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Plasma Dispersion Function for the Kappa Distribution

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Abstract. The plasma dispersion function is computed for a homogeneous isotropic plasma in which the particle velocities are distributed according to a Kappa distribution. An ordinary differential equation is derived for the plasma dispersion function and it is shown that the solution can be written in terms of Gauss' hypergeometric function in the form $Z_\kappa(\xi) = i\kappa^{-3/2}(\kappa-1/2)F(1, 2\kappa, \kappa+1, t)$, where $t = (1+i\xi/\sqrt{\kappa})/2$. Using the extensive theory of the hypergeometric function, various mathematical properties of the plasma dispersion function are derived including symmetry relations, series expansions, integral representations, and closed form expressions for integer and half-integer values of κ.

1 Introduction

Particle distribution functions with power law tails are frequently observed in space plasmas throughout the solar system. In practice, they are often modeled using Kappa distributions (defined below). The existence of such nonthermal, non-Maxwellian distribution functions can have profound effects on wave propagation and transport processes in these plasmas; effects that are of significant interest in space physics. The kinetic theory of such processes depends in an essential way on the plasma dielectric function and, therefore, on the plasma dispersion function.

The purpose of this paper is to derive the plasma dispersion function for the Kappa distribution in a new way and to investigate its mathematical properties. This function was derived independently by Summers and Thorne[1] and by Chateau and Meyer-Vernet[2] for the special case where $\kappa = 1, 2, 3, \ldots$ It was later derived in the more general case $\kappa > 0$ by Mace and Hellberg.[3] These authors all used the technique of contour integration. The approach taken here is different, being based instead on differential equations.

In order to be as consistent as possible with the formulation of the classical plasma dispersion function, the definition given here is slightly different from that of Summers and Thorne[1] and Mace and Hellberg.[3] To clarify the reasons for doing this, the definition of the plasma dispersion function is briefly derived from first principles. The resulting dispersion relation for plasma waves, equation (29), is somewhat simplified in the present formulation.

Section 2 gives a concise list of the main results of this paper. The definition of the Kappa distribution is given in Section 3, and the definition of the plasma dispersion function is given in Section 4. In Section 5, the differential equation for the plasma dispersion function is derived and the solution is expressed in terms of Gauss' hypergeometric function. Symmetry relations and series expansions are derived in Sections 6 and 7. Integral representations are derived in Sections 8 and 9. Closed form expressions for integer and half-integer values of κ are derived in Sections 9 and 10, respectively.

2 Summary of Results

The important properties of the plasma dispersion function are listed here for easy reference. These properties are investigated in detail in the sections that follow. Throughout this paper the power function z^α, $\alpha \in \mathbb{R}$, is defined by its principal branch, that is, $z^\alpha = r^\alpha \exp(i\alpha\theta)$, where $z = r\exp(i\theta)$, $r \geq 0$, and $-\pi < \theta \leq \pi$.

Plasma Dispersion Function

$$Z_\kappa(\xi) = \frac{i(\kappa - \frac{1}{2})}{\kappa^{3/2}} F\left[1, 2\kappa, \kappa+1, \frac{1}{2}\left(1 + \frac{i\xi}{\sqrt{\kappa}}\right)\right], \quad \kappa > 0. \tag{1}$$

If $\kappa = n$ is an integer, this function has a single pole of order n at the point $\xi = -i\sqrt{\kappa}$. Otherwise, it is analytic with a branch cut from $\xi = -i\sqrt{\kappa}$ to $-i\infty$ along the negative imaginary axis.

Behavior Near $\boldsymbol{\xi = -i\sqrt{\kappa}}$

$$Z_\kappa(\xi) \sim \frac{i\pi^{1/2}}{2^{\kappa-1}\sqrt{\kappa}} \frac{\Gamma(\kappa)}{\Gamma(\kappa - \frac{1}{2})} \left(1 - \frac{i\xi}{\sqrt{\kappa}}\right)^{-\kappa} \quad \text{as} \quad \xi \to -i\sqrt{\kappa}. \tag{2}$$

Symmetry Relations

$$Z_\kappa(-\xi^*) = -\left[Z_\kappa(\xi)\right]^*. \tag{3}$$

$$Z_\kappa(\xi) + Z_\kappa(-\xi) = \frac{2i\sqrt{\pi}}{\kappa^{1/2}} \frac{\Gamma(\kappa)}{\Gamma(\kappa - 1/2)} \left(1 + \frac{\xi^2}{\kappa}\right)^{-\kappa}. \tag{4}$$

Integral Representations

$$Z_\kappa(\xi) = \frac{1}{\sqrt{\pi\kappa}} \frac{\Gamma(\kappa)}{\Gamma(\kappa - \frac{1}{2})} \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^\kappa} \cdot \frac{1}{(x-z)}\, dx, \quad \mathrm{Im}(z) > 0, \tag{5}$$

where $z = \xi/\sqrt{\kappa}$,

$$Z_\kappa(\xi) = -\frac{2\kappa - 1}{\kappa^{1/2}} \left(1 + z^2\right)^{-\kappa} \int_i^z \left(1 + t^2\right)^{\kappa - 1} dt, \quad z \neq -i. \tag{6}$$

Differential Equation

$$(\xi^2 + \kappa) Z_\kappa'' + 2(\kappa + 1)\xi Z_\kappa' + 2\kappa Z_\kappa = 0, \tag{7}$$

$$Z_\kappa(i\sqrt{\kappa}) = i\, \frac{\kappa - 1/2}{\kappa^{3/2}}, \qquad Z_\kappa'(i\sqrt{\kappa}) = -\frac{\kappa - 1/2}{\kappa(\kappa + 1)}. \tag{8}$$

Differentiation Formula

$$\frac{dZ_\kappa}{d\xi} = -\frac{2\kappa - 1}{\kappa} \left[1 + \left(\frac{\kappa + 1}{\kappa}\right)^{1/2} \xi Z_{\kappa+1}(\xi)\right]. \tag{9}$$

Closed Form Expression for $\boldsymbol{\kappa = n = 1, 2, 3 \ldots}$

$$Z_n(\xi) = i \frac{(n - \frac{1}{2})}{n^{1/2}} \frac{\Gamma(n)}{\Gamma(2n)} \sum_{m=0}^{n-1} \frac{\Gamma(m + n)}{m!} \left(\frac{1 - iz}{2}\right)^{m-n}, \tag{10}$$

where $z = \xi/\sqrt{n}$.

Closed Form Expression for $\boldsymbol{\kappa = n + \frac{1}{2}}$

$$\begin{aligned} Z_\kappa(\xi) = \frac{1}{(n + \frac{1}{2})^{1/2}} \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n)} \Bigg\{ & \sum_{m=1}^{n} \frac{\Gamma(n + 1 - m)}{\Gamma(n + \frac{3}{2} - m)} z(1 + z^2)^{-m} \\ & - \frac{2}{\sqrt{\pi}} (1 + z^2)^{-n-1/2} \left[\log\left(z + \sqrt{z^2 + 1}\right) - \frac{\pi i}{2}\right] \Bigg\}, \end{aligned} \tag{11}$$

where $z = \xi/\sqrt{\kappa}$ and $n = 1, 2, 3, \ldots$

Series Expansion for $|z| < 1$

$$
\begin{aligned}
Z_\kappa(\xi) = {} & i\frac{\sqrt{\pi}}{\kappa^{1/2}}\frac{\Gamma(\kappa)}{\Gamma(\kappa-\frac{1}{2})}(1+z^2)^{-\kappa} \\
& - \frac{2\kappa-1}{\kappa^{1/2}} z\left\{1 - \left(\frac{2\kappa+1}{3}\right)z^2 + \left(\frac{2\kappa+1}{3}\right)\left(\frac{2\kappa+3}{5}\right)z^4 - \cdots\right\}, \quad (12)
\end{aligned}
$$

where $z = \xi/\sqrt{\kappa}$.

Series Expansion for $|z| > 1$

$$
\begin{aligned}
Z_\kappa(\xi) = {} & \frac{i\sqrt{\pi}}{\sqrt{\kappa}}\frac{\Gamma(\kappa)}{\Gamma(\kappa-\frac{1}{2})}\Big\{1 + i\tan(\pi\kappa)\mathrm{sgn}[\mathrm{Re}(z)]\Big\}(z^2)^{-\kappa}\left(1+\frac{1}{z^2}\right)^{-\kappa} \\
& - \frac{1}{\sqrt{\kappa}}\frac{1}{z}\left\{1 + \left(\frac{1}{2\kappa-3}\right)\frac{1}{z^2} + \left(\frac{1}{2\kappa-3}\right)\left(\frac{3}{2\kappa-5}\right)\frac{1}{z^4} + \cdots\right\}, \quad (13)
\end{aligned}
$$

where $z = \xi/\sqrt{\kappa}$, $\kappa \neq \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, and the function $\mathrm{sgn}[\mathrm{Re}(z)]$ is defined by equation (81).

3 Kappa Distribution

In the astrophysics and space physics literature, the isotropic Kappa distribution in three dimensions is defined by

$$
f_\kappa(\mathbf{v}) = A_\kappa\left(1 + \frac{v^2}{\kappa v_0^2}\right)^{-(\kappa+1)}, \quad \kappa > 1/2, \tag{14}
$$

where $\mathbf{v} = (v_x, v_y, v_z)$ is the velocity vector, $v = |\mathbf{v}|$, and v_0 is some characteristic velocity. The normalization constant is

$$
A_\kappa = \frac{1}{(\pi\kappa v_0^2)^{3/2}} \cdot \frac{\Gamma(\kappa+1)}{\Gamma(\kappa - 1/2)} = \frac{1}{(\pi v_0^2)^{3/2}\sqrt{\kappa}} \cdot \frac{\Gamma(\kappa)}{\Gamma(\kappa-1/2)} \tag{15}
$$

and is chosen such that

$$
4\pi\int_0^\infty f_\kappa(v)v^2\,dv = 1. \tag{16}
$$

The condition $\kappa > 1/2$ is necessary for this integral to converge. The Kappa distribution is closely related to the Beta distribution. In fact, the Beta function arises when computing the normalization factor A_κ and the statistical moments. In the limit as $\kappa \to \infty$, the Kappa distribution approaches the well known Maxwell distribution e^{-v^2/v_0^2}.

The velocity moments of the Kappa distribution are given by

$$\langle v^n \rangle = 4\pi \int_0^\infty f_\kappa(v) v^{n+2}\, dv = \frac{2}{\sqrt{\pi}} (\kappa v_0^2)^{n/2} \frac{\Gamma(\frac{n+3}{2})\Gamma(\kappa - \frac{n+1}{2})}{\Gamma(\kappa - \frac{1}{2})}, \tag{17}$$

where n is an integer and $0 \leq n \leq 2(\kappa - 1)$. For an arbitrary real power $n \geq 0$, this integral is finite if and only if $n < 2\kappa - 1$. In applications, the equivalent temperature of an isotropic Kappa distribution is defined by the relation

$$\frac{1}{2} m \left\langle v^2 \right\rangle = \frac{3}{2} k_B T \tag{18}$$

which implies

$$k_B T = \frac{1}{2} m v_0^2 \left(\frac{\kappa}{\kappa - 3/2} \right), \tag{19}$$

where k_B is Boltzmann's constant. The second moment $\langle v^2 \rangle$ is finite if and only if $\kappa > 3/2$.

4 Plasma Dispersion Function

In the Vlasov theory of plasma waves[4] the propagation of electrostatic waves (having $\mathbf{E} \parallel \mathbf{k}$) in a homogeneous isotropic plasma is governed by the dispersion relation $\varepsilon_L(\mathbf{k}, s) = 0$, where

$$\varepsilon_L(\mathbf{k}, s) = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{k^2} \int_{-\infty}^{\infty} \frac{\partial F_{\alpha 0}/\partial u}{u - is/|k|}\, du \tag{20}$$

is the longitudinal dielectric function and SI units are used throughout. The index α denotes the different particle species, $\omega_{p\alpha}$ is the plasma frequency for species α, $\mathbf{k}$ is the Fourier wavevector with magnitude $k = |\mathbf{k}|$, s is the Laplace transform variable, and $F_{\alpha 0}$ is related to the particle distribution function $f_{\alpha 0}$ by

$$F_{\alpha 0}(u) = \int f_{\alpha 0}(\mathbf{v}) \delta\Big(u - \frac{\mathbf{k} \cdot \mathbf{v}}{|k|}\Big)\, d\mathbf{v}. \tag{21}$$

The distribution function $f_{\alpha 0}(\mathbf{v})$ is normalized to unity.

Spatial isotropy implies $f_0(\mathbf{v}) = f_0(v^2)$, where $v = |\mathbf{v}|$. After a rotation of coordinates which aligns the v_z direction with $\mathbf{k}$, the last equation becomes

$$F_{\alpha 0}(u) = \int\int f_{\alpha 0}(v_x^2 + v_y^2 + u^2)\, dv_x dv_y \tag{22}$$

$$= 2\pi \int_{|u|}^{\infty} f_{\alpha 0}(v^2) v\, dv \tag{23}$$

and, therefore,

$$\frac{\partial F_{\alpha 0}}{\partial u} = -2\pi u f_{\alpha 0}(u^2). \tag{24}$$

It is usually the case that $f_{\alpha 0}$ has the functional form $f_{\alpha 0}(v/v_{\alpha 0})$, where $v_{\alpha 0}$ is some characteristic velocity; consequently, $F_{\alpha 0} = F_{\alpha 0}(u/v_{\alpha 0})$. Changing to normalized velocity variables:

$$u' = \frac{u}{v_{\alpha 0}}, \qquad \tilde{F}(u') = v_{\alpha 0} F(u'), \qquad \frac{\partial F}{\partial u} = \frac{1}{v_{\alpha 0}^2}\frac{\partial \tilde{F}}{\partial u'}, \tag{25}$$

the dispersion relation (20) can be written

$$1 - \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2 v_{\alpha 0}^2} Z'\Big(\alpha, \frac{is}{|k| v_{\alpha 0}}\Big) = 0, \tag{26}$$

where, after dropping the primes on u,

$$Z'(\alpha, \xi) = \int_{-\infty}^{\infty} \frac{\partial \tilde{F}_{\alpha 0}/\partial u}{u - \xi}\, du = \int_{-\infty}^{\infty} \frac{\tilde{F}_{\alpha 0}(u)}{(u-\xi)^2}\, du = \frac{d}{d\xi}\int_{-\infty}^{\infty} \frac{\tilde{F}_{\alpha 0}(u)}{u - \xi}\, du \tag{27}$$

and, by definition,

$$Z(\alpha, \xi) = \int_{-\infty}^{\infty} \frac{\tilde{F}_{\alpha 0}(u)}{u - \xi}\, du, \qquad \mathrm{Im}(\xi) > 0, \tag{28}$$

is the *plasma dispersion function*. It is defined for $\mathrm{Im}(\xi) \leq 0$ by analytic continuation. Introducing the complex frequency $\omega = is$ which is consistent with Fourier modes of the form $e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$, the plasma dispersion relation takes the standard form[5]

$$1 - \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2 v_{\alpha 0}^2} Z'\Big(\frac{\omega}{|k| v_{\alpha 0}}\Big) = 0. \tag{29}$$

If $f_{\alpha 0}$ is a Maxwell distribution

$$f_{\alpha 0}(v) = \frac{1}{(\pi v_0^2)^{3/2}} e^{-v^2/v_0^2}, \tag{30}$$

then the evaluation of (23) and (28) yields the well known plasma dispersion function[6]

$$Z(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-u^2}}{u - \xi}\, du, \qquad \mathrm{Im}(\xi) > 0. \tag{31}$$

If $f_{\alpha 0}$ is a Kappa distribution (14), then the evaluation of (23) and (28) yields

$$F_{\alpha 0}(u) = \frac{1}{\sqrt{\pi \kappa} v_0} \cdot \frac{\Gamma(\kappa)}{\Gamma(\kappa - \frac{1}{2})} \cdot \frac{1}{(1 + u^2/\kappa v_0^2)^{\kappa}} \tag{32}$$

and

$$Z_\kappa(\xi) = \frac{1}{\sqrt{\pi \kappa}} \cdot \frac{\Gamma(\kappa)}{\Gamma(\kappa - \frac{1}{2})} \int_{-\infty}^{\infty} \frac{1}{(u - \xi)\,(1 + u^2/\kappa)^{\kappa}}\, du. \tag{33}$$

This is the dispersion function for the isotropic Kappa distribution. It generalizes the well known result (31) for the isotropic Maxwell distribution. It is different from the definition of Summers and Thorne[1] which was also adopted by Mace and Hellberg.[3] These authors employ an exponent $\kappa + 1$ rather than κ in the integrand. Consequently, the function $Z_\kappa(\xi)$ of Summers and Thorne is more closely related to the derivative $Z'_\kappa(\xi)$ of the function defined here.

5 Representation by Gauss' Hypergeometric Function

For purposes of mathematical manipulation, it is convenient to let $x = u/\sqrt{\kappa}$ in equation (33) and write

$$Z_\kappa(\xi) = \frac{1}{\sqrt{\pi \kappa}} \frac{\Gamma(\kappa)}{\Gamma(\kappa - \frac{1}{2})} \int_{-\infty}^{\infty} \frac{1}{(1 + x^2)^{\kappa}} \cdot \frac{1}{(x - z)}\, dx, \quad \mathrm{Im}(z) > 0, \tag{34}$$

where $z = \xi/\sqrt{\kappa}$. Therefore, the calculation of the plasma dispersion function reduces to the evaluation of the integral

$$I = \int_{-\infty}^{\infty} \frac{1}{(1 + x^2)^{\kappa}} \cdot \frac{1}{(x - z)}\, dx, \qquad \mathrm{Im}(z) > 0. \tag{35}$$

It easy to see that this integral converges for all $\kappa > 0$. There are two cases to consider: the case where $\kappa > 0$ is an integer, and the case when $\kappa > 0$ is not an integer. The solution in the first case was derived independently by Chateau and Meyer-Vernet[2] and by Summers and Thorne.[1] The solution in the second case was derived by Mace and Hellberg.[3] As will now be shown, the solution for arbitrary $\kappa > 0$ is easily obtained by first finding the differential equation for $I(z)$. This method of approach is an alternative to the method of contour integration used by the above authors.

Differentiating equation (35) with respect to z yields

$$I'(z) = \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^\kappa} \cdot \frac{1}{(x-z)^2}\, dx. \tag{36}$$

Multiplying this by z and then using the identity

$$1 + \frac{z}{x-z} = \frac{x}{x-z}, \tag{37}$$

it follows that

$$zI' + I = \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^\kappa} \cdot \frac{x}{(x-z)^2}\, dx. \tag{38}$$

Differentiating this equation with respect to z, multiplying the result by z, and then using the identity (37) yields

$$z^2 I'' + 4zI' + 2I = \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^\kappa} \cdot \frac{2x^2}{(x-z)^3}\, dx. \tag{39}$$

Now differentiate (36) and add the result to the last equation to obtain

$$(1+z^2)I'' + 4zI' + 2I = \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{\kappa-1}} \cdot \frac{2}{(x-z)^3}\, dx. \tag{40}$$

Integrating the remaining integral by parts and comparing the result with equation (38), the right hand side of (40) equals

$$-2(\kappa-1) \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^\kappa} \cdot \frac{x}{(x-z)^2}\, dx = -2(\kappa-1)(zI' + I). \tag{41}$$

Thus, the differential equation for $I(z)$ is

$$(1+z^2)I'' + 2(\kappa+1)zI' + 2\kappa I = 0. \tag{42}$$

To obtain a unique solution, it is necessary to impose the initial conditions

$$I(z=i) = I_0, \tag{43}$$

$$I'(z=i) = \frac{2\kappa}{\kappa+1}\frac{iI_0}{2}, \tag{44}$$

where I_0 is evaluated in Appendix A with the result

$$I_0 = i\sqrt{\pi}\frac{\Gamma(\kappa+\frac{1}{2})}{\Gamma(\kappa+1)}. \tag{45}$$

Making the transformation

$$t = \frac{1+iz}{2} \tag{46}$$

the differential equation (42) takes the form

$$t(1-t)I'' + (\kappa+1)(1-2t)I' - 2\kappa I = 0, \tag{47}$$

where

$$I(t=0) = I_0, \tag{48}$$

$$I'(t=0) = \frac{2\kappa}{\kappa+1}I_0. \tag{49}$$

Equation (47) is now in the standard form of the well known hypergeometric equation.

Recall that the solution of the hypergeometric equation

$$z(1-z)F'' + [c-(a+b+1)z]F' - abF = 0 \tag{50}$$

with initial conditions

$$F(0) = 1, \tag{51}$$

$$F'(0) = \frac{ab}{c} \tag{52}$$

is given by Gauss' hypergeometric function $F(a,b,c,z) = {}_2F_1(a,b,c,z)$. This is a single valued analytic function of z in the complex plane with a branch cut from 1 to ∞ along the positive real axis. For fixed z it is analytic in a,

b, and c separately with the exception of simple poles at $c = 0, -1, -2, \dots$ In the unit disk it has the series representation

$$F(a, b, c, z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots, \quad |z| < 1, \tag{53}$$

which also shows that it is symmetric in a and b. The series terminates if a or $b \in \{0, -1, -2, \dots\}$, in which case $F(a, b, c, z)$ is entire because it reduces to a polynomial in z. The properties of the function F have been thoroughly studied and documented. Therefore, for all intents and purposes it can be considered to be a closed form function in the same way that $\sin(z)$ and $\cos(z)$ are.

Choosing

$$a = 1, \qquad b = 2\kappa, \qquad c = \kappa + 1, \tag{54}$$

equation (47) is equivalent to the hypergeometric equation (50); hence, the unique solution of (47) is given by

$$I(t) = i\sqrt{\pi}\frac{\Gamma(\kappa + \frac{1}{2})}{\Gamma(\kappa + 1)} F(1, 2\kappa, \kappa + 1, t). \tag{55}$$

Using (46), a closed form expression for the integral (35) is

$$I(z) = i\sqrt{\pi}\frac{\Gamma(\kappa + \frac{1}{2})}{\Gamma(\kappa + 1)} F\left(1, 2\kappa, \kappa + 1, \frac{1 + iz}{2}\right), \tag{56}$$

which is valid for all $\kappa > 0$, in fact, for $\mathrm{Re}(\kappa) > 0$. Substituting this into (34), the plasma dispersion function is found to be

$$Z_\kappa(\xi) = \frac{i(\kappa - \frac{1}{2})}{\kappa^{3/2}} F(1, 2\kappa, \kappa + 1, t), \tag{57}$$

where

$$t = \frac{1 + iz}{2} = \frac{1}{2}\left(1 + \frac{i\xi}{\sqrt{\kappa}}\right). \tag{58}$$

It is important to note that the function $Z_\kappa(\xi)$ inherits all the properties of $F(a, b, c, z)$. Because $F(a, b, c, z)$ has a branch cut from 1 to ∞, the function $Z_\kappa(\xi)$ is single valued analytic with a branch cut from $\xi = -i\sqrt{\kappa}$ to $\xi = -i\infty$ along the negative imaginary axis. Here, the ξ-plane is the complex frequency plane, $\xi = is/|k|v_0$, and the lower half plane consists of all the damped modes.

The behavior in the neighborhood of the singularity $\xi = -i\sqrt{\kappa}$ can be obtained from the identity[7]

$$F(a,b,c,z) = (1-z)^{c-a-b} F(c-a, c-b, c, z). \tag{59}$$

This yields

$$F\Big(1, 2\kappa, \kappa+1, \frac{1+iz}{2}\Big) = \Big(\frac{1-iz}{2}\Big)^{-\kappa} F\Big(\kappa, 1-\kappa, \kappa+1, \frac{1+iz}{2}\Big). \tag{60}$$

Using the formula[7]

$$F(a,b,c,1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad c > a+b, \tag{61}$$

it follows that

$$\lim_{z\to -i} \Big(\frac{1-iz}{2}\Big)^{\kappa} F\Big(1, 2\kappa, \kappa+1, \frac{1+iz}{2}\Big) = \frac{\Gamma(\kappa+1)\Gamma(\kappa)}{\Gamma(1)\Gamma(2\kappa)} = \frac{\pi^{1/2}\Gamma(\kappa+1)}{2^{2\kappa-1}\Gamma(\kappa+\frac{1}{2})}. \tag{62}$$

Therefore, one obtains the asymptotic relation

$$Z_\kappa(\xi) \sim \frac{i\pi^{1/2}}{2^{\kappa-1}\sqrt{\kappa}} \frac{\Gamma(\kappa)}{\Gamma(\kappa-\frac{1}{2})} \Big(1 - \frac{i\xi}{\sqrt{\kappa}}\Big)^{-\kappa} \quad \text{as} \quad \xi \to -i\sqrt{\kappa}. \tag{63}$$

6 Symmetry Relations and Taylor Expansion

Using the well established theory of the hypergeometric function, the properties of $Z_\kappa(\xi)$ may be derived with little effort. From the series (53), it is obvious that for a, b, and c real,

$$F(a,b,c,z^*) = \big[F(a,b,c,z)\big]^*. \tag{64}$$

Therefore, replacing ξ with $-\xi^*$ in equations (57) and (58), it follows that

$$Z_\kappa(-\xi^*) = -\big[Z_\kappa(\xi)\big]^*. \tag{65}$$

This relates the values of $Z_\kappa(\xi)$ in the left half-plane to those in the right half-plane. The classical dispersion function for the Maxwell distribution has the same property.

From the identity[8,9]

$$F\Big(2a, 2b, a+b+\tfrac{1}{2}, \frac{1+z}{2}\Big) = \frac{\Gamma(a+b+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(a+\frac{1}{2})\Gamma(b+\frac{1}{2})} F\big(a, b, \tfrac{1}{2}, z^2\big) \\ - z\frac{\Gamma(a+b+\frac{1}{2})\Gamma(-\frac{1}{2})}{\Gamma(a)\Gamma(b)} F\big(a+\tfrac{1}{2}, b+\tfrac{1}{2}, \tfrac{3}{2}, z^2\big) \quad (66)$$

one has

$$F\Big(1, 2\kappa, \kappa+1, \frac{1+iz}{2}\Big) = \sqrt{\pi}\frac{\Gamma(\kappa+1)}{\Gamma(\kappa+\frac{1}{2})} F\big(\tfrac{1}{2}, \kappa, \tfrac{1}{2}, -z^2\big) \\ + 2\kappa iz F\big(1, \kappa+\tfrac{1}{2}, \tfrac{3}{2}, -z^2\big). \quad (67)$$

Substituting the closed form expression[7]

$$F(a, b, b, z) = F(b, a, b, z) = (1-z)^{-a}, \quad (68)$$

equation (67) takes the form

$$F\Big(1, 2\kappa, \kappa+1, \frac{1+iz}{2}\Big) = \sqrt{\pi}\frac{\Gamma(\kappa+1)}{\Gamma(\kappa+\frac{1}{2})}(1+z^2)^{-\kappa} + 2\kappa iz F\big(1, \kappa+\tfrac{1}{2}, \tfrac{3}{2}, -z^2\big). \quad (69)$$

Multiplying this equation by the leading coefficient in equation (57) yields

$$Z_\kappa(\xi) = \frac{i\sqrt{\pi}}{\kappa^{1/2}}\frac{\Gamma(\kappa)}{\Gamma(\kappa-\frac{1}{2})}(1+z^2)^{-\kappa} - \frac{2\kappa-1}{\kappa^{1/2}} z F\big(1, \kappa+\tfrac{1}{2}, \tfrac{3}{2}, -z^2\big), \quad (70)$$

where $z = \xi/\sqrt{\kappa}$. Changing ξ to $-\xi$ (or z to $-z$) and adding the result to the last equation yields the symmetry relation

$$Z_\kappa(\xi) + Z_\kappa(-\xi) = \frac{2i\sqrt{\pi}}{\kappa^{1/2}}\frac{\Gamma(\kappa)}{\Gamma(\kappa-\frac{1}{2})}\left(1+\frac{\xi^2}{\kappa}\right)^{-\kappa}. \quad (71)$$

This relates the values of $Z_\kappa(\xi)$ in the upper half-plane to those in the lower half-plane. In the limit as $\kappa \to \infty$ one recovers the symmetry relation

$$Z(\xi) + Z(-\xi) = 2i\sqrt{\pi}e^{-\xi^2} \quad (72)$$

for the Maxwellian dispersion function.

Using the series expansion (53) for $F(a,b,c,z)$, equation (70) immediately yields the following series expansion valid for $|z|<1$:

$$Z_\kappa(\xi) = i\frac{\sqrt{\pi}}{\kappa^{1/2}}\frac{\Gamma(\kappa)}{\Gamma(\kappa-\frac{1}{2})}(1+z^2)^{-\kappa} \\ -\frac{2\kappa-1}{\kappa^{1/2}}z\left\{1+\left(\frac{2\kappa+1}{3}\right)(-z^2)+\left(\frac{2\kappa+1}{3}\right)\left(\frac{2\kappa+3}{5}\right)(-z^2)^2+\cdots\right\}. \tag{73}$$

Here, the even powers are neatly summed into the first term. If necessary, the first term can be expanded using the binomial theorem. The series expansion can also be written in the more compact form

$$Z_\kappa(\xi) = i\frac{\sqrt{\pi}}{\kappa^{1/2}}\frac{\Gamma(\kappa)}{\Gamma(\kappa-\frac{1}{2})}(1+z^2)^{-\kappa} \\ -\frac{\sqrt{\pi}}{\kappa^{1/2}}\frac{z}{\Gamma(\kappa-\frac{1}{2})}\sum_{n=0}^{\infty}\frac{\Gamma(\kappa+n+1/2)}{\Gamma(n+3/2)}(-z^2)^n, \qquad |z|<1, \tag{74}$$

however, the coefficients in (73) are much more explicit.

7 Series Expansion for Large z

Begin with the identity[7]

$$F(a,b,c,z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}F\left(a, a-c+1, a-b+1, \frac{1}{z}\right) \\ +\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}F\left(b, b-c+1, b-a+1, \frac{1}{z}\right). \tag{75}$$

Applying this to the last term in equation (69) yields

$$F\left(1, 2\kappa, \kappa+1, \frac{1+iz}{2}\right) = \\ \sqrt{\pi}\frac{\Gamma(\kappa+1)}{\Gamma(\kappa+\frac{1}{2})}(1+z^2)^{-\kappa}+\frac{i\kappa}{(\kappa-\frac{1}{2})}z^{-1}F\left(1, \tfrac{1}{2}, \tfrac{3}{2}-\kappa, -\frac{1}{z^2}\right) \\ +i\sqrt{\pi}\kappa\frac{\Gamma(\frac{1}{2}-\kappa)}{\Gamma(1-\kappa)}z(z^2)^{-(\kappa+1/2)}F\left(\kappa+\tfrac{1}{2}, \kappa, \kappa+\tfrac{1}{2}, -\frac{1}{z^2}\right). \tag{76}$$

In addition, using the identity (68), this simplifies to

$$F\Big(1, 2\kappa, \kappa+1, \frac{1+iz}{2}\Big) = \sqrt{\pi}\frac{\Gamma(\kappa+1)}{\Gamma(\kappa+\frac{1}{2})}(1+z^2)^{-\kappa} + \frac{i\kappa}{(\kappa-\frac{1}{2})}z^{-1}F\Big(1, \tfrac{1}{2}, \tfrac{3}{2}-\kappa, -\frac{1}{z^2}\Big) + i\sqrt{\pi}\kappa\frac{\Gamma(\frac{1}{2}-\kappa)}{\Gamma(1-\kappa)}z(z^2)^{-(\kappa+1/2)}\left(1+\frac{1}{z^2}\right)^{-\kappa}. \quad (77)$$

Using the reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad (78)$$

the last equation takes the form

$$F\Big(1, 2\kappa, \kappa+1, \frac{1+iz}{2}\Big) = \sqrt{\pi}\frac{\Gamma(\kappa+1)}{\Gamma(\kappa+\frac{1}{2})}(1+z^2)^{-\kappa} + \frac{i\kappa}{(\kappa-\frac{1}{2})}z^{-1}F\Big(1, \tfrac{1}{2}, \tfrac{3}{2}-\kappa, -\frac{1}{z^2}\Big) + \sqrt{\pi}i\tan(\pi\kappa)\frac{\Gamma(\kappa+1)}{\Gamma(\kappa+\frac{1}{2})}z(z^2)^{-(\kappa+1/2)}\left(1+\frac{1}{z^2}\right)^{-\kappa}. \quad (79)$$

In order to get the phases right special care must be taken when combining the different power functions. This is because, for the principal branch of the power function, it is not always true that $(z^\alpha)^\beta = z^{\alpha\beta}$ or $z_1^\alpha z_2^\alpha = (z_1 z_2)^\alpha$. However, using the identities

$$z^{\alpha+\beta} = z^\alpha z^\beta, \qquad (1+z)^\alpha = z^\alpha\Big(1+\frac{1}{z}\Big)^\alpha, \quad (80)$$

and

$$z(z^2)^{-1/2} = \mathrm{sgn}[\mathrm{Re}(z)] \equiv \begin{cases} +1 & \text{if} \quad \mathrm{Re}(z) \geq 0 \quad \text{and} \quad \arg(z) \neq -\pi/2 \\ -1 & \text{if} \quad \mathrm{Re}(z) \leq 0 \quad \text{and} \quad \arg(z) \neq +\pi/2, \end{cases} \quad (81)$$

which are valid for all $z \neq 0$, equation (79) can be written

$$F\Big(1, 2\kappa, \kappa+1, \frac{1+iz}{2}\Big) = \frac{i\kappa}{(\kappa-\frac{1}{2})}z^{-1}F\Big(1, \tfrac{1}{2}, \tfrac{3}{2}-\kappa, -\frac{1}{z^2}\Big) + \sqrt{\pi}\frac{\Gamma(\kappa+1)}{\Gamma(\kappa+\frac{1}{2})}\Big\{1 + i\tan(\pi\kappa)\mathrm{sgn}[\mathrm{Re}(z)]\Big\}(z^2)^{-\kappa}\left(1+\frac{1}{z^2}\right)^{-\kappa}. \quad (82)$$

It should be noted that for half-integer values of κ the right-hand side is undefined. Otherwise, this equation holds for all $z \neq 0$ in the complex plane with a branch cut from $-i$ to $-i\infty$ along the negative imaginary axis. For $|z| > 1$, the hypergeometric function on the right-hand side of (82) can be expanded using the series (53) to obtain the expansion

$$\begin{aligned} F\Big(1, 2\kappa, \kappa+1, \frac{1+iz}{2}\Big) &= \\ &\sqrt{\pi}\frac{\Gamma(\kappa+1)}{\Gamma(\kappa+\frac{1}{2})}\Big\{1 + i\tan(\pi\kappa)\mathrm{sgn}[\mathrm{Re}(z)]\Big\}(z^2)^{-\kappa}\left(1+\frac{1}{z^2}\right)^{-\kappa} \\ &+ \frac{i\kappa}{(\kappa-\frac{1}{2})}\frac{1}{z}\left\{1 + \frac{\frac{1}{2}}{\frac{3}{2}-\kappa}\left(-\frac{1}{z^2}\right) + \frac{\frac{1}{2}\cdot\frac{3}{2}}{(\frac{3}{2}-\kappa)(\frac{5}{2}-\kappa)}\left(-\frac{1}{z^2}\right)^2 + \cdots\right\}, \quad (83) \end{aligned}$$

$|z| > 1$. The factor $\mathrm{sgn}[\mathrm{Re}(z)]$ ensures that the right-hand side is continuous throughout the upper half plane. This expansion can be arrived at independently using the theory of Laplace transforms as shown in Appendix B. Substituting (83) into (57) yields

$$\begin{aligned} Z_\kappa(\xi) = \frac{i\sqrt{\pi}}{\sqrt{\kappa}}\frac{\Gamma(\kappa)}{\Gamma(\kappa-\frac{1}{2})}&\Big\{1 + i\tan(\pi\kappa)\mathrm{sgn}[\mathrm{Re}(z)]\Big\}(z^2)^{-\kappa}\left(1+\frac{1}{z^2}\right)^{-\kappa} \\ - \frac{1}{\sqrt{\kappa}}\frac{1}{z}&\left\{1 + \frac{\frac{1}{2}}{\frac{3}{2}-\kappa}\left(-\frac{1}{z^2}\right) + \frac{\frac{1}{2}\cdot\frac{3}{2}}{(\frac{3}{2}-\kappa)(\frac{5}{2}-\kappa)}\left(-\frac{1}{z^2}\right)^2 + \cdots\right\}, \quad (84) \end{aligned}$$

which is valid for $|z| > 1$ and $\kappa \neq \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ It is important to note that this expansion is exact, not just asymptotic.

8 Integration of the Differential Equation

The differential equation (42) can be written in the symmetric form

$$\frac{d}{dz}\left\{\left(1+z^2\right)^{1-\kappa}\frac{d}{dz}\Big[\left(1+z^2\right)^{\kappa}I(z)\Big]\right\} = 0. \quad (85)$$

This has the first integral

$$\left(1+z^2\right)^{1-\kappa}\frac{d}{dz}\Big[\left(1+z^2\right)^{\kappa}I(z)\Big] = C_\kappa, \quad (86)$$

where

$$C_\kappa = -2\kappa \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{\kappa+1}}\, dx = -2\sqrt{\pi}\frac{\Gamma(\kappa+\frac{1}{2})}{\Gamma(\kappa)}. \tag{87}$$

Equations (86) and (87) can be verified by substituting (35) into (86), carrying out the differentiation, and then using integration by parts and the identity (37). Therefore, the solution can be written

$$I(z) = C_\kappa \big(1+z^2\big)^{-\kappa} \int_i^z \big(1+t^2\big)^{\kappa-1}\, dt, \tag{88}$$

where the lower limit of integration is chosen such that

$$\lim_{z\to i} I(z) = i\sqrt{\pi}\frac{\Gamma(\kappa+\frac{1}{2})}{\Gamma(\kappa+1)}. \tag{89}$$

If $\kappa = 1/2$, the integral (88) has the closed form solution

$$I(z) = -2\big(1+z^2\big)^{-1/2}\left[\sinh^{-1}(z) - \frac{\pi i}{2}\right], \tag{90}$$

where

$$\sinh^{-1}(z) = \int_0^z \frac{1}{(1+t^2)^{1/2}}\, dt. \tag{91}$$

Equivalently, this can be written

$$\sinh^{-1}(z) = \log\left(z + \sqrt{z^2+1}\right), \tag{92}$$

where $\log(z)$ is the principal branch of the logarithm. Substituting the result (88) into (34), the plasma dispersion function can be written

$$Z_\kappa(\xi) = -\frac{2\kappa-1}{\kappa^{1/2}}\big(1+z^2\big)^{-\kappa} \int_i^z \big(1+t^2\big)^{\kappa-1}\, dt, \tag{93}$$

where $z = \xi/\sqrt{\kappa}$. This holds in both the upper and lower half planes and therefore represents the analytic continuation of Z_κ.

9 Solutions for Half-Integer Values: $\kappa = n + \frac{1}{2}$

Setting $\kappa = n + \frac{1}{2}$ in equation (69) yields

$$F\Big(1, 2\kappa, \kappa + 1, \frac{1 + iz}{2}\Big) = \sqrt{\pi}\frac{\Gamma(n + \frac{3}{2})}{\Gamma(n + 1)}(1 + z^2)^{-n-1/2} + (2n + 1)izF\big(1, n + 1, \tfrac{3}{2}, -z^2\big). \quad (94)$$

By repeated application of the identity[7]

$$(b-a)(1-z)F(a, b, c, z) = (c-a)F(a-1, b, c, z)-(c-b)F(a, b-1, c, z) \quad (95)$$

one may prove the formula

$$F(1, n + 1, \tfrac{3}{2}, -z^2) = \frac{1}{2}\sum_{m=1}^{n}\frac{\Gamma(n + 1 - m)}{\Gamma(n + 1)}\frac{\Gamma(n + \frac{1}{2})}{\Gamma(n + \frac{3}{2} - m)}(1 + z^2)^{-m} + \frac{1}{2}\frac{\Gamma(1)}{\Gamma(n + 1)}\frac{\Gamma(n + \frac{1}{2})}{\Gamma(\frac{3}{2})}(1 + z^2)^{-n}F(1, 1, \tfrac{3}{2}, -z^2), \quad (96)$$

where $n = 1, 2, 3, \ldots$ Inserting this formula into (94) and using the closed form expression[7]

$$F(1, 1, \tfrac{3}{2}, -z^2) = \frac{1}{iz}(1 + z^2)^{-1/2}\sin^{-1}(iz) \quad (97)$$

one finds

$$F\Big(1, 2\kappa, \kappa + 1, \frac{1 + iz}{2}\Big) = \sqrt{\pi}\frac{\Gamma(n + \frac{3}{2})}{\Gamma(n + 1)}(1 + z^2)^{-n-1/2}\left[1 + \frac{2}{\pi}\sin^{-1}(iz)\right] + \frac{\Gamma(n + \frac{3}{2})}{\Gamma(n + 1)}\sum_{m=1}^{n}\frac{\Gamma(n + 1 - m)}{\Gamma(n + \frac{3}{2} - m)}iz(1 + z^2)^{-m}, \quad (98)$$

where $\kappa = n + \frac{1}{2}$. Thus, from (56) one obtains the closed form solutions

$$I(z) = -2(1 + z^2)^{-n-1/2}\left[\sinh^{-1}(z) - \frac{\pi i}{2}\right] - \sqrt{\pi}\sum_{m=1}^{n}\frac{\Gamma(n + 1 - m)}{\Gamma(n + \frac{3}{2} - m)}z(1 + z^2)^{-m}, \quad (99)$$

$n = 1, 2, 3, \ldots$ The first few solutions are as follows; for $\kappa = \frac{3}{2}$:

$$I(z) = -2(1+z^2)^{-3/2}\left[\sinh^{-1}(z) - \frac{\pi i}{2}\right] - 2z(1+z^2)^{-1}, \tag{100}$$

for $\kappa = \frac{5}{2}$:

$$I(z) = -2(1+z^2)^{-5/2}\left[\sinh^{-1}(z) - \frac{\pi i}{2}\right] - \frac{4}{3}z(1+z^2)^{-1} - 2z(1+z^2)^{-2}, \tag{101}$$

and for $\kappa = \frac{7}{2}$:

$$\begin{aligned} I(z) = -2(1+z^2)^{-7/2}&\left[\sinh^{-1}(z) - \frac{\pi i}{2}\right] \\ &- \frac{16}{15}z(1+z^2)^{-1} - \frac{4}{3}z(1+z^2)^{-2} - 2z(1+z^2)^{-3}. \end{aligned} \tag{102}$$

Multiplying equation (99) by the leading coefficient in equation (34), the plasma dispersion function is given by

$$\begin{aligned} Z_\kappa(\xi) = \frac{1}{(n+\frac{1}{2})^{1/2}} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n)} &\left\{ \sum_{m=1}^{n} \frac{\Gamma(n+1-m)}{\Gamma(n+\frac{3}{2}-m)} z(1+z^2)^{-m} \right. \\ &\left. - \frac{2}{\sqrt{\pi}}(1+z^2)^{-n-1/2}\left[\sinh^{-1}(z) - \frac{\pi i}{2}\right] \right\}, \end{aligned} \tag{103}$$

where $z = \xi/\sqrt{\kappa}$ and $n = 1, 2, 3, \ldots$ A more explicit form is obtained by making the substitution given by equation (92). Similar expressions were first obtained by Summers and Thorne[10] using different methods.

10 Solutions for Integer Values: $\kappa = n$

If $\kappa = n$ is a positive integer, a closed form solution can be derived using the identity (61) to obtain a hypergeometric series which terminates; however, this solution lacks mathematical symmetry. It is preferable to proceed as follows. The hypergeometric series (53) can be written

$$F(1,2n,n+1,t) = \sum_{m=0}^{\infty} \frac{\Gamma(1+m)}{\Gamma(1)} \frac{\Gamma(2n+m)}{\Gamma(2n)} \frac{\Gamma(n+1)}{\Gamma(n+1+m)} \frac{t^m}{m!} \tag{104}$$

$$= \frac{\Gamma(n+1)}{\Gamma(2n)} \sum_{m=0}^{\infty} \frac{\Gamma(2n+m)}{\Gamma(n+1+m)} t^m \tag{105}$$

$$= \frac{\Gamma(n+1)}{\Gamma(2n)} \sum_{m=n}^{\infty} \frac{\Gamma(m+n)}{\Gamma(m+1)} t^{m-n}. \tag{106}$$

Splitting the sum into two parts

$$F(1,2n,n+1,t) = \frac{\Gamma(n+1)}{\Gamma(2n)} \left\{ \sum_{m=0}^{\infty} \frac{\Gamma(m+n)}{m!} t^{m-n} - \sum_{m=0}^{n-1} \frac{\Gamma(m+n)}{m!} t^{m-n} \right\} \tag{107}$$

the first part can be summed by the binomial theorem

$$\sum_{m=0}^{\infty} \frac{\Gamma(m+\alpha)}{\Gamma(\alpha)} \frac{z^m}{m!} = (1-z)^{-\alpha}, \qquad \alpha \neq 0, -1, -2, \ldots, \tag{108}$$

to obtain

$$F(1,2n,n+1,t) = \frac{\Gamma(n+1)}{\Gamma(2n)} \left\{ \Gamma(n) t^{-n} (1-t)^{-n} - \sum_{m=0}^{n-1} \frac{\Gamma(m+n)}{m!} t^{m-n} \right\}. \tag{109}$$

Substituting $t = (1+iz)/2$ this yields

$$F\left(1, 2n, n+1, \frac{1+iz}{2}\right) = \frac{\Gamma(n+1)\Gamma(n)}{\Gamma(2n)} \left\{ \left(\frac{1+z^2}{4}\right)^{-n} - \sum_{m=0}^{n-1} \frac{\Gamma(m+n)}{\Gamma(n) m!} \left(\frac{1+iz}{2}\right)^{m-n} \right\}. \tag{110}$$

Using the duplication formula (B6) found in Appendix B to evaluate the coefficient on the right hand side, one obtains

$$F\left(1, 2n, n+1, \frac{1+iz}{2}\right) = 2\pi^{1/2} \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} \left\{ (1+z^2)^{-n} - \frac{1}{4^n} \sum_{m=0}^{n-1} \frac{\Gamma(m+n)}{\Gamma(n) m!} \left(\frac{1+iz}{2}\right)^{m-n} \right\}. \tag{111}$$

Comparing this equation term by term with the symmetry relation [see equation (69)]

$$F\Big(1, 2\kappa, \kappa+1, \frac{1+iz}{2}\Big) + F\Big(1, 2\kappa, \kappa+1, \frac{1-iz}{2}\Big) = 2\sqrt{\pi}\frac{\Gamma(\kappa+1)}{\Gamma(\kappa+\frac{1}{2})}(1+z^2)^{-\kappa} \tag{112}$$

it follows that

$$F\Big(1, 2n, n+1, \frac{1-iz}{2}\Big) = 2\pi^{1/2}\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\frac{1}{4^n}\sum_{m=0}^{n-1}\frac{\Gamma(m+n)}{\Gamma(n)m!}\left(\frac{1+iz}{2}\right)^{m-n} \tag{113}$$

or, equivalently,

$$F\Big(1, 2n, n+1, \frac{1+iz}{2}\Big) = \frac{\pi^{1/2}}{2^{2n-1}}\cdot\frac{n}{\Gamma(n+\frac{1}{2})}\sum_{m=0}^{n-1}\frac{\Gamma(m+n)}{m!}\left(\frac{1-iz}{2}\right)^{m-n}. \tag{114}$$

Substituting this into equation (56) yields

$$I(z) = \frac{i\pi}{2^{2n-1}}\sum_{m=0}^{n-1}\binom{n+m-1}{m}\left(\frac{1-iz}{2}\right)^{m-n}, \tag{115}$$

$n = 1, 2, 3, \ldots$ This same result is obtained by using the theory of residues to evaluate the integral (35) along a large semicircle in the lower half plane.[2] Because (115) is a rational function of z, it immediately gives the analytic continuation of $I(z)$ throughout the complex plane.

Using equation (110) to rewrite (113) in the form

$$F\Big(1, 2n, n+1, \frac{1+iz}{2}\Big) = \frac{\Gamma(n+1)\Gamma(n)}{\Gamma(2n)}\sum_{m=0}^{n-1}\frac{\Gamma(m+n)}{\Gamma(n)m!}\left(\frac{1-iz}{2}\right)^{m-n}, \tag{116}$$

it follows from equation (57) that the plasma dispersion function is given by

$$Z_n(\xi) = i\frac{(n-\frac{1}{2})}{n^{1/2}}\frac{\Gamma(n)}{\Gamma(2n)}\sum_{m=0}^{n-1}\frac{\Gamma(m+n)}{m!}\left(\frac{1-iz}{2}\right)^{m-n}, \tag{117}$$

where $z = \xi/\sqrt{n}$ and $n = 1, 2, 3\ldots$

11 Equivalent Integral Representations

Consider the integral defined for $\mathrm{Im}(z) > 0$ by

$$I(z) = \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^\kappa} \cdot \frac{1}{(x-z)}\, dx \tag{118}$$

and by analytic continuation for $\mathrm{Im}(z) \leq 0$. By means of some elementary changes of variables, one may derive the following equivalent representations:

$$I(z) = \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^\kappa} \cdot \frac{z}{(x^2-z^2)}\, dx, \tag{119}$$

$$I(z) = \int_{-\infty}^{\infty} \frac{1}{(\cosh x)^{2\kappa-1}} \cdot \frac{1}{[\sinh(x)-z]}\, dx, \tag{120}$$

$$I(z) = \int_{-\infty}^{\infty} \frac{1}{(\cosh x)^{2\kappa-1}} \cdot \frac{z}{[\sinh^2(x)-z^2]}\, dx, \tag{121}$$

$$I(z) = \int_{-\infty}^{\infty} \left(\frac{x^2}{1+x^2}\right)^\kappa \frac{z}{1-z^2x^2}\, dx, \tag{122}$$

$$I(z) = \int_{-\infty}^{\infty} (\tanh x)^{2\kappa} \frac{z\cosh x}{1-z^2\sinh^2 x}\, dx, \tag{123}$$

$$I(z) = z\int_0^{\infty} t^{\kappa-1/2}(1+t)^{-\kappa}(1-z^2t)^{-1}\, dt, \tag{124}$$

Thus, by equation (55), all of these integrals are equal to

$$I(z) = i\sqrt{\pi}\frac{\Gamma(\kappa+1/2)}{\Gamma(\kappa+1)} F\left(1, 2\kappa, \kappa+1, \frac{1+iz}{2}\right). \tag{125}$$

In addition, $I(z)$ can be represented as a Laplace transform. If $z = is$ and $\mathrm{Re}(s) > 0$, then one has the identity

$$\frac{1}{x-z} = \frac{i}{s+ix} = i\int_0^{\infty} e^{-(s+ix)t}\, dt. \tag{126}$$

Substituting this into (118) and changing the order of integration yields

$$I(is) = i\int_0^{\infty} f(t)e^{-st}\, dt, \tag{127}$$

where

$$f(t) = 2\int_0^{\infty} \frac{\cos(xt)}{(1+x^2)^\kappa}\, dx. \tag{128}$$

Using the well known integral representation of the modified Bessel function of the second kind:

$$K_\nu(t) = \frac{\Gamma(\nu + 1/2)2^\nu}{\pi^{1/2} t^\nu} \int_0^\infty \frac{\cos(xt)}{(1 + x^2)^{\nu+1/2}}\, dx, \qquad t > 0, \tag{129}$$

it follows that

$$f(t) = \frac{2\sqrt{\pi}}{\Gamma(\kappa)} \left(\frac{t}{2}\right)^{\kappa-1/2} K_{\kappa-1/2}(t), \qquad t > 0. \tag{130}$$

This establishes the identity

$$I(z = is) = i\mathscr{L}[f(t)], \qquad \mathrm{Re}(s) > 0, \tag{131}$$

where $f(t)$ is defined by the previous equation and $\mathscr{L}$ is the Laplace transform operator.

12 Conclusions

Starting with the definition of the plasma dispersion function in terms of the integral (35), a second order differential equation is derived that immediately allows the dispersion function to be expressed in terms of Gauss' hypergeometric function. Using the well known properties of the hypergeometric function, the many properties of the plasma dispersion function are then derived in a unified manner. A summary of the results is provided in Section 2.

Appendix A: Evaluation of the Integral I_0

By definition

$$I_0 = I(z = i) = \int_{-\infty}^{\infty} \frac{1}{(1 + x^2)^\kappa} \cdot \frac{1}{(x - i)}\, dx. \tag{A1}$$

Multiplying the integrand by $(x + i)/(x + i)$, this becomes

$$I_0 = 2i \int_0^\infty \frac{1}{(1 + x^2)^{\kappa+1}}\, dx, \tag{A2}$$

and with the change of variable $t = x^2$ one obtains

$$I_0 = i \int_0^\infty \frac{t^{-1/2}}{(1+t)^{\kappa+1}}\, dx. \tag{A3}$$

The remaining integral is the Beta function

$$B(p,q) = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}\, dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \tag{A4}$$

with $p = 1/2$ and $q = \kappa + 1/2$. Therefore, it follows that

$$I_0 = i\sqrt{\pi}\, \frac{\Gamma(\kappa+1/2)}{\Gamma(\kappa+1)}. \tag{A5}$$

The calculation of $I'(z = i)$ is similar.

Appendix B: Alternative Derivation of the Expansion for $|z| > 1$

The purpose of this appendix is to give an independent derivation of the expansion (83) for $|z| > 1$. The starting point is the representation of that function as a Laplace transform. Using the definition

$$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\pi\nu)}, \tag{B1}$$

together with the series expansion for the modified Bessel function $I_\nu(x)$, one obtains

$$\begin{aligned} \left(\frac{x}{2}\right)^\nu K_\nu(x) = \frac{\pi}{2\sin(\pi\nu)} \Bigg\{ &\sum_{n=0}^\infty \frac{1}{\Gamma(n-\nu+1)n!} \left(\frac{x}{2}\right)^{2n} \\ &- \sum_{n=0}^\infty \frac{1}{\Gamma(n+\nu+1)n!} \left(\frac{x}{2}\right)^{2n+2\nu} \Bigg\}, \end{aligned} \tag{B2}$$

where $x > 0$ and $\nu > 0$ is not an integer. By a well known theorem,[11] the Laplace transform of this function has the asymptotic expansion

$$\mathscr{L}\left\{\left(\frac{t}{2}\right)^{\nu} K_\nu(t)\right\} \sim \frac{\pi}{2\sin(\pi\nu)}\left\{\sum_{n=0}^{\infty}\frac{\Gamma(2n+1)}{\Gamma(n-\nu+1)}\frac{1}{n!2^{2n}}\frac{1}{s^{2n+1}}\right.$$
$$\left. -\sum_{n=0}^{\infty}\frac{\Gamma(2n+2\nu+1)}{\Gamma(n+\nu+1)}\frac{1}{n!2^{2n+2\nu}}\frac{1}{s^{2n+2\nu+1}}\right\}, \quad \text{(B3)}$$

where $s \to \infty$ along any ray $e^{i\theta}$ with $|\theta| < \pi/2$. In fact, the following analysis shows that this series converges in the right half-plane whenever $|s| > 1$ and, therefore, equality holds throughout this region. However, this fact is not needed now. Setting $\nu = \kappa - 1/2$, then from the relations (131) and (56) one obtains

$$F\left(1, 2\kappa, \kappa+1, \frac{1-s}{2}\right) = \frac{1}{i\sqrt{\pi}}\frac{\Gamma(\kappa+1)}{\Gamma(\kappa+\frac{1}{2})}I(is)$$
$$\sim \frac{\pi}{2\sin[(\kappa-\frac{1}{2})\pi]}\frac{\kappa}{\Gamma(\kappa+\frac{1}{2})}\left\{\sum_{n=0}^{\infty}\frac{\Gamma(2n+1)}{\Gamma(n-\kappa+\frac{3}{2})}\frac{1}{n!2^{2n}}\frac{1}{s^{2n+1}}\right.$$
$$\left. -\sum_{n=0}^{\infty}\frac{\Gamma(2n+2\kappa)}{\Gamma(n+\kappa+\frac{1}{2})}\frac{1}{n!2^{2n+2\kappa-1}}\frac{1}{s^{2n+2\kappa}}\right\}, \quad \text{(B4)}$$

where $s \to \infty$ along any ray $e^{i\theta}$ with $|\theta| < \pi/2$. It will now be shown that this agrees with the expansion (83), that is:

$$F\left(1, 2\kappa, \kappa+1, \frac{1-s}{2}\right) =$$
$$\sqrt{\pi}\frac{\Gamma(\kappa+1)}{\Gamma(\kappa+\frac{1}{2})}\left[1 + i\tan(\pi\kappa)\mathrm{sgn}[\mathrm{Re}(is)]\right](-s^2)^{-\kappa}\left(1-\frac{1}{s^2}\right)^{-\kappa}$$
$$+\frac{\kappa}{(\kappa-\frac{1}{2})}\frac{1}{s}\left\{1 + \frac{\frac{1}{2}}{\frac{3}{2}-\kappa}\left(\frac{1}{s^2}\right) + \frac{\frac{1}{2}\cdot\frac{3}{2}}{(\frac{3}{2}-\kappa)(\frac{5}{2}-\kappa)}\left(\frac{1}{s^2}\right)^2 + \cdots\right\}. \quad \text{(B5)}$$

The even and odd power terms are compared separately. First the even terms. Using the duplication formula

$$\Gamma(2z) = \pi^{-1/2}2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \quad \text{(B6)}$$

and the binomial theorem, the even terms in (B4) can be summed as follows

$$
\begin{aligned}
&-\frac{\pi}{\sin[(\kappa-\frac{1}{2})\pi]}\frac{\kappa}{\Gamma(\kappa+\frac{1}{2})}\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\Gamma(2n+2\kappa)}{\Gamma(n+\kappa+\frac{1}{2})}\frac{1}{2^{2n+2\kappa-1}s^{2n+2\kappa}} \\
&\qquad\qquad = \frac{\sqrt{\pi}}{\cos(\kappa\pi)}\frac{\kappa\Gamma(\kappa)}{\Gamma(\kappa+\frac{1}{2})}\sum_{n=0}^{\infty}\frac{\Gamma(n+\kappa)}{\Gamma(\kappa)n!}\frac{1}{s^{2n+2\kappa}} \\
&\qquad\qquad\qquad\qquad = \frac{\sqrt{\pi}}{\cos(\kappa\pi)}\frac{\kappa\Gamma(\kappa)}{\Gamma(\kappa+\frac{1}{2})}s^{-2\kappa}\left(1-\frac{1}{s^2}\right)^{-\kappa}. \qquad \text{(B7)}
\end{aligned}
$$

This will now be compared with the even terms in (B5):

$$
\sqrt{\pi}\frac{\Gamma(\kappa+1)}{\Gamma(\kappa+\frac{1}{2})}\big[1+i\tan(\pi\kappa)\mathrm{sgn}[\mathrm{Re}(is)]\big](-s^2)^{-\kappa}\left(1-\frac{1}{s^2}\right)^{-\kappa}. \qquad \text{(B8)}
$$

It is easy to show that

$$
\frac{(-s^2)^{-\kappa}}{s^{-2\kappa}} = \begin{cases} e^{+i\pi\kappa} & \text{if} \quad 0<\arg(s)<\pi/2 \\ e^{-i\pi\kappa} & \text{if} \quad -\pi/2<\arg(s)\le 0, \end{cases} \qquad \text{(B9)}
$$

therefore,

$$
\frac{s^{-2\kappa}}{\cos(\kappa\pi)} = \big[1+i\tan(\pi\kappa)\mathrm{sgn}[\mathrm{Re}(is)]\big](-s^2)^{-\kappa}, \qquad \text{(B10)}
$$

which establishes the equality of (B7) and (B8), that is, for the even power terms in (B4) and (B5). Similarly, using the duplication formula (B6) the odd terms in (B4) are easily shown to be equal to the odd terms in (B5). This gives an independent proof of (83) in the upper half plane which, by means of the symmetry relations, extends to the whole plane.

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Author's Note

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14. ABSTRACT
The plasma dispersion function is computed for a homogeneous isotropic plasma in which the particle velocities are distributed according to a kappa distribution. An ordinary differential equation is derived for the plasma dispersion function and it is shown that the solution can be written in terms of Gauss' hypergeometric function. Using the extensive theory of the hypergeometric function, various mathematical properties of the plasma dispersion function are derived including symmetry relations, series expansions, integral representations, and closed form expressions for integer and half-integer values of kappa.

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