration constants B, C, D and F can be determined from the conditions that the portions of the deflection curve have the same deflection and slope at $x = \ell_1$ and ℓ_2, respectively. Thus

$$\begin{bmatrix} C \\ D \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \beta d \tag{9}$$

$$B = \frac{1}{\sin k_1 \ell_1} (C \cos k_2 \ell_1 + D \sin k_2 \ell_1 - \beta d) \tag{10}$$

$$F = \frac{-\cos k_1 \ell}{\sin k_1 \ell_1} (C \cos k_2 \ell_2 + D \sin k_2 \ell_2 - \beta d) \tag{11}$$

where

$$A = \begin{bmatrix} \cos k_2 \ell_1 + \gamma \sin k_2 \ell_1 & \sin k_2 \ell_1 - \gamma \cos k_2 \ell_1 \\ \cos k_2 \ell_2 - \gamma \sin k_2 \ell_2 & \sin k_2 \ell_2 + \gamma \cos k_2 \ell_2 \end{bmatrix} \tag{12}$$

$$\gamma = \frac{k_2}{k_1} \tan k_1 \ell_1$$

Note that the equation $\det A = |\, A \,| = 0$ gives

$$(1 - \gamma^2) \sin k_2 a + 2\gamma \cos k_2 a = 0 \tag{13}$$

Eq. (13) provides the buckling loads for the host beam with inactive actuators ($V = 0$).

(b) Bending deformation of the beam with initial geometric imperfections

In general, an initial imperfection shape on a simply supported beam can be given by a series of sine functions

$$w_o = \sum_{n=1,2,3}^{\infty} a_n \sin \frac{n \pi x}{2\ell} \tag{14}$$

The series can be made to represent any initial curve with a degree of accuracy in which depends upon the number of terms taken.

Let w denote the deflection produced by the external axial compressive force. Then w due to deformation is determined in the usual way from the differential equations

$$E_b I_b w'' = - P(w + w_o)\ , \qquad 0 < x < (\ell - a)/2$$

$$(E_a I_a + E_b I_b) w'' = - P(w + w_o)\ , \qquad (\ell - a)/2 < x < (\ell + a)/2 \tag{15}$$

$$E_b I_b w'' = -P(w + w_o), \qquad (\ell + a)/2 < x < \ell$$

Without loss of generosity, let w_0 be the arbitrary term

$$w_o = a_n \sin n\pi x/\ell \tag{16}$$

Considering the boundary conditions

$$w(0) = w(\ell) = 0 \tag{17}$$

solutions of the equations can be written in the form

$$\begin{aligned} w &= B\sin k_1 x + a_n f_1 \sin n\pi x/\ell, && 0 < x < \ell_1 \\ w &= C\cos k_2 x + D\sin k_2 x + a_n f_2 \sin n\pi x/\ell, && \ell_1 < x < \ell_2 \\ w &= F(-\tan k_1\ell \cos k_1 x + \sin k_1 x) + a_n f_1 \sin n\pi x/\ell, && \ell_2 < x < \ell \end{aligned} \tag{18}$$

where

$$f_1 = \frac{\alpha_1}{n^2 - \alpha_1}, \qquad f_2 = \frac{\alpha_2}{n^2 - \alpha_2},$$

$$\alpha_1 = \frac{P}{\pi^2 (E_b I_b)/\ell^2}, \qquad \alpha_2 = \frac{P}{\pi^2 (E_a I_a + E_b I_b)/\ell^2}.$$

Applying the conditions which indicate the parts of the deflection curve have the same deflection and common tangent at $x = \ell_1$ and ℓ_2, the constants of integration are given by

$$\begin{bmatrix} C \\ D \end{bmatrix} = A^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} a_n (f_2 - f_1) \tag{19}$$

$$B = \frac{1}{\sin k_1 \ell_1} [\, C\cos k_2 \ell_1 + D\sin k_2 \ell_1 + a_n (f_2 - f_1)\sin n\pi \ell_1/\ell \,] \tag{20}$$

$$F = \frac{-\cos k_1 \ell}{\sin k_1 \ell_1} [\, C\cos k_2 \ell_2 + D\sin k_2 \ell_2 + a_n (f_2 - f_1)\sin n\pi \ell_2/\ell \,] \tag{21}$$

where

$$b_1 = \frac{n\pi}{k_2 \ell} \gamma \cos n\pi \ell_1/\ell - \sin n\pi \ell_1/\ell,$$

$$b_2 = -\frac{n\pi}{k_2 \ell} \gamma \cos n\pi \ell_2/\ell - \sin n\pi \ell_2/\ell.$$

(c) Bending deformation of the beam with axial load eccentricity

If the axial compressive forces P are applied at the ends of the beam with the same amount of load eccentricity e, the deflection curve may be written in the form

$$
\begin{aligned}
&w = e(\cos k_1 x - 1) + B \sin k_1 x\,, && 0 < x < \ell_1 \\
&w = C \cos k_2 x + D \sin k_2 x - e\,, && \ell_1 < x < \ell_2 \\
&w = F(-\tan k_1 \ell \cos k_1 x + \sin k_1 x) + e\left(\frac{\cos k_1 x}{\cos k_1 \ell} - 1\right), && \ell_2 < x < \ell
\end{aligned}
\tag{22}
$$

Applying the continuity conditions at $x = \ell_1$ and ℓ_2, we can find the constants B, C, D and F:

$$\begin{bmatrix} C \\ D \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{e}{\cos k_1 \ell_1} \tag{23}$$

$$B = \frac{1}{\sin k_1 \ell_1}(C \cos k_2 \ell_1 + D \sin k_2 \ell_1 - e \cos k_1 \ell_1) \tag{24}$$

$$F = \frac{-\cos k_1 \ell}{\sin k_1 \ell_1}\left(C \cos k_2 \ell_2 + D \sin k_2 \ell_2 - e\frac{\cos k_1 \ell_2}{\cos k_1 \ell}\right) \tag{25}$$

To increase the critical buckling load for the beam a combination of sensors and actuators can be used. The sensor can detect small deformation in the beam, while the actuators are utilized to provide a restoring moment within the beam if necessary. The moment will push the beam towards its equilibrium position when the deflection at some particular points is beyond the equilibrium position. Hence, the combination may prevent (eliminate) buckling in the first buckling mode, and increase the load carrying capability. The active control may be outlined below.

Under the assumption of small deflections, if both compressive load and voltage are applied to the beam, according to superposition principle, the total deflection w is

$$w = w_a + w_m \tag{26}$$

where w_m is the deflection of the beam due to the compressive load P ($V = 0$) and w_a the deflection due to the voltage calculated for a straight beam, which was discussed in the previous section.

Actually, we may control the deflection at the center of the beam to stabilize the beam against buckling in the first mode.

Let δ_d be the maximum deflection at the center of the beam, $x = \ell/2$, under design load, δ_m be the deflection at the center measured by the sensor before the actuators are active.

If $|\delta_m| > |\delta_d|$, the actuators are to be active such that the resultant deflection at the center is δ_d, i. e.,

$$\delta_a + \delta_m = \delta_d \tag{27}$$

Here δ_a is the deflection at the center of the host beam due to the voltage V, and δ_a was given in the previous discussion.

Expressing δ_a as

$$\delta_a = \beta d \tilde{\delta}_a \tag{28}$$

From the eq. (27), we have

$$\beta d = -\frac{\delta_m - \delta_d}{\tilde{\delta}_a} \tag{29}$$

The above equation provides a controlled voltage to remove unacceptable deflections and restore the beam center to its original position with the design threshold when the deflection is beyond the acceptable value. For a simply supported beam with imperfections, or eccentricity of loading, it can be proved that the first buckling mode is eliminated, and the critical load for the beam with active control can reach the second buckling mode without control. Similarly, preventing buckling in the second mode is possible if another pair of actuators is used to control of deflection at another point. Thus, the first two buckling modes can be stabilized.

Enhanced First Buckling Load of Initially Straight Beam using Piezoelectric Actuators

It is assumed that the beam was initially perfectly straight, and the compressive load P is applied through the centroids of the cross section. A cantilever beam shown in the Fig. 1 is partially covered by the actuator and is subjected to an axial compressive load P. The differential equations for the two portions of the deflection curve then become

$$E_b I_b w_1'' = P(\delta - w_1), \qquad \ell_2 < x < \ell \tag{30}$$

$$(E_a I_a + E_b I_b) w_2'' = P(\delta - w_2) - Q\delta, \qquad 0 < x < \ell_2 \tag{31}$$

where δ is the deflection at the end of the beam.

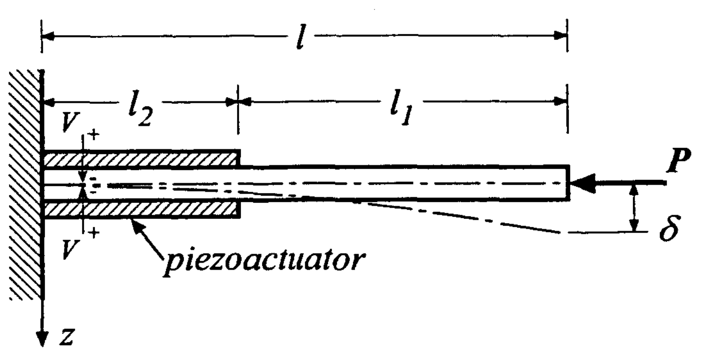

Fig. 1 A clamped beam with piezoelectric actuators under axial compressive load

The equivalent force due to the voltage applied to the piezo-actuators is defined as

$$Q = 2E_a \frac{d_{31}}{t_a} GA_a d \tag{32}$$

In the above equation, the input voltage V required for actuator has been calculated using a constant gain feedback control algorithm, and can be expressed as

$$V = G\delta \tag{33}$$

Solving eqs. (30) and (31) and satisfying the following two boundary conditions eq. (34) and (35a)

$$w_1 = \delta \qquad \text{at } x = \ell \tag{34}$$

$$w_1 = w_2, \quad w_1' = w_2' \qquad \text{at } x = \ell_2 \tag{35}$$

we have

$$\begin{aligned} w_1 &= A\cos k_1 x + B\sin k_1 x + \delta \\ w_2 &= \delta\,(1-\alpha)\,(1-\cos k_2 x) \end{aligned} \tag{36}$$

where

$$A = -B\tan k_1 \ell$$

$$B = \frac{[\cos k_2\ell_2 + (1-\cos k_2\ell_2)\alpha]\cos k_1\ell}{\sin k_1\ell}\delta$$

$$\alpha = \frac{Q}{P}$$

Using the second equation in eq. (35), the transcendental equation for calculating the critical load is given by

$$\frac{k_1}{k_2} = \tan k_1\ell_1 \frac{\sin k_2\ell_2}{\cos k_2\ell_2 + \dfrac{\alpha}{1-\alpha}} \tag{37}$$

where

$$k_1^2 = P/(E_b I_b) = P/(EI)_1 \qquad k_2^2 = P/(\,E_a I_a + E_b I_b\,) = P/(EI)_2 \tag{38}$$

In the case of the beam without voltage applied from piezoelectric actuators, $\alpha = 0$, the critical buckling load is calculated from

$$k_1 / k_2 = \tan k_1\ell_1 \, \tan k_2\ell_2 \tag{39}$$

The eq. (39) has been reported in a book by Timoshenko and Gere (1961).

If the piezoelectric actuators cover the entire beam length, $\ell_2 = \ell$, the buckling equation reduces to

$$k_2\ell = \cos^{-1}\left(\frac{\alpha}{\alpha - 1}\right) \tag{40}$$

For a simply supported beam covered partially by a piezoactuator and subjected an axial compressive load P, the buckling load for the odd buckling mode shapes can be calculated from eq. (39) by letting $\ell_2 = a/2, \ell_1 = (l-a)/2$.

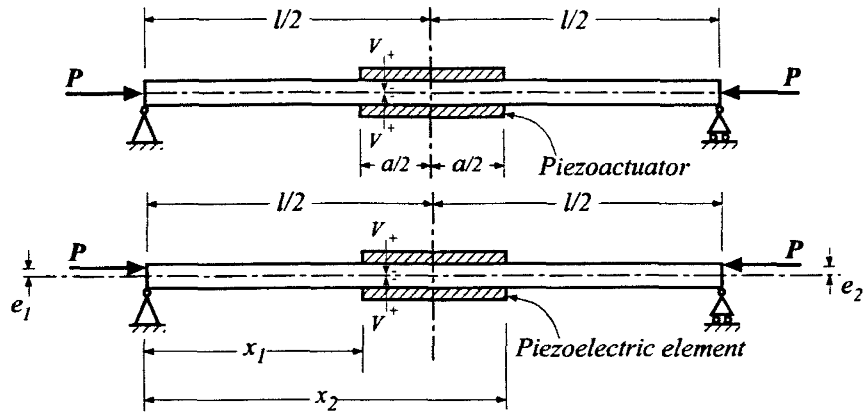

Fig. 2 A simply supported beam with piezoelectric actuators under axial compressive load

Numerical Results and Discussion

The buckling load calculation described by eq. (13) is functions of bending rigidity ratio, *$(EI)_2/(EI)_1$, a/l*, and the piezoelectric force parameter expressed by *Q*. Fig. 3 shows the buckling increase as a function of Q/P_{cr}^0 with three different bending rigidity ratios for a/l = ¼ and 1/8 respectively. P_{cr}^0 is the critical buckling load for the host beam. For Q = 0, the buckling load increase is due to the added piezoelectric materials only. For the range of piezoelectric force parameter studied, the enhancement of critical buckling load can reach 25%.

The buckling load calculation described by eq. (13) is functions of bending rigidity ratio, $(EI)_2/(EI)_1$, a/l, and the piezoelectric force parameter expressed by Q. Fig. 3 shows the buckling load increase as a function of Q/P_{cr}^0 with three different bending rigidity ratios for a/l = ¼ and 1/8 respectively. For Q = 0, the buckling load increase is due to the added piezoelectric materials only. For the range of piezoelectric force parameter studied, the enhancement of critical buckling load can reach 25%.

Using active control, numerical calculations are carried out for a simply supported beam with

$$a/L = 1/4, \quad (EI)_2/(EI)_1 = 2$$

The first two buckling loads of the beam with V = 0 are given by

$$P_{cr}^1/P_{cr}^0 = 1.2999, \quad P_{cr}^2/P_{cr}^0 = 4.1775$$

Based on eq. (27), Figs. 5 – 7 show the curves of load-deflection, voltage-load for geometric imperfection and load eccentricity, respectively. The results indicate that the first buckling load is eliminated. In these calculations, the following values of δ_d are used:

$$\delta_d / e = \delta_d / a_1 = \delta_d / a_2 = 0.2$$

Fig. 5 is for the simply supported beam with initial geometric imperfection $w_0 = a_1 \sin \pi\, x / L$. Without active control, V = 0, as the load P approached to $P = P'_{cr} = 1.2999\, P'_{cr}$, the deflection $w(\ell/4)$ increases without limit. With the active control, the deflection $w(\ell/4)$ increased indefinitely when P (s) approaches $P = 5.124\, P_{cr}^0$. Note that when $P = 5.124\, P_{cr}^0$, the deflection due to the active actuators, $w_a(\ell/2) = 0$, and $w_a(\ell/4) \neq 0$. That means that the deflection $w(\ell/2)$ can not be controlled by the active actuators. Therefore $w(\ell/4)$ increases without limit at $P = 5.124\, P_{cr}^0$.

Fig. 6 shows the active control of the simply supported beam with geometric imperfection $w_0 = a_2 \sin 2\pi x / L$. As $P \to P_{cr}^2 = P_{cr}^0\, 4.1775$, $w(\ell/4) \to \infty$. Fig. 7 is for the beam with load eccentricity e. If active control is applied, it indicates that as $P \to 5.124\, P_{cr}^0$, $w(\ell/4) \to \infty$. It is easy to show that, with active control when $P < P_{cr}^2 = 4.1775 = P_{cr}^0$, the deflection at any location of $0 < x\ L\ \ell$ is limited. Therefore, the critical load of the beam with active control is $P_{cr}^0 = 4.1775\, P_{cr}^0$.

Similarly, numerical calculations are preformed for cantilever beam with

$$a / L = 1/4, \quad (EI)_2 / (EI)_1 = 2$$

The first two buckling loads of the beam with $V = 0$ are given by

$$P_{cr}^1 / P_{cr}^0 = 1.2999, \quad P_{cr}^2 / P_{cr}^0 = 11.2533$$

Based on eq. (A15), Figs. 8-10 show the curves of load-deflection, voltage-load for geometric imperfection and load eccentricity, respectively. The results indicate that the first buckling load is eliminated. In these calculations, the following values of δ_d are used:

$$\delta_d / e = \delta_d / a_1 = \delta_d / a_3 = 0.2$$

Figs. 8-10 show the active control of the cantilever beam with a variety of imperfections. In the case, $w_0 = a_1\left(1 - \cos\dfrac{\pi x}{2\ell}\right)$, Fig. 8 indicates $w(3\ell/4) \to \infty$ as $P \to 5.124\, P_{cr}^0$. This is due to $w_a(\ell) = 0$ and $w_a(3\ell/4) \neq 0$ as $P \to 5.124\, P_{cr}^0$. Since $w(\ell)$ can not be controlled by the active actuators at this value of load, the deflection $w(3\ell/4)$ increased without limit as $P \to 5.124\, P_{cr}^0$. Figs. 9 and 10 demonstrate $w(3\ell/4) \to \infty$ as $P \to 5.124\, P_{cr}^0$ for the imperfections:

$$w_0 = a_3(1 - \cos 3\pi x / 2\ell)$$

and eccentricity e.

It can be shown that the deflection any location of the beam is limited with active control, if $P < 5.124\, P_{cr}^0$. Hence, using the active control, the critical load of the cantilever beam can reach $P = 5.124\, P_{cr}^0$.

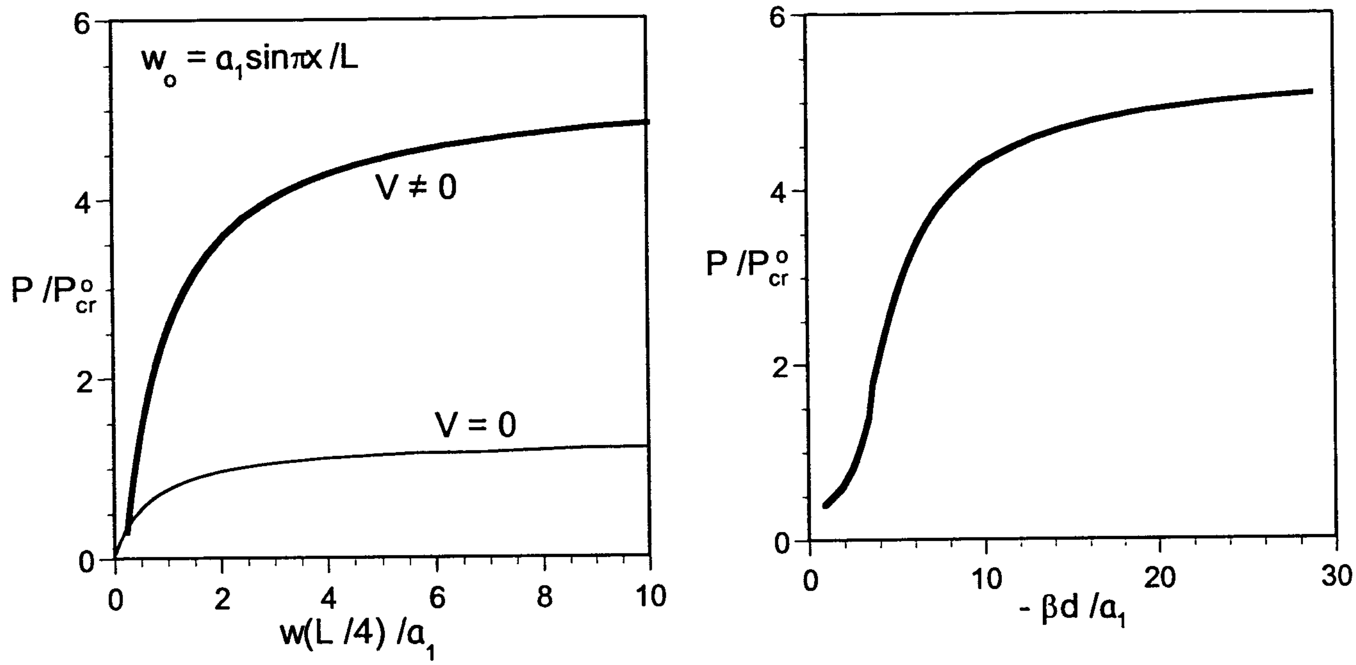

Fig. 4 (a) Load P and deflection w(L /4) diagram (b) P - β for a simply supported beam with initial geometric imperfection $w_o = a_1 \sin\pi x/L$, $a/L = 1/4$, $(EI)_2/(EI)_1 = 2$

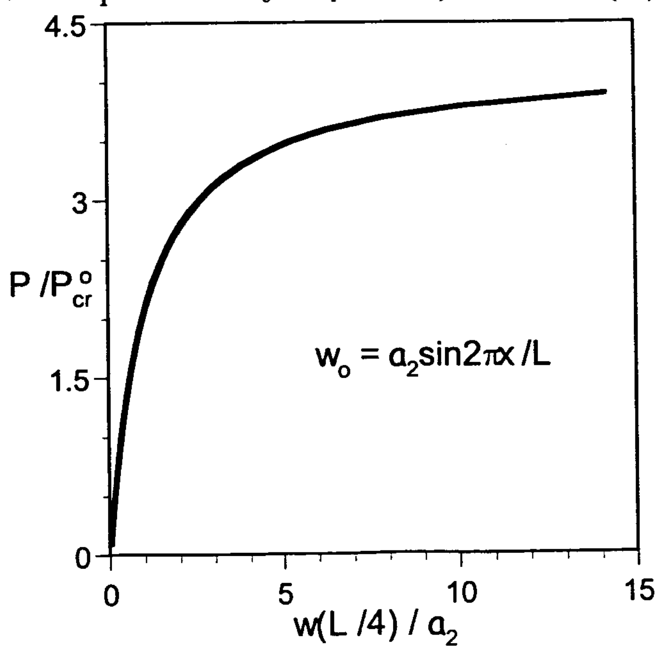

Fig. 5 Load P and deflection w(L /4) diagram for a simply supported beam with initial geometric imperfection, $w_o = a_2 \sin 2\pi x/L$, a /L =1 /4, $(EI)_2 /(EI)_1 = 2$

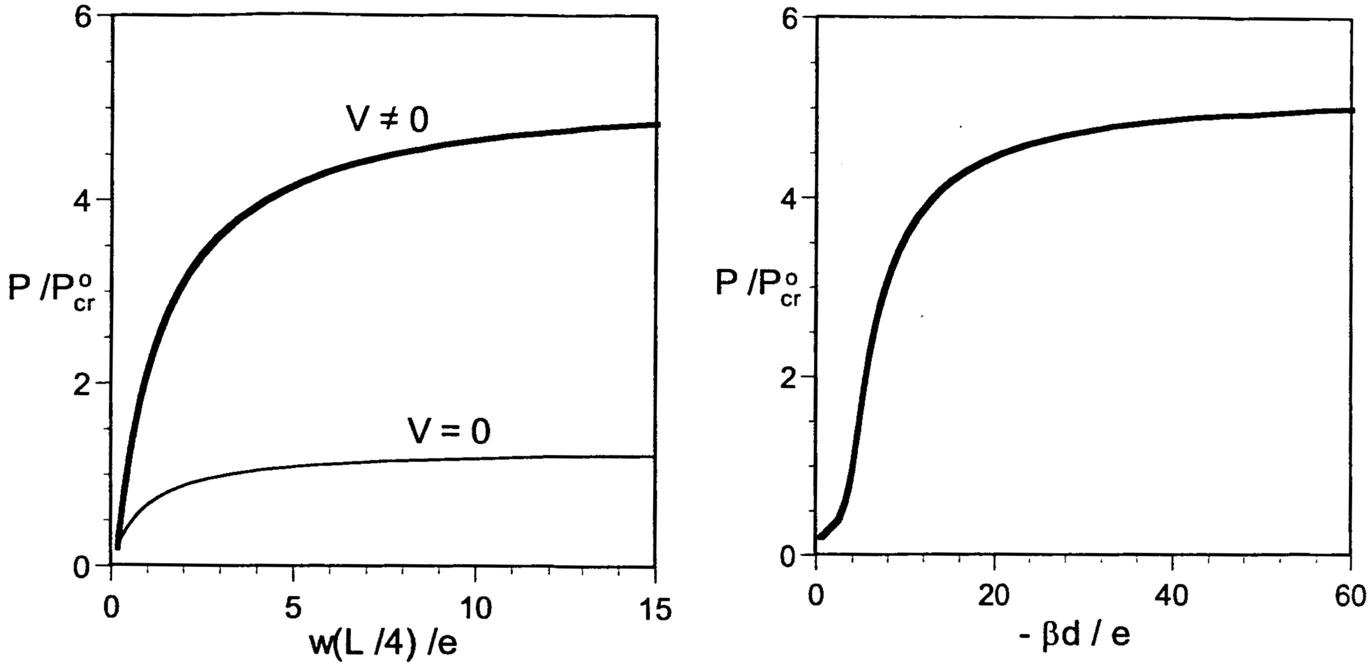

Fig. 6 Simply supported beam with load eccentricity for a /L =1 /4, $(EI)_2 /(EI)_1 = 2$, (a) Load P and deflection w(L /4) diagram, and (b) P - β diagram for a /L =1 /4, $(EI)_2 /(EI)_1 = 2$

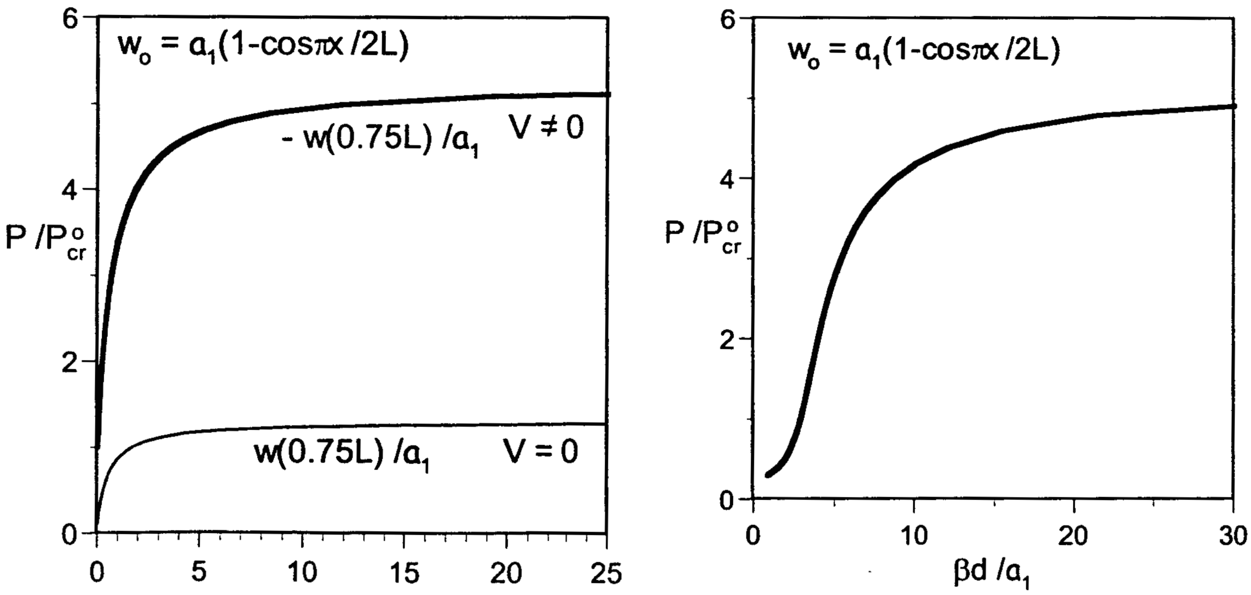

Fig. 7 (a) Load P and deflection w(3L /4) diagram and (b) P - β diagram for cantilever beam with initial geometric imperfection $w_o = a_1(1-\cos\pi x/2L)$, a /L =1 /4, $(EI)_2 /(EI)_1 = 2$

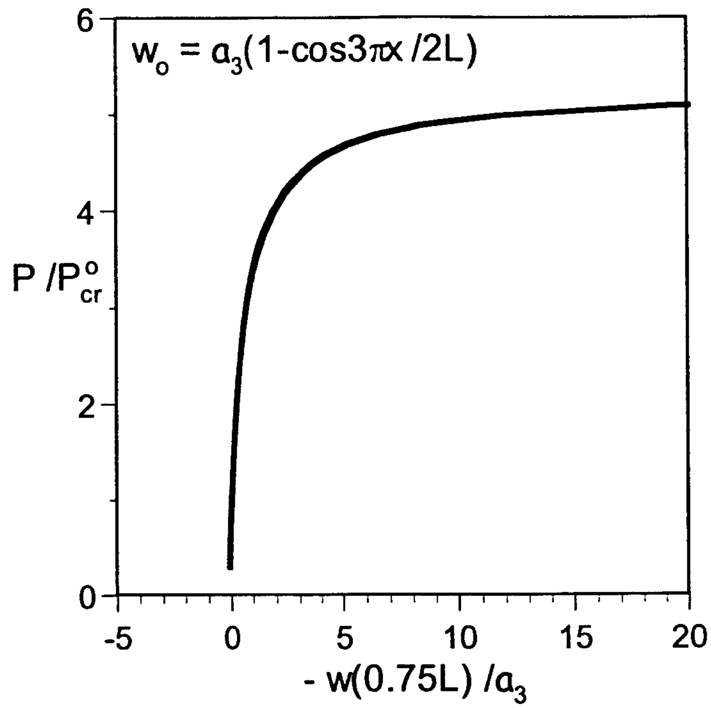

Fig. 8 Load P and deflection w(3L /4) diagram for cantilever beam with initial geometric imperfection $w_o = a_3(1\text{-}\cos 3\pi x/2L)$, a /L =1 /4, $(EI)_2/(EI)_1 = 2$

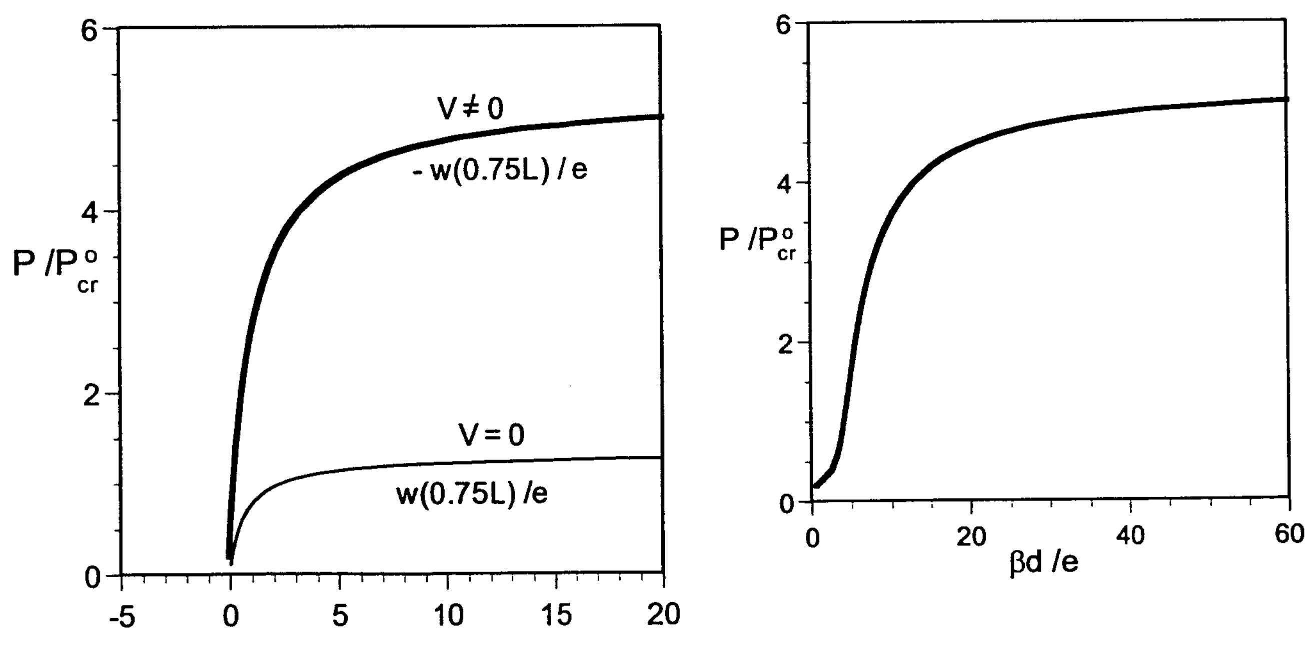

Fig. 9 (a) Load P and deflection w(3L /4) diagram and (b) P - β diagram for cantilever beam with load eccentricity e, a /L =1 /4, $(EI)_2/(EI)_1 = 2$

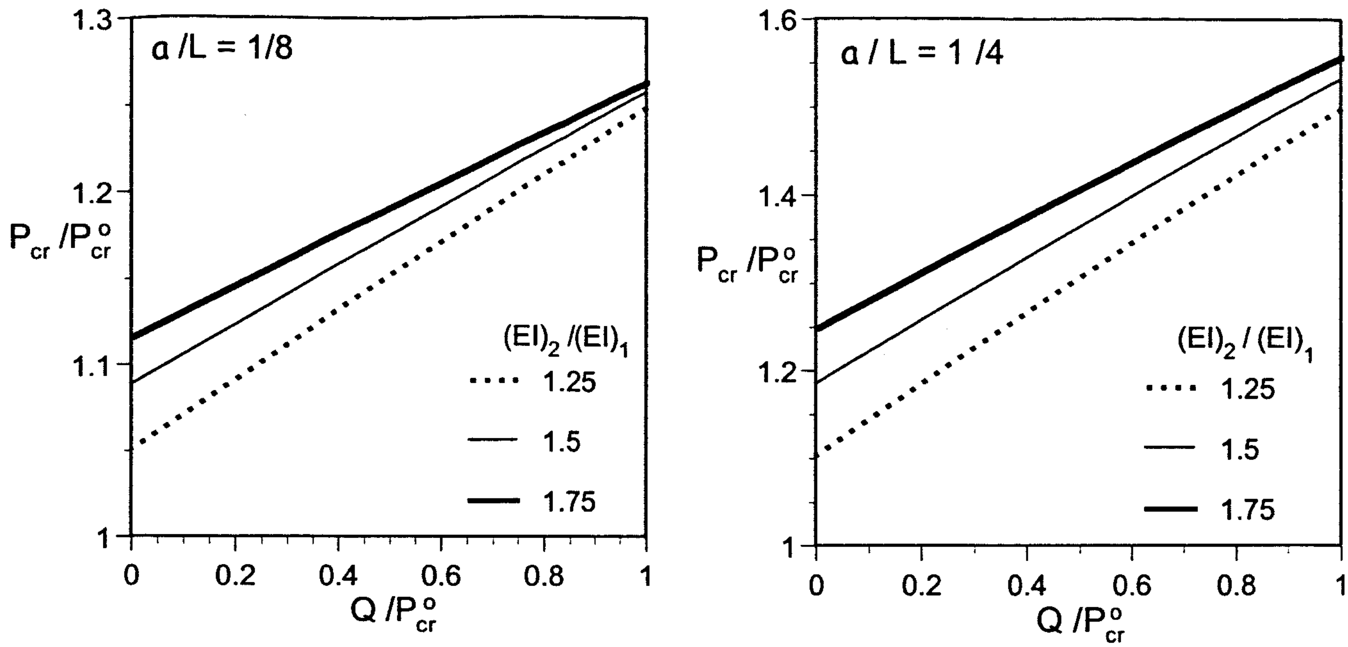

Fig. 10 The critical buckling load P_{cr} vary with piezoelectric force parameter Q for different values of *a* /L and $(EI)_2/(EI)_1$

Conclusions

Using piezoactuators, a technique for active buckling control of a beam with imperfections and eccentricity of loading has been developed. The theoretical results obtained show that the active control dramatically increases the load-carrying capability of a beam under compressive load. The critical buckling load for a simply supported beam can reach the second buckling load, while the critical buckling load for a cantilever beam can be several times of the first buckling load.

Acknowledgment

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Appendix A

Bending of a Cantilever Beam with Active Piezoelectric Actuators.

Consider a straight cantilever beam with two piezoelectric elements which are bonded symmetrically to the structure about its neutral axis. Under an axial compressive load P without eccentricity, and moments applied by the piezoactuators on part of the beam. The differential equations for the deflection curve are

$$(E_a I_a + E_b I_b)w'' = P(\delta - w) - M_a, \qquad for \quad 0 < x < \ell_2 \tag{A1}$$

$$E_b I_b w'' = P(\delta - w), \qquad for \quad \ell_2 < x < \ell \tag{A2}$$

where δ is the deflection at the end $x = \ell$.

Taking into account of the conditions at the ends of the beam, the solution can be expressed in the form

$$w = (\delta - \beta d)(1 - \cos k_2 x), \qquad for \quad 0 < x < \ell_2 \tag{A3}$$

$$w = D(-\tan k_1 \ell \cos k_1 x + \sin k_1 x) + \delta, \qquad for \quad \ell_2 < x < \ell \tag{A4}$$

where $\beta = 2E_a A_a \Lambda / P$. Because the deflection curve is continuous at point $x = l_2$, the continuity conditions determine the constants D and δ, that is,

$$\begin{bmatrix} D \\ \delta \end{bmatrix} = A^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \beta d \tag{A5}$$

where

$$A = \begin{bmatrix} -\dfrac{\sin k_1 \ell_1}{\cos k_1 \ell} & \cos k_2 \ell_2 \\ \dfrac{k_1 \cos k_1 \ell_1}{k_2 \cos k_1 \ell} & -\sin k_2 \ell_2 \end{bmatrix}, \qquad \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 - \cos k_2 \ell_2 \\ \sin k_2 \ell_2 \end{bmatrix} \tag{A6}$$

and $\ell_1 = \ell - \ell_2$.

The equation $\det A = |A| = 0$ leads to

$$\sin k_1 \ell_1 \sin k_2 \ell_2 - \frac{k_1}{k_2} \cos k_1 \ell_1 \cos k_2 \ell_2 = 0 \tag{A7}$$

which yields the buckling loads for the idea cantilever beam with non active actuators ($V = 0$).

Bending of Cantilever Beam with Initial Imperfections

An initial shape of a cantilever beam can be approximated by a series form

$$w_o = \sum_{n=1,3,5}^{\infty} a_n (1 - \cos\frac{n\pi x}{2\ell}) \tag{A8}$$

For simplicity, let w_o be the arbitrary term

$$w_o = a_n (1 - \cos\frac{n\pi x}{2\ell}), \quad n = 1, 3, 5, \cdots \tag{A9}$$

The deflection of the beam due to deformation, w, can be expressed as

$$w = (a_n f_2 - \delta)\cos k_2 x \; - \; a_n f_2 \cos\frac{n\pi x}{2\ell} + \delta, \qquad for \quad 0 < x < \ell_2 \tag{A10}$$

$$w = D(-\tan k_1 l \cos k_1 x \; + \; \sin k_1 x) \; - \; a_n f_1 \cos\frac{n\pi x}{2\ell} + \delta, \quad for \quad \ell_2 < x < \ell \tag{A11}$$

where

$$\begin{bmatrix} D \\ \delta \end{bmatrix} = A^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} a_n$$

$$b_1 = -(f_2 - f_1)\cos\frac{n\pi l_2}{2\ell} + f_2 \cos k_2 \ell_2$$

$$b_2 = (f_2 - f_1)\frac{n\pi}{2k_2\ell}\sin\frac{n\pi l_2}{2\ell} - f_2 \sin k_2 \ell_2$$

$$f_1 = \frac{\alpha_1}{n^2 - \alpha_1}, \qquad f_2 = \frac{\alpha_2}{n^2 - \alpha_2},$$

$$\alpha_1 = \frac{P}{\pi^2 (E_b I_b)/4\ell^2}, \qquad \alpha_2 = \frac{P}{\pi^2 (E_a I_a + E_b I_b)/4\ell^2}$$

Bending of Cantilever Beam with Axial Load Eccentricity

Letting e be the eccentricity in loading, the deflection curve is

$$w = (\delta + e)(1 - \cos k_2 x), \qquad for \quad 0 < x < \ell_2 \tag{A12}$$

$$w = -\frac{D \sin k_1 \ell + e}{\cos k_1 \ell}\cos k_1 x + D \sin k_1 x + \delta + e, \quad for \quad \ell_2 < x < \ell \tag{A13}$$

where

$$\begin{bmatrix} D \\ \delta \end{bmatrix} = A^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} e$$

$$b_1 = -\cos k_2 \ell_2 + \frac{\cos k_1 \ell_2}{\cos k_1 \ell}$$

$$b_2 = \sin k_2 \ell_2 - \frac{k_1}{k_2} \frac{\sin k_1 \ell_2}{\cos k_1 \ell}$$

Active Buckling Control of Cantilever Beam

Using piezoactuators, control of the deflection of the center of a clamped beam can be carried out. For a cantilever beam with imperfections, or eccentricity of loading, we may control the deflection at the end $x = \ell$ to eliminate the first buckling load, and increase the load-bearing capability of the beam. The procedure is given below.

Suppose both compressive load and voltage are applied to the beam, according to superposition, the total deflection δ at $x = \ell$ is

$$\delta = \delta_a + \delta_m \tag{A14}$$

where δ_m is the deflection of the end due to the compressive load P ($V = 0$), which can be measured by the sensor, δ_a the deflection due to the voltage calculated for a straight beam. Based on the above equation, an active buckling control of buckling of a cantilever beam may be stated in the following:

If $|\delta_m| > |\delta_d|$, δ_d is the allowable deflection at the end $x = \ell$ under design load, then the actuators are to be active, and the applied voltage is given by

$$\delta_a + \delta_m = \delta_d \tag{A15}$$

The equation indicates that the active actuators enforce the end of the beam to back to the allowable position. It may be rewritten as

$$\beta d = -\frac{\delta_m - \delta_d}{\tilde{\delta}_a} \tag{A16}$$

if δ_a is expressed as $\delta_a = \beta d \tilde{\delta}_a$.

For a cantilever beam with imperfections, or eccentricity in loading, it can be proved that the first buckling mode is eliminated. The deflection w of the beam is

$$w = w_a + w_m \tag{A17}$$

where w_m is the deflection of the real beam due to the compressive load P ($V = 0$), w_a the deflection due to the voltage calculated for a straight beam. With the value of voltage provided by the controlling equation, the above equation shows that the buckling load can be increased as several times of the critical loading of the idea beam ($V = 0$). If compressive load is below the buckling load, the deflection of the beam is finite, and the applied voltage is finite too.

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