An operator method for field moments from the extended parabolic wave equation and analytical solutions of the first and second moments for atmospheric electromagnetic wave propagation

Robert M. Manning
National Aeronautics and Space Administration
Glenn Research Center
Cleveland, OH 44135

ABSTRACT

The extended wide-angle parabolic wave equation applied to electromagnetic wave propagation in random media is considered. A general operator equation is derived which gives the statistical moments of an electric field of a propagating wave. This expression is used to obtain the first and second order moments of the wave field and solutions are found that transcend those which incorporate the full paraxial approximation at the outset. Although these equations can be applied to any propagation scenario that satisfies the conditions of application of the extended parabolic wave equation, the example of propagation through atmospheric turbulence is used. It is shown that in the case of atmospheric wave propagation and under the Markov approximation (i.e., the δ-correlation of the fluctuations in the direction of propagation), the usual parabolic equation in the paraxial approximation is accurate even at millimeter wavelengths. The comprehensive operator solution also allows one to obtain expressions for the longitudinal (generalized) second order moment. This is also considered and the solution for the atmospheric case is obtained and discussed. The methodology developed here can be applied to any qualifying situation involving random propagation through turbid or plasma environments that can be represented by a spectral density of p e r m i t t i v i t y   f l u c t u a t i o n s .

I. INTRODUCTION

Since its inception by Leontovitch and Fock[1], the parabolic wave equation has found application in all aspects of electromagnetic wave propagation scenarios[2-4] as well as those in acoustic wave propagation[5-7]. In the case of electromagnetic wave propagation in random media, the parabolic wave equation was originally applied by Klyatskin and Tatarskii[8,9] to describe the statistical moments of the attendant electric field. This type of parabolic wave equation, however, is accurate only to within the paraxial approximation and is therefore suited to treat small-angle wave scattering about a preferential direction of propagation. Thus, in the case of propagation through random media, the parabolic wave equation can strictly be applied only when the smallest size l_0 of the permittivity fluctuations within the medium and the wavelength λ satisfy the condition $l_0 \gg \lambda$, i.e., the wavelength must be the smallest spatial scale in the problem. In the case of the open atmosphere, $l_0 \sim 1mm$. As shown by Klyatskin and Tatarskii[10], this is indeed a good approximation in the case of atmospheric propagation at optical wavelengths. However, there are situations in the applications, e.g., atmospheric millimeter wave propagation, in which one has $l_0 \geq \lambda$ and the paraxial approximation strictly no longer holds. In these cases, one can have wide-angle scattering about the preferential direction of propagation. Again, in the scenario of electromagnetic wave propagation through random media, such considerations have been made[11]. Much more recently, the same situation has been met with in acoustical wave propagation[6][12-15]. In what is to follow, the wide-angle modification of the parabolic wave equation will be called the extended parabolic wave equation for reasons that will be apparent in the next section. Although extensions of this theory have been advanced within the realm of stochastic propagation through random media[11,16], none have been analytically treated at the level where corrections to the paraxial approximation have been quantitatively identified and compared to paraxial results.

It is the purpose of this work to analytically derive solutions for the first and second order moments of a plane wave field propagating through a random medium within the context of the extended parabolic wave equation. Approximations must, of course, be made in order to obtain analytical results. However, the main point here is that such approximations are not made at the outset, as is usually done. Such solutions will afford a comparison between the results of the parabolic wave equation within the paraxial approximation and those of the extended wide-angle theory. In Section 2, the extended parabolic wave equation for electromagnetic wave propagation is derived, for completeness, from the stochastic Helmholtz equation. Since this will be an operator equation in the random electric field, a statistical operator method is developed in Section 3, which will give a general equation for an arbitrary spatial statistical moment of the wave field. From this, expressions are obtained as special cases for the first and second spatial moments of the field. The first field moment is just the average field and the second field moment is known as the mutual coherence function (MCF). Analytical solutions to these equations are derived for the Kolmogorov spectrum of atmospheric permittivity fluctuations within the Markov approximation. Unlike earlier treatments of propagation using the parabolic wave equation[8-10], the generality of the operator method also allows one to obtain an expression for the second order field moment in the direction

longitudinal to the direction of propagation; this is known as the generalized MCF. A solution for this quantity is also obtained.

II. THE EXTENDED PARABOLIC WAVE EQUATION FOR RANDOM MEDIA

Consider the scalar stochastic Helmholtz equation for an electric field $E(x,\vec{\rho})$ propagating principally along an x-axis and perpendicular to the $\vec{\rho}$-plane of an otherwise arbitrary coordinate system

$$\frac{\partial^2 E(x,\vec{\rho})}{\partial x^2} + \nabla_\rho^2 E(x,\vec{\rho}) + k^2 E(x,\vec{\rho}) = k^2 \tilde{\varepsilon}(x,\vec{\rho}) E(x,\vec{\rho}) \tag{1}$$

where $\tilde{\varepsilon}(x,\vec{\rho})$ is the random part of the total permittivity $\varepsilon(x,\vec{\rho}) = 1 + \tilde{\varepsilon}(x,\vec{\rho})$ of the propagation medium. Using the decomposition of the total field into a forward propagating field $E^+(x,\vec{\rho})$ and a backward propagating field $E^-(x,\vec{\rho})$, i.e.,

$$E(x,\vec{\rho}) = E^+(x,\vec{\rho}) + E^-(x,\vec{\rho}), \qquad \frac{\partial E(x,\vec{\rho})}{\partial x} = \frac{\partial E^+(x,\vec{\rho})}{\partial x} + \frac{\partial E^-(x,\vec{\rho})}{\partial x} \tag{2}$$

with the expansion into inhomogeneous plane waves

$$E^\pm(x,\vec{\rho}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^\pm(\vec{q}) \exp\left[i\vec{q}\cdot\vec{\rho} \pm i\left(k^2 - q^2\right)^{1/2} x\right] d^2 q \tag{3}$$

one can obtain the following operator expressions for the component fields

$$2i\frac{\partial E^\pm(x,\vec{\rho})}{\partial x} \pm 2\left(k^2 + \nabla_\rho^2\right)^{1/2} E^\pm(x,\vec{\rho}) = \pm k^2 \left(k^2 + \nabla_\rho^2\right)^{-1/2} \tilde{\varepsilon}(x,\vec{\rho})\left[E^+(x,\vec{\rho}) + E^-(x,\vec{\rho})\right] \tag{4}$$

Such an equation (or its Fourier transform) was considered by Malakov and Saichev[17] as well as by Klyatskin[18] and Frankenthal and Beran[19].

In the event that the backscattered field is insignificant with respect to the forward scattered field, i.e., $E^+(x,\vec{\rho}) >> E^-(x,\vec{\rho})$, one can formally set $E^-(x,\vec{\rho}) = 0$; Eq.(4) thus becomes a single equation for the forward propagating field

$$2i\frac{\partial E^+(x,\vec{\rho})}{\partial x} + 2\left(k^2 + \nabla_\rho^2\right)^{1/2} E^+(x,\vec{\rho}) = k^2 \left(k^2 + \nabla_\rho^2\right)^{-1/2} \tilde{\varepsilon}(x,\vec{\rho}) E^+(x,\vec{\rho}). \tag{5}$$

This equation, which can be called the "extended parabolic equation"[18], was previously analyzed by Saichev[16] using a method different from the operator formulation in what is to follow. Its integral formulation is also known as the method of multiple forward scatter[11,20]. However, it was pointed out[11] that this method is applicable to situations

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when[21] $\lambda \leq l_0$, the penalty for ignoring the backscattered component; The classical parabolic wave equation in the paraxial approximation holds for cases where $\lambda << l_0$. Hence, although Eq.(5) is capable of describing propagation situations in which the wave is scattered at angles up to $\pi/2$ with respect to the x axis, its application is limited to cases where $\lambda \leq l_0$.

The form of Eq.(5) can be simplified by defining the differential operator

$$U_\rho \equiv 2k^2 \left(1 + \frac{\nabla_\rho^2}{k^2}\right)^{1/2}$$ (6)

and the corresponding integral operator

$$V_\rho = \frac{1}{2k^2}\left(1 + \frac{\nabla_\rho^2}{k^2}\right)^{-1/2}.$$ (7)

Equation (4) then becomes

$$2ik\frac{\partial E(x,\vec{\rho})}{\partial x} + U_\rho E(x,\vec{\rho}) - 2k^4 V_\rho \tilde{\varepsilon}(x,\vec{\rho}) E(x,\vec{\rho}) = 0.$$ (8)

(Hereafter, the superscript '+' will be dropped.) This equation can be reduced to the well-known stochastic parabolic wave equation in the paraxial approximation by expanding the operators to give $U_\rho \approx 2k^2 + \nabla_\rho^2$ and $V_\rho \approx 1/2k^2$ and transforming the field via $E(x,\vec{\rho}) = W(x,\vec{\rho})\exp(ikx)$. It must be noted that these operator expansions, which are necessary conditions for the paraxial approximation, applied to Eq.(8) at this juncture yields the paraxial form of the parabolic wave equation. In what is to follow, the full form of Eq.(8) will be dealt with in forming an expression for the statistical moments of the electric field. Only during the solutions of these equations for the first and second moments will these operator expansions be judiciously applied. It will turn out that a geometrical optics approximation which is implicitly made in the scattering term of these solutions can be lifted which will essentially allow diffractive effects to be introduced into the scattering term.

In what is to follow, Eq.(8) will be employed to yield an operator equation for the generalized statistical moments of the field. Here, the term 'generalized' connotes moments at differing transverse coordinates $\vec{\rho}$ as well as differing longitudinal coordinates x.

III. OPERATOR SOLUTIONS FOR THE GENERALIZED FIELD MOMENTS
A. An expression for the generalized nm[th] field moments

Defining the stochastic operator

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$$D_{x,\rho} \equiv 2ik\frac{\partial}{\partial x} + U_{\rho} + 2k^4 V_{\rho}\tilde{\varepsilon}(x,\vec{\rho}) \tag{9}$$

Eq.(8) simply becomes

$$D_{x,\rho}E(x,\vec{\rho}) = 0 \tag{10}$$

which is easily amenable to further statistical analysis. To this end, one defines the generalized moment of the electric field

$$\Gamma_{nm}\left(x_1,\vec{\rho}_1;x_2,\vec{\rho}_2;\cdots x_n,\vec{\rho}_n;x_{n+1},\vec{\rho}_{n+1};\cdots x_{n+m},\vec{\rho}_{n+m}\right) \equiv \langle g_{nm} \rangle, \tag{11}$$

$$g_{nm} \equiv \prod_{j=1}^{n} E(x_j,\vec{\rho}_j) \prod_{l=n+1}^{n+m} E^*(x_l,\vec{\rho}_l). \tag{12}$$

Using a modification of previous prescriptions[22,23], one can employ Eq.(10) for each of the field points $(x,\vec{\rho})$ and obtain for the product of fields

$$L_{nm}g_{nm}\left(x_1,\vec{\rho}_1;x_2,\vec{\rho}_2;\cdots x_n,\vec{\rho}_n;x_{n+1},\vec{\rho}_{n+1};\cdots x_{n+m},\vec{\rho}_{n+m}\right) = 0 \tag{13}$$

where

$$L_{nm} \equiv 2ik\left(\sum_{j=1}^{n}\frac{\partial}{\partial x_j} + \sum_{l=n+1}^{n+m}\frac{\partial}{\partial x_l}\right) + \sum_{j=1}^{n}\left(U_{\rho_j} + 2k^4 V_{\rho_j}\tilde{\varepsilon}(x_j,\rho_j)\right) - \\ - \sum_{l=n+1}^{n+m}\left(U_{\rho_l}^* + 2k^4 V_{\rho_l}^*\tilde{\varepsilon}^*(x_l,\rho_l)\right) \tag{14}$$

In order to isolate the quantity $\langle g_{nm} \rangle = \Gamma_{nm}$ from this relation, it is expedient[24,25] to decompose the operator L_{nm} into its average and random parts, i.e.,

$$\langle L_{nm} \rangle \equiv 2ik\left(\sum_{j=1}^{n}\frac{\partial}{\partial x_i} + \sum_{l=n+1}^{n+m}\frac{\partial}{\partial x_j}\right) + \sum_{j=1}^{n} U_{\rho_j} - \sum_{l=n+1}^{n+m} U_{\rho_l}^* \tag{15}$$

and

$$\tilde{L}_{nm} \equiv 2k^4\left[\sum_{j=1}^{n} V_{\rho_j}\tilde{\varepsilon}(x_j,\rho_j) - \sum_{l=n+1}^{n+m} V_{\rho_l}^*\tilde{\varepsilon}^*(x_l,\rho_l)\right], \qquad \langle \tilde{L}_{nm} \rangle = 0 \tag{16}$$

Hence, Eq.(13) becomes

$$\left(\langle L_{nm} \rangle + \tilde{L}_{nm}\right)g_{nm} = 0. \tag{17}$$

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Ensemble averaging this relation yields

$$\langle L_{nm} \rangle \Gamma_{nm} + \langle \tilde{L}_{nm} g_{nm} \rangle = 0. \tag{18}$$

Similarly writing

$$g_{nm} = \Gamma_{nm} + \tilde{g}_{nm}, \qquad \langle \tilde{g}_{nm} \rangle = 0, \tag{19}$$

and substituting into Eq.(18) gives

$$\langle L_{nm} \rangle \Gamma_{nm} + \langle \tilde{L}_{nm} \tilde{g}_{nm} \rangle = 0. \tag{20}$$

Remembering that it is the goal of this development to obtain an expression for the general field moment Γ_{nm}, one subtracts Eq.(20) from Eq.(13) and using Eq.(19) obtains[25]

$$L_{nm} \Gamma_{nm} + L_{nm} \tilde{g}_{nm} - \langle L_{nm} \rangle \Gamma_{nm} - \langle \tilde{L}_{nm} \tilde{g}_{nm} \rangle = 0. \tag{21}$$

Combining the first and third members of this equation using the fact that $\left[L_{nm} - \langle L_{nm} \rangle \right] \Gamma_{nm} = \tilde{L}_{nm} \Gamma_{nm}$ gives

$$L_{nm} \tilde{g}_{nm} + \tilde{L}_{nm} \Gamma_{nm} - \langle \tilde{L}_{nm} \tilde{g}_{nm} \rangle = 0. \tag{22}$$

One must now isolate the random quantity $\tilde{g}_{nm}$ by defining an operator L_{nm}^{-1} inverse to L_{nm}, i.e., $L_{nm}^{-1} L_{nm} = 1$. Thus, operating on Eq.(22) with L_{nm}^{-1} yields

$$\tilde{g}_{nm} + L_{nm}^{-1} \tilde{L}_{nm} \Gamma_{nm} - L_{nm}^{-1} \langle \tilde{L}_{nm} \tilde{g}_{nm} \rangle = 0. \tag{23}$$

Finally, operating on this relation with $\tilde{L}_{nm}$, ensemble averaging and solving the resulting expression for $\langle \tilde{L}_{nm} \tilde{g}_{nm} \rangle$ gives

$$\langle \tilde{L}_{nm} \tilde{g}_{nm} \rangle = -\left[1 - \langle \tilde{L}_{nm} L_{nm}^{-1} \rangle \right]^{-1} \langle \tilde{L}_{nm} L_{nm}^{-1} \tilde{L}_{nm} \rangle \Gamma_{nm}. \tag{24}$$

Substituting this result back into Eq.(20), one obtains for the equation governing Γ_{nm}

$$\left\{ \langle L_{nm} \rangle - \left[1 - \langle \tilde{L}_{nm} L_{nm}^{-1} \rangle \right]^{-1} \langle \tilde{L}_{nm} L_{nm}^{-1} \tilde{L}_{nm} \rangle \right\} \Gamma_{nm} = 0. \tag{25}$$

The solution of this operator equation gives an exact solution for the arbitrary field moments for wide-angle propagation through a random medium characterized by the stochastic permittivity $\tilde{\varepsilon}(x, \vec{\rho})$ and the assumption that $\langle \tilde{L}_{nm} \rangle = 0$.

The general relation given by Eq.(25) can be reduced to the parabolic equation for the field moments in the paraxial approximation in the case where $\lambda \ll l_0$. In this instance, one employs the approximations for the operators $U_\rho \approx 2k^2 + \nabla_\rho^2$ and $V_\rho \approx 1/2k^2$ used earlier. In addition, the classical parabolic equation considers statistical moments in the same transverse plane, i.e., $x_j = x_l = x$. Thus, the partial differential operators in Eq.(15) collapse into the single operator $\partial/\partial x$. Equations (15) and (16) then become

$$\langle L_{nm} \rangle \approx 2ik \frac{\partial}{\partial x} + \sum_{j=1}^{n} \nabla_{\rho_j}^2 - \sum_{l=1+n}^{n+m} \nabla_{\rho_l}^2 \qquad (26)$$

and

$$\tilde{L}_{nm} \equiv k^2 \left[+ \sum_{j=1}^{n} \tilde{\varepsilon}(x, \vec{\rho}_j) - \sum_{l=1+n}^{n+m} \tilde{\varepsilon}^*(x, \vec{\rho}_l) \right]. \qquad (27)$$

Two related palatable approximations must now be made; since $L_{nm} \equiv \langle L_{nm} \rangle + \tilde{L}_{nm}$, and it is usually assumed that $\left\| \tilde{L}_{nm} \right\| \ll 1$, one has

$$L_{nm} = \left[\langle L_{nm} \rangle + \tilde{L}_{nm} \right]^{-1} \approx \langle L_{nm} \rangle^{-1}, \quad \left[1 - \langle \tilde{L}_{nm} L_{nm}^{-1} \rangle \right]^{-1} \approx 1 \qquad (28)$$

where

$$\langle L_{nm} \rangle^{-1} = \left[2ik \frac{\partial}{\partial x} + \sum_{j=1}^{n} \nabla_{\rho_j}^2 - \sum_{l=1+n}^{n+m} \nabla_{\rho_l}^2 \right]^{-1}$$

$$\approx \left[2ik \frac{\partial}{\partial x} \right]^{-1}$$

$$= \frac{1}{2ik} \int_0^x dx'. \qquad (29)$$

Equation (25) then becomes

$$\left[2ik \frac{\partial}{\partial x} + \sum_{j=1}^{n} \nabla_{\rho_j}^2 - \sum_{l=1+n}^{n+m} \nabla_{\rho_l}^2 - \langle \tilde{L}_{nm} \langle L_{nm}^{-1} \rangle \tilde{L}_{nm} \rangle \right] \Gamma_{nm} = 0, \qquad (30)$$

where

$$\langle \tilde{L}_{nm} \langle L_{nm}^{-1} \rangle \tilde{L}_{nm} \rangle \approx \frac{k^3}{2i} \int_0^x \left[\sum_{j=1}^{n} \sum_{j'=1}^{n} \langle \tilde{\varepsilon}(x, \vec{\rho}_j) \tilde{\varepsilon}(x', \vec{\rho}_{j'}) \rangle - \sum_{j=1}^{n} \sum_{l'=1+n}^{n+m} \langle \tilde{\varepsilon}(x, \vec{\rho}_j) \tilde{\varepsilon}^*(x', \vec{\rho}_{l'}) \rangle - \right.$$

$$-\sum_{j'=1}^{n} \sum_{l=1+n}^{n+m} \left\langle \tilde{\varepsilon}^*(x,\bar{\rho}_{j'})\tilde{\varepsilon}(x',\bar{\rho}_l') \right\rangle + \sum_{l=1+n}^{n+m} \sum_{l'=1+n}^{n+m} \left\langle \tilde{\varepsilon}^*(x,\bar{\rho}_l)\tilde{\varepsilon}^*(x',\bar{\rho}_{l'}) \right\rangle \Bigg] dx', \qquad (31)$$

which is the well known classical paraxial form for the problem[23]. It is interesting to note the 'geometrical optics' approximation made in Eq.(29) which gives the usual scattering term of Eq.(31). One can envision a substantial extension of this development beyond that of the classical treatment if one is to use the entire form of the operator $\left\langle L_{nm} \right\rangle^{-1}$, i.e., use the inverse of the operator $\left\langle L_{nn} \right\rangle$ as solved in its entirety rather than in the geometrical optics approximation which was used above. The inclusion of the ∇_ρ^2 operators will allow diffraction phenomena to be incorporated into the scattering term. This will form the subject of the next section in which the first order moment and the generalized second order moment (i.e., the generalized MCF) are derived using the above formalism.

B. Solution for the first moment

The first order moment or average field of the random electric field in a plane transverse to the direction of propagation is defined through Eq.(11) to be given by

$$\Gamma_{10}(x;\bar{\rho}_1) = \left\langle E(x;\bar{\rho}_1) \right\rangle \qquad (32)$$

which is a solution of the operator relation of Eq.(25), viz,

$$\left\{ \left\langle L_{10} \right\rangle - \left[1 - \left\langle \tilde{L}_{10} L_{10}^{-1} \right\rangle \right]^{-1} \left\langle \tilde{L}_1 L_{10}^{-1} \tilde{L}_{10} \right\rangle \right\} \Gamma_{10} = 0 \qquad (33)$$

where, from Eqs.(15) and (16),

$$\left\langle \hat{L}_{10} \right\rangle \equiv 2ik\frac{\partial}{\partial x} + U_\rho, \qquad \tilde{L}_{10} \equiv 2k^4 V_{\rho_1} \tilde{\varepsilon}(x,\bar{\rho}), \qquad \bar{\rho} \equiv \bar{\rho}_1. \qquad (34)$$

The ability to proceed in an analytical fashion is dependent upon the simplifying approximations used earlier, viz., one has that

$$L_{10}^{-1} = \left[\left\langle L_{10} \right\rangle + \tilde{L}_{10} \right]^{-1} \approx \left\langle L_{10} \right\rangle^{-1}, \qquad \left[1 - \left\langle \tilde{L}_{10} L_{10}^{-1} \right\rangle \right]^{-1} \approx 1. \qquad (35)$$

Using this result, Eq.(33) becomes

$$\left\{ \left\langle L_{10} \right\rangle - \left\langle \tilde{L}_{10} \left\langle L_{10} \right\rangle^{-1} \tilde{L}_{10} \right\rangle \right\} \Gamma_{10} = 0. \qquad (36)$$

Employing the appropriate definitions of the operators, Eq.(36) gives

$$\left[2ik\frac{\partial}{\partial x}+U_\rho-\left\langle\left(2k^4V_\rho\,\tilde{\varepsilon}(x,\vec{\rho})\right)\left\{2ik\frac{\partial}{\partial x}+U_\rho\right\}^{-1}\left(2k^4V_\rho\tilde{\varepsilon}(x,\vec{\rho})\right)\right\rangle\right]\Gamma_{10}(x,\vec{\rho})=0. \quad (37)$$

This differential equation in the operators U_ρ and V_ρ must now be simplified and solved for the first-order moment $\Gamma_{10}(x,\vec{\rho})$.

To this end, one must first deal with the 'diffraction propagator'

$$\left\{2ik\frac{\partial}{\partial x}+U_\rho\right\}^{-1}\equiv G(x,\vec{\rho}) \quad (38)$$

It is at this point that this method has its greatest effect on the description of the propagation mechanism. Within the geometrical optics approximation, the U_ρ term in Eq.(38) is neglected with respect to the $2ik\,\partial/\partial x$ term. Hence, potentially important diffractive effects are neglected within the scattering term $\langle\cdots\rangle$ of Eq.(37) by use of this approximation (see Eq.(29)). As mentioned earlier, the retention of the U_ρ term, even in its approximate form, will include diffractive phenomena inherent in the random scattering mechanism that have hitherto been neglected. The Green function of the operators $2ik\,\partial/\partial x+U_\rho$ given by Eq.(38) is defined by

$$\left\{2ik\frac{\partial}{\partial x}+2k^2\left(1+\frac{\nabla_\rho^2}{k^2}\right)^{1/2}\right\}G(x,\vec{\rho})=\delta(x-x')\delta(\vec{\rho}-\vec{\rho}') \quad (39)$$

where the definition of U_ρ is used. Applying the approximation $U_\rho\approx2k^2+\nabla_\rho^2$ and solving for the Green function $G(x,\vec{\rho})$ yields,

$$G(x,\vec{\rho})=G(x,\vec{\rho};x',\vec{\rho}')=\left(\frac{1}{4\pi}\right)\exp\left[-ik(x-x')\right]\frac{\exp\left[-ik(\vec{\rho}-\vec{\rho}')^2/2(x-x')\right]}{x-x'}. \quad (40)$$

Thus, the third term within the brackets of Eq.(37) can be written

$$\langle\cdots\rangle\equiv\left\langle\left(2k^4V_\rho\,\tilde{\varepsilon}(x,\vec{\rho})\right)\left\{2ik\frac{\partial}{\partial x}+U_\rho\right\}^{-1}\left(2k^4V_\rho\tilde{\varepsilon}(x,\vec{\rho})\right)\right\rangle=$$

$$=\left(2k^4\right)^2\int_0^x\int_{-\infty}^\infty G(x,\vec{\rho};x',\vec{\rho}')\left\langle V_\rho\tilde{\varepsilon}(x,\vec{\rho})V_\rho\tilde{\varepsilon}(x,\vec{\rho}')\right\rangle d^2\rho'dx'. \quad (41)$$

Proceeding further, one now must deal with the operator products

$$V_\rho \tilde{\varepsilon}(x,\vec{\rho}) = \left(\frac{1}{2k^2}\right)\left(1 + \frac{\nabla_\rho^2}{k^2}\right)^{-1/2} \tilde{\varepsilon}(x,\vec{\rho}). \tag{42}$$

Since $\tilde{\varepsilon}(x,\vec{\rho})$ is a random function, it can be represented in the form of a Fourier-Stieltjes integral[26], i.e.,

$$\tilde{\varepsilon}(x,\vec{\rho}) = \int \exp(i\vec{\kappa}\cdot\vec{\rho})dZ(x,\vec{\kappa}) \tag{43}$$

in which the spectral amplitude $dZ(x,\vec{\kappa})$ is endowed with the same statistical properties as is the random function $\tilde{\varepsilon}(x,\vec{\rho})$ as will be shown in what is to follow. Applying Eq.(43) to Eq.(42) gives

$$V_\rho \tilde{\varepsilon}(x,\vec{\rho}) = \left(\frac{1}{2k^2}\right)\int\left(1 - \frac{\kappa^2}{k^2}\right)^{-1/2} \exp(i\vec{\kappa}\cdot\vec{\rho})dZ(x,\vec{\kappa}). \tag{44}$$

Thus, the ensemble averaged product appearing in right side of Eq.(41) becomes,

$$\left\langle V_\rho\tilde{\varepsilon}(x,\vec{\rho})V_{\rho'}\tilde{\varepsilon}(x',\vec{\rho}')\right\rangle = \left(\frac{1}{2k^2}\right)^2 \int\int\left(1 - \frac{\kappa^2}{k^2}\right)^{-1/2}\left(1 - \frac{\kappa'^2}{k^2}\right)^{-1/2} \cdot$$

$$\cdot \exp(i\vec{\kappa}\cdot\vec{\rho} + i\vec{\kappa}'\cdot\vec{\rho}')\left\langle dZ(x,\vec{\kappa})dZ(x',\vec{\kappa}')\right\rangle. \tag{45}$$

One now makes use of the fact that the atmospheric permittivity fluctuation field $\tilde{\varepsilon}(x,\vec{\rho})$ is taken to be statistically homogeneous, characterized by a power spectral density $\Phi_\varepsilon(\vec{\kappa})$ and δ-correlated in the longitudinal direction (the Markov approximation); these circumstances allow one to write[26]

$$\left\langle dZ(x,\vec{\kappa})dZ(x',\vec{\kappa}')\right\rangle = \delta(\vec{\kappa} + \vec{\kappa}')F_\varepsilon(x - x',\vec{\kappa})d^2\kappa d^2\kappa' \tag{46a}$$

where for δ-correlated fluctuations in the x direction, the two-dimensional spectrum $F_\varepsilon(x - x',\vec{\kappa})$ is given by

$$F_\varepsilon(x - x',\vec{\kappa}) = 2\pi\delta(x - x')\Phi_\varepsilon(\vec{\kappa}) \tag{46b}$$

in which $\Phi_\varepsilon(\vec{\kappa})$ is the three dimensional spectrum of permittivity fluctuations. Using these relations in Eq.(45) and performing the integrations where possible yields

$$\left\langle V_\rho\tilde{\varepsilon}(x,\vec{\rho})V_{\rho'}\tilde{\varepsilon}(x',\vec{\rho}')\right\rangle = 2\pi\left(\frac{1}{2k^2}\right)^2\delta(x - x')\int_{-\infty}^{\infty}\left(1 - \frac{\kappa^2}{k^2}\right)^{-1}\exp[i\vec{\kappa}\cdot\vec{\rho}_d]\Phi_\varepsilon(\vec{\kappa})d^2\kappa$$

$$\tag{47}$$

where $\bar{\rho}_d \equiv \bar{\rho} - \bar{\rho}'$ is the difference coordinate.

Taking the statistics governing the random field $\tilde{\varepsilon}(x, \bar{\rho})$ to be also isotropic, i.e., $\Phi_\varepsilon(\bar{\kappa}) = \Phi_\varepsilon(\kappa)$, Eq.(41) can now finally be evaluated by substituting into it Eqs.(40) and (47); converting the integration in the $\bar{\rho}_d$-plane into one in plane polar coordinates and performing the associated integrations gives

$$\langle \cdots \rangle = -i\pi^2 k^3 \int_0^\infty \left(1 - \frac{\kappa^2}{k^2}\right)^{-1} \Phi_\varepsilon(\kappa)\kappa d\kappa \qquad (48)$$

where the δ-function relation

$$\int_0^x \delta(x - x')dx' = \frac{1}{2} \qquad (49)$$

is employed.

Returning to Eq.(37) and, substituting Eq.(48) into Eq.(37), one obtains

$$\left[2ik\frac{\partial}{\partial x} + 2k^2\left(1 + \frac{\nabla_\rho^2}{k^2}\right)^{1/2} + i\pi^2 k^3 \int_0^\infty \left(1 - \frac{\kappa^2}{k^2}\right)^{-1} \Phi_\varepsilon(\kappa)\kappa d\kappa\right]\Gamma_{10}(x, \bar{\rho}) = 0 \qquad (50)$$

The general solution to this equation is unknown. However, for the plane-wave case, one has that

$$\left(1 + \frac{\nabla_\rho^2}{k^2}\right)^{1/2} \Gamma_{10}(x, \bar{\rho}) = \left(1 + \frac{\nabla_\rho^2}{k^2}\right)^{1/2} \Gamma_{10}(x) = \Gamma_{10}(x) \qquad (51)$$

since the plane wave will not possess any transverse variations. In this special case, Eq.(50) becomes

$$\left[2ik\frac{\partial}{\partial x} + 2k^2 + i\pi^2 k^3 \int_0^\infty \left(1 - \frac{\kappa^2}{k^2}\right)^{-1} \Phi_\varepsilon(\kappa)\kappa d\kappa\right]\Gamma_{10}(x) = 0 \qquad (52)$$

the solution of which is

$$\Gamma_{10}(x) = \Gamma_{10}(0)\exp\left[ikx - \frac{\pi^2}{2}k^2 x \int_0^\infty \left(1 - \frac{\kappa^2}{k^2}\right)^{-1} \Phi_\varepsilon(\kappa)\kappa d\kappa\right] \qquad (53)$$

This result differs by the factor $\left(1 - \kappa^2/k^2\right)^{-1}$ from that of the parabolic equation in the paraxial approximation. Of course, the later is obtained from the former by retaining the first term in the series expansion of this factor. The presence of this factor tends to

accentuate the spatial frequencies near the value of the wave number k. Since this theory is applicable to those situations in which $\lambda \leq l_0$, most of the contribution of this factor to wave scattering will occur at the largest spatial frequencies of the inhomogeneities. Hence, in the case of atmospheric turbulence, one can employ the von Karman spectral density (i.e., the Kolmogorov spectral density modified by a lower cutoff frequency K_0)

$$\Phi_\varepsilon(\kappa) = 0.033 C_\varepsilon^2 (\kappa^2 + K_0^2)^{-11/6} \tag{54}$$

which is not bounded at the high spatial frequencies, to compare the result of Eq.(53) with that of the paraxial approximation. Here, C_ε^2 is the 'structure parameter' giving the strength of permittivity fluctuations and $K_0 \equiv 2\pi/L_0$ where L_0 is the outer scale (or the largest spatial size) of permittivity inhomogeneity. Substituting Eq.(54) into Eq.(53) and evaluating the integral using *Mathematica*[27] yields

$$\Gamma_{10}(x) = \Gamma_{10}(0) \exp\left[ikx - \left\{ (1.772 + 1.023i)k^{-5/3} + 0.3908 K_0^{-5/3} {}_2F_1\left[1,1;\frac{1}{6};-\frac{K_0^2}{k^2}\right] \right\} C_n^2 k^2 x \right] \tag{55}$$

where the structure parameters for the permittivity C_ε^2 and the associated refractive index C_n^2 are related by $C_\varepsilon^2 = 4C_n^2$. In the case of the open atmosphere, one always has within the bounds of the extended parabolic wave equation, $k >> K_0$; thus, the hypergeometric function reduces to unity and $k^{-5/3} << K_0^{-5/3}$, allowing Eq.(55) to be approximated by

$$\Gamma_{10}(x) = \Gamma_{10}(0) \exp\left[ikx - 0.3908 K_0^{-5/3} C_n^2 k^2 x \right] \tag{56}$$

which is the result of the parabolic equation in the paraxial approximation. Hence, the use of the extended parabolic wave equation only makes negligible amplitude and phase corrections to the first order moment of the wave field propagating through atmospheric turbulence. This result extends and establishes the accuracy of the parabolic equation in the paraxial approximation for the first order moment (mean field) as it applies to atmospheric turbulence at all acceptable wavelengths of application, $\lambda \leq l_0$. The next section will consider the calculation of the generalized second-order moment (generalized MCF) of the wave field from the extended parabolic equation and compare its result to that of the paraxial approximation. In addition, due to the completeness of the operator analysis, one naturally obtains expressions for the MCF along the longitudinal axis.

C. Solution for the generalized second moment

The generalized second order moment or generalized MCF of the random electric field in two planes transverse to the direction of propagation is, from Eq.(11),

$$\Gamma_{11}(x_1,\bar{\rho}_1;x_2,\bar{\rho}_2) \equiv \left\langle E(x_1;\bar{\rho}_1) E^*(x_2;\bar{\rho}_2) \right\rangle \tag{57}$$

which is a solution of Eq.(25), in this case given by

$$\left\{ \langle L_{11} \rangle - \left[1 - \langle \tilde{L}_{11} L_{11}^{-1} \rangle \right]^{-1} \langle \tilde{L}_{11} L_{11}^{-1} \tilde{L}_{11} \rangle \right\} \Gamma_{11} = 0 \tag{58}$$

where, from the definitions of Eqs.(15) and (16),

$$\langle L_{11} \rangle \equiv 2ik\frac{\partial}{\partial x_1} + 2ik\frac{\partial}{\partial x_2} + U_{\rho_1} - U^*_{\rho_2} \tag{59}$$

and

$$\tilde{L}_{11} \equiv 2k^4\left[V_{\rho_1}\tilde{\varepsilon}(x_1,\bar{\rho}_1) - V^*_{\rho_2}\tilde{\varepsilon}^*(x_2,\bar{\rho}_2) \right], \qquad \langle \tilde{L}_{11} \rangle = 0. \tag{60}$$

As with the case for the first order moment, two related approximations must be made at the outset to render the problem analytically tractable. In particular, so long as $\left\|\tilde{L}_{11}\right\| \ll 1$,

$$L_{11}^{-1} = \left[\langle L_{11} \rangle + \tilde{L}_{11} \right]^{-1} \approx \langle L_{11} \rangle^{-1}, \qquad \left[1 - \langle \tilde{L}_{11} L_{11}^{-1} \rangle \right]^{-1} \approx 1, \tag{61}$$

which allows Eq.(58) to become

$$\left\{ \langle L_{11} \rangle - \langle \tilde{L}_{11} \langle L_{11}^{-1} \rangle \tilde{L}_{11} \rangle \right\} \Gamma_{11} = 0; \tag{62}$$

At this point, it is suggested to connect the longitudinal coordinates x_1 and x_2 to the related centroid x_c and difference x_d coordinates,

$$x_c \equiv \frac{x_1 + x_2}{2}, \qquad x_d \equiv x_1 - x_2. \tag{63}$$

The operator expressions of Eqs.(59) and (60) then become

$$\langle L_{11} \rangle \equiv 2ik\frac{\partial}{\partial x_c} + U_{\rho_1} - U^*_{\rho_2}, \tag{64}$$

$$\tilde{L}_{11} \equiv 2k^4\left[V_{\rho_1}\tilde{\varepsilon}\left(x_c + \frac{x_d}{2},\bar{\rho}_1\right) - V^*_{\rho_2}\tilde{\varepsilon}^*\left(x_c - \frac{x_d}{2},\bar{\rho}_2\right) \right], \qquad \langle \tilde{L}_{11} \rangle = 0. \tag{65}$$

Hence, Eq.(62) can be written as

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$$\left[2ik\frac{\partial}{\partial x_c}+U_{\rho_1}-U_{\rho_2}^*-\left\langle\left(2k^4\left\{V_{\rho_1}\bar{\varepsilon}\left(x_c+\frac{x_d}{2},\vec{\rho}_1\right)-V_{\rho_2}^*\bar{\varepsilon}^*\left(x_c-\frac{x_d}{2},\vec{\rho}_2\right)\right\}\right)\cdot\right.\right.$$

$$\left.\left.\cdot\left\{2ik\frac{\partial}{\partial x_c}+U_{\rho_1}-U_{\rho_2}^*\right\}^{-1}\left(2k^4\left\{V_{\rho_1}\bar{\varepsilon}\left(x_c'+\frac{x_d}{2},\vec{\rho}_1\right)-V_{\rho_2}^*\bar{\varepsilon}^*\left(x_c'-\frac{x_d}{2},\vec{\rho}_2\right)\right\}\right)\right\rangle\right]\Gamma_{11}=0.$$

(66)

The solution of this equation commences with obtaining an expression for the Green function, i.e., the diffraction factor

$$\left\{2ik\frac{\partial}{\partial x_c}+U_{\rho_1}-U_{\rho_2}^*\right\}^{-1}\equiv G(x_c,\vec{\rho}_1,\vec{\rho}_2).$$

(67)

Proceeding as in the last section and using the approximation $U_\rho\approx 2k^2+\nabla_\rho^2$, this requires the solution of

$$\left\{2ik\frac{\partial}{\partial x_c}+\nabla_{\rho_1}^2-\nabla_{\rho_2}^2\right\}G(x,\vec{\rho}_1,\vec{\rho}_2)=\delta(x_c-x_c')\delta(\vec{\rho}_1-\vec{\rho}_1')\delta(\vec{\rho}_2-\vec{\rho}_2')$$

(68)

which is given by

$$G(x_c,\vec{\rho}_1,\vec{\rho}_2)=-\left(\frac{ik}{2}\right)\left(\frac{1}{2\pi}\right)^2\left(\frac{1}{x_c-x_c'}\right)^2\exp\left[ik\left((\vec{\rho}_1-\vec{\rho}_1')-(\vec{\rho}_2-\vec{\rho}_2')\right)/2(x_c-x_c')\right]$$

$$=G(x_c-x_c',\vec{\rho}_1-\vec{\rho}_1',\vec{\rho}_2-\vec{\rho}_2')$$

(69)

Therefore, the fourth term in Eq.(66) becomes

$$\langle\cdots\rangle=\left(2k^4\right)^2\int_0^x\int_{-\infty}^\infty\int_{-\infty}^\infty G(x,\vec{\rho}_1,\vec{\rho}_2;x',\vec{\rho}_1',\vec{\rho}_2')\left\langle V_{\rho_1}V_{\rho_1'}\bar{\varepsilon}\left(x_c+\frac{x_d}{2},\vec{\rho}_1\right)\bar{\varepsilon}\left(x_c'+\frac{x_d}{2},\vec{\rho}_1'\right)-\right.$$

$$-V_{\rho_1}V_{\rho_2'}^*\bar{\varepsilon}\left(x_c+\frac{x_d}{2},\vec{\rho}_1\right)\bar{\varepsilon}^*\left(x_c'-\frac{x_d}{2},\vec{\rho}_2'\right)-V_{\rho_2}^*V_{\rho_1'}\bar{\varepsilon}^*\left(x_c-\frac{x_d}{2},\vec{\rho}_2\right)\bar{\varepsilon}\left(x_c'+\frac{x_d}{2},\vec{\rho}_1'\right)+$$

$$\left.+V_{\rho_2}^*V_{\rho_2'}^*\bar{\varepsilon}^*\left(x_c-\frac{x_d}{2},\vec{\rho}_2\right)\bar{\varepsilon}^*\left(x_c'-\frac{x_d}{2},\vec{\rho}_2'\right)\right\rangle d^2\rho_1'd^2\rho_2'dx_c'$$

(70)

One now employs the Fourier-Stieltjes transform as before to represent the products of $V_\rho\varepsilon$, i.e.,

$$V_\rho \tilde{\varepsilon}\left(x_c \pm \frac{x_d}{2}, \vec{\rho}_{1,2}\right) = \frac{1}{2k^2} \int \left(1 - \frac{\kappa_{1,2}^2}{k^2}\right)^{-1/2} \exp\left(i\vec{\kappa}_{1,2} \cdot \vec{\rho}_{1,2}\right) dZ\left(x_c \pm \frac{x_d}{2}, \vec{\kappa}_{1,2}\right), \text{ etc.}, \quad (71)$$

and obtains the following relations (functional arguments have been suppressed but the correspondence to those in Eq.(70) follows)

$$\left\langle V_{\rho_1} V_{\rho_1'} \tilde{\varepsilon}\tilde{\varepsilon} \right\rangle = \left(\frac{1}{2k^2}\right)^2 \int \left(1 - \frac{\kappa_1^2}{k^2}\right)^{-1} \exp\left[i\vec{\kappa}_1 \cdot (\vec{\rho}_1 - \vec{\rho}_1')\right] F_\varepsilon(x_c - x_c', \vec{\kappa}_1) d^2\kappa_1, \quad (72a)$$

$$\left\langle V_{\rho_1} V_{\rho_2'}^* \tilde{\varepsilon}\tilde{\varepsilon}^* \right\rangle = \left(\frac{1}{2k^2}\right)^2 \int \left(1 - \frac{\kappa_1^2}{k^2}\right)^{-1} \exp\left[i\vec{\kappa}_1 \cdot (\vec{\rho}_1 - \vec{\rho}_2')\right] F_\varepsilon(x_c - x_c' + x_d, \vec{\kappa}_1) d^2\kappa_1, \quad (72b)$$

$$\left\langle V_{\rho_2}^* V_{\rho_1'} \tilde{\varepsilon}^*\tilde{\varepsilon} \right\rangle = \left(\frac{1}{2k^2}\right)^2 \int \left(1 - \frac{\kappa_2^2}{k^2}\right)^{-1} \exp\left[-i\vec{\kappa}_2 \cdot (\vec{\rho}_2 - \vec{\rho}_1')\right] F_\varepsilon(x_c - x_c' - x_d, \vec{\kappa}_2) d^2\kappa_2, \quad (72c)$$

$$\left\langle V_{\rho_2}^* V_{\rho_2'}^* \tilde{\varepsilon}^*\tilde{\varepsilon}^* \right\rangle = \left(\frac{1}{2k^2}\right)^2 \int \left(1 - \frac{\kappa_2^2}{k^2}\right)^{-1} \exp\left[-i\vec{\kappa}_2 \cdot (\vec{\rho}_2 - \vec{\rho}_2')\right] F_\varepsilon(x_c - x_c', \vec{\kappa}_2) d^2\kappa_2. \quad (72d)$$

Substituting Eqs.(69) and (72a)-(72d) into Eq.(70), performing the required integrals over $\vec{\rho}_1$ and $\vec{\rho}_2$ in plane polar coordinates, and taking all the spectra $F_\varepsilon(x, \vec{\kappa}_{1,2}) = F_\varepsilon(x, \kappa_{1,2})$, i.e., to be isotropic in the frequency $\vec{\kappa}$, one obtains

$$\langle \cdots \rangle = -i\pi k^3 \int_0^{x_c} \int_0^\infty \left(1 - \frac{\kappa^2}{k^2}\right)^{-1} \left\{ \exp\left[-i\frac{\kappa^2(x_c - x_c')}{2k}\right] F_\varepsilon(x_c - x_c', \kappa) - \right.$$
$$- \exp\left[i\frac{\kappa^2(x_c - x_c')}{2k}\right] J_0(\kappa\rho_d) F_\varepsilon(x_c - x_c' + x_d, \kappa) -$$
$$- \exp\left[-i\frac{\kappa^2(x_c - x_c')}{2k}\right] J_0(\kappa\rho_d) F_\varepsilon(x_c - x_c' - x_d, \kappa) +$$
$$\left. + \exp\left[i\frac{\kappa^2(x_c - x_c')}{2k}\right] F_\varepsilon(x_c - x_c', \kappa) \right\} \kappa \, d\kappa \, dx_c. \quad (73)$$

where $\rho_d \equiv |\vec{\rho}_2 - \vec{\rho}_1|$. Using Eq.(46b) in Eq.(73) and performing the x_c integration, remembering Eq.(49), finally gives

$$\langle \cdots \rangle = -2\pi^2 i k^3 \int_0^\infty \left(1 - \frac{\kappa^2}{k^2}\right)^{-1} \left\{ 1 - \exp\left(-\frac{i\kappa^2 x_d}{2k}\right) J_0(\kappa\rho_d) \right\} \Phi_\varepsilon(\kappa) \kappa \, d\kappa. \quad (74)$$

Hence, Eq.(66) becomes

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$$\left[2ik\frac{\partial}{\partial x_c} + U_{\rho_1} - U_{\rho_2}^* + 2\pi^2 ik^3 \int_0^\infty \left(1 - \frac{\kappa^2}{k^2}\right)^{-1} \left\{1 - \exp\left(-\frac{i\kappa^2 x_d}{2k}\right) J_0(\kappa\rho_d)\right\} \Phi_\varepsilon(\kappa)\kappa d\kappa \right]\Gamma_{11} = 0$$

(75)

where $\Gamma_{11} = \Gamma_{11}(x_c, x_d, \rho_d)$. Making the plane wave approximation of Eq.(75), analogous to that done earlier for Eq.(50), the resulting differential equation has as a solution

$$\Gamma(x_c, x_d, \rho_d) = \Gamma(0,0,\rho_d)\exp\left[-\pi^2 k^2 x_c \int_0^\infty \left(1 - \frac{\kappa^2}{k^2}\right)^{-1} \left\{1 - \exp\left(-\frac{i\kappa^2 x_d}{2k}\right) J_0(\kappa\rho_d)\right\} \Phi_\varepsilon(\kappa)\kappa d\kappa\right].$$

(76)

It is important to note the initial condition $\Gamma(0,0,\rho_d)$; since one necessarily must take $x_c = 0$, one must also have $x_d = 0$ by Eqs.(63) since $x_1, x_2 \geq 0$.

This result cannot be analytically studied in its entirety and hence will be considered in two special cases. The first case is defined by $x_d = 0$ in which one deals with the usual transverse MCF $\Gamma(x_c, \rho_d) \equiv \Gamma(x_c, 0, \rho_d)$. Thus, the extended parabolic equation solution for the MCF in a transverse plane at a distance x_c from the source is, from Eq.(76)

$$\Gamma(x_c, \rho_d) = \Gamma(0,0,\rho_d)\exp\left[-\pi^2 k^2 x_c \int_0^\infty \left(1 - \frac{\kappa^2}{k^2}\right)^{-1} \left\{1 - J_0(\kappa\rho_d)\right\} \Phi_\varepsilon(\kappa)\kappa d\kappa\right].$$

(77)

As with the case of the first-order moment, this result differs from that of the paraxial approximation in the presence of the factor $\left(1 - \kappa^2/k^2\right)^{-1}$ which serves to accentuate spectral contributions for frequencies near k. The integral indicated in this expression cannot be analytically evaluated using the von Karman spectrum, Eq.(54). However, since the integrand is such that no singularities exist in the use of the unbounded Kolmogorov spectrum, viz., Eq.(54) with $K_0 = 0$, one can use such a spectrum in Eq.(77) and, upon evaluating the integral via *Mathematica*, obtain

$$\Gamma_{11}(x_c, \rho_d) = \Gamma_{11}(0, \rho_d)\exp\left[-\left\{(3.544 - 2.048i)k^{-5/3}\left(1 - J_0(k\rho_d)\right) \right.\right.$$
$$\left.\left. + 0.108k^2\rho^{1/3}{}_1F_2\left(1; \frac{17}{6}, \frac{17}{6}; -\frac{k^2\rho_d^2}{4}\right)\right\}k^2 C_n^2 x_c\right]$$

(78)

where ${}_1F_2(\cdots)$ is a generalized hypergeometric function and, as noted earlier, $C_\varepsilon^2 = 4C_n^2$. At the outset, since $\lambda \ll \rho_d$ in most applications, the first term within the braces of Eq.(78), although quite interesting in structure, is negligible with respect to the second

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term. The hypergeometric function of the second term is most easily dealt with by first converting it to a Lommel function[28] $s_{\mu,0}(\cdots)$, i.e.,

$$_1F_2\left(1;\frac{17}{6},\frac{17}{6};-\frac{k^2\rho_d^2}{4}\right)=\left(\frac{11}{3}\right)^2(k\rho_d)^{-11/3}s_{8/3,0}(k\rho_d). \quad (79)$$

The Lommel function reduces[28], in the case where $k\rho_d \gg 1$, to the simple approximate result $s_{8/3,0}(k\rho_d)\approx(k\rho_d)^{5/3}$. Using this in Eq.(79), Eq.(78) becomes the well-known paraxial result

$$\Gamma_{11}(x_c,\rho_d)=\Gamma_{11}(0,\rho_d)\exp\left[-1.457k^2C_n^2\rho_d^{5/3}x_c\right]. \quad (80)$$

Again, as with the first-order moment, corrections to the second moment afforded by the extended parabolic wave equation are negligible in the case of atmospheric turbulence.

The second special case in which Eq.(76) will be examined is in the instance where the factor $\left(1-\kappa^2/k^2\right)^{-1}$ can be neglected; as shown above, this is a good approximation for atmospheric turbulence. Thus, one is now dealing with the solution for the generalized MCF for field locations at different transverse and longitudinal points,

$$\Gamma(x_c,x_d,\rho_d)=\Gamma(0,0,\rho_d)\exp\left[-\pi^2k^2x_c\int_0^\infty\left\{1-\exp\left(-\frac{i\kappa^2x_d}{2k}\right)J_0(\kappa\rho_d)\right\}\Phi_\varepsilon(\kappa)\kappa d\kappa\right]. \quad (81)$$

Longitudinal correlations of the wave field have been previously considered[29,30] using different methods and obtaining different results. Setting $x_d=0$ in Eq.(81) gives the well known paraxial result for the transverse MCF $\Gamma(x_c,0,\rho_d)$. Unlike the cases studied above, the form of Eq.(81) suggests the use of the Kolmogorov spectrum as modified by Tatarskii, which incorporates a cutoff at high spatial frequencies by allowing the introduction of the inner scale of turbulence l_0, viz.,

$$\Phi_\varepsilon(\kappa)=0.033C_\varepsilon^2\kappa^{-11/3}\exp\left(-\frac{\kappa^2}{\kappa_m^2}\right),\qquad \kappa_m\equiv\frac{5.92}{l_0} \quad (82)$$

Substituting Eq.(82) into Eq.(81) and evaluating the resulting integral yields for the generalized MCF

$$\Gamma(x_c,x_d,\rho_d)=\Gamma(0,0,\rho_d)\exp\left[-4.352k^2C_n^2x_c\left\{B^{5/6}\,_1F_1\left(-\frac{5}{6};1;-\frac{\rho_d^2}{4B}\right)-\kappa_m^{-5/3}\right\}\right] \quad (83)$$

where

$$B \approx \frac{1}{\kappa_m^2} + i\frac{x_d}{2k} \tag{84}$$

Thus, as previously noted[30], the presence of the diffraction factor on Eq.(81) modifies the effect of the cutoff frequency κ_m. In the case where $\rho_d > \left|1/\kappa_m^2 + i\,x_d/2k\right|^{1/2}$, Eq.(83) reduces to Eq. (80) upon employing the asymptotic representation of the confluent hypergeometric function $_1F_1(\cdots)$. When $\rho_d = 0$, Eq.(83) becomes

$$\Gamma(x_c, x_d, 0) = \Gamma(0, 0, \rho_d)\exp\left[-4.352k^2 C_n^2 x_c \kappa_m^{-7/3}\left\{\left(1 + \frac{ix_d\kappa_m^2}{2k}\right)^{5/6} - 1\right\}\right] \tag{85}$$

In the appropriate limits, this expression gives

$$\Gamma(x_c, x_d, 0) = \Gamma(0, 0, \rho_d)\begin{cases}\exp\left[-1.8133\,ikC_n^2\kappa_m^{1/3}x_c x_d\right], & x_d\lambda \ll l_0^2 \\[12pt] \exp\left[-(0.5631 + 2.102i)k^{7/6}C_n^2 x_c x_d^{5/6}\right], & x_d\lambda \gg l_0^2\end{cases} \tag{86}$$

Hence, there is a phase variation in the longitudinal direction, as expected, with an attendant attenuation as the longitudinal separation is increased. It must be noted, however, that one needs to realize the condition $x_d > l_0$ in all cases so as to satisfy the assumption of δ-correlation of the fluctuations along the longitudinal axis[30].

IV. SUMMARY AND EXTENSIONS OF THE FOREGOING

A general operator equation for the moments of an electromagnetic wave field propagating through a random medium as described by the extended parabolic wave equation, i.e., Eq.(5), has been derived and is given by Eq.(25). Unlike the usual parabolic equation approximation in the paraxial approximation which holds only for $\lambda \ll l_0$, the extended version of this equation holds for cases up to $\lambda \leq l_0$; this limitation is dictated by the neglect of the backscattered field since the extended parabolic wave equation describes scattering at angles up to $\pi/2$ about the preferred direction of propagation. An example application of this theory was chosen to be that of the scenario of propagation through atmospheric turbulence in which the first and generalized second order statistical moments are given by Eqs.(50) and (75), respectively. In the course of deriving these equations, use of the paraxial approximation is avoided at the outset, allowing one to be able to include diffractive effects within the scattering terms of these equations. These relations, which assume a δ-function of permittivity fluctuations in the preferred direction of propagation (i.e., the Markov approximation) are then solved in the plane wave case yielding Eqs.(53) and (76) in terms of the spectrum $\Phi_\varepsilon(\kappa)$ of homogeneous and isotropic fluctuations. Kolmogorov type spectra (i.e., spectra $\sim \kappa^{-11/3}$) are employed to obtain analytical expressions for the first and generalized second

moments. In this atmospheric example, this extended theory showed that the paraxial approximation holds even for millimeter wavelengths in which $\lambda \sim l_0$. This is due to the fact that the spectra which is used to represent atmospheric turbulence (all variations of the Kolmogorov spectrum for atmospheric turbulent fluctuations) is such that the contribution of the spectral frequencies, which approximately correspond to millimeter wavelengths, is relatively small as compared to those at the smaller spatial frequencies (which correspond the outer scale of turbulence). This gives rise to the insignificant levels of the obtained correction terms at nominal operating wavelengths. However, the novel feature of this application example is the description of the longitudinal second moment which naturally follows from the comprehensive operator solution and the incorporation of diffraction within the scattering term. A solution for this quantity, given by Eq.(83), is discussed as well as its limiting forms.

The operator method presented here from which the statistical moments are obtained is general enough to allow the use of assumptions less restrictive than the Markov approximation. The use of fluctuation spectra that do not decay as rapidly as that of the Kolmogorov case (e.g., that of a turbid medium or some turbulent plasmas) will give rise to non-negligible correction terms for the first and second moments of the wave field.

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