AIAA 98-2375
A Study of Supersonic Surface Sources—The Ffowcs Williams-Hawkings Equation and the Kirchhoff Formula

F. Farassat and Kenneth S. Brentner
NASA Langley Research Center
Hampton, Virginia

M. H. Dunn
Old Dominion University
Norfolk, Virginia

Corrected September 21, 1998

4th AIAA/CEAS Aeroacoustics Conference
June 2–4, 1998 / Toulouse, France

A STUDY OF SUPERSONIC SURFACE SOURCES—THE FFOWCS WILLIAMS-HAWKINGS EQUATION AND THE KIRCHHOFF FORMULA[#]

F. Farassat[*] and Kenneth S. Brentner[†]
NASA Langley Research Center,
Hampton, Virginia

M.H. Dunn[‡]
Old Dominion University,
Norfolk, Virginia

Abstract

In this paper we address the mathematical problem of noise generation from high speed moving surfaces. The problem we are solving is the linear wave equation with sources on a moving surface. The Ffowcs Williams-Hawkings (FW-H) equation as well as the governing equation for deriving the Kirchhoff formula for moving surfaces are both this type of partial differential equation. We give a new exact solution of this problem here in closed form which is valid for subsonic and supersonic motion of the surface but it is particularly suitable for supersonically moving surfaces. This new solution is the simplest of all high speed formulations of Langley and is denoted formulation 4 following the tradition of numbering of our major results for the prediction of the noise of rotating blades. We show that for a smooth surface moving at supersonic speed, our solution has only removable singularities.Thus it can be used for numerical work.

1. Introduction

The problem of noise generation from moving bodies is very important in aeroacoustics. Two current methods of attacking this problem are the acoustic analogy and the Kirchhoff formula for moving surfaces. The acoustic analogy method is based on different forms of the solution of the Ffowcs Williams-Hawkings (FW-H) equation[1]. This is a linear wave equation with sources on a moving surface. The Kirchhoff formula is also derived from a linear wave equation with sources on a moving surface[2]. For simplicity, we refer to this wave equation here as the Kirchhoff (K) equation. Using generalized function theory, both the FW-H and the K equations can be written with inhomogeneous source terms involving the Dirac delta function with support on the moving surface $f = 0$ and the first derivatives of this delta function. The method of solution of these two equations are, thus, identical.

Obtaining various forms of the solution of these equations for subsonic surfaces is fairly easy. These solutions (Formulations 1 and 1A of Farassat) have been published elsewhere[2-5]. We will not, therefore, address the subsonic case here. We mention that the common forms of the solution for subsonic surfaces have a Doppler singularity which make them unsuitable for supersonically moving surfaces. To obtain new forms of solution of the FW-H and the K equations for supersonic surfaces, we must integrate the Green's function of the wave equation in a different way than the subsonic case[1-3]. This was fully recognized by Ffowcs Williams and Hawkings[1] and they laid the foundation for the work we present here. The solution of the supersonic problem is considerably more difficult than the subsonic case and it has taken a lot longer to fully overcome the many mathematical obstacles.

To understand the nature of the complexities involved, one must recognize that the problem as treated here is four dimensional. We are interested in formulations which are suitable for efficient numerical noise prediction from rotating machinery. This requirement puts a restriction on what forms of the solution of the FW-H and the K equation are acceptable to us. In practice, it has been found that the common formulations for subsonic surfaces are much more efficient than the supersonic formulations even if the latter can also be used for subsonic surfaces. Thus, one is forced to use more than one formulation in any noise prediction code based on the FW-H and K equations.

In noise calculation, a moving surface, such as a blade, is divided into panels and the noise generated by each panel is summed up to get the total noise from the surface. This means that the FW-H and K equation must be solved for an open surface, e.g., a panel on the moving surface. Thus we must solve these two equations with inhomogeneous source terms that have a Heaviside function multiplying the Dirac delta functions which describe the open surface. The mathematical

[*] Senior Research Scientist, AIAA Associate Fellow
[†] Research Scientist, AIAA Member
[‡] Associate Professor, AIAA Member

treatment of these inhomogeneous source terms produces an additional complexity in obtaining solutions of these equations for the supersonic case.

In the subsonic case, the Doppler factor appears as the result of integration of the source time variable in the Green's function solution of the wave equation after a Lagrangian frame is introduced in which the surface is time independent. This step makes the problem essentially three dimensional. To get other forms of the solution suitable for a supersonically moving surface, one must use integration over the influence surface of the observer space-time variables $(\boldsymbol{x}, t)$ which we call the $\Sigma-\mathrm{surface}$. This surface is more fundamental to the solution of the wave equation than the actual moving surface over which the subsonic formulations are integrated.

In the next section the governing equations of the problem under consideration are presented. In Section 3, we will give a new solution of the FW-H and the K equations in closed form for supersonic surfaces. In Section 4 we will show that the singularities of the solution are integrable. The concluding remarks follow.

2. The Governing Equations

The Ffowcs Williams-Hawkings Equation and the Kirchhoff equation are quite well-known in aeroacoustics. We will need a special form of these equations suitable for our work. The FW-H equation for a moving surface $f(\boldsymbol{x}, t) = 0$ where $f > 0$ outside the body is

$$\begin{aligned}\Box^2 p' &= \frac{\partial}{\partial t}\{[\rho u_n - (\rho - \rho_0)v_n]\delta(f)\} \\ &\quad - \frac{\partial}{\partial x_i}\{[\rho(u_n - v_n)u_i + pn_i]\delta(f)\} \\ &\quad + \frac{\partial^2}{\partial x_i \partial x_j}[T_{ij}H(f)] \qquad (1)\end{aligned}$$

where p' is $(\rho-\rho_0)c^2$, ρ is the density, and ρ_0 and c are the density and speed of sound of undisturbed medium. The local normal fluid and body velocities are denoted by u_n and v_n, respectively. The Lighthill stress tensor is denoted T_{ij} and p is the surface pressure on $f = 0$. Note that we assume that the surface f is defined such that $\nabla f = \boldsymbol{n}$ where $\boldsymbol{n}$ is the unit outward normal to this surface. The Heaviside function is denoted $H(f)$. As proposed by Ffowcs Williams and Hawkings[1], the moving surface $f = 0$ can be penetrable and we assume so here.

We next find all the surface contributions of the last term of Eq. (1) by taking the space derivatives explicitly and using the rules of generalized differentiation[2,6,7]. We get

$$\begin{aligned}\Box^2 p' &= \frac{\partial}{\partial t}\{[\rho u_n - (\rho-\rho_o)v_n]\delta(f)\} \\ &\quad + \frac{\partial}{\partial x_i}\left\{[\rho v_n u_i - (\rho-\rho_0)c^2 n_i]\delta(f)\right\} \\ &\quad + \frac{\partial T_{ij}}{\partial x_j} n_i\ \delta(f) + \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}\ H(f) \\ &\equiv Q_1 + Q_2 + Q_3 + Q_4 \qquad (2)\end{aligned}$$

We will then consider an open surface described by $f = 0$, $\tilde{f} > 0$ where $f = \tilde{f} = 0$ is the equation of the edge of this open surface[2,6]. We define $\tilde{f}$ such that $\nabla \underset{\sim}{\tilde{f}} = \nu$ where ν is the unit inward geodesic normal to the edge $f = \tilde{f} = 0$ [2,6]. To calculate the noise from this open surface, we must multiply $\delta(f)$ in Q_1, Q_2 and Q_3 by the Heaviside function $H(\tilde{f})$. We will next use the concept of restriction of a variable to the surface $f = 0$ and then take the derivatives of Q_1 and Q_2 terms explicitly[6]. We use a tilde under a symbol to signify restriction.

Introducing the notations

$$E = \rho u_n - (\rho - \rho_0)v_n \qquad (3a)$$

$$E_i = \rho v_n u_i - (\rho-\rho_0)c^2 n_i, \qquad (3b)$$

we have

$$\begin{aligned}Q_1 &= \frac{\partial}{\partial t}[\underset{\sim}{E}\ H(\tilde{f})\delta(f)] \\ &= \dot{\underset{\sim}{E}}\ H(\tilde{f})\delta(f) - E\ v_\nu\ \delta(\tilde{f})\delta(f) \\ &\quad - \underset{\sim}{E}\ \underset{\sim}{v}_n\ H(\tilde{f})\delta'(f) \qquad (4)\end{aligned}$$

and

$$\begin{aligned}Q_2 &= \frac{\partial}{\partial x_i}[\underset{\sim}{E}_i\ H(\tilde{f})\delta(f)] \\ &= \nabla_2 \cdot \boldsymbol{E}_T\ H(\tilde{f})\delta(f) + E_i \nu_i \delta(\tilde{f})\delta(f) \\ &\quad - 2\ H_f E_n n_i\ H(\tilde{f})\delta(f) \\ &\quad + \underset{\sim}{E}_i \underset{\sim}{n}_i\ H(\tilde{f})\delta'(f) \qquad (5)\end{aligned}$$

Here, v_n is the local velocity of the edge along the geodesic normal ν with components ν_i, $\boldsymbol{E}_T$ is the projection vector E_i on the surface of $f = 0$ and H_f is the local mean curvature of the surface $f = 0$[8,9]. The surface divergence of $\boldsymbol{E}_T$ is $\nabla_2 \cdot \boldsymbol{E}_T$[9]. Using Eqs. (4) and (5), we get

$$Q_1 + Q_2 + Q_3 = q_1 H(\tilde{f})\delta(f) + \underset{\sim}{q_2} H(\tilde{f})\delta'(f) + q_3 \delta(\tilde{f})\delta(f) \tag{6}$$

where we have defined the following symbols

$$q_1 = \frac{\partial T_{ij}}{\partial x_j} n_i + \dot{\underset{\sim}{E}} + \nabla_2 \cdot \boldsymbol{E}_T - 2 H_f E_n \tag{7}$$

$$q_2 = c^2(\rho - \rho_0)(M_n^2 - 1) \tag{8}$$

$$q_3 = \rho(v_n u_\nu - u_n v_\nu) + (\rho - \rho_0) v_n v_\nu \tag{9}$$

In Eq.Eq.Eq. (7), $E_n = E_i n_i$ and $M_n = v_n/c$ is the local Mach number on $f = 0$. Note that q_2 in Eq. (6) is restricted to the surface $f = 0$. Also note that in Eq. (5), we have dropped the restriction on any variable that multiplies $\delta(f)$ if it is not differentiatedPhus we write $\dot{\underset{\sim}{E}}$ and not $\dot{E}$. There is also no need to use restriction sign on $\nabla_2 \cdot \boldsymbol{E}_T$ since $\boldsymbol{E}_T$ is already restricted to $f = 0$. It is important to recognize that $\dot{\underset{\sim}{E}}$ is the rate of change of E as measured by an observer on the surface.

The FW-H equation for an open penetrable surface moving at supersonic speed is:

$$\Box^2 p' = q_1 H(\tilde{f})\delta(f) + \underset{\sim}{q_2} H(\tilde{f})\delta'(f) + q_3 \delta(\tilde{f})\delta(f) \tag{10}$$

A similar equation is also obtained for derivation of the Kirchhoff formula[2] for an open surface $f = 0$, $\tilde{f} > 0$

where we have:

$$q_1 = -\frac{\partial p'}{\partial n} - \frac{1}{c} M_n \frac{\partial p'}{\partial t} - \frac{1}{c}\frac{\partial}{\partial t}(\underset{\sim}{M_n} \underset{\sim}{p'}) + 2 H_f p' \tag{11}$$

$$q_2 = (M_n^2 - 1)p' \tag{12}$$

$$q_3 = M_n M_\nu p' \tag{13}$$

We will go one further step here in preparation of obtaining the new formulation. We note that for a surface $f = 0$, $|\nabla f| = 1$, we have the following results[2,6]:

$$\int Q(\boldsymbol{y})\,\delta'(f)\,d\boldsymbol{y} = \int_{f=0} \left[-\frac{\partial Q}{\partial n} + 2H_f Q \right] dS \tag{14}$$

$$\int Q(\boldsymbol{y})\,\delta(f)\,d\boldsymbol{y} = \int_{f=0} Q\,dS \tag{15}$$

where H_f is the mean curvature of $f = 0$[8,9]. Introduce a new generalized function (distribution) $\delta'_s(f)$ by the following relation:

$$\int Q(\boldsymbol{y})\,\delta'_s(f)\,d\boldsymbol{y} = -\int_{f=0} \frac{\partial Q}{\partial n} dS \tag{16}$$

The subscript s in $\delta'_s(f)$ stands for "simple" which emphasizes the similarity of $\delta'_s(f)$ to the one dimensional $\delta'(x)$ that behaves as follows:

$$\int \phi(x)\,\delta'(x)\,dx = -\phi'(0) \tag{17}$$

Now using the results of eqs. (15) and (16) in Eq. (14), we see that the following relation holds:

$$\delta'(f) = \delta'_s(f) + 2 H_f \delta(f) \tag{18}$$

Equation (18) is next used in Eq. (10) which is written as

$$\Box^2 p' = q_1 H(\tilde{f})\,\delta(f) + \underset{\sim}{q_2} H(f)\,\delta'_s(f) + q_3 \delta(\tilde{f})\,\delta(f) \tag{19}$$

where now only the definition of q_1 is changed as follows:

$$q_1 = \frac{\partial T_{ij}}{\partial x_j} n_i + \dot{\underset{\sim}{E}} + \nabla_2 \cdot \boldsymbol{E}_T + 2H_f[\rho v_n(v_n - u_n) - \rho_0 v_n^2] \quad \text{(FW-H eq.)} \tag{20}$$

and

$$q_1 = -\frac{\partial p'}{\partial n} - \frac{1}{c} M_n \frac{\partial p'}{\partial t} - \frac{1}{c}\frac{\partial}{\partial t}(\underset{\sim}{M_n} \underset{\sim}{p'}) + 2H_f M_n^2 \quad \text{(K eq.)} \tag{21}$$

In the following section, we give the full solution of Eq. (19).

<u>Remark</u>. We will not address the solution of the FW-H equation with the pure quadrupole term alone:

$$\Box^2 p' = Q_4 = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} H(f) \tag{22}$$

The solution using the collapsing sphere approach is singularity free and is given elsewhere[1,6].

3. Solution of Wave Equation With Sources on A Moving Open Surface

In this section, we give the solution to the following three wave equations:

$$\Box^2\phi_1 = q_1(\boldsymbol{x}, t)\, H(\tilde{f})\delta(f) \tag{23}$$

$$\Box^2\phi_2 = q_2(\boldsymbol{x}, t)\, H(\tilde{f})\delta'_s(f) \tag{24}$$

$$\Box^2\phi_3 = q_3(\boldsymbol{x}, t)\, \delta(\tilde{f})\delta(f) \tag{25}$$

The source terms here are similar to those of Eq. (19). The treatment of these equations are discussed in two references by Farassat[2,6]. We will use the solutions to eqs. (23) and (25) given in these references but will give here a new and particularly simple solution of Eq. (24). The materials presented in these references are essential in understanding what follows.

Let $\tilde{F}(\boldsymbol{y}; \boldsymbol{x}, t) = \tilde{f}(\boldsymbol{y}, t - r/c) = [\tilde{f}(\boldsymbol{y}, \tau)]_{ret}$, and $F(\boldsymbol{y}; \boldsymbol{x}, t) = [f(\boldsymbol{y}, \tau)]_{ret}$, where the subscript *ret* stands for retarded time. The influence surface of the open surface $f = 0$, $\tilde{f} > 0$ is called the Σ – surface and is described by $F = 0$, $\tilde{F} > 0$ [2,6]. The edge of this surface is the L-curve described by $F = \tilde{F} = 0$. Below, we use $(\boldsymbol{x}, t)$ and $(\boldsymbol{y}, \tau)$ as the observer and the source space-time variables, respectively. The solution of Eq. (23) is[2,6]:

$$\boxed{4\pi\phi_1(\boldsymbol{x}, t) = \int_{\substack{F = 0\\ \tilde{F} > 0}} \frac{1}{r\Lambda}[q_1]_{ret}\, d\Sigma} \tag{26}$$

where

$$\Lambda^2 = 1 + M_n^2 - 2\, M_n \cos\theta \tag{27}$$

Here, M_n is the local normal Mach number of the surface $f = 0$ and $\cos\theta = \boldsymbol{n} \cdot \hat{\boldsymbol{r}}$ where n is the unit outward normal to $f = 0$, $\hat{\boldsymbol{r}} = (\boldsymbol{x} - \boldsymbol{y})/r$ is the unit radiation vector from the source to the observer and $r = |\boldsymbol{x} - \boldsymbol{y}|$.

We now consider Eq. (24). The formal solution of this equation using the Green's function method is

$$4\pi\phi_2(\boldsymbol{x}, t) = \int \frac{1}{r} q_2\, (\boldsymbol{y}, \tau)\, H(\tilde{f})\delta'_s(f)\delta(g)\, d\boldsymbol{y} d\tau \tag{28}$$

where $g = \tau - t + r/c$. We now introduce a new local frame (u^1, u^2, u^3) where $u^3 = f$ and u^1 and u^2 are the Gaussian coordinates on $f =$ constant, extended from $f = 0$ along local normal. We assume that u^1 is the length variable along the projection of $\hat{\boldsymbol{r}}$ on the local tangent plane and u^2 is the length variable along $\hat{\boldsymbol{r}} \times \hat{\boldsymbol{n}}$. Let $g_{(2)}$ be the determinant of the coefficients of the first fundamental form in the new variables. Let $u^1 \to g$, then Eq. (28) becomes

$$\begin{aligned} 4\pi\phi_2(\boldsymbol{x}, t) &= \int \frac{q_2(\boldsymbol{y}, \tau)}{r} H(f)\, \delta'_s(u^3) \\ &\quad \times \delta(g)\sqrt{g_{(2)}}\, du^1 du^2 du^3 d\tau \\ &= c\int \left[\frac{q_2\sqrt{g_{(2)}}\; H(f)}{r\sin\theta}\right]_{g = 0} \\ &\quad \times \delta'_s(u^3)\, du^2 du^3 d\tau \end{aligned} \tag{29}$$

Note that we have used

$$\frac{\partial g}{\partial u^1} = \frac{1}{c}\sin\theta \tag{30}$$

and $g_{(2)}$ must be restricted to $f = 0$ because we are dealing with $\delta'_s(u^1)$ where the curvature term of $\delta'(u^1)$ has already been removed (see Eq. (18)) and added to q_1. We must mention here that, in new variables $q_2(\boldsymbol{y}, \tau) = q_2[\boldsymbol{y}\,(u^1, u^2, 0, \tau), \tau]$.

The condition $g = 0$ in Eq. (29) implies that $u^1 = u^1(u^2, u^3, \tau)$. We will use this result in the integration of $\delta'_s(u^1)$ in Eq. (29). We get

$$\begin{aligned} &4\pi\phi_2(\boldsymbol{x}, \tau) \\ &= -c\int \left\{\frac{\partial}{\partial u^3}\left[\frac{q_2\sqrt{g_{(2)}}H(\tilde{f})}{r\sin\theta}\right]_{g = 0}\right\}_{u^3 = 0} du^2 d\tau \end{aligned} \tag{31}$$

We have the following results

$$\left.\frac{\partial u^1}{\partial u^3}\right|_{g = 0} = -\cot\theta \tag{32}$$

$$\left.\frac{\partial \sin\theta}{\partial u^3}\right|_{g = 0} = \frac{\cot\theta}{r} \tag{33}$$

$$\frac{\partial\sqrt{g_{(2)}}}{\partial u^3} = -\Gamma^{\,i}_{1i}\sqrt{g_{(2)}}\cot\theta \tag{34}$$

$$\frac{\partial H(\tilde{f})}{\partial u^3} = -\cot\theta\, \mathrm{v} \cdot \boldsymbol{t}_1\, \delta(\tilde{f}) \tag{35}$$

where $\boldsymbol{t}_1$ is the unit vector along the projection of $\hat{\boldsymbol{r}}$ on the local tangent plane and $\Gamma^{\,i}_{1i}$ (sum on i) is the Christoffel symbol of second kind in the new coordinate system. Note that we have a locally orthogonal frame which gives $g_{(2)} = 1$ at the origin but because we have a curved surface, we get Eq. (34). Therefore, in the rest of the algebraic manipulations, we will set $g_{(2)} = 1$.

We can now write Eq. (31) in the following form after taking the derivative with respect to u^3 of the integrand:

$$
\begin{aligned}
&4\pi\phi_2(\boldsymbol{x}, t) \\
&= c\int\left[\frac{\cos\theta}{r\sin^2\theta}\boldsymbol{t}_1\cdot\nabla_2 q_2 + \frac{\cos\theta}{r^2\sin^3\theta}q_2\right]du^2 d\tau \\
&\quad + c\int\frac{\cos\theta}{r\sin^2\theta}\Gamma^i_{1i}\, q_2\, du^2 d\tau \\
&\quad + c\int\frac{\nu\cdot\boldsymbol{t}_1\ \cos\theta}{r\sin^2\theta|\cos\psi|}q_2\, d\tau
\end{aligned}
\qquad (36)
$$

where ψ is the angle between $\hat{\boldsymbol{r}}$ and the edge of the open surface described by $f = \tilde{f} = 0$.

In our previous work, we have used $d\Gamma$ for du^2 where $d\Gamma$ is the element of the curve of intersection of $f = 0$ with the collapsing sphere $g = 0$. We have shown also that[6,10]

$$\frac{c\,d\tau\,d\Gamma}{\sin\theta} = \frac{d\Sigma}{\Lambda} \qquad (37)$$

$$\frac{d\tau}{|\cos\psi|} = \frac{dL}{\Lambda_0} \qquad (38)$$

where L is the edge of the Σ – surface described by $F = \tilde{F} = 0$ and

$$\Lambda_0 = |\nabla F\times\nabla\tilde{F}| = \Lambda\tilde{\Lambda}\sin\theta' \qquad (39)$$

$$\cos\theta' = \boldsymbol{N}\cdot\boldsymbol{N}' \qquad (40)$$

$$\tilde{\boldsymbol{N}} = \frac{\nabla\tilde{F}}{|\nabla\tilde{F}|} = \frac{\nu - \boldsymbol{M}_\nu\hat{\boldsymbol{r}}}{\tilde{\Lambda}} \qquad (41)$$

$$\tilde{\Lambda}^2 = 1 + M_\nu^2 - 2M_\nu\cos\tilde{\theta} \qquad (42)$$

$$\cos\tilde{\theta} = \nu\cdot\hat{\boldsymbol{r}} = \nu\cdot\boldsymbol{t}_1 \qquad (43)$$

Therefore, Eq. (36) can be written as follows:

$$
\boxed{
\begin{aligned}
&4\pi\phi_2(\boldsymbol{x}, t) \\
&= \int_{\substack{F = 0\\ \tilde{F} > 0}}\left[\frac{\cos\theta}{r\sin\theta}\boldsymbol{t}_1\cdot\nabla_2 q_2 + \frac{\cos\theta}{r^2\sin^2\theta}q_2\right]_{ret}\frac{d\Sigma}{\Lambda} \\
&\quad + \int_{\substack{F = 0\\ \tilde{F} > 0}}\frac{\cos\theta}{r\Lambda\sin\theta}\Gamma^i_{1i}\,[q_2]_{ret}\, d\Sigma \\
&\quad + \int_{\substack{F = 0\\ \tilde{F} = 0}}\frac{\cos\tilde{\theta}\cos\theta}{r\Lambda_0\sin^2\theta}[q_2]_{ret}dL
\end{aligned}
}
\qquad (44)
$$

We have separated the integrals over the Σ – surface in this equation to simplify the analysis of the singularities.

Finally, the solution of Eq. (25) was also given by Farassat[2,6] as follows:

$$\boxed{4\pi\phi_3(\boldsymbol{x}, t) = \int_{\substack{F = 0\\ \tilde{F} = 0}}\frac{1}{r\Lambda_0}[q_3]_{ret}\, dL} \qquad (45)$$

We have thus given the full solution of Eq. (19) which we refer to as *formulation 4*. Only Eq. (44) in our analysis is new. In the next section, we will discuss the important question of the singularity of the solution of Eq. (19).

4. A Study of the Singularities of the Solution for Supersonic Surfaces

We will now address the question of the singularities in the solution of the FW-H and K equations. There has been a general belief among the researchers, the authors of this paper included, that the solution of these equations lead to nonintegrable singularities for some observer space-time variables ($\boldsymbol{x}$, t). We will show here that for a smooth surface, all the singularities of the new solution are integrable. We assume that $f = 0$ is not an open surface and thus we only consider the integrals over the Σ – surface: $F(\boldsymbol{y}; \boldsymbol{x}, t) = f(\boldsymbol{y}, t - r/c) = 0$. See articles by Farassat, De Bernardis and Myers for the analysis of the singularities of the line integrals[11,12]. As will be seen below, the analysis of the singularities of the surface integrals is very difficult.

There are two kinds of singularities in the surface integrals: i) $\sin\theta = 0$, i. e., $\theta = 0°$ or $180°$ but $\Lambda \neq 0$, and ii) $\Lambda = 0$. We will show below that $\Lambda = 0$ also implies $\theta = 0°$ or $180°$. Condition i) means that at some source time, the collapsing sphere $g = 0$ is tangent to the surface $f = 0$ at a point where $M_n \neq \pm 1$. Condition ii) means that at some source time $g = 0$ is tangent to $f = 0$ at a point where $M_n = \pm 1$. We will prove these assertions for appearance of singularity $\Lambda = 0$ below. We assume that $f = 0$ is convex with no saddle points.

Consider a rotating surface part of which moves at supersonic speed. We can write Λ^2 as follows:

$$\Lambda^2 = (1 - M_n \cos\theta)^2 + M_n^2 \sin^2\theta. \tag{46}$$

Therefore, $\Lambda = 0$ if $\sin\theta = 0$ and $1 - M_n \cos\theta = 0$ simultaneously. This means that we must have $M_n = 1$ and $\theta = 0°$ or $M_n = -1$ and $\theta = 180°$. The geometrical interpretation of these conditions is obvious. Since $f = 0$ is assumed to be moving supersonically on part of its surface, there is a curve Ψ on $f = 0$ on which $M_n = 1$ or $M_n = -1$. If at any source time τ_0, the collapsing sphere $g = \tau - t + r/c = 0$ is tangent to $f = 0$ at a point on Ψ, then the integrands of the surface integrals of formulation 4 are singular for the time $t_0 = \tau_0 + r/c$. Note that, in general, the curve Ψ is time dependent but it is not so for a hovering rotor operating at supersonic tip speed. This is the case we study here.

We must study the two conditions of appearance of singularities separately. The reason becomes apparent below. Let us first write the kinds of integrals we have:

$$I_1 = \int_{\substack{F=0\\ \tilde{F}>0}} \frac{1}{r\Lambda}[q_1]_{ret} d\Sigma \tag{47}$$

$$I_2 = \int_{\substack{F=0\\ \tilde{F}>0}} \frac{\cos\theta}{r\Lambda\sin\theta}\Gamma^i_{1i}[q_2]_{ret} d\Sigma \tag{48}$$

$$I_3 = \int_{\substack{F=0\\ \tilde{F}>0}} \left[\frac{\cos\theta}{r\sin\theta}\boldsymbol{t}_1 \cdot \nabla_2 q_2 + \frac{\cos\theta}{r^2\sin^2\theta} q_2\right]_{ret} \frac{d\Sigma}{\Lambda} \tag{49}$$

We will show that all the singularities are integrable.

Condition i): $\sin\theta = 0, \Lambda \neq 0$

Near this point, the intersection of $g = 0$ and $f = 0$ is a circle of radius b. We can show that if this condition appears at $\tau = 0$, then as $\tau \to 0$, $b = C\sqrt{|\tau|}$ where C is a constant. We next use the relation[10]

$$\frac{d\Sigma}{\Lambda} = \frac{c\, d\Gamma\, d\tau}{\sin\theta} \tag{50}$$

where Γ is the curve of intersection of $g = 0$ with $f = 0$. Near the point A, above, $d\Gamma = b d\varphi$, $\sin\theta = b/r$, where b is the radius of Γ which is a small circle, and φ is the azimuthal angle around Γ. Thus, we have

$$\frac{d\Sigma}{\Lambda} = cr d\varphi d\tau \tag{51}$$

which means if $q(\boldsymbol{y}, \tau)$ is continuous, then

$$I_1 = c\int_\tau d\tau \int_0^{2\pi} [q_2]_{ret} d\varphi \tag{52}$$

is integrable. Therefore, we have shown the integral

$$I_1 = \int_{\substack{F=0\\ \tilde{F}>0}} \frac{1}{r\Lambda}[q_1]_{ret} d\Sigma \tag{53}$$

is integrable.

For I_2, we use the fact that $\sin\theta = b/r$ and $b = C\sqrt{|\tau|}$ to write the integral in the form

$$I_2 = \int_\tau \frac{c\, a\, d\tau}{C\sqrt{|\tau|}} \int_0^{2\pi} \Gamma^i_{1i}[q_2]_{ret} d\varphi \tag{54}$$

where $a = r(\tau = 0)$. This integral is convergent.

The study of convergence of I_3 is very interesting. It appears that this integral is not convergent. We manipulate the integrand as follows near the condition $\sin\theta = 0$:

$$E = \frac{\cos\theta}{r\sin\theta}\boldsymbol{t}_1 \cdot \nabla_2 q_2 + \frac{\cos\theta}{r^2\sin^2\theta} q_2 \approx \frac{a}{rb^2}[b\,\boldsymbol{t}_1 \cdot \nabla_2 q_2 + q_2] \tag{55}$$

We note that $\boldsymbol{t}_1 \cdot \nabla_2 q_2 = -\partial q_2/\partial b$ and

$$b\,\boldsymbol{t}_1 \cdot \nabla_2 q_2 + q_2 = b^2 \frac{\partial^2 q_2}{\partial b^2} \tag{56}$$

and, thus, near the point of tangency of $f = 0$ and $g = 0$:

$$E \approx \frac{a}{r}\frac{\partial^2 q_2}{\partial b^2} \tag{57}$$

The convergence of I_3, therefore, depends on the value of the following integral when $\sin\theta \to 0$:

$$I_3 = c\,a\int_\tau d\tau \int_0^{2\pi}\left[\frac{\partial^2 q_2}{\partial b^2}\right]_{\tau=0} d\varphi \tag{58}$$

We can easily show that

$$\int_0^{2\pi}\left[\frac{\partial^2 q_2}{\partial b^2}\right]_{\tau=0} d\varphi = [\nabla_2^2 q_2]_{\tau=0} \tag{59}$$

The convergence of I_3 is, thus, guaranteed. We conclude that when $\sin\theta \to 0$ and $\Lambda \neq 0$, formulation 4 has only removable singularities.

Condition ii): $\Lambda = 0$ (implies $\sin\theta = 0$)

As we have shown above, when $g = 0$ is tangent to $f = 0$ at a point where $M_n = \pm 1$, we have $\Lambda = 0$ and from the tangency condition $\sin\theta = 0$. The first thing we study here is the structure of the Σ – surface near a point where $\Lambda = 0$. We then study the problem of the singularities of formulation 4.

We consider the condition $\theta = 0°$ and $M_n = 1$ for a hovering rotor. Figure 1 shows the tangent plane T to a point on Ψ looking edgewise at the moment of tangency of the collapsing sphere with $f = 0$ at the point A. Note that $\boldsymbol{M}_n = \boldsymbol{n} = \hat{\boldsymbol{r}}$ at this moment so that the observer is in the plane shown at the center of the collapsing sphere $g = 0$. We assume $\tau_0 = 0$ and $r = a$ at the moment of tangency. Since $M_n = 1$ at A, we have $\sin\mu = 1/M$ where μ is the angle that T makes with Mach number vector $\boldsymbol{M}$ shown in Fig. 1. We now consider the plane T in motion for $|\tau| < \varepsilon$ where $\varepsilon > 0$ is a small number. In the frame fixed to T with origin at A as shown in Fig. 2, the curve of intersection of $g = 0$ with T, which is a circle is given by the relation

$$(x - \beta V_a\tau)^2 + (y + \gamma V_R\tau)^2 = V_a^2\beta^2\tau^2 \tag{60}$$

Where $V_a = a\omega$, $V_R = R\omega$ and β and γ are defined in fig. 2. We use R for the distance from A to the axis of rotation. The center of the circle is at the point $(x, y) = (\beta V_a\tau, -\gamma V_R\tau)$ and its radius is $V_a\beta|\tau|$. It is clear that as $|\tau| \to 0$, the Σ – surface looks like a vertex of a cone and has no tangent at the vertex coinciding with A. The condition of $\Lambda = 0$ is thus equivalent to the Σ – surface becoming pointed and having no tangent plane at the point A.

Let us see what the intersection of $g = 0$ and T will appear to an observer on the tangent plane T. Figure 3 shows the envelope of the circles of Eq. (60). We reject the part of the envelope in the region M < 1 because it would imply multiple emission from subsonic region which is impossible. This means that we have no intersection of $g = 0$ and T for $\tau < 0$. This figure clearly shows that the Σ – surface looks like a cone near A.

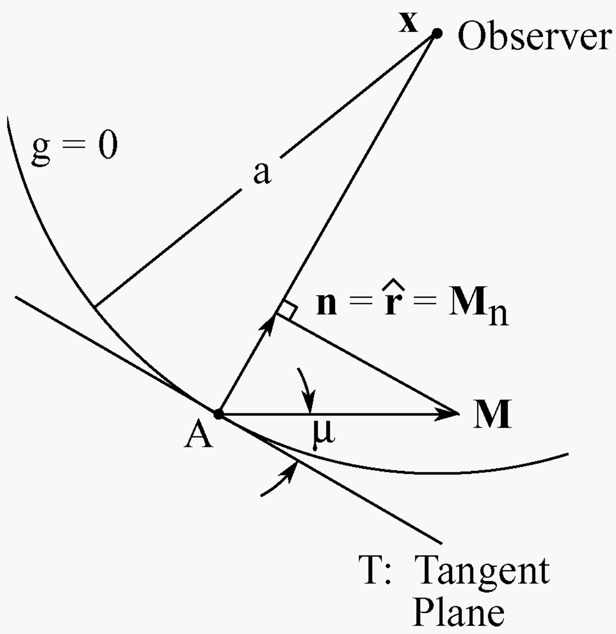

Fig. 1. The geometric condition for the appearance of $\Lambda = 0$: $\theta = 0°$, $M_n = 1$, i.e., $\boldsymbol{M}_n = \hat{\boldsymbol{r}} = \boldsymbol{n}$.

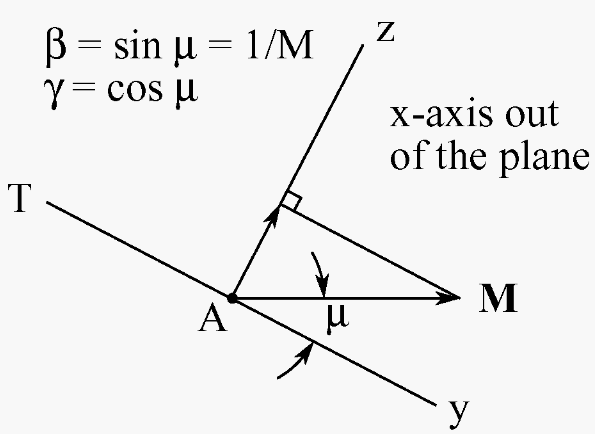

Fig. 2. The coordinate system used to study the intersection of $g = 0$ with $f = 0$ near the condition $\Lambda = 0$.

For convergence of I_1, the analysis of condition i) applies exactly so that I_1 has removable singularity. The

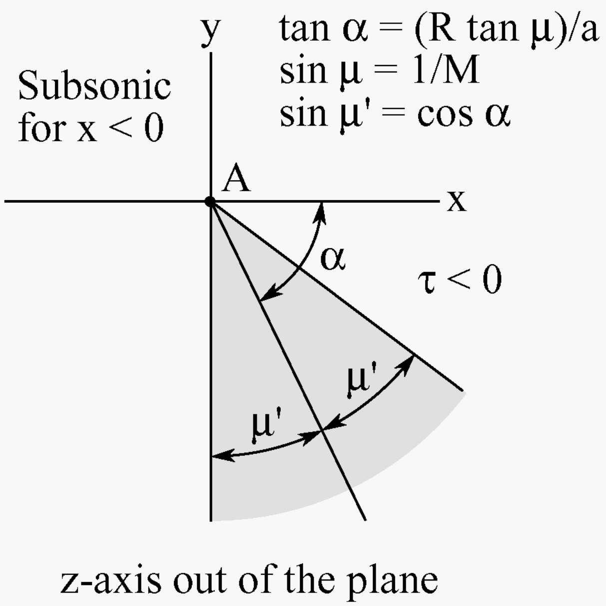

Fig. 3. The shaded area is the trace of the intersection of $g = 0$ in the xy-plane of figure 2 near the condition $\Lambda = 0$. It is assumed that $\tau = 0$ for $\Lambda = 0$.

study of the convergence of I_2 is different since from Eq. (60) we see that as $\Lambda \to 0$, we have $b = C_1\tau$ where C_1 is a constant. However, we note that for both the FW-H and K equations, q_2 is proportional to $M_n^2 - 1$. We can easily show that as $\Lambda \to 0$, $M_n^2 - 1 = C_2\tau$ where C_2 is another constant. Therefore, the convergence of I_2 is guaranteed because near $\Lambda = 0$, we can write I_2 as

$$I_2 = \frac{aC_2}{C_1}\int_\tau d\tau \int_0^{2\pi} \Gamma^i_{1i}\,[q'_2]_{ret}\,d\varphi \qquad (61)$$

where $q'_2 = q_2/(M_n^2 - 1)$. As a matter of fact, I_2, is better behaved for condition ii) than condition i)!

For I_3, the convergence study of condition i) applies exactly for condition ii). Therefore formulation 4 has only removable singularities for condition ii) also.

We conclude that *for a smooth surface, the solution of FW-H and the K equations as given here have only integrable (removable) singularities for a supersonic surface $f = 0$.* The solution of the K equation is, of course, known as *the Kirchhoff formula for supersonic surfaces*.

Remark 1. The solutions of the FW-H and the K equation here are valid for all range of the surface speed. But we do not recommend to use the present results for subsonic surfaces since much more efficient solutions for numerical method are available[2,5,10].

Remark 2. It can be shown that had we not added the surface terms from the quadrupole source term of the FW-H equation to the thickness and loading source terms, the resulting solution would be singular when the condition $\Lambda = 0$ appears. The acoustic pressure signature will have a logarithmic singularity which will appear as an infinite pulse. Our analysis shows that when all the surface sources from thickness, loading and quadrupole terms are included in the analysis, there is no infinite singularities in the acoustic field.

5. Concluding Remarks

We have given the closed form solution of the FW-H and the K equations for an open surface. Although these solutions are valid for all range of Mach numbers, we recommend them for the supersonic motion of the surface because of their complexity. We have shown that for a smooth surface, the singularities of the solutions of both of these equations are integrable. The nature of these singularities is explained in this paper. It is very interesting to note that for the FW-H equation, the thickness and loading source terms alone have nonintegrable singularities in the solution. However, the addition of the surface source terms from the quadrupole source term removes this singularity. This is, of course, expected on intuitive grounds.

We hope that the present work gives further impetus to numerical applications of our results in high speed rotating blade noise prediction. The closed form analytic results of this paper open up two other areas of application which could help aeroacousticians in their endeavor to reduce the noise of aeronautical machines. These areas are: i) qualitative analysis of noise generation mechanisms by the analytic study of the appropriate integrals in our solutions of the FW-H and K equations, and ii) approximate analysis of the radiation field from ducted fan inlet and exhaust and other openings that radiate sound to an infinite medium. The analysis is similar to the use of the conventional Kirchhoff formula for the study of diffraction by an aperture.

Much work is ahead of us in the order of magnitude analysis of the terms in the solution of the FW-H and the K equation. Can a deformable body $f = 0$ be used to control noise radiation at high speed? Can we design a rigid body with desirable noise radiation property in a given direction?

References

1. J. E. Ffowcs Williams and D. L. Hawkings: Sound Generation by Turbulence and Surfaces in Arbitrary Motion, Phil. Trans. Roy. Soc. London, Vol. A264, 1969, 321-342

2. F. Farassat: The Kirchhoff Formulas for Moving Surfaces in Aeroacoustics- The Subsonic and Supersonic Cases, NASA Technical Memorandum 110285, September 1996

3. F. Farassat: Linear Acoustic Formulas for Calculation of Rotating Blade Noise, AIAA J., Vol. 19, 1981, 1122-1130

4. F. Farassat and M. K. Myers: Extension of Kirchhoff's Formula to Radiation from Moving Surfaces, J. Sound and Vib., Vol. 123, 1988, 451-460

5. F. Farassat and G. P. Succi: The Prediction of Helicopter Rotor Discrete Frequency Noise, Vertica, Vol. 9, 1983, 309-320

6. F. Farassat: Introduction to Generalized Functions with Applications in Aerodynamics and Aeroacoustics, NASA Technical Paper 3428, May 1994, Corrected Copy, April 1996

7. Ram P. Kanwal: Generalized Functions-Theory and Technique, Second Edition, 1998, Birkhauser, Boston

8. Erwin Kreyszig: Differential geometry, Dover Books, 1991

9. A.J. McConnell: Applications of Tensor Analysis, Dover Books, 1957

10. F. Farassat: Theory of Noise Generation from Moving Bodies With an Application to Helicopter Rotors, NASA Technical Report R-451, 1975

11. E. De Bernardis and F. Farassat: On the Possibility of Singularities in the Acoustic Field of Supersonic Sources when BEM is Applied to Wave Equation, in Boundary Element Methods in Engineering, B. S. Annigeri and K. Tseng (Eds.), Springer-Verlag, 1990, 522-528

12. F. Farassat and M. K. Myers: Line Source Singularity in Wave Equation and Its Removal by Quadrupole Sources- A Supersonic Propeller Noise Problem, in Structural Acoustics, Scattering and Propagation, International Conference on Theoretical and Computational Acoustics, Vol. 1, John E. Ffowcs Williams, Ding Lee and Allan Pierce (Eds.), World Scientific, 1994, 29-43

Errata For AIAA 98-2375: A Study of Supersonic Sources- The Ffowcs Williams-Hawkings Equation and the Kirchhoff Formula

F. Farassat, Kenneth S. Brentner and M. H. Dunn

There is an error in Eq. (30) of this paper presented at the 4th AIAA/CEAS Aeroacoustics Conference in Toulouse, France resulting in a number of changes in the solution of the wave Eq.(28). Equation (30) must be corrected to

$$\frac{\partial g}{\partial u^1} = \frac{1}{c}[\sin\theta + \kappa_1(u^1\cos\theta - u^3\sin\theta)] \qquad (30)$$

where κ_1 is the normal curvature of the surface $f = 0$ in the direction of $\boldsymbol{t}_1$ which is the unit vector in the direction of the projection of the radiation vector $\hat{\boldsymbol{r}}$ on the local tangent plane. This relation is valid to the first order in u^1 and u^3. Equation (29) must be written as follows:

$$\begin{aligned} 4\pi\phi_2(\boldsymbol{x}, t) &= \int \frac{q_2(\boldsymbol{y}, \tau)}{r} H(\tilde{f})\,\delta'_s(u^3)\delta(g)\sqrt{g_{(2)}}\,du^1\,du^2\,du^3\,d\tau \\ &= \int \left[\frac{q_2\sqrt{g_{(2)}}\,H(\tilde{f})}{r\left|\partial g/\partial u^1\right|}\right]_{g=0} \delta'_s(u^3)\,du^2\,du^3\,d\tau \end{aligned} \qquad (29)$$

Equation (31) becomes

$$4\pi\phi_2(\boldsymbol{x}, t) = -\int\left\{\frac{\partial}{\partial u^3}\left[\frac{q_2 H(\tilde{f})\sqrt{g_{(2)}}}{r\left|\partial g/\partial u^1\right|}\right]_{g=0}\right\}_{u^3=0} du^2\,d\tau \qquad (31)$$

Initially we assume that the collapsing sphere is not tangent to the panel $f = 0, \tilde{f} > 0$ as it crosses the panel. It can be shown that

$$\left\{\frac{\partial}{\partial u^3}\left[\frac{1}{r|\partial g/\partial u^1|}\right]_{g=0}\right\}_{u^3=0} = c\left[-\frac{\cos\theta}{r^2(\sin\theta)^3} + \frac{\kappa_1}{r(\sin\theta)^3}\right]$$

Then, using the above result and after some algebraic manipulations, Eq. (36) becomes

$$4\pi\phi_2(\boldsymbol{x}, t) = c\int \frac{1}{r}\left[\frac{\cos\theta}{\sin^2\theta}\boldsymbol{t}_1 \cdot \nabla_2 q_2 - \frac{\kappa_1}{\sin\theta}q_2\right] du^2 d\tau + c\int \frac{\nu \cdot \boldsymbol{t}_1 \cot\theta}{r|\cos\psi|} q_2 \, d\tau \tag{36}$$

Equation (44) becomes

$$4\pi\phi_2(\boldsymbol{x}, t) = \int \frac{1}{r}[\cot\theta\, \boldsymbol{t}_1 \cdot \nabla_2 q_2 - \kappa_1 q_2]_{ret} \frac{\mathrm{d}\Sigma}{\Lambda} + \int \frac{1}{r}\left[\frac{\nu \cdot \boldsymbol{t}_1 \cot\theta}{\Lambda_0} q_2\right]_{ret} \mathrm{dL} \tag{44}$$

If, however, the collapsing sphere leaves the panel tangentially, another line integral similar to that in eq.(44) around the edge of a hole enclosing the point of tangency must be added. The limit of this line integral, as the maximum diameter of the hole goes to zero, adds the following term to Eq.(44) which is not in the paper:

$$I = \left[\frac{4q_2}{r|1 - M_r|}\int_0^{\pi/2} \frac{sig[k(\varphi)]}{k_r - k(\varphi)} d\varphi\right]_T$$

Here we have defined $k_r = 1/r$, $sig(.)$ is the signum function and $k(\varphi)$ is the normal curvature of the surface $f = 0$ at the point of tangency T as a function of the azimuthal angle φ. The sign convention for the curvature is based on assuming $\boldsymbol{n}_T = \hat{\boldsymbol{r}}_T$, i.e., $k(\varphi) > 0$ if the center of curvature is on the same side that $\boldsymbol{n}_T$ points into. We assume that the surface $f = 0$ has nonnegative Gaussian curvature everywhere. This is not a severe restriction on the surface $f = 0$ since, in general, one avoids a surface with saddle points. Note that at the point of tangency T we have $M_r = \pm M_n$. Since q_2 has a factor of $M_n^2 - 1$, the above equation is not singular even when the collapsing equation leaves the panel at a point where there is a Doppler singularity. Using Euler's formula $k(\varphi) = \tilde{\kappa}_1(\cos\varphi)^2 + \tilde{\kappa}_2(\sin\varphi)^2$, where $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ are the principal curvatures at the point of tangency, the above integral can be integrated in closed form with respect to φ. Under some conditions, e.g., when $0 < \tilde{\kappa}_2 < k_r < \tilde{\kappa}_1$, this integral must be interpreted as the principal value integral. The limit for a flat point or a cylindrical point can also be obtained. The above integral appears in geometrical acoustics and geometric diffraction theory. The full discussion of this point as well as the verification of the final results will be published soon[1,2]. The conclusions of the paper are correct. Specifically, we claim that we have presented the simplest possible formula (designated Formulation 4) for prediction of the noise from high speed (transonic and supersonic) moving surfaces . The discussion of singularities in the paper must be changed in light of the above results. However, the corrections given here improve the behavior of the integrals in Formulation 4 at the singularities which are all removable.

We take this opportunity to bring to the attention of the readers the following misprints in this paper:

1. Equation (29), p 4, replace f with $\tilde{f}$

2. Page 4, second column $\delta'(u^1)$ and $\delta'_s(u^1)$ must be replaced everywhere by $\delta'(u^3)$ and $\delta'_s(u^3)$, respectively

3. Equation (31), p 4, replace $g_{(2)}$ by $\tilde{g}_{(2)}$

4. The left sides of Eqs. (34) and (35) are evaluated at $g = 0$

5. Equation (38), p 5, replace $d\tau$ with $cd\tau$

6. Equation (40), p 5, replace $\boldsymbol{N}'$ with $\tilde{\boldsymbol{N}}$

7. Equation (43), p 5, replace $\upsilon \cdot \boldsymbol{t}_1$ with $\upsilon \cdot \boldsymbol{t}_1 \sin\theta$

8. First paragraph of Sec. 4, p 5, third line, 'brief' must be replaced by 'belief'

The authors thank Professor Mark Farris of Midwestern State University, Wichita Falls, Texas who pointed out the error in Eq.(30) and independently verified our results in detail. He was the 1998 ASEE Summer Faculty Fellow at NASA Langley Research Center working with F. Farassat.

References

1. F. Farassat, Kenneth S. Brentner and M. H. Dunn: A New Formula for the Study of the Noise From High Speed Surfaces- The Ffowcs Williams-Hawkings Equation and the Kirchhoff Formula, submitted to the Journal of Sound and Vibration

2. F. Farassat and Mark Farris: Verification and Analysis of Formulation 4 of Langley for the Study of Noise Radiation From High Speed Surfaces, submitted to the 5th AIAA/CEAS Aeroacoustics Conference, 10-12 May, 1999