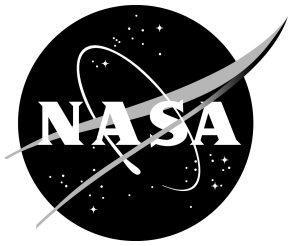
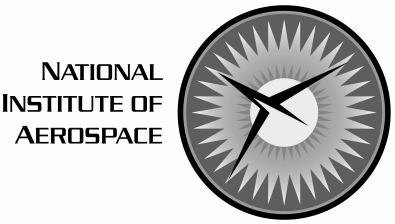
NASA/CR-2004-213035
NIA Report No. 2004-06





Continuation Methods for Qualitative Analysis of Aircraft Dynamics

Peter A. Cummings
National Institute of Aerospace, Hampton, Virginia

July 2004

The NASA STI Program Office . . . in Profile

Since its founding, NASA has been dedicated to the advancement of aeronautics and space science. The NASA Scientific and Technical Information (STI) Program Office plays a key part in helping NASA maintain this important role.

The NASA STI Program Office is operated by Langley Research Center, the lead center for NASA's scientific and technical information. The NASA STI Program Office provides access to the NASA STI Database, the largest collection of aeronautical and space science STI in the world. The Program Office is also NASA's institutional mechanism for disseminating the results of its research and development activities. These results are published by NASA in the NASA STI Report Series, which includes the following report types:

- TECHNICAL PUBLICATION. Reports of completed research or a major significant phase of research that present the results of NASA programs and include extensive data or theoretical analysis. Includes compilations of significant scientific and technical data and information deemed to be of continuing reference value. NASA counterpart of peer-reviewed formal professional papers, but having less stringent limitations on manuscript length and extent of graphic presentations.

- TECHNICAL MEMORANDUM. Scientific and technical findings that are preliminary or of specialized interest, e.g., quick release reports, working papers, and bibliographies that contain minimal annotation. Does not contain extensive analysis.

- CONTRACTOR REPORT. Scientific and technical findings by NASA-sponsored contractors and grantees.

- CONFERENCE PUBLICATION. Collected papers from scientific and technical conferences, symposia, seminars, or other meetings sponsored or co-sponsored by NASA.

- SPECIAL PUBLICATION. Scientific, technical, or historical information from NASA programs, projects, and missions, often concerned with subjects having substantial public interest.

- TECHNICAL TRANSLATION. English-language translations of foreign scientific and technical material pertinent to NASA's mission.

Specialized services that complement the STI Program Office's diverse offerings include creating custom thesauri, building customized databases, organizing and publishing research results . . . even providing videos.

For more information about the NASA STI Program Office, see the following:

- Access the NASA STI Program Home Page at ***http://www.sti.nasa.gov***

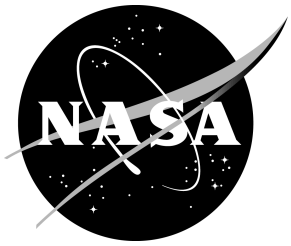
- Email your question via the Internet to help@sti.nasa.gov

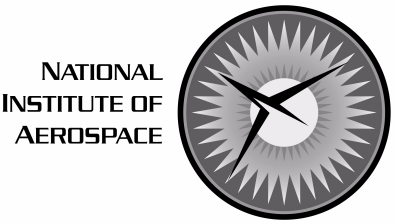
- Fax your question to the NASA STI Help Desk at (301) 621-0134

- Telephone the NASA STI Help Desk at (301) 621-0390

- Write to:
  NASA STI Help Desk
  NASA Center for AeroSpace Information
  7121 Standard Drive
  Hanover, MD 21076-1320

NASA/CR-2004-213035
NIA Report No. 2004-06





Continuation Methods for Qualitative Analysis of Aircraft Dynamics

Peter A. Cummings
National Institute of Aerospace, Hampton, Virginia

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23681-2199

Prepared for Langley Research Center
under Contract NCC-1-02043

July 2004

Available from the following:

NASA Center for AeroSpace Information (CASI)
7121 Standard Drive
Hanover, MD 21076-1320
(301) 621-0390

National Technical Information Service (NTIS)
5285 Port Royal Road
Springfield, VA 22161-2171
(703) 487-4650

CONTINUATION METHODS FOR QUALITATIVE ANALYSIS OF AIRCRAFT DYNAMICS

Peter A. Cummings*

ABSTRACT

A class of numerical methods for constructing bifurcation curves for systems of coupled, nonlinear ordinary differential equations is presented. Foundations are discussed, and several variations are outlined along with their respective capabilities. Appropriate background material from dynamical systems theory is presented.

1 INTRODUCTION

Development of a model for the macroscopic motion of aircraft begins with the rigid body equations of motion, a system of coupled, nonlinear ordinary differential equations (ODEs). The final model, called the *aircraft equations of motion*, is of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u) \tag{1}$$

Both $\mathbf{x}$ and $\mathbf{u}$ are multi–dimensional, and the exact number of components depends on the aircraft and on the reference frame in which the model is developed. In any case, components of x are commonly referred to as the *state variables*, and components of u as the *control variables*.

Quantitative analysis of the equations of motion often involves numerical solution of a simplified version of (1). A typical approach is to assume that the aircraft's motion is limited to small variations from a reference, or trim, condition. Mathematically, this is equivalent to linearizing the equations of motion, or assuming that the state variables are of the form $\mathbf{x}_0 + \Delta\mathbf{x}$, where $\mathbf{x}_0$ characterizes the reference condition and $\Delta\mathbf{x}$ represents a small perturbation from the reference condition.

Simplifying assumptions are useful and effective when applied to perturbations from a reference flight condition. In order to achieve successful and consistent predictions of dynamical behavior of aircraft for large amplitude general motions, however, one must eschew the conventional linearity assumptions and incorporate a suitable nonlinear description.

Because aircraft motion is modeled with a system of nonlinear ODEs, techniques from dynamical systems theory show promise for expanding the understanding of nonlinear aspects of flight dynamics. The dynamical systems approach is *qualitative*; one does not seek to numerically or explicitly calculate particular solutions. Rather, one endeavors to glean information about the qualitative nature of all solutions. Moreover, one wishes to determine if and how qualitative characteristics change when the system parameters are subject to perturbation. For flight dynamics problems, treating control surface deflections as parameters in qualitative analysis can reveal information about aircraft stability regimes.

Mehra, Kessel and Carroll [13] were among the first to suggest a dynamical systems approach to achieving a global, qualitative understanding of nonlinear aircraft dynamics. They christened their technique the *bifurcation analysis and catastrophe theory methodology*, or BACTM, and used tools rooted in dynamical systems theory to analyze and construct stability boundaries for aircraft

*Staff Scientist, National Institute of Aerospace (NIA), Hampton, VA 23666. Email: cummings@nianet.org.

such as the F100A and F-80A. Other studies of note include Jahnke and Culick's work on the F-14 [7]; Goman, Zagainov and Khramtsovsky's applications of the qualitative methodology to roll-coupling, stall and spin problems [5]; and Ananthkrishnan and Sinha's extended bifurcation analysis procedure [2].

In all of the above studies, a primary goal is to construct equilibrium curves and/or surfaces for the relevant aircraft models. Indeed, equilibrium and bifurcation information is the cornerstone of qualitative analysis for any system of ODEs. Constructing the equilibrium curves poses a considerable challenge, and a mathematical technique known as the *continuation method* has emerged as the technique of choice in this venture. The continuation method has its pedigree in a 1953 work of D.F. Davidenko [1], whose fundamental observation forms the theoretical foundation of most contemporary continuation methods. The method has evolved considerably since the publication of Davidenko's seminal work; contributions from mathematicians such as R.W. Klopfenstein and H.B. Keller have been instrumental in refining the technique to its present sophistication.

The purpose of this report is to provide a mathematical investigation of continuation methods, their foundations, capabilities and variations. Section 2 gives a brief discussion of dynamical systems to provide context. A simple example illustrates the main ideas of bifurcation curves. Section 3 proceeds to a detailed presentation of continuation methods. Sections 3.1 and 3.2 discuss embedding methods; section 3.1 introduces the concepts, and section 3.2 presents polynomial continuation - a particular type of embedding method - along with some simple examples. Section 3.3 outlines the two varieties of parameter continuation methods: natural parameter and artificial parameter continuation. Finally section 3.4 discusses bifurcation detection and branch switching.

2 DYNAMICAL SYSTEMS

The theory of dynamical systems is a broad methodology for the study of deterministic processes. This section highlights those aspects of dynamical systems theory that are relevant to the study of flight models, i.e. those that are generally used in the qualitative analysis of systems of coupled, nonlinear ordinary differential equations. For a more complete treatment of dynamical systems, see [6] or [12].

The general idea of qualitative analysis of ODEs is to predict long–term behavior of solutions without actually solving the differential equations. To that end, two main tasks are paramount – calculating equilibrium solutions, and determining their stability. If the system is of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{2}$$

then all equilibria must satisfy the algebraic equation

$$\mathbf{f}(\mathbf{x}) = 0, \tag{3}$$

and finding the equilibrium for system (2) therefore amounts to solving a nonlinear algebraic system. Once equilibrium solutions are found, determining stability is straightforward. A famous result of Lyapunov implies that an equilibrium point of (2) is stable if all eigenvalues of the Jacobian of $\mathbf{f}$ evaluated at the equilibrium have negative real parts. Because determining stability is relatively easy once the equilibria are known, finding equilibria is the more difficult task and typically receives more attention.

For systems of the form (2), the fundamental tasks of qualitative analysis reduce to numerical calculations for which there are many well–known numerical methods. If the system includes a

parameter, however, the situation becomes more involved. Consider a system of the form

$$(label: "brimstone")\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \lambda). \tag{4}$$

To find the equilibria of system (4), one must solve the algebraic equation

$$(label: "jupiter")\mathbf{f}(\mathbf{x}, \lambda) = 0. \tag{5}$$

The presence of parameter λ complicates the task of finding equilibria. A complete qualitative study of the dynamics of (4) must now include equilibria as a function of the problem parameter. That is, one must find a function $\mathbf{x}(\lambda)$ such that $\mathbf{f}(\mathbf{x}(\lambda), \lambda) = 0$ for some appropriate range of values for the parameter. A simple but detailed example is illustrative.

Example (Bead on a rotating hoop): Suppose a bead of mass m is threaded on a circular hoop of radius r. The hoop rotates with a constant angular velocity ω about a vertical axis that corresponds to a diameter of the hoop. The bead slides freely, but its motion is damped by friction and gravitational forces. The motion of the bead on the hoop is governed by the following system of ODEs [17]:

$$(label: "bolero") \quad \begin{array}{ll} \dot{x_1} & = x_2 \\ \dot{x_2} & = \frac{-b}{mr} x_2 + \omega^2 \sin x_1 \left(\cos x_1 - \frac{g}{r\omega^2}\right) \end{array} \tag{6}$$

where x_1 is the counter–clockwise angle of the bead's position from the bottom, measured in radians. For example, if the bead is at the 3 o'clock position, then $x_1 = \pi/2$. Note that x_2 is the angular velocity of the bead, g is the acceleration due to gravity and b is a damping constant; see figure 1.

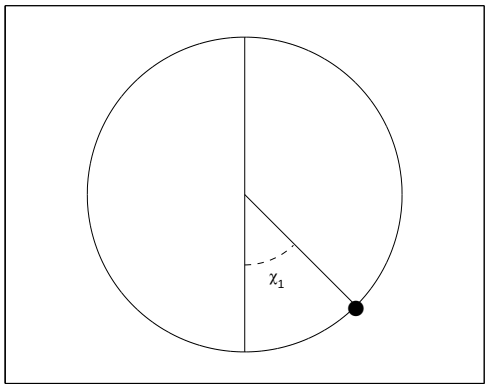


Figure 1: *Bead on a Rotating Hoop*

Assuming for simplicity that $r = g$, system (6) is of the form (4) where $\mathbf{x} := [x_1, x_2]^T$ and $\lambda := \omega^2$. To calculate equilibrium solutions, set the derivatives in (6) equal to zero and solve the resulting algebraic system. In this simple example, the solutions can be found explicitly. At an equilibrium solution, x_2 must be zero (the bead is stationary) and x_1 must satisfy

$$\sin x_1 \left(\cos x_1 - \frac{1}{\lambda}\right) = 0. \tag{7}$$

It is clear from equation (7) that the number of equilibria depend on the parameter λ. When $\lambda \leq 1$, $x_1 = 0$ and $x_1 = -\pi$ are the only two equilibria in the interval $[-\pi, \pi)$. When $\lambda > 1$, two additional equilibria appear at $x_1 = \pm \arccos(1/\lambda)$.

It is also straightforward to determine the stability of equilibria. The Jacobian of the system is

$$J := \begin{bmatrix} 0 & 1 \\ \omega^2 \cos x_1 \left(\cos x_1 - \frac{1}{\lambda}\right) - \omega^2 \sin^2 x_1 & \frac{-b}{mr} \end{bmatrix}, \tag{8}$$

and therefore the characteristic equation is

$$s^2 + \frac{b}{mr} s - \left(\omega^2 \cos x_1 \left(\cos x_1 - \frac{1}{\lambda}\right) - \omega^2 \sin^2 x_1\right) = 0. \tag{9}$$

Note that the Jacobian and the characteristic equation are independent of x_2. When $x_1 = 0$, the roots of the characteristic equation are

$$s = \frac{\frac{-b}{mr} \pm \sqrt{\frac{b^2}{m^2 r^2} - 4\omega^2 \left(1 - \frac{1}{\lambda}\right)}}{2}. \tag{10}$$

Stability of $\mathbf{x} = 0$ can thus determined by inspection. If $\lambda > 1$, or equivalently if $4\omega^2(1-1/\lambda) > 0$, then J has exactly one positive real eigenvalue, and $x_1 = 0$ is unstable. If $\lambda < 1$, or equivalently if $4\omega^2(1 - 1/\lambda) < 0$, then J has either two complex eigenvalues with negative real parts (when the discriminant is negative), or two negative real eigenvalues (when the discriminant is positive). In either case, the equilibrium $x_1 = 0$ is stable. A similar analysis shows that $x_1 = -\pi$ is always unstable, and the equilibria at $x_1 = \pm \arccos(1/\lambda)$ are always stable when they exist.

After determining all information about equilibria and their associated stabilities, the results can be interpreted in the context of the original problem. That is, one can use equilibria information to predict the behavior of the bead in terms of the parameter λ. When $\lambda \leq 1$, the bottom position represents a stable equilibrium, and the top represents an unstable equilibrium. If the bead's initial position is the top or the bottom, it will remain at its initial position for all time. All other initial positions will approach the bottom. In this situation, the angular velocity ω is less than the critical value $\sqrt{g/r}$; the centrifugal force of the bead is not enough to overcome the force of gravity, and the bead tends toward the bottom.

As the angular velocity of the hoop increases beyond the critical value $\sqrt{g/r}$, λ increases past 1. The bottom position becomes unstable, and two stable equilibria appear at symmetric points on the hoop. Centrifugal force is now sufficient to overcome the force of gravity.

It is useful to summarize the results graphically by plotting a *bifurcation diagram*, which consists of curves that depict equilibria in terms of the parameter λ. By convention, solid lines indicate stable equilibria and dashed lines to indicate unstable equilibria. Figure 2 shows the bifurcation diagram for system (6). The curve bifurcates at the critical value $\lambda = 1$. At this point, two additional branches appear (corresponding to the two additional equilibria), and the equilibrium $x_1 = 0$ changes stability.

Constructing the bifurcation diagram is an effective technique for comprehensively describing the dynamics of a system of nonlinear ODEs. The diagram shows not only the type and number of equilibria in terms of the problem parameter, but also critical or *bifurcation* values where stability changes and/or new equilibria appear. This is the essence of qualitative study of systems of ODEs.

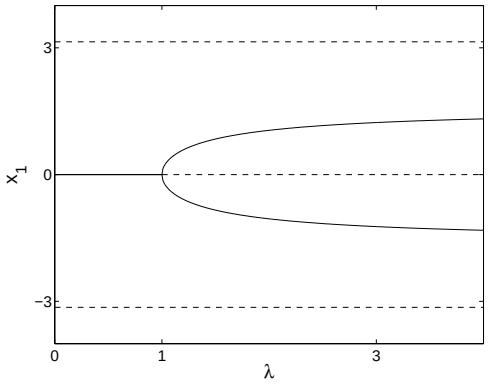


Figure 2: *Bifurcation Diagram, Bead on a Rotating Hoop*

Unfortunately, constructing the bifurcation diagram is rarely as straightforward as it is for system (6). In general, it will not be possible to solve the algebraic system (5) analytically, and one must therefore develop appropriate numerical techniques to construct the curves. One such class of techniques are continuation methods.

3 CONTINUATION METHODS

Continuation methods are numerical methods for solving nonlinear equations of the form

$$\mathbf{f}(\mathbf{x}, \lambda) = 0 \tag{11}$$

where $\mathbf{f} : \Re^n \times \Re \rightarrow \Re^n$. In the context of continuation, "solve" does not mean merely to find a single solution. Rather, a continuation method seeks to construct one or more paths or curves of solutions, i.e. sets of the form

$$(label : "mandibles")\Gamma := \{(\mathbf{x}(\lambda), \lambda) : \lambda \in I, \mathbf{f}(\mathbf{x}(\lambda), \lambda) = 0\} \tag{12}$$

where I is an interval of real numbers.

The need to construct the set Γ may arise from the need to solve an equation of the form

$$(label : "nopar")\mathbf{g}(\mathbf{x}) = 0 \tag{13}$$

where $\mathbf{g} : \Re^n \rightarrow \Re^n$ and $\mathbf{g}(\mathbf{x}) = \mathbf{f}(\mathbf{x}, \lambda)|_{\lambda=\lambda_0}$. In this case, the solution to $\mathbf{g}(\mathbf{x}) = 0$ represents an endpoint of the curve Γ, and one is interested in Γ only insofar as it provides a means for determining the solution to $\mathbf{g}(\mathbf{x}) = 0$. When the solution is found, the curve Γ is discarded. For this type of application, continuation methods simply form another class of methods for solving nonlinear equations. It is worth noting, however, that for certain classes of problems, continuation methods can have significant advantages over conventional alternatives such as iterative nonlinear solvers. For example, although Newton's method is a rapidly convergent scheme, it requires a good initial approximation and finds only one solution. Certain formulations of continuation methods, on the other hand, find all solutions without an initial guess. We will discuss examples in sections 3.1 and 3.2.

Though continuation methods sometimes provide an attractive alternative to conventional nonlinear solvers, the real power of continuation methods lies in their ability to solve problems that

appear explicitly in the form (11). Among such problems is, of course, the problem of constructing a bifurcation curve for a system of ODEs. In this case, no information is discarded, and no computational work is wasted, because Γ itself is the solution.

In all cases, the general idea of a continuation method is that of a *predictor–corrector* scheme. Starting with an initial point on the "continuation path," the goal is to trace the remainder of the path in steps. At each step, the algorithm first predicts the next point on the path, and subsequently corrects the predicted point toward the solution curve. Newton's method – or some variant of Newton's method – is nearly always used for the corrector step. The purpose of the predictor step is to supply an adequate initial guess to the Newton corrector. Consequently, success of a continuation method depends critically on the appropriate choice of a predictor. That is, the predictor must successfully choose a good approximation to the next point on the continuation curve *at every step* along the curve in order for the overall continuation scheme to be effective. The other main consideration is bifurcation detection; an effective method must detect and follow branches. The sections that discuss several continuation methods, from the simple to the sophisticated.

Before proceeding to discuss specific examples of continuation algorithms, it is appropriate to turn briefly to the issue of existence. When solving a problem numerically (or analytically, for that matter), it is expedient to know *a priori* that the problem in question has a solution. In the case of continuation methods, the existence question can be framed as follows. Given a solution $(\mathbf{x}_0, \lambda_0)$ to $\mathbf{f}(\mathbf{x}, \lambda) = 0$, when does a curve of the form Γ exist that passes through $(\mathbf{x}_0, \lambda_0)$? A simple application of the Implicit Function Theorem (IFT) answers the question definitively. The IFT can be stated and proved in abstract settings, but the following version in Euclidean $n-$space is sufficient for our purposes.

Theorem 1 (Implicit Function Theorem) *Suppose* $\mathbf{f} : \Re^n \times \Re \to \Re^n$, $x_0 \in \Re^n$, $\lambda_0 \in \Re$ *such that*

1. $\mathbf{f}(\mathbf{x}_0, \lambda_0) = 0$*;*
2. $\mathbf{f}_\mathbf{x}(\mathbf{x}_0, \lambda_0)$ *is nonsingular;*
3. $\mathbf{f}, \mathbf{f}_\mathbf{x}(\mathbf{x}_0, \lambda_0)$ *are continuous on the set* $\{(\mathbf{x}, \lambda) : \|\mathbf{x} - \mathbf{x}_0\| < \epsilon_1, |\lambda - \lambda| < \epsilon_2\}$ *for some* $\epsilon_1, \epsilon_2 > 0$*.*

Then there exists a continuous function $\varphi : (\lambda_0 - \epsilon_2, \lambda_0 + \epsilon_2) \to \Re$ *such that*

i. $\mathbf{x}_0 = \varphi(\lambda_0)$*;*

ii. $\mathbf{f}(\varphi(\lambda), \lambda) = 0$ *for all* $\lambda \in (\lambda_0 - \epsilon_2, \lambda_0 + \epsilon_2)$*;*

iii. *For each* $\lambda \in (\lambda_0 - \epsilon_2, \lambda_0 + \epsilon_2)$, $\varphi(\lambda)$ *is a unique solution of* $\mathbf{f}(\mathbf{x}, \lambda) = 0$ *in the set* $\{\mathbf{x} : \|\mathbf{x} - \mathbf{x}_0\| < \epsilon_1\}$*.*

In other words, when the function $\mathbf{f}$ satisfies conditions 1-3 in the hypothesis of the IFT, one can solve equation (5) locally for $\mathbf{x}$ as a function of the parameter λ. Furthermore, the function that represents the solution is unique in an appropriate neighborhood. In practice, one generally monitors the status of hypothesis 2 while tracing a continuation curve.

3.1 Embedding Methods

The most basic continuation methods are embedding methods, a class of methods for solving $\mathbf{g}(\mathbf{x}) = 0$ where $\mathbf{g} : \Re^n \rightarrow \Re^n$. Though this method has limited application, a brief discussion is warranted because it gives a gentle introduction to the main concepts of continuation.

The idea is simple. To solve $\mathbf{g}(\mathbf{x}) = 0$, *embed* $\mathbf{g}(\mathbf{x}) = 0$ in a family of problems, $\mathbf{h}(\mathbf{x}, \lambda)$ (for $\lambda \in [0, 1]$) where

- $\mathbf{h}(\mathbf{x}, 0) = 0$ is "easy" to solve;
- $\mathbf{h}(\mathbf{x}, 1) = \mathbf{g}(\mathbf{x})$; and
- the solution of $\mathbf{h}(\mathbf{x}, \lambda) = 0$ changes smoothly for $\lambda \in [0, 1]$.

The function $\mathbf{h}$ is variously referred to as a *homotopy, deformation,* or *embedding*. In order to calculate a solution to $\mathbf{g}(\mathbf{x}) = 0$, start with a solution $\mathbf{x}_0$ to $\mathbf{f}(\mathbf{x}, \lambda_0) = 0$ ($\lambda_0 = 0$) and vary λ monotonically through $[0, 1]$, updating the solution with each new value of λ. When λ reaches 1, the solution is found.

Example (Convex embedding): Define

$$(label : "convex")\mathbf{h}(\mathbf{x}, \lambda) = (1 - \lambda)\mathbf{f}(\mathbf{x}) + \lambda\mathbf{g}(\mathbf{x}) \tag{14}$$

where at least one solution, $\mathbf{x}_0$ of $\mathbf{f}(\mathbf{x}) = 0$ is known. The continuation algorithm for the convex embedding is as follows:

Algorithm For $i = 1, 2, ..., m$; do

Step 1 Put $\lambda_i = \lambda_{i-1} + \Delta\lambda$ ($\Delta\lambda = \frac{1}{m}, \lambda_0 = 0$)

Step 2 Solve $\mathbf{h}(\mathbf{x}, \lambda_i)$ iteratively, using $(\mathbf{x}_{i-1}, \lambda_i)$ as the initial guess.

The predictor step increments the parameter, and uses the $\mathbf{x}-$coordinate from the previous step as the initial guess for the corrector. The convex embedding is simple to understand and to implement, but unfortunately works for only the most contrived problems. It is easy to anticipate a variety of difficulties with the convex embedding algorithm. If the continuation path is too steep, for example, the predictor may stray too far from the path to provide an adequate initial guess for the corrector. If continuation paths cross, the method may trace the wrong path at the crossing point; this situation may occur in problems that have mathematical solutions (in addition to physical solutions) which do not make sense in the physical context of the problem. Other more serious problems such as backtracking and path divergence can cause the convex embedding method to fail. See figure 4 for schematics of bad path behaviors.

Elementary modifications to the convex embedding algorithm can handle, or even remove bad path behaviors in some cases. The cure nearly always amounts to designing an "intelligent" predictor that can trace the path more tightly. Modifying the convex embedding to include an adaptive step length scheme, for example, may handle difficulties with steep paths or path crossing. Other cures for steep paths and path crossing include the use of tangent or higher order predictors. For polynomial systems, complex embeddings preclude path divergence and path crossing situations. The next section discusses these methods briefly.

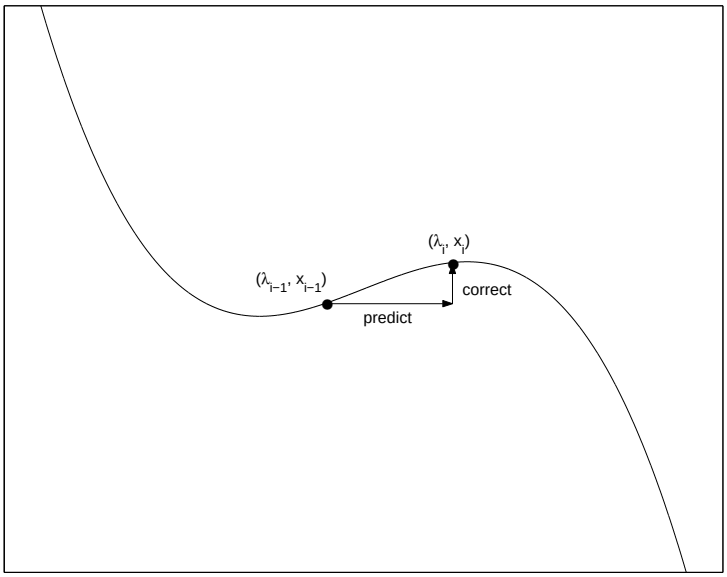


Figure 3: *Convex Embedding*

3.2 Polynomial Continuation

Polynomial continuation is an embedding method. In order to solve the polynomial system $\mathbf{g}(\mathbf{x}) = 0$, embed the system in a family of problems $\mathbf{h}(\mathbf{x}, \lambda) = 0$ as described at the beginning of section 3.1. By exploiting properties of polynomial systems, the method finds all solutions without an initial guess while avoiding backtracking, path crossing, and path divergence. Alexander Morgan has done extensive research in the field of polynomial continuation and the content of this section borrows heavily from his work; see [15] and [14].

In this section, assume that $\mathbf{g}(\mathbf{x})$ is a polynomial function mapping $\Re^n$ to $\Re^n$ whose solutions are geometrically isolated. That is, $\mathbf{g}(\mathbf{x})$ is of the form

$$\mathbf{g}(\mathbf{x}) = \begin{cases} g_1(x_1, x_2, ..., x_n) \\ g_2(x_1, x_2, ..., x_n) \\ \vdots \\ g_n(x_1, x_2, ..., x_n) \end{cases} \tag{15}$$

where each g_j is a polynomial in n–variables (i.e. $g_j : \Re^n \longrightarrow \Re$), and the solution set of $\mathbf{g}(\mathbf{x}) = 0$ is discrete, that is, each solution is geometrically isolated. The terms of g_j are of the form

$$c \prod_{k=1}^{n} x_k^{m_k}$$

where c is a constant and each m_k is a non–negative integer The degree of a term of g_j is by definition

$$\sum_{k=1}^{n} m_k$$

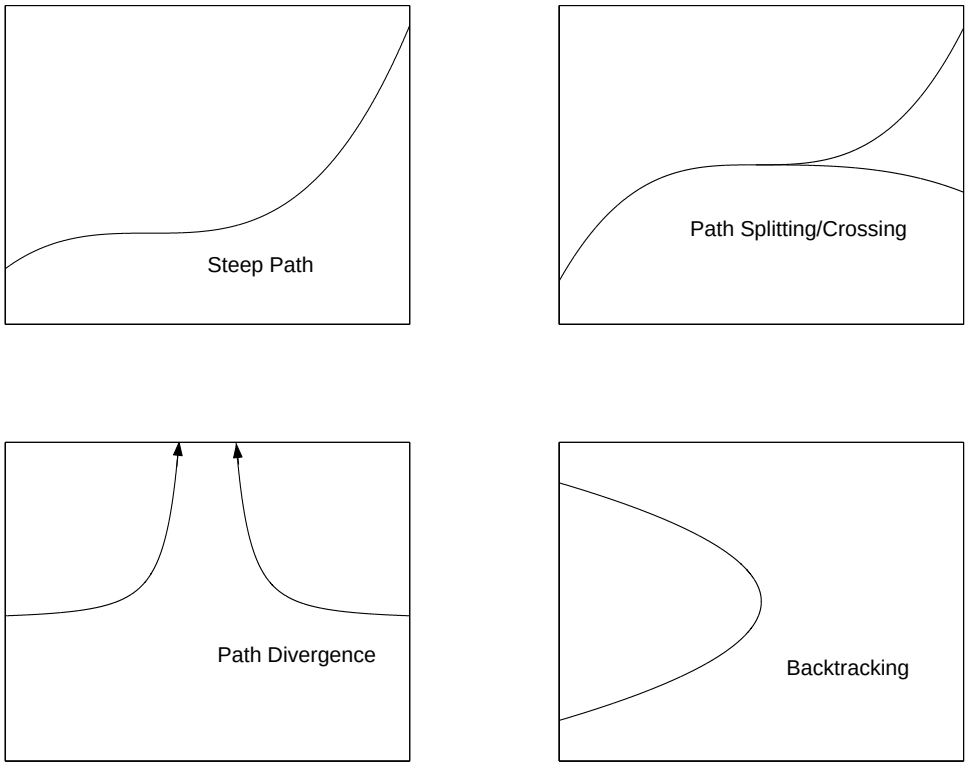


Figure 4: *Bad Path Behaviors*

and the degree of g_j is the maximum of the degrees of all its terms. The *total degree* of the system is defined as

$$d := \prod_{j=1}^{n} d_j$$

where d_j is the degree of g_j. Since the roots of g are isolated, d is an upper bound on the number of distinct roots [14].

The homotopy for polynomial continuation takes the form of equation (14), and determining the homotopy amounts to appropriately constructing the function $\mathbf{f}$. The idea is to choose a function with roots that are easy to find and which has at least as many solutions as the function $\mathbf{g}$. Each solution of $\mathbf{f}$ will then serve as a starting point for a continuation path that converges to a solution of $\mathbf{g}$. By tracing all continuation paths, the method theoretically finds all solutions of $\mathbf{g}(\mathbf{x}) = 0$.

Consider the function

$$f_j(x_j) = p_j^{d_j} x_j^{d_j} - q_j^{d_j} \tag{16}$$

where p_j and q_j are randomly chosen complex constants. Treating each f_j as a function from $\Re^n$ to $\Re$, define an *initial system* $\mathbf{f}(\mathbf{x})$ in terms of the f_j functions as follows:

$$f(x) := \begin{cases} f_1(x_1) \\ f_2(x_2) \\ \vdots \\ f_n(x_n) \end{cases} \tag{17}$$

For each j, the equation $f_j(x_j) = 0$ has d_j distinct solutions which are easy to calculate using the d_j^{th} roots of unity. The solutions are

$$\frac{q_j}{p_j} e^{i2\pi k/d_j}$$

where $k = 1, 2, ..., d_j$. It is easy to see, therefore, that the equation $\mathbf{f}(\mathbf{x}) = 0$ has d solutions, and each solution is of the form

$$(label : "dynamo") \left(\frac{q_1}{p_1} e^{i2\pi k_1/d_1}, \frac{q_2}{p_2} e^{i2\pi k_2/d_2}, ..., \frac{q_n}{p_n} e^{i2\pi k_n/d_n} \right) \tag{18}$$

where $k_j = 1, ..., d_j$ for each $j = 1, ..., n$. We now define the homotopy $\mathbf{h}(\mathbf{x}, \lambda)$ in terms of $\mathbf{f}$ and $\mathbf{g}$ as in equation (14); that is $\mathbf{h}(\mathbf{x}, \lambda) = (1 - \lambda)\mathbf{f}(\mathbf{x}) + \lambda \mathbf{g}(\mathbf{x})$.

A homotopy defined as above has the desired properties: $\mathbf{h}(\mathbf{x}, 0) = \mathbf{f}(x) = 0$ is easy to solve (the solutions are given by (18)); $\mathbf{h}(\mathbf{x}, 1) = g(x)$; and the solution of $\mathbf{h}(\mathbf{x}, \lambda) = 0$ changes smoothly with respect to λ. Moreover, by embedding the original real–valued problem in a family of complex–valued problems, backtracking and path crossing are eliminated. The following theorem, whose proof is in [14], summarizes the points.

Theorem 2 (Morgan) *Suppose $\mathbf{g}(\mathbf{x}) = 0$ is a polynomial system whose solutions are geometrically isolated, and $\mathbf{h}(\mathbf{x}, \lambda)$ is defined as above. Then there are sets of measure zero A_p and A_q such that if $(p_1, p_2, ...p_n) \notin A_p$ and $(q_1, q_2, ...q_n) \notin A_q$, then*

1. *The set $\Gamma := \{(\mathbf{x}, \lambda) \in C^n \times [0, 1) : \mathbf{h}(\mathbf{x}, \lambda) = 0\}$ is a collection of d smooth paths that do not cross or backtrack; and*

2. *at least one continuation path converges to each geometrically isolated solution of $\mathbf{g}(\mathbf{x}) = 0$.*

For purposes of implementation, this result essentially means that the complex constants p_i and q_i can be chosen at random, and the method will find all solutions of the problem (15). The method will only fail if the constants are chosen from the sets A_p and A_q, but the properties of the sets assures that such a choice occurs with probability zero.

The number of paths that converge to a given geometrically isolated solution is, in fact, equal to the multiplicity of the solution. A general definition of multiplicity for multi–variate polynomial functions requires algebraic geometry and is beyond the scope of this report. However, systems with only geometrically isolated solutions share an important property with scalar polynomials of one variable. To wit, the sum of the multiplicities of the roots must equal the total degree of the system. The implication is that every continuation path must converge to some solution of $\mathbf{g}(\mathbf{x}) = 0$, and therefore no paths are divergent.

Because $\mathbf{h}(\mathbf{x}, \lambda)$ consists of paths that do not cross or backtrack, the only remaining bad path behavior to address is that of steep paths. A simple adaptive step length scheme is sufficient. In designing an algorithm, one chooses a tolerance, a maximum step size and a maximum number of corrector iterations. If the corrector does not converge to within the given tolerance after executing the specified number of iterations, halve the step size, re–calculate the initial guess, and try again. If several consecutive correctors succeed, double the step size, staying under the maximum, and continue. In practice, the maximum iterations is set at 3–5. This simple adaptive scheme works well for the polynomial embedding. The following examples apply the method to two–dimensional systems.

Example: Consider the system

$$(label : "roundup") \begin{cases} x_1^2 & +x_2^2 - 25 & = 0 \\ x_1^2 & -9 & = 0 \end{cases} \tag{19}$$

The total degree of the system is 4, and it has four distinct geometrically isolated solutions, $(3, 4)$, $(-3, 4)$, $(3, -4)$ and $(-3, -4)$. The continuation method generates four continuation paths, each of which converges to a distinct solution of the system. Figure 5 shows the continuation paths. The x_1 coordinates are depicted with solid red lines, x_2 coordinates with dotted blue lines. Note that the vertical plane perpendicular to the parameter axis is the complex plane.

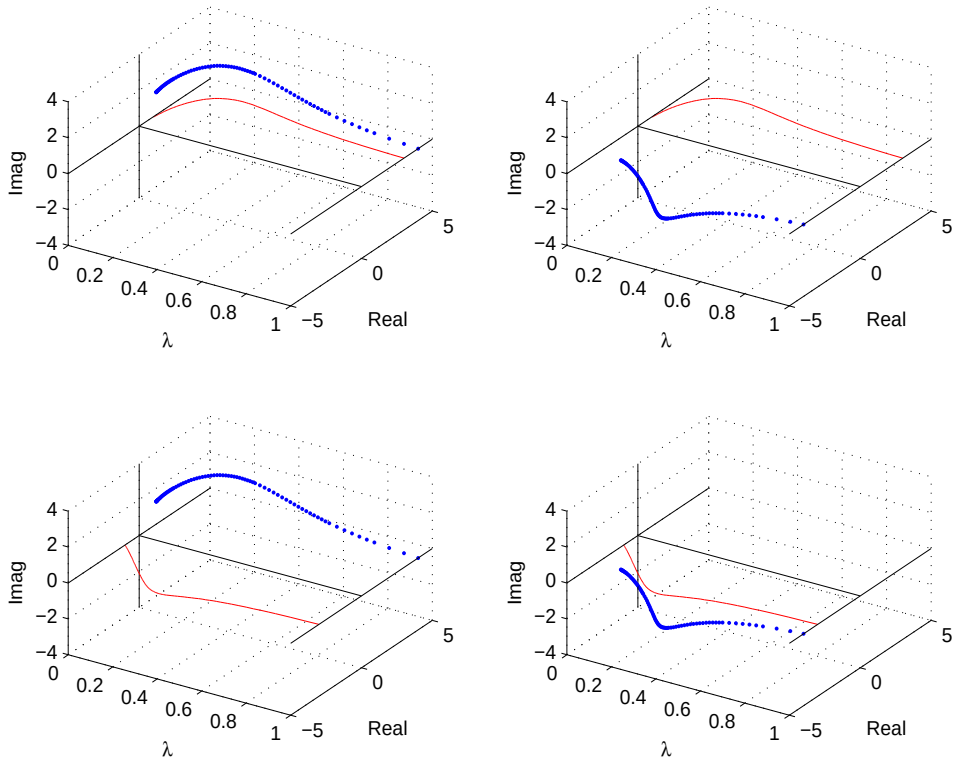


Figure 5: *Continuation Paths for System* (19)

Example: As a second example, consider the system

$$\begin{cases} x_1^2 & +x_2^2 & -25 & = 0 \\ x_1^2 & -x_2 & -5 & = 0 \end{cases} \tag{20}$$

Again, the total degree of the system is 4, but now there are only three distinct solutions, $(3, 4)$, $(-3, 4)$ and $(0, -5)$. All solutions, however, are geometrically isolated, hence by theorem 2 all continuation paths converge to some solution of (20). In this case, two continuation paths converge to the solution (0,-5), therefore (0,-5) is of multiplicity 2.

Polynomial continuation is a highly effective method for solving polynomial systems, particularly those systems with geometrically isolated solutions. Many examples of such problems occur

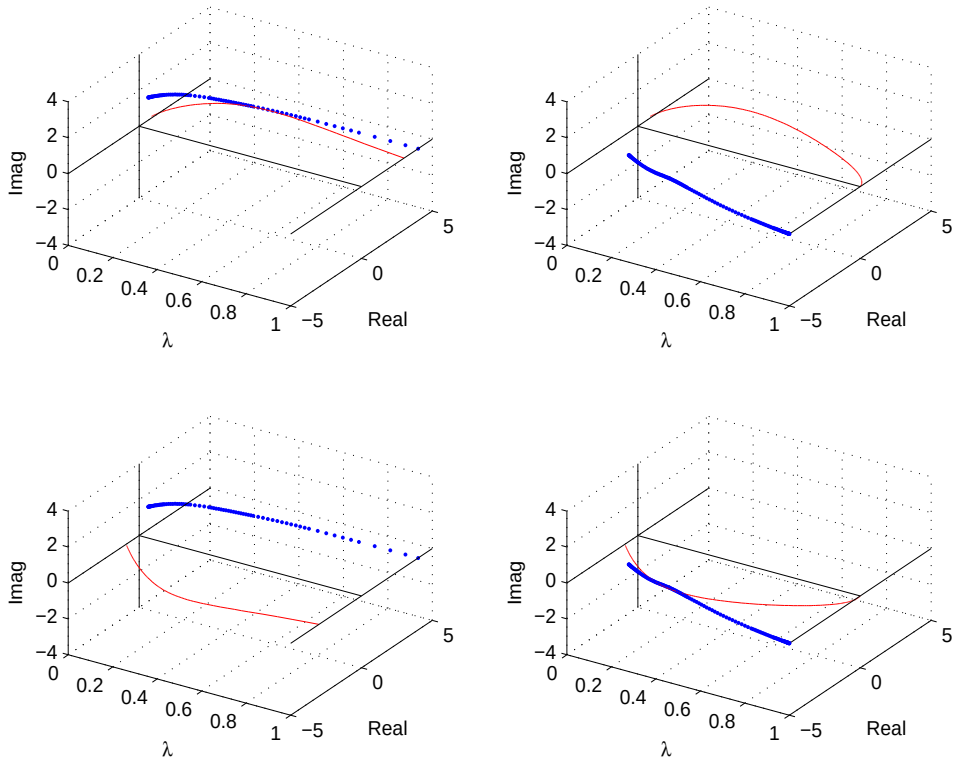


Figure 6: *Continuation Paths for System* (20)

in kinematics (see [18] and chapter 10 of [14]) and in chemical equilibrium systems; see chapter 9 of [14]. Though beyond the scope of this report, polynomial continuation can also be applied to systems with non–discrete solution sets. Successful application often amounts to, in some sense, an *a priori* accounting of the system's solution set.

3.3 Parameter Continuation

While embedding methods can be attractive alternatives to conventional nonlinear solvers for certain classes of algebraic systems, the effectiveness for systems involving a parameter is questionable at best. The only conceivable approach is to specify a discrete set of values for the parameter, and apply an embedding for each separate value of the parameter. In other words, to solve $\mathbf{f}(\mathbf{x}, \lambda) = 0$ using the embedding approach, one first must specify a set of values for the parameter, say $\lambda_1, \lambda_2, ..., \lambda_M$, and apply the method to each problem $\mathbf{f}(\mathbf{x}, \lambda_j) = 0$ separately, introducing an artificial embedding parameter each time. This approach is grossly inefficient at best, as it requires a set of continuation paths for each value of the parameter, and each path is discarded once its endpoints are located. It is also ineffective because bifurcation detection and branch–switching are problematic. The challenge for parameter dependent systems is to contrive a method whose continuation paths coincide with the problem's solution paths.

Henceforth, the term "parameter continuation" will describe such methods, i.e. those that use a naturally occurring parameter to define the continuation paths. Though the term is in a sense redundant (all continuation methods use a parameter), the term is convenient because it distinguishes parameter methods from embedding methods.

3.3.1 Natural Parameter Continuation

In developing a parameter continuation method, one is faced with the problem of how to use a natural parameter to define continuation paths. In his seminal paper [1], Davidenko sought to solve this problem when he observed that solving (4) is equivalent to solving the differential equation

$$(label: "market")\mathbf{f}_{\mathbf{x}}(\mathbf{x},\lambda)\frac{d\mathbf{x}}{d\lambda} + \mathbf{f}_{\lambda}(\mathbf{x},\lambda) = 0, \tag{21}$$

with the initial condition $\mathbf{x}(\lambda_0) = \mathbf{x}_0$ where $\mathbf{f}(\mathbf{x}_0, \lambda_0) = 0$. It is therefore possible to construct curves of the form (12) by numerically solving the solving the differential equation (21) with an appropriate initial value. This differential approach is promising because its solution curves correspond to solution curves of the original problem (4). Of course, any such method is likely to break down at bifurcation points because of singularities in the Jacobian. The approach is also suspect because it does not make explicit use of equation (4).

Natural parameter continuation is a method that exploits Davidenko's observation in order to define continuation paths, but also retains equation (4). As the name suggests, continuation paths are defined using the natural parameter, λ. The idea is to derive a predictor–corrector scheme using equation (21) to motivate the predictor. Starting with a point on the solution curve, $(\mathbf{x}_0, \lambda_0)$, use equation (21) to first determine a *direction vector* $\mathbf{x}'(\lambda_0)$, that is,

$$\mathbf{f}_{\mathbf{x}}(\mathbf{x}_0,\lambda_0)\mathbf{x}'(\lambda_0) = -\mathbf{f}_{\lambda}(\mathbf{x}_0,\lambda_0),$$

where $\mathbf{x}'$ denotes the derivative of $\mathbf{x}$ with respect to λ. The predictor then increments the continuation parameter λ, and predicts the next value of $\mathbf{x}$ on the solution curve by linear extrapolation along the direction vector, $\mathbf{x}'(\lambda_0)$. In summary, the predictor takes the form

$$\begin{aligned}
\lambda_p &= \lambda_0 + \Delta\lambda \\
\mathbf{x}_p &= \mathbf{x}_0 + \Delta\lambda\mathbf{x}'(\lambda_0) \quad \left(\mathbf{x}'(\lambda_0) = -\left[\mathbf{f}_{\mathbf{x}}(\mathbf{x}_0,\lambda_0)\right]^{-1}\mathbf{f}_{\lambda}(\mathbf{x}_0,\lambda_0)\right).
\end{aligned}$$

After calculating the predictor, correct it toward the solution curve by holding λ fixed at λ_p and applying an iterative method to the problem $f(x, \lambda_p) = 0$ using (x_p, λ_p) as an initial guess. In this case, a Newton corrector takes the form

$$\mathbf{x}_p^{k+1} = \mathbf{x}_p^k - \left[\mathbf{f}_{\mathbf{x}}(\mathbf{x}_p^k,\lambda_p)\right]^{-1}\mathbf{f}(\mathbf{x}_p^k,\lambda_p). \tag{22}$$

An equivalent form of the Newton corrector that does not require matrix inversion is

$$[\mathbf{f}_{\mathbf{x}}(\mathbf{x}_p^k,\lambda_p)](\mathbf{x}_p^{k+1} - \mathbf{x}_p^k) = -\mathbf{f}(\mathbf{x}_p^k,\lambda_p). \tag{23}$$

In both cases, $\mathbf{x}_p^k$ denotes the k^{th} iterate of Newton's method for $\mathbf{f}(\mathbf{x}, \lambda_p) = 0$ with initial guess $\mathbf{x}_p$. Figure 7 shows a graphical interpretation of natural parameter continuation. For obvious reasons, the predictor in this method is commonly known as a *tangent predictor*. Continuation paths in the natural parameter method correspond to the original system's solution curves. The natural parameter method is well suited to trace steep paths and paths that cross or split because the tangent predictor allows for tight path tracing, particularly when implemented with an adaptive step length scheme. Unfortunately, the problem of backtracking remains, and the natural parameter method fails at any such folds in the continuation curves.

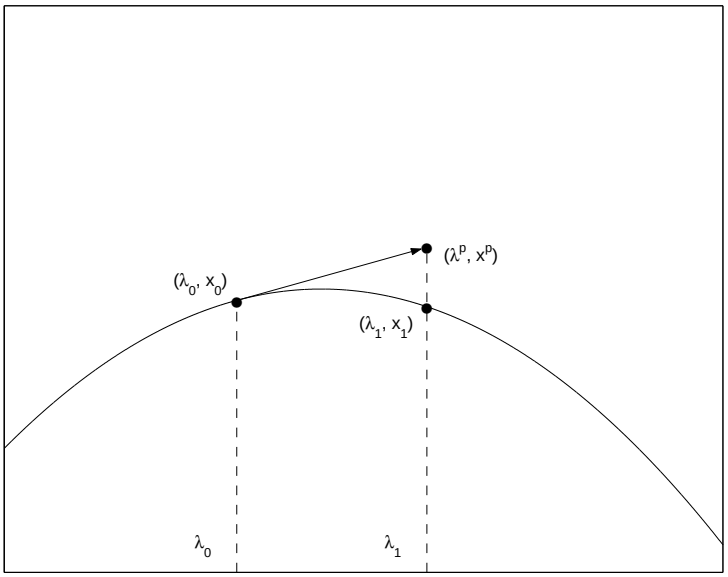


Figure 7: *Natural Parameter Continuation*

3.3.2 Artificial Parameter Continuation

Artificial parameter continuation extends the fundamental observation of Davidenko by exploiting more appropriate curve parameterizations to develop a method that can trace past folds in the continuation curves. Since the problem is to find a function of the form $\mathbf{x}(\lambda)$ such that $\mathbf{f}(\mathbf{x}(\lambda), \lambda) = 0$, Davidenko observed that it is natural to differentiate (5) with respect to λ in order to obtain (21). R.W. Klopfenstein showed [10] that Davidenko's idea could be modified to solve the problem of tracing solution curves past folds if one instead assumes that the curve is parameterized by an artificial parameter rather than the natural parameter λ. That is, Klopfenstein's method introduces an artificial parameter, s, and differentiates (5) with respect to s to obtain the differential equation

$$(label : "garden") f_{\mathbf{x}}(\mathbf{x}, \lambda)\dot{\mathbf{x}} + \mathbf{f}_{\lambda}(\mathbf{x}, \lambda)\dot{\lambda} = 0. \tag{24}$$

As in the natural parameter method, solving (5) can now be accomplished by solving the differential equation (24) with initial conditions $\lambda(0) = \lambda_0$ and $\mathbf{x}(0) = \mathbf{x}_0$.

The artificial parameter approach overcomes a major obstacle that the natural parameter approach fails to address. An appropriate choice of the artificial parameter, such as arclength, allows the method to trace continuation curves around folds. The following definition and theorem characterize paths that Klopfenstein's method succeeds in tracing.

Definition 3 *A solution* $(\mathbf{x}_0, \lambda_0)$ *of* $\mathbf{f}(\mathbf{x}, \lambda) = 0$ *is* **regular** *if*

$$(label : "fullrank")\text{rank } [\mathbf{f}_{\mathbf{x}}(\mathbf{x}_0, \lambda_0) \;\; \mathbf{f}_{\lambda}(\mathbf{x}_0, \lambda_0)] = n. \tag{25}$$

A path $\Gamma := \{(\mathbf{x}(s), \lambda(s)) : \mathbf{f}(\mathbf{x}(s), \lambda(s)) = 0, s \in I\}$ *is regular if all points on* Γ *are regular.*

The Implicit Function Theorem guarantees that a unique solution branch passes through any regular point, i.e., any point that satisfies the full-rank condition (25). Notice that if $\mathbf{f}_{\mathbf{x}}$ is nonsingular at

a solution $(\mathbf{x}_0, \lambda)$ of (5), then $f_x(\mathbf{x}_0, \lambda_0)$ has rank n, and therefore the full rank condition must hold. It is possible, however, for $\mathbf{f}_\mathbf{x}(\mathbf{x}_0, \lambda_0)$ to be singular at a regular solution. In fact, this is exactly the case at folds. At a regular solution which is also fold, $f_x(\mathbf{x}_0, \lambda_0)$ has rank $n-1$ and $\mathbf{f}_\lambda(\mathbf{x}_0, \lambda_0)$ must be linearly independent from the columns of $f_\mathbf{x}(\mathbf{x}_0, \lambda_0)$. Equivalently, the null space of $f_\mathbf{x}(\mathbf{x}_0, \lambda_0)$ has dimension 1, and $\mathbf{f}_\lambda(\mathbf{x}_0, \lambda_0)$ does not lie in the range of $\mathbf{f}_\mathbf{x}(\mathbf{x}_0, \lambda_0)$. The following theorem summarizes.

Theorem 4 *Suppose* $(\mathbf{x}_0, \lambda_0)$ *is a solution of* $\mathbf{f}(\mathbf{x}, \lambda) = 0$. *Then* $(\mathbf{x}_0, \lambda_0)$ *is regular if and only if*

i. $\mathbf{f}_\mathbf{x}(\mathbf{x}_0, \lambda_0)$ *is nonsingular; OR*

ii. dim $N(\mathbf{f}_\mathbf{x}(\mathbf{x}_0, \lambda_0)) = 1$ *AND* $\mathbf{f}_\lambda(\mathbf{x}_0, \lambda_0) \notin$ Range $\mathbf{f}_\mathbf{x}(\mathbf{x}_0, \lambda_0)$.

The proof is easy, and only requires application of the familiar result from linear algebra,

$$(label: "totalrank")\text{rank } T + \text{dim } N(T) = n \tag{26}$$

where T is a linear operator on an $n-$dimensional vector space and $N(T)$ denotes the null space of T. Note that folds are regular points where condition ii holds.

As in the natural parameter method, constructing solution curves by choosing ξ and solving the differential equation (21) is of questionable utility because it does not make explicit use of equation (5). Again, instead of solving the differential equations, the goal is to develop a predictor–corrector continuation scheme that uses Klopfenstein's idea to motivate the predictor. The methodology first appeared in [8].

The development of the artificial parameter method is analogous to the procedure for the natural parameter method. The key is to recognize that specifying a curve parameterization such as arclength is equivalent to augmenting (5) with a constraint equation which exhibits explicit dependence on the artificial parameter. The augmented system then assumes the role of the original system, and the artificial parameter s assumes the role of λ. The advantage is that the constraint equation can be chosen so as to preclude backtracking in the continuation parameter.

To that end, let $\xi(\mathbf{x}, \lambda, s) = 0$ denote the constraint equation, and augment (5) with the constraint. The method now seeks to solve the augmented system

$$(label: "augmented") \begin{cases} \mathbf{f}(\mathbf{x}, \lambda) = 0 \\ \xi(\mathbf{x}, \lambda, s) = 0 \end{cases} . \tag{27}$$

For clarity, define $B := \Re^n \times \Re$, $v := [x \ \ \lambda]^T$, and put

$$(label: "bigF")\mathbf{F}(\mathbf{v}, s) := \begin{cases} \mathbf{f}(\mathbf{x}, \lambda) \\ \xi(\mathbf{x}, \lambda, s) \end{cases} . \tag{28}$$

Then $\xi : B \times \Re \longrightarrow \Re$, and therefore $\mathbf{F} : B \times \Re \longrightarrow B$. It is easy to develop a continuation scheme for $\mathbf{F}$ in analogy with the natural parameter method. Given a point $(\mathbf{v}_0, s_0)$ that satisfies $\mathbf{F}(\mathbf{v}, s) = 0$, predict the next point on the curve in three steps. First, increment the artificial parameter s; let Δs denote the size of the increment, i.e., the step length. Next, define the direction vector $\dot{\mathbf{v}}_0$ according to

$$\mathbf{F}_\mathbf{v}(\mathbf{v}_0, s_0)\dot{\mathbf{v}}_0 = -\mathbf{F}_s(\mathbf{v}_0, s_0). \tag{29}$$

Finally, predict the dependent variable $\mathbf{v}$ by linear extrapolation along the direction vector, that is,

$$\mathbf{v}_p := \mathbf{v}_0 + \Delta s \dot{\mathbf{v}}_0. \tag{30}$$

Equivalent expressions in terms of the original function $\mathbf{f}$, and the original variables $\mathbf{x}$ and λ are now easy to derive. Given the points $\mathbf{v}_0 = (\mathbf{x}_0, \lambda_0)$ and s_0 which satisfy (27), predict the next point $(\mathbf{x}_p, \lambda_p)$ on the path as

$$s_p = s_0 + \Delta s \tag{31}$$

$$\begin{bmatrix} \mathbf{x}_p \\ \lambda_p \end{bmatrix} = \begin{bmatrix} \mathbf{x}_0 \\ \lambda_0 \end{bmatrix} - \Delta s \begin{bmatrix} \mathbf{f}_{\mathbf{x}}(\mathbf{x}_0, \lambda_0) & \mathbf{f}_{\lambda}(\mathbf{x}_0, \lambda_0) \\ \xi_{\mathbf{x}}(\mathbf{x}_0, \lambda_0, s_0)^T & \xi_{\lambda}(\mathbf{x}_0, \lambda_0, s_0) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \xi_s(x_0, \lambda_0, s_0) \end{bmatrix}. \tag{32}$$

The Newton corrector, in terms of $\mathbf{v}$, is

$$\mathbf{v}_p^{k+1} = \mathbf{v}_p^k - \left[\mathbf{F}_{\mathbf{v}}(\mathbf{v}_p^k, s_p)\right]^{-1} \mathbf{F}(\mathbf{v}_p^k, s_p), \tag{33}$$

or equivalently, in terms of $\mathbf{x}$ and λ, is

$$\begin{bmatrix} \mathbf{x}_p^{k+1} \\ \lambda_p^{k+1} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_p^k \\ \lambda_p^k \end{bmatrix} - \begin{bmatrix} \mathbf{f}_{\mathbf{x}}(\mathbf{x}_p^k, \lambda_p^k) & \mathbf{f}_{\lambda}(\mathbf{x}_p^k, \lambda_p^k) \\ \xi_{\mathbf{x}}(\mathbf{x}_p^k, \lambda_p^k, s_p)^T & \xi_{\lambda}(\mathbf{x}_p^k, \lambda_p^k, s_p) \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{f}(\mathbf{x}_p^k, \lambda_p^k) \\ \xi(\mathbf{x}_p^k, \lambda_p^k, s_p) \end{bmatrix}. \tag{34}$$

where $s_p = s_0 + \Delta s$.

Recall that the Implicit Function Theorem guarantees the existence of a unique solution curve through any point at which the full rank condition (25) holds. A fold is a regular point, but the natural parameter method fails to trace this curve at a fold because of a singularity in the Jacobian of $\mathbf{f}$. If the function ξ is chosen appropriately, however, the function F in the artificial parameter method will be nonsingular at all regular points, even folds. The remaining issue, therefore, is to choose the function ξ thusly. Theorem 5 below is a special case of a result that appears in [9], and it informs the choice of ξ because the Jacobian of $\mathbf{F}$ in (28) takes the form of the matrix $\mathbf{M}$ in theorem (5).

Theorem 5 *Suppose* $\mathbf{A}$, $\mathbf{b}$, c *and* d *are linear operators,* $\mathbf{A} : \Re^n \longrightarrow \Re^n$, $\mathbf{b} : \Re \longrightarrow \Re^n$, $\mathbf{c}^T : \Re^n \longrightarrow \Re$ *and* $d : \Re \longrightarrow \Re$ *and define*

$$\mathbf{M} := \begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{c}^T & d \end{bmatrix}. \tag{35}$$

i. *If* $\mathbf{A}$ *is nonsingular, then* M *is nonsingular iff* $d - \mathbf{c}^T\mathbf{A}^{-1}\mathbf{b} \neq 0$.

ii. *If* $\mathbf{A}$ *is singular with* dim $N(\mathbf{A}) = 1$, *then* $\mathbf{M}$ *is nonsingular iff* $\mathbf{b} \notin$ Range $(\mathbf{A})$ *and* $\mathbf{c}^T \notin$ Range $(\mathbf{A})$.

iii. *If* dim $N(\mathbf{A}) \geq 2$, *then* $\mathbf{M}$ *is singular.*

After choosing $\xi(\mathbf{x}, \lambda, s)$, we can apply the theorem to determine the efficacy of the parameterization. A common choice [8], [9], [4] is the so-called *pseudoarclength* parameterization,

$$(label : "pseudoarc")\xi(\mathbf{x}, \lambda, s) = \dot{\mathbf{x}}_0^T(\mathbf{x} - \mathbf{x}_0) + \dot{\lambda}_0(\lambda - \lambda_0) - s. \tag{36}$$

where the directions $\dot{\mathbf{x}}_0$ and $\dot{\lambda}_0$ are defined according to the direction vector $\dot{\mathbf{v}}_0$ in (30). Also, the directions are normalized, i.e., rescaled at each step so that

$$\|\dot{\mathbf{x}}_0^T\|^2 + |\dot{\lambda}_0|^2 = 1.$$

The Jacobian of the pseudoarclength system is nonsingular along a regular path. The following theorem summarizes the formal result. The proof [8] is based on application of theorems 5 and 4.

Theorem 6 *Suppose* $(\mathbf{x}_0, \lambda_0)$ *is a regular point for* $\mathbf{f}(\mathbf{x}, \lambda) = 0$. *Then* $\mathbf{F}_\mathbf{v}$ *is nonsingular for* ξ *defined as in (36).*

The pseudoarclength method is the state of the art in continuation methods. When implemented with adaptive step length schemes and efficient matrix algorithms, it is a highly effective method for tracing regular solution paths of (5). Pseudoarclength continuation therefore constitutes an essential component to a numerical scheme for constructing bifurcation curves for a system of ODEs such as the aircraft equations of motion. Other necessary components are bifurcation detection and branch switching. The report concludes with a brief discussion of these ideas.

3.4 Branch Switching and Bifurcation Detection

A rigorous mathematical definition of a bifurcation requires the concept of topological equivalence of phase portraits [6]. For simplicity, however, one can think of a bifurcation as a point on a solution curve at which two or more solution branches intersect. As noted, the Implicit Function Theorem implies that a unique solution branch passes through any regular point, therefore a bifurcation can only occur at solutions which do not satisfy the full rank condition (25). That is, a necessary condition for bifurcation at the solution $(\mathbf{x}_0, \lambda_0)$ is

$$\text{rank } [\mathbf{f}_\mathbf{x}(\mathbf{x}_0, \lambda_0) \;\; \mathbf{f}_\lambda(\mathbf{x}_0, \lambda_0)] < n. \tag{37}$$

Basis vectors for the null space of $[\mathbf{f}_\mathbf{x}(\mathbf{x}_0, \lambda_0) \;\; \mathbf{f}_\lambda(\mathbf{x}_0, \lambda_0)]$ and its transpose can be used to determine if bifurcation occurs at a singular point. Furthermore, the basis vectors of $N([\mathbf{f}_\mathbf{x}(\mathbf{x}_0, \lambda_0) \;\; \mathbf{f}_\lambda(\mathbf{x}_0, \lambda_0)])$ are used to calculate direction vectors of the branches which appear. The remainder of this section outlines the procedure for a *simple singular point*, which is by definition a point where the rank deficiency in (37) is exactly 1. A discussion of the procedure for higher order singularities appears in [3].

Suppose $(\mathbf{x}_0, \lambda_0)$ is a simple singular point. Then (26) implies that

$$\begin{aligned} \dim N\left([\mathbf{f}_\mathbf{x}(\mathbf{x}_0, \lambda_0) \;\; \mathbf{f}_\lambda(\mathbf{x}_0, \lambda_0)]\right) &= 2 \\ \dim N([\mathbf{f}_\mathbf{x}(\mathbf{x}_0, \lambda_0) \;\; \mathbf{f}_\lambda(\mathbf{x}_0, \lambda_0)]^T) &= 1. \end{aligned}$$

Hence, there exist vectors ϕ_1, ϕ_2 that span the null space of $[\mathbf{f}_\mathbf{x}(\mathbf{x}_0, \lambda_0) \;\; \mathbf{f}_\lambda(\mathbf{x}_0, \lambda_0)]$, and a vector ψ that spans the null space of its transpose. It can be shown [3] that bifurcation at a simple singular point occurs if the so–called *algebraic bifurcation equation* (ABE)

$$\psi^T \mathbf{f}_{\mathbf{vv}}(\mathbf{x}_0, \lambda_0)\phi_1\phi_1\alpha^2 + 2\psi^T \mathbf{f}_{\mathbf{vv}}(\mathbf{x}_0, \lambda_0)\phi_1\phi_2\alpha\beta + \psi^T \mathbf{f}_{\mathbf{vv}}(\mathbf{x}_0, \lambda_0)\phi_2\phi_2\beta^2 = 0 \tag{38}$$

has two distinct real solutions, (α_1, β_1) and (α_2, β_2). The ABE is derived by calculating second order s–derivatives of (5) and recognizing that certain terms vanish when the resulting equation is

left–multiplied by the transpose of ψ [4]. Note that $\mathbf{v} = [\mathbf{x} \quad \lambda]^T$. For brevity, put

$$a_{11} = \psi^T \mathbf{f}_{\mathbf{vv}}(\mathbf{x}_0, \lambda_0)\phi_1\phi_2$$
$$a_{12} = \psi^T \mathbf{f}_{\mathbf{vv}}(\mathbf{x}_0, \lambda_0)\phi_1\phi_2$$
$$a_{22} = \psi^T \mathbf{f}_{\mathbf{vv}}(\mathbf{x}_0, \lambda_0)\phi_2\phi_2.$$

Then (38) has two distinct real solutions when $a_{12}^2 - a_{11}a_{22} > 0$, and a two–branch bifurcation occurs at (x_0, λ_0).

An effective continuation method should trace branches that appear at bifurcation points. In order to do so, one must have a direction vector for each new branch that emerges from the bifurcation. Keller shows [8] that if a simple singular point is a bifurcation, then there are two branches and the direction vectors satisfy

$$(\dot{\mathbf{x}}, \dot{\lambda})^T = \alpha_i\phi_1 + \beta_i\phi_2, \qquad i = 1, 2. \tag{39}$$

One can therefore use the direction vectors in (39) to define two separate continuation paths by applying equation (30) in an artificial parameter method for each direction vector.

If a simple singular point is given, computing the bifurcation directions is straightforward. The continuation method traces the curve in steps, however, and it is therefore likely that, in practice, the method will "skip over" bifurcations. Hence, a continuation method must also include a capability for detection of singular points. In general, one monitors a test function which changes sign at a singular point. For the pseudoarclength method, the test function is simply the determinant of the Jacobian of the augmented system (28); this determinant changes sign at a simple singular point [8]. After determining the existence of a singular point, one can apply a numerical method such as that of Seydel [16] to calculate the point, then proceed to find the direction vectors as above.

4 SOFTWARE

Some continuation software packages are available. CONSOL is a FORTRAN package for solving polynomial systems using the embedding procedure outlined in section 3.2. CONSOL is available in [14]. AUTO is a package for limited bifurcation analysis of general ODE systems. Information and documentation, as well as FORTRAN and C versions are available through the AUTO website, http://cmvl.cs.concordia.ca/auto/

REFERENCES

[1] D.F. Davidenko, "On a New Method of Numerical Solution of Systems of Nonlinear Equations," *Doklady Akademii Nauk SSSR*, Vol. 88 (1953), 601-602 (in Russian).

[2] N. Ananthkrishnan and Nandan K. Sinha, "Level Flight Trim Stability Analysis Using Extended Bifurcation and Continuation Procedure," *Journal of Guidance*, Vol. 24, No. 6 (1997), 1225-1228.

[3] D.W. Decker and H. B. Keller, "Multiple Limit Point Bifurcation," *Journal of Mathematical Analysis*, Vol. 75, No. 2 (1980) 417-430.

[4] Eusebius Doedel, Herbert B. Keller and Jean Pierre Kernevez, "Numerical Analysis and Control of Bifurcation Problems," *International Journal of Bifurcation and Chaos*, Vol. 1 (1991) 493-520.

[5] M.G. Goman, G.I. Zagainov and A.V Khramtsovsky, "Application of Bifurcation Methods to Nonlinear Flight Dynamics Problems," *Progress in Aerospace Sciences*, Vol. 33,(1997), 539-586.

[6] Jack Hale and Huseyin Ko**c**cak, *Dynamics and Bifurcations*, Springer-Verlag, New York, 1991.

[7] C.C. Jahnke and F.E.C. Culick "Application of Bifurcation Theory to the High-Angle-of-Attack Dynamics of the F-14," *Journal of Aircraft*, Vol. 31, No. 1 (1994), 26-34.

[8] Herbert B. Keller, "Numerical Solution of Bifurcation and Nonlinear Eigenvalue Problems," in *Applications of Bifurcation Theory*, Academic Press, New York, 1977.

[9] Herbert B. Keller, *Lecture Notes on Numerical Methods in Bifurcation Problems*, Springer-Verlag, New York, 1987.

[10] R.W. Klopfenstein, "Zeros of Nonlinear Functions," *Journal of the Association for Computing Machinery*, Vol. 8 (1961), 366-373.

[11] Milan Kubí**v**cek, "Algorithm 502 Dependence of Solution of Nonlinear Systems on a Parameter" *ACM Transactions on Mathematical Software*, Vol. 2 (1976), 98-107.

[12] Yuri A. Kuznetsov, *Elements of Applied Bifurcation Theory*, Springer-Verlag, New York, 1998.

[13] R.K. Mehra, W.C. Kessel and J.V. Carroll, "Global Stability and Control Analysis of Aircraft at High Angles of Attack," Report ONR-CR215-248-1, 1977.

[14] Alexander Morgan, *Solving Polynomial Systems Using Continuation for Engineering and Scientific Problems*, Prentice–Hall, Inc., Englewood Cliffs, N.J., 1987.

[15] A.P. Morgan and A.J. Sommese, "Computing All Solutions to Polynomial Systems Using Homotopy Continuation," *Applied Mathematics of Computation*, Vol. 24 (1987), 115-138.

[16] Rudiger Seydel "Numerical Computation of Branch Points in Nonlinear Equations," *Numerische Mathematik*, Vol. 33 (1979), 339-352.

[17] Steven H. Strogatz, *Nonlinear Dynamics and Chaos*, Addison Wesley (1994).

[18] C.W. Wampler, A.P. Morgan and A.J. Sommese, "Numerical Continuation Methods for Solving Polynomial Systems Arising in Kinematics," *Journal of Mechanical Design*, Vol. 112 (1990), 59-68.

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. REPORT DATE *(DD-MM-YYYY)*	2. REPORT TYPE	3. DATES COVERED *(From - To)*
01/07/2004	Contractor Report	

4. TITLE AND SUBTITLE

Continuation Methods for Qualitative Analysis of Aircraft Dynamics

5a. CONTRACT NUMBER

5b. GRANT NUMBER

NCC-1-02043

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)

Cummings, Peter A.

5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER

23-762-60-31

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

NASA Langley Research Center
Hampton, VA 23681-2199

National Institute of Aerospace (NIA)
144 Research Drive
Hampton, VA 23666

8. PERFORMING ORGANIZATION REPORT NUMBER

NIA Report No. 2004-06

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

National Aeronautics and Space Administration
Washington, DC 20546-0001

10. SPONSORING/MONITOR'S ACRONYM(S)

NASA

11. SPONSORING/MONITORING REPORT NUMBER

NASA/CR-2004-213035

12. DISTRIBUTION/AVAILABILITY STATEMENT

Unclassified - Unlimited
Subject Category 08
Availability: NASA CASI (301) 621-0390 Distribution: Nonstandard

13. SUPPLEMENTARY NOTES

Langley Technical Monitor: Patrick C. Murphy

14. ABSTRACT

A class of numerical methods for constructing bifurcation curves for systems of coupled, non-linear ordinary differential equations is presented. Foundations are discussed, and several variations are outlined along with their respective capabilities. Appropriate background material from dynamical systems theory is presented.

15. SUBJECT TERMS

Continuation Methods; Bifurcation Analysis; Flight Dynamics

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19b. NAME OF RESPONSIBLE PERSON
a. REPORT	b. ABSTRACT	c. THIS PAGE			STI Help Desk (email: help@sti.nasa.gov)
U	U	U	UU	24	19b. TELEPHONE NUMBER *(Include area code)* (301) 621-0390