The Application of Nonstandard Analysis to the Study of Inviscid Shock Wave Jump Conditions

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Abstract

The use of conservation laws in nonconservative form for deriving shock jump conditions by Schwartz distribution theory leads to ambiguous products of generalized functions. Nonstandard analysis is used to define a class of Heaviside functions where the jump from zero to one occurs on an infinitesimal interval. These Heaviside functions differ by their microstructure near x = 0, i.e., by the nature of the rise within the infinitesimal interval it is shown that the conservation laws in nonconservative form can relate the different Heaviside functions used to define jumps in different flow parameters. There are no mathematical or logical ambiguities in the derivation of the jump conditions. An important result is that the microstructure of the Heaviside function of the jump in entropy has a positive peak greater than one within the infinitesimal interval where the jump occurs. This phenomena is known from more sophisticated studies of the structure of shock waves using viscous fluid assumption. However, the present analysis is simpler and more direct.

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Outline of the Talk

- Shock jump conditions using Schwartz distributions
- Ambiguities of products of distributions
- The source and solution of the problem
- A new class of Heaviside functions
- Some properties of the new class of Heaviside functions
- Shock jump conditions- Nonconservative form
- Summary of the results
- Concluding Remarks

Shock Jump Conditions Using Schwartz Distributions

Assuming that the shock surface is given by $f(\boldsymbol{x}, t) = 0$ and using mass continuity equation in conservative form, we have

$$\underbrace{\frac{\bar{\partial}\rho}{\partial t} + \bar{\ }\ (\rho \boldsymbol{u})}_{=0} = \underbrace{\frac{\partial\rho}{\partial t} + \ (\rho \boldsymbol{u})}_{=0} - |\ f|\ \underbrace{[v_n \Delta\rho - \Delta(\rho u_n)]}_{=0 \text{ Jump Cond.}}\ \delta(f)$$

$\bar{\partial}$ and $\bar{\ }$ are generalized derivatives, $\Delta(\ \ )$ is the jump, $v_n = -\partial f / \partial t / |\ f|$ and $u_n = \boldsymbol{u}\ \ f$ are shock and fluid normal velocities. *This method fails for equations in nonconservative form because ambiguous products of distributions appear.*

Ambiguities of Products of Distributions

Let $H(x)$ **be the Heaviside function, then** $H^n(x) = H(x)$**. Taking the derivatives of both sides, we get**

$$nH^{n-1}(x)H'(x) = H'(x)$$

It follows that $H^{n-1}(x) = 1/n$**.**

Taking $n = 2$**, we get** $H(x) = 1/2$

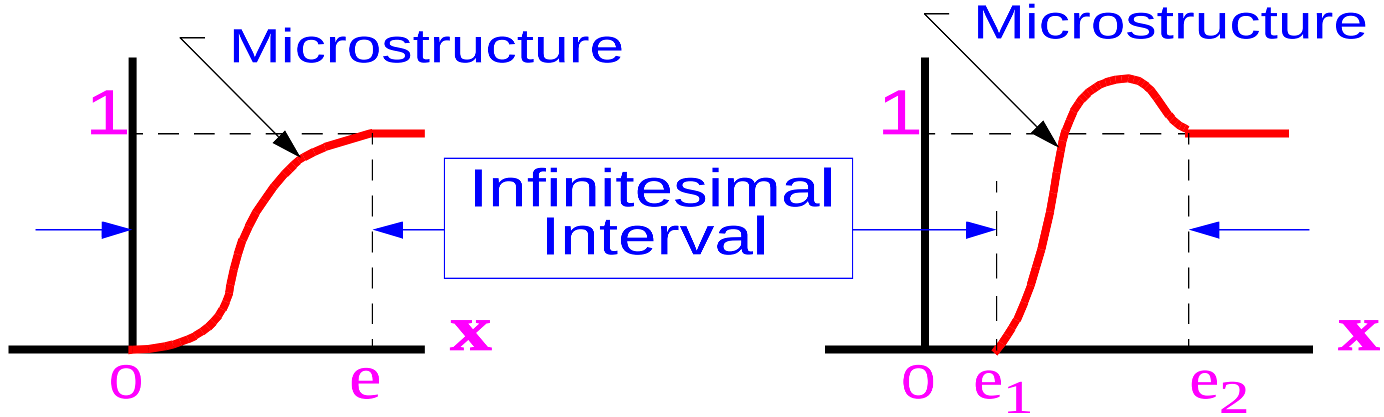
CONTRADICTION!

The Source and Solution of the Problem

The jump of the Heaviside function occurs at $x = 0$ which coincides with the support of its derivative, the Dirac delta function. In fact $H(x)\delta(x)$ is ***not*** defined. This gives us a ***solution to this problem***

Let the jump of the Heaviside function occur on an infinitesimal interval near $x = 0$ and give a microstructure to the jump. This rules out $H^n(x) = H(x)$ and we have a new class of Heaviside functions each different from others.

A New Class of Heaviside Functions



The jump of the Heaviside functions occurs on a true infinitesimal interval in the halo of $x = 0$. We, thus, have a class of Heaviside functions all basically behaving as the classical Heaviside function and all satisfying $H^n(x) \neq H(x)$. We claim that the conservation laws in nonconservative form give the relations between the various Heaviside functions associated with the jump.

Some Properties of the New Class of Heaviside Functions

i) Two Heaviside functions are equal if they have the same *microstructure* and infinitesimal interval of the jump. The microstructure is assumed to be as smooth as needed in a given problem.

ii) The derivative of a Heaviside function behaves as the Dirac delta function. The derivative takes *hyperreal infinite* values on the infinitesimal interval of the jump.

Some Properties of the New Class of Heaviside Functions (Cont'd)

iii) The usual **algebraic manipulations** of smooth functions can be applied to the Heaviside functions in the new class.

iv) **Product** of a Heaviside function $H(x)$ and its derivative $H'(x)$ is always defined.

v) **Cancellation law** holds: $Q(x)$ and $P(x)$ smooth, then $H^{(n)}(x)Q(x) = H^{(n)}(x)P(x)$ implies that $Q(x) = P(x)$.

Shock Jump Conditions- Nonconservative Form

Assumptions:

i) Different Heaviside functions $H(\xi), K(\xi), L(\xi)$ and $T(\xi)$ describe the jumps of specific volume v, velocity u, pressure p and entropy S, respectively.

ii) The jump for all flow parameters occur on the same infinitesimal interval (o, ε).

Shock Jump Conditions- Nonconservative Form (Cont'd)

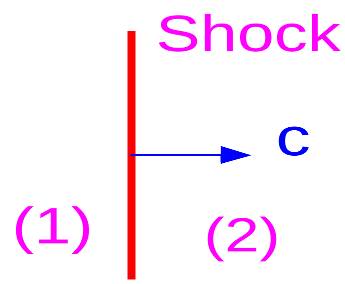
Let $\xi = x - ct$ and $\Delta(\ ) = (\ )_2 - (\ )_1$ the jump in a fluid parameter. Define

$v = v_1 + \Delta v H(\xi)$ Specific volume

$u = u_1 + \Delta u K(\xi)$ Fluid velocity

$p = p_1 + \Delta p L(\xi)$ Pressure

$S = S_1 + \Delta S T(\xi)$ Entropy



c is the shock speed

Shock Jump Conditions: Nonconservative Form (Cont'd)

Mass conservation law $v_t - vu_x + uv_x = 0$

gives $(u_1 - c)\Delta v H' - v_1 \Delta u K' - \Delta v \Delta u H K' + \Delta v \Delta u H' K = 0$

resulting in the ODE $\left(K + \frac{u_1 - c}{\Delta u}\right)\frac{dH}{dK} - H = \frac{v_1}{\Delta v}$.

The solution is $H(\xi) = -\frac{v_1}{\Delta v} + b\left(K + \frac{u_1 - c}{\Delta u}\right)$, b const..

From $H(0) = K(0) = 0$, we get $b = v_1 \Delta u / [\Delta v (u_1 - c)]$

and from $H(\varepsilon) = K(\varepsilon) = 1$, we get $b = 1$. It follows that $H(\xi) = K(\xi)$ and $c = (u_1 \Delta v - v_1 \Delta u)/\Delta u$.

Shock Jump Conditions- Nonconservative Form (Cont'd)

Momentum equation $u_t + uu_x + vp_x = 0$ gives

$\left(K + \frac{u_1 - c}{\Delta u}\right)(\Delta u)^2 K' + \left(K + \frac{v_1}{\Delta v}\right)\Delta v \Delta p L' = 0$. We have used $H(\xi) = K(\xi)$ here. This gives the ODE

$\frac{dL}{dK} + \frac{(\Delta u)^2}{\Delta v \Delta p} = 0$. We used $(u_1 - c)/\Delta u = v_1/\Delta v$ from $b = 1$. The solution of this ODE is

$L(\xi) = -\frac{(\Delta u)^2}{\Delta v \Delta p} K(\xi) + d, d \text{ const.}$. From $L(0) = K(0) = 0$, we get $d = 0$. From $L(\varepsilon) = K(\varepsilon) = 1$, we get $(\Delta u)^2 / \Delta v \Delta p = -1$. It follows that $L(\xi) = K(\xi)$.

Shock Jump Conditions- Nonconservative Form (Cont'd)

The **constitutive relation** $S = c_v \log(p v^{\gamma})$

gives $S = S_1 + \Delta S T(\xi) = c_v \log[(1 + \Delta p H)(1 + \Delta v H)^{\gamma}]$

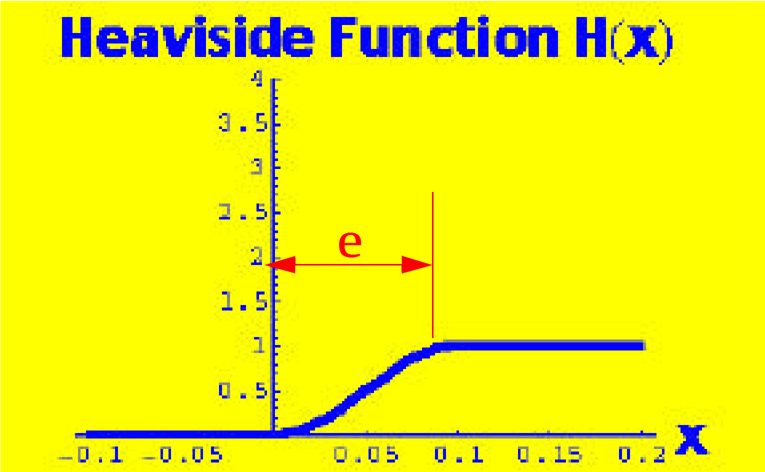
$$= c_v \log\left(p_1 v_1^{\gamma}\right) + c_v \log\left[\left(1 + \frac{\Delta p}{p_1} H\right)\left(1 + \frac{\Delta v}{v_1} H\right)^{\gamma}\right]$$

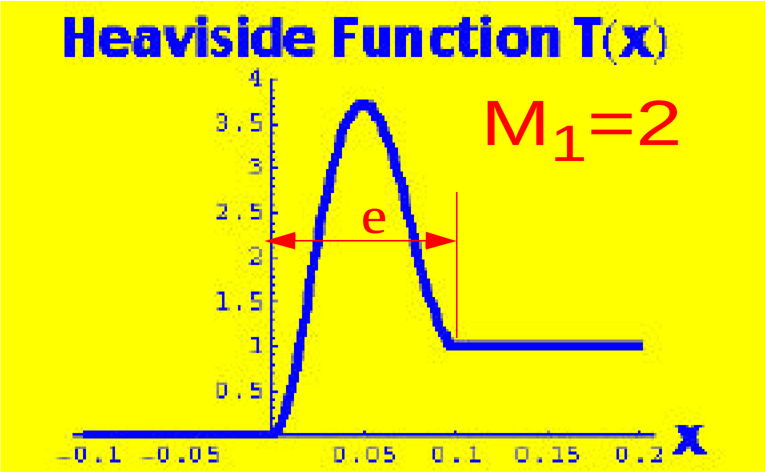
Using $S_1 = c_v \log(p_1 v_1^{\gamma})$, we get

$$T(\xi) = \frac{c_v}{\Delta S} \log\left[\left(1 + \frac{\Delta p}{p_1} H(\xi)\right)\left(1 + \frac{\Delta v}{v_1} H(\xi)\right)^{\gamma}\right] \quad \textit{Nonlinear!}$$

The Behavior of $T(x)$

If we assume that the Heaviside function $H(x)$ is monotonic, then $T(x)$ will have a maximum above 1 within the shock layer as predicted by more complex analysis.





Summary of the Results

We have shown that

$$H(\xi) = K(\xi) = L(\xi)$$

$$T(\xi) = \frac{c_v}{\Delta S}\log\left[\left(1 + \frac{\Delta p}{p_1}H(\xi)\right)\left(1 + \frac{\Delta v}{v_1}H(\xi)\right)^{\gamma}\right]$$

Shock speed: $c = (u_1 \Delta v - v_1 \Delta u)/\Delta u$

The relation $(\Delta u)^2/\Delta v \Delta p = -1$ gives the Prandtl's relation $(u_1 - c)(u_2 - c) = \Delta p/\Delta\rho$. All other known shock jump conditions are also obtained.

Concluding Remarks

– We defined a new class of Heaviside functions with smooth jump on an infinitesimal interval in the halo of $x = 0$ and satisfying $H^n(x) \neq H(x)$.

– Using this class of Heaviside functions, we obtain the shock speed and jump conditions from conservation laws in nonconservative form.

– The relative shape of the microstructure of the jump of different fluid parameters can be obtained by rigorous algebraic manipulations.

– The unusual behavior of entropy within a shock can be obtained by using the new class of Heaviside functions.