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Direct and Inverse Kinematics of a Novel Tip-Tilt-Piston Parallel Manipulator

F. Tahmasebi

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Farhad Tahmasebi,
NASA Goddard Spacce Flight Center, Greenbelt, MD

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ABSTRACT

Closed-form direct and inverse kinematics of a new three degree-of-freedom (DOF) parallel manipulator with inextensible limbs and base-mounted actuators are presented. The manipulator has higher resolution and precision than the existing three DOF mechanisms with extensible limbs. Since all of the manipulator actuators are base-mounted; higher payload capacity, smaller actuator sizes, and lower power dissipation can be obtained. The manipulator is suitable for alignment applications where only tip, tilt, and piston motions are significant. The direct kinematics of the manipulator is reduced to solving an eighth-degree polynomial in the square of tangent of half-angle between one of the limbs and the base plane. Hence, there are at most 16 assembly configurations for the manipulator. In addition, it is shown that the 16 solutions are eight pairs of reflected configurations with respect to the base plane. Numerical examples for the direct and inverse kinematics of the manipulator are also presented.

INTRODUCTION

In the past few years, several researchers have shown a great deal of interest in studying kinematic synthesis and analysis of parallel manipulators. Such mechanisms are most suitable for applications in which the requirements for precision, rigidity, load-to-weight ratio, and load distribution are more important than the need for a large workspace.

The Stewart-Gough platform [1] is probably the first six degree-of-freedom (DOF) parallel mechanism which has been studied in the literature. Waldron and Hunt [2] showed that kinematic behavior of parallel mechanisms has many inverse characteristics to that of serial mechanisms. For example, direct kinematics of a parallel manipulator is much more difficult than its inverse kinematics; whereas, for a serial manipulator, the opposite is true.

Several researchers have analyzed the direct kinematics of the Stewart-Gough platform. Griffis and Duffy [3] as well as Nanua et al. [4] studied direct kinematics of special cases of the Stewart-Gough platform, in which pairs of spherical joints are concentric on either the platform or both the base and the platform. They were able to reduce the problem to an eighth-degree polynomial in the square of a single variable (total degree of 16). However, as mentioned by Griffis and Duffy [3], pairs of concentric spherical joints may very well present design problems. Lin et al. [5] solved direct kinematics of another class of the Stewart-Gough platforms, in which there are two concentric spherical joints on the base and two more concentric spherical joints on the platform. The latter class of the Stewart-Gough platforms suffer from lack of symmetry and concentric spherical joints are still needed in their construction. Parenti-Castelli and Innocenti have also been able to obtain closed-form solutions for other special forms of the Stewart-Gough platform [6]. Raghavan used a numerical technique known as polynomial continuation to show that there are forty solutions for the direct kinematics of the Stewart-Gough platform of general geometry [7]. Other types of six-DOF parallel manipulators have been introduced and studied in the literature by Tahmasebi and Tsai [8,9], Merlet [10], Ben-Horin and Shoham [11], and Hudgens and Tesar [12].

Some researchers have also shown interest in three DOF parallel mechanism. For example, Gosselin and Angeles have studied optimal kinematic design of planar and spherical parallel manipulators [13,14]. Tsai analyzed the kinematics of a three DOF platform manipulator with three extensible limbs [15]. Song and Zhang studied a three DOF mechanism with three

RPS legs [16]. Ceccarelli introduced a new three DOF spatial parallel mechanism [17].

In this paper, closed-form direct kinematic solution for a new three DOF parallel manipulator is presented. It will be shown that direct kinematics of the minimanipulator involves solving an eighth-degree polynomial in the square of a single variable. The simpler inverse kinematics of the three DOF manipulators will also be presented.

The manipulator, which is being analyzed in this article is suitable for optical (and other types of) alignment applications where only tip, tilt, and piston motions are significant (e.g., alignment of segmented spherical mirrors, alignment of Fabry-Perot interferometers).

DESCRIPTION OF THE MANIPULATOR

The mechanism described here is a three DOF parallel alignment manipulator with three inextensible limbs and base-mounted actuators. Figure 1 shows the details of the manipulator. The picture of a manipulator prototype is shown in Figure 2.

The three inextensible limbs R_1P_1, R_2P_2, and R_3P_3 are connected to the output moving platform through spherical joints P_1, P_2, and P_3. The lower ends of the limbs are connected to links R_1T_1, R_2T_2, and R_3T_3 through revolute joints at R_1, R_2, and R_3. Slider Links R_1T_1, R_2T_2, and R_3T_3 are connected to the fixed base through base-mounted prismatic actuators N_1T_1, N_2T_2, and N_3T_3, respectively.

The manipulator has three degrees of freedom. Tip, tilt, and piston motions of the moving platform (output link) can be obtained by using the prismatic actuators to vary the O_1R_1, O_2R_2, and O_3R_3 lengths. Note that the prismatic actuators can be inside or outside of the $R_1R_2R_3$ triangle formed by the lower ends of the limbs.

Examples of prismatic actuators that can be used in the manipulator include: (1) lead screws; (2) linear hydraulic motors; (3) inch worm linear stepper motors; (4) piezoelectric linear drives; (5) linear flexure (compliant) drives.

Let subscript i in this section and the rest of this work represent numbers 1, 2, and 3 in a cyclic manner. The angle between the lines ON_i and ON_{i+1} is equal to 120 degrees.

Compared to the existing three DOF platforms with extensible limbs and limb-mounted actuators, the manipulator being introduced here has the following advantages:

• Its power and sensor lines need not be routed through its joints at the lower ends of its limbs.

• It has higher resolution and precision.

⋆ The prismatic actuators move the lower ends of the limbs on the fixed base. Large movements at the lower ends of the limbs are needed to generate smaller movements at the top ends of the limbs, which are connected to the moving platform. This “motion reduction” feature results in higher mechanical advantage.

• Weight of any of its base-mounted actuators is not a load for its other two actuators.

Note that the manipulator limb configurations shown in Figures 1, 2, and 3 are different from those used in the six DOF minimanipulator introduced by Tahmasebi and Tsai [8,9]. Compliant (flexured) joints and linear actuators can be used in construction of the manipulator to obtain very small movements.

Since there are no joints on the limbs, the manipulator can also be used as an inflatable space device.

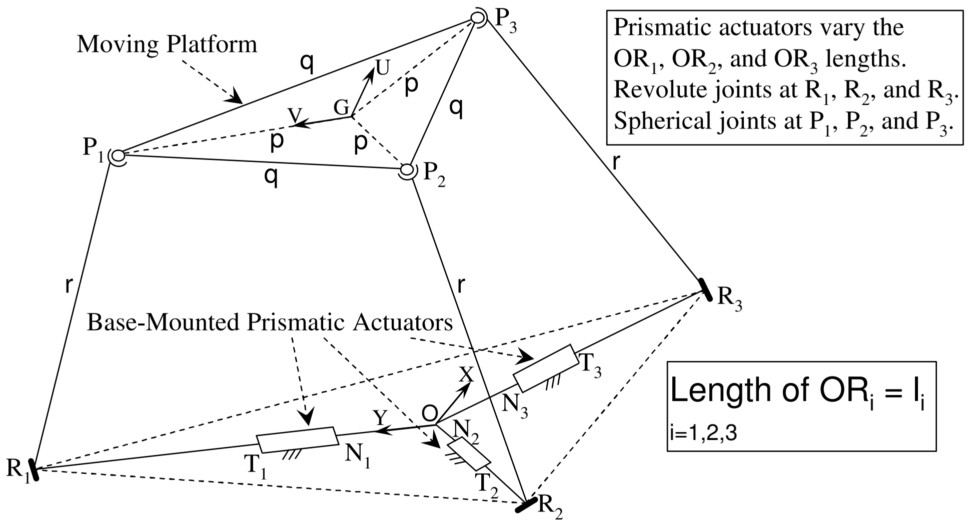

Figure 1: The new manipulator with base-mounted actuators and inextensible limbs.

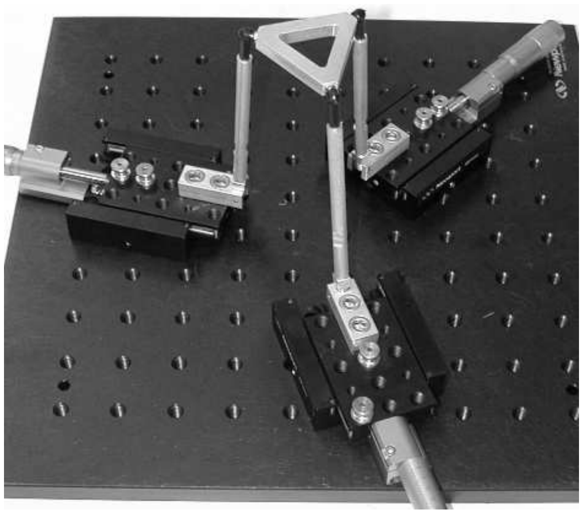

Figure 2: The new manipulator prototype.

If rotary actuation is desired, slider-crank mechanisms can be used as drivers for the manipulators (see Figure 4). Links $C_i'A_i'$ and $A_i'B_i'$ transform the rotary actuation at C_i' to linear motion of the slider link T_iR_i. Link variables a and b in Figure 4 can be chosen properly to add additional mechanical advantage (motion reduction) to the manipulator.

INVERSE KINEMATICS

Solving the inverse kinematics of the manipulator involves finding l_i (length of the vector $\overline{OR_i}$), given three position and/or orientation variables of the moving platform.

As mentioned earlier, the manipulator is suitable for optical (and other types of) alignment applications where only tip (rotation about the X axis), tilt (rotation about the Y axis), and piston (translation along the Z axis) motions are significant. In this paper, we choose the tip, tilt, and piston variables as the three known inputs for the inverse kinematics of the manipulator. As shown below, given the tip, tilt, and piston degrees of freedoms of the

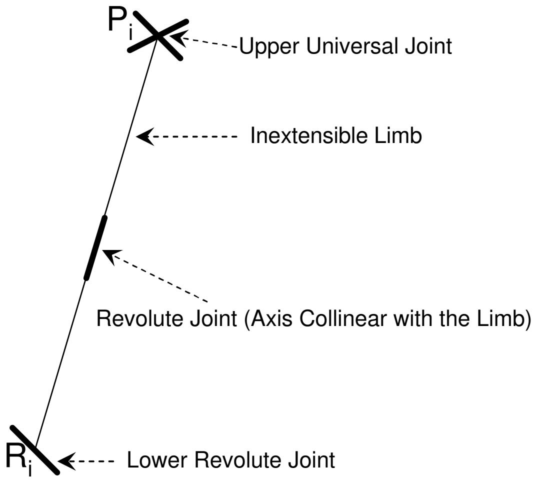

Figure 3: An alternative limb configuration.

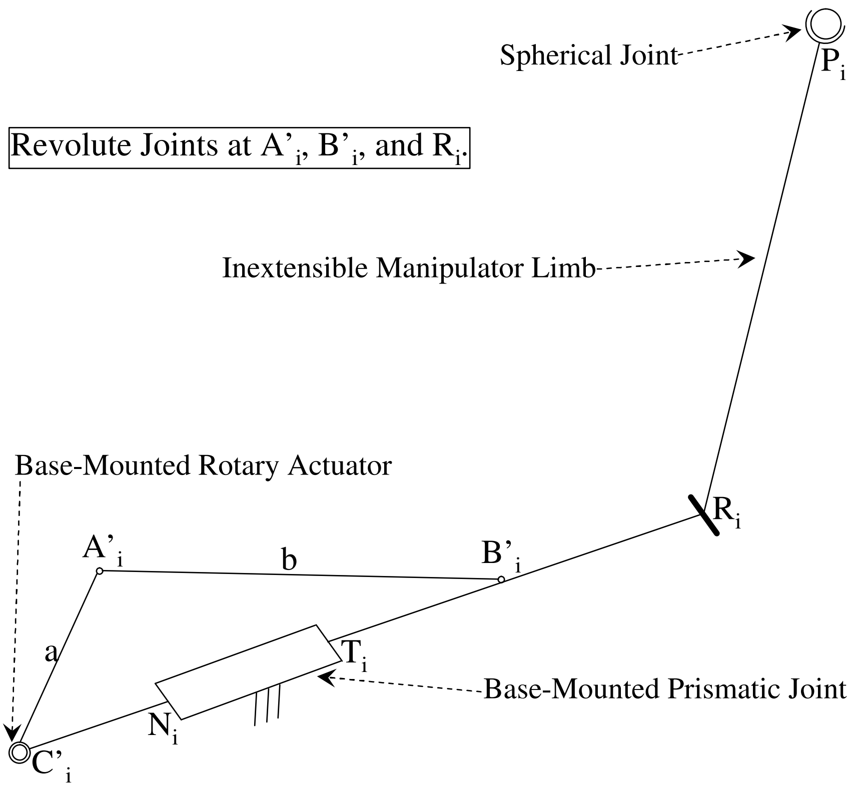

Figure 4: A slider-crank driver for the manipulator.

moving platform, its complete location can be determined.

Let us define the fixed base reference frame (XYZ) and the moving platform reference frame (UVW) in detail. The origin of the platform reference frame (point G) is placed at the centroid of triangle $P_1P_2P_3$ (see Figure 1). The positive U-axis is parallel to and points in the direction of vector $\overline{P_2P_3}$. The positive V-axis points from point G to point P_1. The W-axis is defined by the right-hand-rule. To keep the minimanipulator symmetric, triangle $P_1P_2P_3$ is made equilateral. Let the "home" or reference configuration of the manipulator be the one in which the limbs are perpendicular to the fixed base plane. The origin of the base reference frame (point O) is placed at the projection of point G onto the base plane at the reference configuration. The positive X-axis is parallel to and points in the direction of vector $\overline{R_2R_3}$ at the reference configuration. The positive Y-axis points from point O to point R_1. The Z-axis is defined by the right-hand-rule.

<u>*Revolute Joint Constraints*</u>

The revolute joints at R_1, R_2, and R_3 impose the following constraints on the coordinates of points P_1, P_2, and P_3 in the fixed reference frame.

$$X_{P,1} = 0 \quad , \quad X_{P,2} = \sqrt{3}Y_{P,2} \quad , \quad X_{P,3} = -\sqrt{3}Y_{P,3}, \tag{1}$$

where $X_{P,i}$ and $Y_{P,i}$ are the X and Y coordinates of point P_i, respectively. Let u_x, u_y, u_z be the XYZ components of a unit vector along the U axis. Similarly, let v_x, v_y, v_z represent the XYZ components of a unit vector along the V axis. Finally, let w_x, w_y, w_z denote the XYZ components of a unit vector along the W axis. Then

$$\begin{bmatrix} X_{P,i} \\ Y_{P,i} \\ Z_{P,i} \end{bmatrix} = \begin{bmatrix} X_G \\ Y_G \\ Z_G \end{bmatrix} + \begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix} \begin{bmatrix} U_{P,i} \\ V_{P,i} \\ W_{P,i} \end{bmatrix}, \tag{2}$$

where $U_{P,i}$, $V_{P,i}$, and $W_{P,i}$ are the U, V, and W coordinates of point P_i. Similarly, X_G, Y_G, and Z_G are the X, Y, and Z coordinates of point G. Let p denote the length of vector $\overline{GP_i}$. Then

$$\begin{bmatrix} U_{P,1} \\ V_{P,1} \\ W_{P,1} \end{bmatrix} = \begin{bmatrix} 0 \\ p \\ 0 \end{bmatrix} \quad , \quad \begin{bmatrix} U_{P,2} \\ V_{P,2} \\ W_{P,2} \end{bmatrix} = \begin{bmatrix} -\sqrt{3}p/2 \\ -0.5p \\ 0 \end{bmatrix} \quad , \quad \begin{bmatrix} U_{P,3} \\ V_{P,3} \\ W_{P,3} \end{bmatrix} = \begin{bmatrix} \sqrt{3}p/2 \\ -0.5p \\ 0 \end{bmatrix}. \tag{3}$$

Combining equations (2) and (3), we obtain

$$X_{P,1} = X_G + pv_x \ , \ X_{P,2} = X_G - \sqrt{3}pu_x/2 - 0.5pv_x \ , \ X_{P,3} = X_G + \sqrt{3}pu_x/2 - 0.5pv_x \tag{4}$$

$$Y_{P,1} = Y_G + pv_y \ , \ Y_{P,2} = Y_G - \sqrt{3}pu_y/2 - 0.5pv_y \ , \ Y_{P,3} = Y_G + \sqrt{3}pu_y/2 - 0.5pv_y \tag{5}$$

$$Z_{P,1} = Z_G + pv_z \ , \ Z_{P,2} = Z_G - \sqrt{3}pu_z/2 - 0.5pv_z \ , \ Z_{P,3} = Z_G + \sqrt{3}pu_z/2 - 0.5pv_z. \tag{6}$$

Substituting for $X_{P,1}$, $X_{P,2}$, $X_{P,3}$, $Y_{P,2}$, and $Y_{P,3}$ from equations (4) and (5) into equation (1) yields the following equations

$$X_G = -pv_x, \tag{7}$$

$$X_G - \sqrt{3}pu_x/2 - 0.5pv_x = \sqrt{3}(Y_G - \sqrt{3}pu_y/2 - 0.5pv_y), \textit{and} \tag{8}$$

$$X_G + \sqrt{3}pu_x/2 - 0.5pv_x = -\sqrt{3}(Y_G + \sqrt{3}pu_y/2 - 0.5pv_y). \tag{9}$$

Subtracting equation (9) from equation (8) and simplifying, we get

$$Y_G = 0.5p(v_y - u_x). \tag{10}$$

Adding equations (8) and (9) results in

$$2X_G - pv_x = -3pu_y. \tag{11}$$

Subtracting equation (11) from two times equation (7) and simplifying, we get

$$v_x = u_y. \tag{12}$$

Equations (7), (10), and (12) represent the constraints imposed by the revolute joints at R_1, R_2, and R_3 on the platform motion.

Analytical Solution

Let ψ, θ, and ϕ represent the rotations of the moving platform about the X, Y, and Z axes, respectively. The rotation matrix of the platform with respect to the fixed base ($\mathbf{R}$) can be expressed as [18]

$$\mathbf{R} = \begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix} = \begin{bmatrix} C\phi C\theta & C\phi S\theta S\psi - S\phi C\psi & C\phi S\theta C\psi + S\phi S\psi \\ S\phi C\theta & S\phi S\theta S\psi + C\phi C\psi & S\phi S\theta C\psi - C\phi S\psi \\ -S\theta & C\theta S\psi & C\theta C\psi \end{bmatrix}, \tag{13}$$

where C and S denote the cosine and sine trigonometric functions, respectively. Using the constraint expressed in equation (12) and equation (13), we can write

$$C\phi S\theta S\psi - S\phi C\psi = S\phi C\theta. \tag{14}$$

Equation (14) can be rearranged to obtain

$$\tan\phi = S\phi/C\phi = S\theta S\psi/(C\theta + C\psi). \tag{15}$$

Given ψ (tip) and θ (tilt), equation (15) can be solved for ϕ (twist). Two solutions are possible. Only one of the solutions is feasible. The other solution, which is 180 degrees greater than the feasible solution is not practical.

Now that we have all 3 platform rotation angles (i.e., ψ, θ, and ϕ); we can obtain u_x, u_y, and v_y from equation (13); and solve for X_G and Y_G using equations (7) and (10).

Having found the location (position and orientation) of the platform from its tip, tilt, and piston degrees of freedom; we can now turn our attention to determining l_i. The XYZ coordinates of point R_i is expressed in the following equation:

$$\begin{bmatrix} X_{R,1} \\ Y_{R,1} \\ Z_{R,1} \end{bmatrix} = \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} X_{R,2} \\ Y_{R,2} \\ Z_{R,2} \end{bmatrix} = \begin{bmatrix} -\sqrt{3}l_2/2 \\ -0.5l_2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} X_{R,3} \\ Y_{R,3} \\ Z_{R,3} \end{bmatrix} = \begin{bmatrix} \sqrt{3}l_3/2 \\ -0.5l_3 \\ 0 \end{bmatrix}. \tag{16}$$

Let the length of each inextensible limb be equal to r. Then

$$(X_{P,i} - X_{R,i})^2 + (Y_{P,i} - Y_{R,i})^2 + (Z_{P,i} - Z_{R,i})^2 = r^2. \tag{17}$$

Substituting from equations (4), (5), (6), and (16) into equation (17) yields

$$(X_G + pv_x)^2 + (Y_G + pv_y - l_1)^2 + (Z_G + pv_z)^2 = r^2 \tag{18}$$

$$(X_G - \sqrt{3}pu_x/2 - 0.5pv_x + \sqrt{3}l_2/2)^2 + (Y_G - \sqrt{3}pu_y/2 - 0.5pv_y + 0.5l_2)^2 + (Z_G - \sqrt{3}pu_z/2 - 0.5pv_z)^2 = r^2 \tag{19}$$

$$(X_G + \sqrt{3}pu_x/2 - 0.5pv_x - \sqrt{3}l_3/2)^2 + (Y_G + \sqrt{3}pu_y/2 - 0.5pv_y + 0.5l_3)^2 + (Z_G + \sqrt{3}pu_z/2 - 0.5pv_z)^2 = r^2. \tag{20}$$

Equations (18), (19), and (20) represent three quadratic equations in l_1, l_2, and l_3, respectively. Two solutions exist for each of the three unknowns. Geometrically, these two solutions represent intersections of a sphere, which is centered at P_i and has the radius of r, with the line OR_i.

Numerical Example

In this example, the above procedure is demonstrated. Let the length of each side of triangle $P_1P_2P_3$ be equal to q. The manipulator dimensions in this sample problem are

$$q = 1.5 \quad , \quad r = 1.$$

Using the relationship $p = q/\sqrt{3}$, we find p to be equal to 0.866. Note that only the ratios are important; therefore, r is set equal to 1.The tip, tilt, and piston variables for this example are

$$\psi = 5^\circ \quad , \quad \theta = 5^\circ \quad , \quad Z_G = 0.7.$$

Rotation of the platform about the Z axis (ϕ), X_G, and Y_G are found to be

$$\phi = 0.22^\circ \quad , \quad X_G = -0.0033 \quad , \quad Y_G = 0.0000.$$

The calculated results for l_i are

$$l_1 = 1.49 \ \ or \ \ 0.23 \quad , \quad l_2 = 1.55 \ \ or \ \ 0.18 \quad , \quad l_3 = 1.66 \ \ or \ \ 0.05.$$

The results of the above numerical example have been verified by performing a direct kinematics analysis.

DIRECT KINEMATICS

Solving the direct kinematics of the manipulator involves finding the location (position and orientation) of the moving platform, given the l_1, l_2, and l_3 lengths.

Angles between the Limbs and the Base

As shown in Figure 5, let η_i be the angle from vector $\overline{OR_i}$ to vector $\overline{R_iP_i}$. Also, let α_i be the angle from the positive X-axis to vector $\overline{OR_i}$. Angle α_i can be found (in radians) from

$$\alpha_i = \pi/2 + (i-1)2\pi/3. \tag{21}$$

The X and Y coordinates of point R_i in the fixed reference frame XYZ can be found from

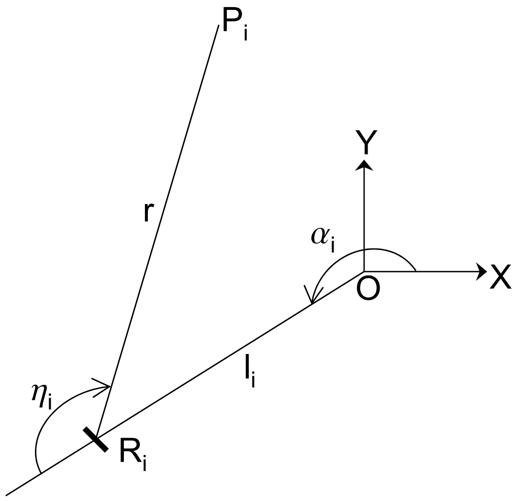

Figure 5: Depiction of angles α_i and η_i.

the following relationships

$$X_{R,i} = l_i C\alpha_i \quad , \quad Y_{R,i} = l_i S\alpha_i. \tag{22}$$

The coordinates of point P_i in the fixed reference frame XYZ are

$$X_{P,i} = rC\alpha_i C\eta_i + X_{R,i}, \tag{23}$$
$$Y_{P,i} = rS\alpha_i C\eta_i + Y_{R,i}, \tag{24}$$
$$Z_{P,i} = rS\eta_i. \tag{25}$$

Referring to Figure 1, we can write

$$(X_{P,i} - X_{P,i+1})^2 + (Y_{P,i} - Y_{P,i+1})^2 + (Z_{P,i} - Z_{P,i+1})^2 = q^2. \tag{26}$$

Substituting from equations (21), (22) into equations (23 and (24); and substituting the resulting expressions for $X_{P,i}$ and $X_{P,i}$ as well as equation (25) into equation (26) and simplifying, we obtain

$$A_i S\eta_i S\eta_{i+1} + B_i C\eta_i C\eta_{i+1} + D_i C\eta_i + E_i C\eta_{i+1} + F_i = 0, \tag{27}$$

where $A_i = -2r^2, B_i = r^2, D_i = 2rl_i + rl_{i+1}, E_i = rl_i + 2rl_{i+1}$, and $F_i = 2r^2 + l_i^2 + l_{i+1}^2 + l_i l_{i+1} - q^2$. Let $t_i = \tan(\eta_i/2)$. Then $C\eta_i = (1 - t_i^2)/(1 + t_i^2)$, and $S\eta_i = 2t_i/(1 + t_i^2)$. Substituting these expressions into equation (27) and simplifying, we obtain

$$G_i t_i^2 t_{i+1}^2 + H_i t_i^2 + I_i t_{i+1}^2 + J_i t_i t_{i+1} + K_i = 0, \tag{28}$$

where $G_i = B_i - D_i - E_i + F_i, H_i = -B_i - D_i + E_i + F_i, I_i = -B_i + D_i - E_i + F_i, J_i = 4A_i$, and $K_i = B_i + D_i + E_i + F_i$. Equation (28) can be rewritten, for $i = 1, 2, 3$, in the following forms

$$(G_1 t_1^2 + I_1)t_2^2 + (J_1 t_1)t_2 + (H_1 t_1^2 + K_1) = 0, \tag{29}$$

$$(G_2 t_3^2 + H_2)t_2^2 + (J_2 t_3)t_2 + (I_2 t_3^2 + K_2) = 0, \tag{30}$$

$$(G_3t_1^2 + H_3)t_3^2 + (J_3t_1)t_3 + (I_3t_1^2 + K_3) = 0. \tag{31}$$

Equations (29), (30), and (31) represent 3 equations in the three unknowns t_1, t_2, and t_3. In what follows, we will reduce these equations to an eighth-degree polynomial in the square of t_1.

•*Step 1 - Elimination of* t_2

We can think of equations (29) and (30) as two equations in the variable t_2. Let

$$L_1 = G_1t_1^2 + I_1 \quad , \quad M_1 = J_1t_1 \quad , \quad N_1 = H_1t_1^2 + K_1,$$

and

$$L_2 = G_2t_3^2 + H_2 \quad , \quad M_2 = J_2t_3 \quad , \quad N_2 = I_2t_3^2 + K_2,$$

then equations (29) and (30) can be written as

$$L_1t_2^2 + M_1t_2 + N_1 = 0, \tag{32}$$

and

$$L_2t_2^2 + M_2t_2 + N_2 = 0. \tag{33}$$

Multiplying equation (32) by L_2 and equation (33) by L_1, and subtracting, we obtain

$$(M_1L_2 - M_2L_1)t_2 + (N_1L_2 - N_2L_1) = 0. \tag{34}$$

Multiplying equation (32) by N_2 and equation (33) by N_1, subtracting, and dividing by t_2, we obtain

$$(L_1N_2 - L_2N_1)t_2 + (M_1N_2 - M_2N_1) = 0. \tag{35}$$

Equations (34) and (35) represent two linear equations in one unknown. Vanishing of their eliminant yields [19]

$$\begin{vmatrix} M_1L_2 - M_2L_1 & N_1L_2 - N_2L_1 \\ L_1N_2 - L_2N_1 & M_1N_2 - M_2N_1 \end{vmatrix} = 0. \tag{36}$$

Expanding equation (36) and substituting the expressions for L_1, M_1, N_1, L_2, M_2, and N_2 results in the following equation.

$$O_1t_3^4 + O_2t_3^3 + O_3t_3^2 + O_4t_3 + O_5 = 0, \tag{37}$$

where

$$\begin{aligned} O_1 &= U_1t_1^4 + U_2t_1^2 + U_3, \\ O_2 &= U_4t_1^3 + U_5t_1, \\ O_3 &= U_6t_1^4 + U_7t_1^2 + U_8, \\ O_4 &= U_9t_1^3 + U_{10}t_1, \\ O_5 &= U_{11}t_1^4 + U_{12}t_1^2 + U_{13}. \end{aligned}$$

Symbolic algebra program Macsyma [20] was used to find the following expressions for U_1 through U_{13}.

$$\begin{aligned} U_1 &= G_1^2I_2^2 - 2G_1G_2H_1I_2 + G_2^2H_1^2 \\ U_2 &= -2G_1G_2I_2K_1 + 2G_2^2H_1K_1 + G_2I_2J_1^2 + 2G_1I_1I_2^2 - 2G_2H_1I_1I_2 \end{aligned}$$

$$
\begin{aligned}
U_3 &= G_2^2K_1^2 - 2G_2I_1I_2K_1 + I_1^2I_2^2 \\
U_4 &= -G_1I_2J_1J_2 - G_2H_1J_1J_2 \\
U_5 &= -G_2J_1J_2K_1 - I_1I_2J_1J_2 \\
U_6 &= 2G_1^2I_2K_2 - 2G_1G_2H_1K_2 + G_1H_1J_2^2 - 2G_1H_1H_2I_2 + 2G_2H_1^2H_2 \\
U_7 &= -2G_1G_2K_1K_2 + G_2J_1^2K_2 + 4G_1I_1I_2K_2 - 2G_2H_1I_1K_2 + G_1J_2^2K_1 - \\
&\quad 2G_1H_2I_2K_1 + 4G_2H_1H_2K_1 + H_1I_1J_2^2 + H_2I_2J_1^2 - 2H_1H_2I_1I_2 \\
U_8 &= -2G_2I_1K_1K_2 + 2I_1^2I_2K_2 + 2G_2H_2K_1^2 + I_1J_2^2K_1 - 2H_2I_1I_2K_1 \\
U_9 &= -G_1J_1J_2K_2 - H_1H_2J_1J_2 \\
U_{10} &= -I_1J_1J_2K_2 - H_2J_1J_2K_1 \\
U_{11} &= G_1^2K_2^2 - 2G_1H_1H_2K_2 + H_1^2H_2^2 \\
U_{12} &= 2G_1I_1K_2^2 - 2G_1H_2K_1K_2 + H_2J_1^2K_2 - 2H_1H_2I_1K_2 + 2H_1H_2^2K_1 \\
U_{13} &= I_1^2K_2^2 - 2H_2I_1K_1K_2 + H_2^2K_1^2
\end{aligned}
$$

•*Step 2 - Elimination of t_3*
Equation (31) can be rewritten as

$$V_1t_3^2 + V_2t_3 + V_3 = 0, \tag{38}$$

where $V_1 = G_3t_1^2 + H_3, V_2 = J_3t_1$, and $V_3 = H_3t_1^2 + K_3$. We can think of equations (37) and (38) as two equations in the variable t_3. Multiplying equation (37) by V_1 and equation (38) by $O_1t_3^2$, and subtracting, we obtain

$$(O_2V_1 - O_1V_2)t_3^3 + (O_3V_1 - O_1V_3)t_3^2 + O_4V_1t_3 + O_5V_1 = 0. \tag{39}$$

Multiplying equation (37) by $V_1t_3 + V_2$ and equation (38) by $O_1t_3^3 + O_2t_3^2$, and subtracting, we obtain

$$(O_3V_1 - O_1V_3)t_3^3 + (O_4V_1 + O_3V_2 - O_2V_3)t_3^2 + (O_5V_1 + O_4V_2)t_3 + O_5V_2 = 0. \tag{40}$$

Multiplying equation (38) by t_3, we obtain

$$V_1t_3^3 + V_2t_3^2 + V_3t_3 = 0. \tag{41}$$

We can think of equations (39), (40), (41), and (38) as four linear equations in three unknowns t_3^3, t_3^2, and t_3. Vanishing of their eliminant yields [19]

$$\begin{vmatrix} O_2V_1 - O_1V_2 & O_3V_1 - O_1V_3 & O_4V_1 & O_5V_1 \\ O_3V_1 - O_1V_3 & O_4V_1 + O_3V_2 - O_2V_3 & O_5V_1 + O_4V_2 & O_5V_2 \\ V_1 & V_2 & V_3 & 0 \\ 0 & V_1 & V_2 & V_3 \end{vmatrix} = 0. \tag{42}$$

Expansion of equation (42) results in

$$
\begin{aligned}
&-O_5V_1[(O_3V_1 - O_1V_3)(V_2^2 - V_1V_3) - V_1V_2(-O_2V_3 + O_3V_2 + O_4V_1) + \\
&V_1^2(O_4V_2 + O_5V_1)] + (O_2V_1 - O_1V_2)[O_5V_2(V_2^2 - V_1V_3) + \\
&V_3^2(-O_2V_3 + O_3V_2 + O_4V_1) - V_2(O_4V_2 + O_5V_1)V_3] + \\
&O_4V_1[-V_1V_3(-O_2V_3 + O_3V_2 + O_4V_1) + V_2V_3(O_3V_1 - O_1V_3) + O_5V_1^2V_2] - \\
&(O_3V_1 - O_1V_3)[V_3^2(O_3V_1 - O_1V_3) - V_1(O_4V_2 + O_5V_1)V_3 + O_5V_1V_2^2] = 0.
\end{aligned} \tag{43}
$$

Equation (43) is an eighth-degree polynomial in the square of t_1 (see the Appendix). An example of such an eighth-degree polynomial is equation (44), which is shown in the following numerical sample. It follows that there are at most eight pairs of solutions for t_1. In each pair, one solution is the negative of the other one. The elimination procedure described above (Sylvester dialytic elimination) has also been used by many others including Roth [21]; Tahmasebi and Tsai [8]; and Ben-Horin and Shoham [11].

Location of the Platform

Having found t_1; t_2, and t_3 can be determined by back substituting t_1 into equations (29) and (31). The angle η_i can easily be determined from t_i. The XYZ coordinates of point P_i can then be found by substituting the values of η_i into equations (23), (24), and (25). The XYZ coordinates of points P_1, P_2, and P_3 completely define the location of the moving platform.

In summary, solving the direct kinematics problem results in at most eight pairs of manipulator locations. In each pair, one location is the mirror image of the other one with respect to the $R_1R_2R_3$ base plane (see the numerical example below).

Numerical Example

In this example, the direct kinematics procedure described above is demonstrated. The same parameters used in the inverse kinematics numerical example is used here. Namely, $r = 1$, $q = 1.5$, and $p = 0.866$. Let the input variables be

$$l_1 = 1.49 \quad , \quad l_2 = 0.18 \quad , \quad l_3 = 1.66.$$

Then, equation (43) reduces to

$$t_1^{16} - 130.1813t_1^{14} + 674.5842t_1^{12} + 4127.9989t_1^{10} - 18783.1681t_1^{8} - 55477.6465t_1^{6} + 134975.0739t_1^{4} + 330780.9874t_1^{2} + 11863.8182 = 0. \tag{44}$$

The 16 solutions for t_1 are

$$\pm ï0.1908,\ \pm ï1.5479,\ \pm ï1.68196,\ \pm 2.1049,\ \pm 2.3239,\ \pm ï1.8714,$$
$$\pm 2.1464,\ \pm 11.1583$$

where $ï=\sqrt{-1}$. The eight real solutions yield the values shown in Tables 1 and 2 for angles η_1, η_2, and η_3 (in degrees) and the coordinates of points P_1, P_2, P_3, and G. As mentioned earlier, triangle $P_1P_2P_3$ is equilateral. Therefore, the XYZ coordinates of point G in Tables 1 and 2 are calculated using the following relationships.

$$X_G = (X_{P,1} + X_{P,2} + X_{P,3})/3,\ Y_G = (Y_{P,1} + Y_{P,2} + Y_{P,3})/3,\ Z_G = (Z_{P,1} + Z_{P,2} + Z_{P,3})/3$$

The results of the numerical example have been verified by performing an inverse kinematics analysis. Note that pairs of solutions for points P_i and G are symmetric with respect to the base plane, as predicted. Also note that solution 1 corresponds to the inverse kinematics numerical example.

SUMMARY

In this paper, closed-form solutions for the direct and inverse kinematics of a new three DOF parallel manipulator, which uses base-mounted actuators and inextensible limbs, are

No.	1	2	3	4
η_1	129.1776	-129.1776	133.4338	-133.4338
η_2	46.6998	-46.6998	-144.4482	144.4482
η_3	143.3420	-143.3420	137.4258	-137.4258
$X_{P,1}$	0.0000	0.0000	0.0000	0.0000
$Y_{P,1}$	0.8628	0.8628	0.8070	0.8070
$Z_{P,1}$	0.7752	-0.7752	0.7262	-0.7262
$X_{P,2}$	-0.7521	-0.7521	0.5465	0.5465
$Y_{P,2}$	-0.4342	-0.4342	0.3155	0.3155
$Z_{P,2}$	0.7278	-0.7278	-0.5814	0.5814
$X_{P,3}$	0.7422	0.7422	0.7992	0.7992
$Y_{P,3}$	-0.4285	-0.4285	-0.4614	-0.4614
$Z_{P,3}$	0.5970	-0.5970	0.6765	-0.6765
X_G	-0.0033	-0.0033	0.4486	0.4486
Y_G	0.0000	0.0000	0.2203	0.2203
Z_G	0.7000	-0.7000	0.2738	-0.2738

Table 1: First four real solutions of the direct kinematics sample problem.

presented. The manipulator is suitable for alignment applications in which only tip, tilt, and piston motions are significant. It is shown that there are at most 16 solutions for the direct kinematics of the manipulator. To obtain these solutions, only an eighth-degree polynomial in the square of a single variable has to be solved. It is also shown that the 16 solutions are eight pairs of reflected configurations with respect to the plane passing through the lower ends of the manipulator's three limbs. Direct and inverse kinematics numerical examples are also presented.

APPENDIX - Expansion of Equation (43)

If we substitute the expressions for $V_1, V_2, V_3, O_1, O_2, O_3, O_4$, and O_5 into equation (43), and expand, we obtain

$$\Xi_1 t_1^{16} + \Xi_2 t_1^{14} + \Xi_3 t_1^{12} + \Xi_4 t_1^{10} + \Xi_5 t_1^{8} + \Xi_6 t_1^{6} + \Xi_7 t_1^{4} + \Xi_8 t_1^{2} + \Xi_9 = 0 \qquad (45)$$

where

$$\Xi_1 = -\mathrm{G_3}^2\,\mathrm{H_3}^2\,\mathrm{U_6}^2 + 2\,\mathrm{G_3}^3\,\mathrm{H_3}\,\mathrm{U_{11}}\,\mathrm{U_6} + 2\,\mathrm{G_3}\,\mathrm{H_3}^3\,\mathrm{U_1}\,\mathrm{U_6} - \mathrm{G_3}^4\,\mathrm{U_{11}}^2 - \\ 2\,\mathrm{G_3}^2\,\mathrm{H_3}^2\,\mathrm{U_1}\,\mathrm{U_{11}} - \mathrm{H_3}^4\,\mathrm{U_1}^2$$

$$\Xi_2 = -\mathrm{G_3}^3\,\mathrm{H_3}\,\mathrm{U_9}^2 + \mathrm{G_3}^2\,\mathrm{H_3}\,\mathrm{J_3}\,\mathrm{U_6}\,\mathrm{U_9} + 2\,\mathrm{G_3}^2\,\mathrm{H_3}^2\,\mathrm{U_4}\,\mathrm{U_9} + \mathrm{G_3}^3\,\mathrm{J_3}\,\mathrm{U_{11}}\,\mathrm{U_9} - \\ 3\,\mathrm{G_3}\,\mathrm{H_3}^2\,\mathrm{J_3}\,\mathrm{U_1}\,\mathrm{U_9} - 2\,\mathrm{G_3}^2\,\mathrm{H_3}^2\,\mathrm{U_6}\,\mathrm{U_7} + 2\,\mathrm{G_3}^3\,\mathrm{H_3}\,\mathrm{U_{11}}\,\mathrm{U_7} + 2\,\mathrm{G_3}\,\mathrm{H_3}^3\,\mathrm{U_1}\,\mathrm{U_7} - \\ 2\,\mathrm{G_3}^2\,\mathrm{H_3}\,\mathrm{K_3}\,\mathrm{U_6}^2 - 2\,\mathrm{G_3}\,\mathrm{H_3}^3\,\mathrm{U_6}^2 + \mathrm{G_3}\,\mathrm{H_3}^2\,\mathrm{J_3}\,\mathrm{U_4}\,\mathrm{U_6} + 2\,\mathrm{G_3}\,\mathrm{H_3}^3\,\mathrm{U_2}\,\mathrm{U_6} + \\ 2\,\mathrm{G_3}^3\,\mathrm{H_3}\,\mathrm{U_{12}}\,\mathrm{U_6} + 2\,\mathrm{G_3}^3\,\mathrm{K_3}\,\mathrm{U_{11}}\,\mathrm{U_6} - \mathrm{G_3}^2\,\mathrm{J_3}^2\,\mathrm{U_{11}}\,\mathrm{U_6} + 6\,\mathrm{G_3}^2\,\mathrm{H_3}^2\,\mathrm{U_{11}}\,\mathrm{U_6} + \\ 6\,\mathrm{G_3}\,\mathrm{H_3}^2\,\mathrm{K_3}\,\mathrm{U_1}\,\mathrm{U_6} - \mathrm{H_3}^2\,\mathrm{J_3}^2\,\mathrm{U_1}\,\mathrm{U_6} + 2\,\mathrm{H_3}^4\,\mathrm{U_1}\,\mathrm{U_6} - \mathrm{G_3}\,\mathrm{H_3}^3\,\mathrm{U_4}^2 -$$

No.	5	6	7	8
η_1	130.0379	-130.0379	169.7577	-169.7577
η_2	45.7923	-45.7923	-50.5084	50.5084
η_3	177.7693	-177.7693	-138.7068	138.7068
$X_{P,1}$	0.0000	0.0000	0.0000	0.0000
$Y_{P,1}$	0.8512	0.8512	0.5104	0.5104
$Z_{P,1}$	0.7656	-0.7656	0.1778	-0.1778
$X_{P,2}$	-0.7620	-0.7620	-0.7089	-0.7089
$Y_{P,2}$	-0.4399	-0.4399	-0.4093	-0.4093
$Z_{P,2}$	0.7168	-0.7168	-0.7717	0.7717
$X_{P,3}$	0.5716	0.5716	0.7863	0.7863
$Y_{P,3}$	-0.3300	-0.3300	-0.4539	-0.4539
$Z_{P,3}$	0.0389	-0.0389	-0.6599	0.6599
X_G	-0.0635	-0.0635	0.0258	0.0258
Y_G	0.0271	0.0271	-0.1176	-0.1176
Z_G	0.5071	-0.5071	-0.4179	0.4179

Table 2: Last four real solutions of the direct kinematics sample problem.

$$
\begin{aligned}
&3\,G_3{}^2\,H_3\,J_3\,U_{11}\,U_4 + H_3{}^3\,J_3\,U_1\,U_4 - 2\,G_3{}^2\,H_3{}^2\,U_{11}\,U_2 - 2\,H_3{}^4\,U_1\,U_2 - \\
&2\,G_3{}^4\,U_{11}\,U_{12} - 2\,G_3{}^2\,H_3{}^2\,U_1\,U_{12} - 4\,G_3{}^3\,H_3\,U_{11}{}^2 - 4\,G_3{}^2\,H_3\,K_3\,U_1\,U_{11} + \\
&4\,G_3\,H_3\,J_3{}^2\,U_1\,U_{11} - 4\,G_3\,H_3{}^3\,U_1\,U_{11} - 4\,H_3{}^3\,K_3\,U_1{}^2
\end{aligned}
$$

$$
\begin{aligned}
\Xi_3 = {} & -G_3{}^3\,K_3\,U_9{}^2 - 3\,G_3{}^2\,H_3{}^2\,U_9{}^2 + G_3{}^2\,H_3\,J_3\,U_7\,U_9 + G_3{}^2\,J_3\,K_3\,U_6\,U_9 + \\
&2\,G_3\,H_3{}^2\,J_3\,U_6\,U_9 + 2\,G_3{}^2\,H_3{}^2\,U_5\,U_9 + 4\,G_3{}^2\,H_3\,K_3\,U_4\,U_9 - G_3\,H_3\,J_3{}^2\,U_4\,U_9 + \\
&4\,G_3\,H_3{}^3\,U_4\,U_9 - 3\,G_3\,H_3{}^2\,J_3\,U_2\,U_9 + G_3{}^3\,J_3\,U_{12}\,U_9 + 3\,G_3{}^2\,H_3\,J_3\,U_{11}\,U_9 - \\
&2\,G_3{}^3\,H_3\,U_{10}\,U_9 - 6\,G_3\,H_3\,J_3\,K_3\,U_1\,U_9 + H_3\,J_3{}^3\,U_1\,U_9 - 3\,H_3{}^3\,J_3\,U_1\,U_9 - \\
&2\,G_3{}^2\,H_3{}^2\,U_6\,U_8 + 2\,G_3{}^3\,H_3\,U_{11}\,U_8 + 2\,G_3\,H_3{}^3\,U_1\,U_8 - G_3{}^2\,H_3{}^2\,U_7{}^2 - \\
&4\,G_3{}^2\,H_3\,K_3\,U_6\,U_7 - 4\,G_3\,H_3{}^3\,U_6\,U_7 + G_3\,H_3{}^2\,J_3\,U_4\,U_7 + 2\,G_3\,H_3{}^3\,U_2\,U_7 + \\
&2\,G_3{}^3\,H_3\,U_{12}\,U_7 + 2\,G_3{}^3\,K_3\,U_{11}\,U_7 - G_3{}^2\,J_3{}^2\,U_{11}\,U_7 + 6\,G_3{}^2\,H_3{}^2\,U_{11}\,U_7 + \\
&6\,G_3\,H_3{}^2\,K_3\,U_1\,U_7 - H_3{}^2\,J_3{}^2\,U_1\,U_7 + 2\,H_3{}^4\,U_1\,U_7 - G_3{}^2\,K_3{}^2\,U_6{}^2 - \\
&4\,G_3\,H_3{}^2\,K_3\,U_6{}^2 - H_3{}^4\,U_6{}^2 + G_3\,H_3{}^2\,J_3\,U_5\,U_6 + 2\,G_3\,H_3\,J_3\,K_3\,U_4\,U_6 + \\
&H_3{}^3\,J_3\,U_4\,U_6 + 2\,G_3\,H_3{}^3\,U_3\,U_6 + 6\,G_3\,H_3{}^2\,K_3\,U_2\,U_6 - H_3{}^2\,J_3{}^2\,U_2\,U_6 + \\
&2\,H_3{}^4\,U_2\,U_6 + 2\,G_3{}^3\,H_3\,U_{13}\,U_6 + 2\,G_3{}^3\,K_3\,U_{12}\,U_6 - G_3{}^2\,J_3{}^2\,U_{12}\,U_6 + \\
&6\,G_3{}^2\,H_3{}^2\,U_{12}\,U_6 + 6\,G_3{}^2\,H_3\,K_3\,U_{11}\,U_6 - 2\,G_3\,H_3\,J_3{}^2\,U_{11}\,U_6 + 6\,G_3\,H_3{}^3\,U_{11}\,U_6 + \\
&G_3{}^2\,H_3\,J_3\,U_{10}\,U_6 + 6\,G_3\,H_3\,K_3{}^2\,U_1\,U_6 - 2\,H_3\,J_3{}^2\,K_3\,U_1\,U_6 + 6\,H_3{}^3\,K_3\,U_1\,U_6 - \\
&2\,G_3\,H_3{}^3\,U_4\,U_5 - 3\,G_3{}^2\,H_3\,J_3\,U_{11}\,U_5 + H_3{}^3\,J_3\,U_1\,U_5 - 3\,G_3\,H_3{}^2\,K_3\,U_4{}^2 - \\
&H_3{}^4\,U_4{}^2 + H_3{}^3\,J_3\,U_2\,U_4 - 3\,G_3{}^2\,H_3\,J_3\,U_{12}\,U_4 - 3\,G_3{}^2\,J_3\,K_3\,U_{11}\,U_4 + \\
&G_3\,J_3{}^3\,U_{11}\,U_4 - 6\,G_3\,H_3{}^2\,J_3\,U_{11}\,U_4 + 2\,G_3{}^2\,H_3{}^2\,U_{10}\,U_4 + 3\,H_3{}^2\,J_3\,K_3\,U_1\,U_4 - \\
&2\,G_3{}^2\,H_3{}^2\,U_{11}\,U_3 - 2\,H_3{}^4\,U_1\,U_3 - H_3{}^4\,U_2{}^2 - 2\,G_3{}^2\,H_3{}^2\,U_{12}\,U_2 -
\end{aligned}
$$

$$4\,G_3{}^2\,H_3\,K_3\,U_{11}\,U_2+4\,G_3\,H_3\,J_3{}^2\,U_{11}\,U_2-4\,G_3\,H_3{}^3\,U_{11}\,U_2-8\,H_3{}^3\,K_3\,U_1\,U_2-$$
$$2\,G_3{}^4\,U_{11}\,U_{13}-2\,G_3{}^2\,H_3{}^2\,U_1\,U_{13}-G_3{}^4\,U_{12}{}^2-8\,G_3{}^3\,H_3\,U_{11}\,U_{12}-$$
$$4\,G_3{}^2\,H_3\,K_3\,U_1\,U_{12}+4\,G_3\,H_3\,J_3{}^2\,U_1\,U_{12}-4\,G_3\,H_3{}^3\,U_1\,U_{12}-6\,G_3{}^2\,H_3{}^2\,U_{11}{}^2+$$
$$G_3{}^3\,J_3\,U_{10}\,U_{11}-2\,G_3{}^2\,K_3{}^2\,U_1\,U_{11}+4\,G_3\,J_3{}^2\,K_3\,U_1\,U_{11}-8\,G_3\,H_3{}^2\,K_3\,U_1\,U_{11}-$$
$$J_3{}^4\,U_1\,U_{11}+4\,H_3{}^2\,J_3{}^2\,U_1\,U_{11}-2\,H_3{}^4\,U_1\,U_{11}-3\,G_3\,H_3{}^2\,J_3\,U_1\,U_{10}-$$
$$6\,H_3{}^2\,K_3{}^2\,U_1{}^2$$

$$\Xi_4=-3\,G_3{}^2\,H_3\,K_3\,U_9{}^2-3\,G_3\,H_3{}^3\,U_9{}^2+G_3{}^2\,H_3\,J_3\,U_8\,U_9+G_3{}^2\,J_3\,K_3\,U_7\,U_9+$$
$$2\,G_3\,H_3{}^2\,J_3\,U_7\,U_9+2\,G_3\,H_3\,J_3\,K_3\,U_6\,U_9+H_3{}^3\,J_3\,U_6\,U_9+4\,G_3{}^2\,H_3\,K_3\,U_5\,U_9-$$
$$G_3\,H_3\,J_3{}^2\,U_5\,U_9+4\,G_3\,H_3{}^3\,U_5\,U_9+2\,G_3{}^2\,K_3{}^2\,U_4\,U_9-G_3\,J_3{}^2\,K_3\,U_4\,U_9+$$
$$8\,G_3\,H_3{}^2\,K_3\,U_4\,U_9-H_3{}^2\,J_3{}^2\,U_4\,U_9+2\,H_3{}^4\,U_4\,U_9-3\,G_3\,H_3{}^2\,J_3\,U_3\,U_9-$$
$$6\,G_3\,H_3\,J_3\,K_3\,U_2\,U_9+H_3\,J_3{}^3\,U_2\,U_9-3\,H_3{}^3\,J_3\,U_2\,U_9+G_3{}^3\,J_3\,U_{13}\,U_9+$$
$$3\,G_3{}^2\,H_3\,J_3\,U_{12}\,U_9+3\,G_3\,H_3{}^2\,J_3\,U_{11}\,U_9-2\,G_3{}^3\,K_3\,U_{10}\,U_9-6\,G_3{}^2\,H_3{}^2\,U_{10}\,U_9-$$
$$3\,G_3\,J_3\,K_3{}^2\,U_1\,U_9+J_3{}^3\,K_3\,U_1\,U_9-6\,H_3{}^2\,J_3\,K_3\,U_1\,U_9-2\,G_3{}^2\,H_3{}^2\,U_7\,U_8-$$
$$4\,G_3{}^2\,H_3\,K_3\,U_6\,U_8-4\,G_3\,H_3{}^3\,U_6\,U_8+G_3\,H_3{}^2\,J_3\,U_4\,U_8+2\,G_3\,H_3{}^3\,U_2\,U_8+$$
$$2\,G_3{}^3\,H_3\,U_{12}\,U_8+2\,G_3{}^3\,K_3\,U_{11}\,U_8-G_3{}^2\,J_3{}^2\,U_{11}\,U_8+6\,G_3{}^2\,H_3{}^2\,U_{11}\,U_8+$$
$$6\,G_3\,H_3{}^2\,K_3\,U_1\,U_8-H_3{}^2\,J_3{}^2\,U_1\,U_8+2\,H_3{}^4\,U_1\,U_8-2\,G_3{}^2\,H_3\,K_3\,U_7{}^2-$$

$$2\,G_3\,H_3{}^3\,U_7{}^2-2\,G_3{}^2\,K_3{}^2\,U_6\,U_7-8\,G_3\,H_3{}^2\,K_3\,U_6\,U_7-2\,H_3{}^4\,U_6\,U_7+$$
$$G_3\,H_3{}^2\,J_3\,U_5\,U_7+2\,G_3\,H_3\,J_3\,K_3\,U_4\,U_7+H_3{}^3\,J_3\,U_4\,U_7+2\,G_3\,H_3{}^3\,U_3\,U_7+$$
$$6\,G_3\,H_3{}^2\,K_3\,U_2\,U_7-H_3{}^2\,J_3{}^2\,U_2\,U_7+2\,H_3{}^4\,U_2\,U_7+2\,G_3{}^3\,H_3\,U_{13}\,U_7+$$
$$2\,G_3{}^3\,K_3\,U_{12}\,U_7-G_3{}^2\,J_3{}^2\,U_{12}\,U_7+6\,G_3{}^2\,H_3{}^2\,U_{12}\,U_7+6\,G_3{}^2\,H_3\,K_3\,U_{11}\,U_7-$$
$$2\,G_3\,H_3\,J_3{}^2\,U_{11}\,U_7+6\,G_3\,H_3{}^3\,U_{11}\,U_7+G_3{}^2\,H_3\,J_3\,U_{10}\,U_7+6\,G_3\,H_3\,K_3{}^2\,U_1\,U_7-$$
$$2\,H_3\,J_3{}^2\,K_3\,U_1\,U_7+6\,H_3{}^3\,K_3\,U_1\,U_7-2\,G_3\,H_3\,K_3{}^2\,U_6{}^2-2\,H_3{}^3\,K_3\,U_6{}^2+$$
$$2\,G_3\,H_3\,J_3\,K_3\,U_5\,U_6+H_3{}^3\,J_3\,U_5\,U_6+G_3\,J_3\,K_3{}^2\,U_4\,U_6+2\,H_3{}^2\,J_3\,K_3\,U_4\,U_6+$$
$$6\,G_3\,H_3{}^2\,K_3\,U_3\,U_6-H_3{}^2\,J_3{}^2\,U_3\,U_6+2\,H_3{}^4\,U_3\,U_6+6\,G_3\,H_3\,K_3{}^2\,U_2\,U_6-$$
$$2\,H_3\,J_3{}^2\,K_3\,U_2\,U_6+6\,H_3{}^3\,K_3\,U_2\,U_6+2\,G_3{}^3\,K_3\,U_{13}\,U_6-G_3{}^2\,J_3{}^2\,U_{13}\,U_6+$$
$$6\,G_3{}^2\,H_3{}^2\,U_{13}\,U_6+6\,G_3{}^2\,H_3\,K_3\,U_{12}\,U_6-2\,G_3\,H_3\,J_3{}^2\,U_{12}\,U_6+6\,G_3\,H_3{}^3\,U_{12}\,U_6+$$

$$6\,G_3\,H_3{}^2\,K_3\,U_{11}\,U_6-H_3{}^2\,J_3{}^2\,U_{11}\,U_6+2\,H_3{}^4\,U_{11}\,U_6+G_3{}^2\,J_3\,K_3\,U_{10}\,U_6+$$
$$2\,G_3\,H_3{}^2\,J_3\,U_{10}\,U_6+2\,G_3\,K_3{}^3\,U_1\,U_6-J_3{}^2\,K_3{}^2\,U_1\,U_6+6\,H_3{}^2\,K_3{}^2\,U_1\,U_6-$$
$$G_3\,H_3{}^3\,U_5{}^2-6\,G_3\,H_3{}^2\,K_3\,U_4\,U_5-2\,H_3{}^4\,U_4\,U_5+H_3{}^3\,J_3\,U_2\,U_5-$$
$$3\,G_3{}^2\,H_3\,J_3\,U_{12}\,U_5-3\,G_3{}^2\,J_3\,K_3\,U_{11}\,U_5+G_3\,J_3{}^3\,U_{11}\,U_5-6\,G_3\,H_3{}^2\,J_3\,U_{11}\,U_5+$$
$$2\,G_3{}^2\,H_3{}^2\,U_{10}\,U_5+3\,H_3{}^2\,J_3\,K_3\,U_1\,U_5-3\,G_3\,H_3\,K_3{}^2\,U_4{}^2-3\,H_3{}^3\,K_3\,U_4{}^2+$$
$$H_3{}^3\,J_3\,U_3\,U_4+3\,H_3{}^2\,J_3\,K_3\,U_2\,U_4-3\,G_3{}^2\,H_3\,J_3\,U_{13}\,U_4-3\,G_3{}^2\,J_3\,K_3\,U_{12}\,U_4+$$
$$G_3\,J_3{}^3\,U_{12}\,U_4-6\,G_3\,H_3{}^2\,J_3\,U_{12}\,U_4-6\,G_3\,H_3\,J_3\,K_3\,U_{11}\,U_4+H_3\,J_3{}^3\,U_{11}\,U_4-$$
$$3\,H_3{}^3\,J_3\,U_{11}\,U_4+4\,G_3{}^2\,H_3\,K_3\,U_{10}\,U_4-G_3\,H_3\,J_3{}^2\,U_{10}\,U_4+4\,G_3\,H_3{}^3\,U_{10}\,U_4+$$
$$3\,H_3\,J_3\,K_3{}^2\,U_1\,U_4-2\,H_3{}^4\,U_2\,U_3-2\,G_3{}^2\,H_3{}^2\,U_{12}\,U_3-4\,G_3{}^2\,H_3\,K_3\,U_{11}\,U_3+$$

$$4\,G_3\,H_3\,J_3{}^2\,U_{11}\,U_3 - 4\,G_3\,H_3{}^3\,U_{11}\,U_3 - 8\,H_3{}^3\,K_3\,U_1\,U_3 - 4\,H_3{}^3\,K_3\,U_2{}^2 -$$
$$2\,G_3{}^2\,H_3{}^2\,U_{13}\,U_2 - 4\,G_3{}^2\,H_3\,K_3\,U_{12}\,U_2 + 4\,G_3\,H_3\,J_3{}^2\,U_{12}\,U_2 - 4\,G_3\,H_3{}^3\,U_{12}\,U_2 -$$
$$2\,G_3{}^2\,K_3{}^2\,U_{11}\,U_2 + 4\,G_3\,J_3{}^2\,K_3\,U_{11}\,U_2 - 8\,G_3\,H_3{}^2\,K_3\,U_{11}\,U_2 - J_3{}^4\,U_{11}\,U_2 +$$
$$4\,H_3{}^2\,J_3{}^2\,U_{11}\,U_2 - 2\,H_3{}^4\,U_{11}\,U_2 - 3\,G_3\,H_3{}^2\,J_3\,U_{10}\,U_2 - 12\,H_3{}^2\,K_3{}^2\,U_1\,U_2 -$$
$$2\,G_3{}^4\,U_{12}\,U_{13} - 8\,G_3{}^3\,H_3\,U_{11}\,U_{13} - 4\,G_3{}^2\,H_3\,K_3\,U_1\,U_{13} + 4\,G_3\,H_3\,J_3{}^2\,U_1\,U_{13} -$$
$$4\,G_3\,H_3{}^3\,U_1\,U_{13} - 4\,G_3{}^3\,H_3\,U_{12}{}^2 - 12\,G_3{}^2\,H_3{}^2\,U_{11}\,U_{12} + G_3{}^3\,J_3\,U_{10}\,U_{12} -$$
$$2\,G_3{}^2\,K_3{}^2\,U_1\,U_{12} + 4\,G_3\,J_3{}^2\,K_3\,U_1\,U_{12} - 8\,G_3\,H_3{}^2\,K_3\,U_1\,U_{12} - J_3{}^4\,U_1\,U_{12} +$$
$$4\,H_3{}^2\,J_3{}^2\,U_1\,U_{12} - 2\,H_3{}^4\,U_1\,U_{12} - 4\,G_3\,H_3{}^3\,U_{11}{}^2 + 3\,G_3{}^2\,H_3\,J_3\,U_{10}\,U_{11} -$$
$$4\,G_3\,H_3\,K_3{}^2\,U_1\,U_{11} + 4\,H_3\,J_3{}^2\,K_3\,U_1\,U_{11} - 4\,H_3{}^3\,K_3\,U_1\,U_{11} - G_3{}^3\,H_3\,U_{10}{}^2 -$$
$$6\,G_3\,H_3\,J_3\,K_3\,U_1\,U_{10} + H_3\,J_3{}^3\,U_1\,U_{10} - 3\,H_3{}^3\,J_3\,U_1\,U_{10} - 4\,H_3\,K_3{}^3\,U_1{}^2$$

$$\Xi_5 = -3\,G_3\,H_3{}^2\,K_3\,U_9{}^2 - H_3{}^4\,U_9{}^2 + G_3{}^2\,J_3\,K_3\,U_8\,U_9 + 2\,G_3\,H_3{}^2\,J_3\,U_8\,U_9 +$$
$$2\,G_3\,H_3\,J_3\,K_3\,U_7\,U_9 + H_3{}^3\,J_3\,U_7\,U_9 + H_3{}^2\,J_3\,K_3\,U_6\,U_9 + 2\,G_3{}^2\,K_3{}^2\,U_5\,U_9 -$$
$$G_3\,J_3{}^2\,K_3\,U_5\,U_9 + 8\,G_3\,H_3{}^2\,K_3\,U_5\,U_9 - H_3{}^2\,J_3{}^2\,U_5\,U_9 + 2\,H_3{}^4\,U_5\,U_9 +$$
$$4\,G_3\,H_3\,K_3{}^2\,U_4\,U_9 - H_3\,J_3{}^2\,K_3\,U_4\,U_9 + 4\,H_3{}^3\,K_3\,U_4\,U_9 - 6\,G_3\,H_3\,J_3\,K_3\,U_3\,U_9 +$$
$$H_3\,J_3{}^3\,U_3\,U_9 - 3\,H_3{}^3\,J_3\,U_3\,U_9 - 3\,G_3\,J_3\,K_3{}^2\,U_2\,U_9 + J_3{}^3\,K_3\,U_2\,U_9 -$$
$$6\,H_3{}^2\,J_3\,K_3\,U_2\,U_9 + 3\,G_3{}^2\,H_3\,J_3\,U_{13}\,U_9 + 3\,G_3\,H_3{}^2\,J_3\,U_{12}\,U_9 + H_3{}^3\,J_3\,U_{11}\,U_9 -$$
$$6\,G_3{}^2\,H_3\,K_3\,U_{10}\,U_9 - 6\,G_3\,H_3{}^3\,U_{10}\,U_9 - 3\,H_3\,J_3\,K_3{}^2\,U_1\,U_9 - G_3{}^2\,H_3{}^2\,U_8{}^2 -$$
$$4\,G_3{}^2\,H_3\,K_3\,U_7\,U_8 - 4\,G_3\,H_3{}^3\,U_7\,U_8 - 2\,G_3{}^2\,K_3{}^2\,U_6\,U_8 - 8\,G_3\,H_3{}^2\,K_3\,U_6\,U_8 -$$
$$2\,H_3{}^4\,U_6\,U_8 + G_3\,H_3{}^2\,J_3\,U_5\,U_8 + 2\,G_3\,H_3\,J_3\,K_3\,U_4\,U_8 + H_3{}^3\,J_3\,U_4\,U_8 +$$
$$2\,G_3\,H_3{}^3\,U_3\,U_8 + 6\,G_3\,H_3{}^2\,K_3\,U_2\,U_8 - H_3{}^2\,J_3{}^2\,U_2\,U_8 + 2\,H_3{}^4\,U_2\,U_8 +$$

$$2\,G_3{}^3\,H_3\,U_{13}\,U_8 + 2\,G_3{}^3\,K_3\,U_{12}\,U_8 - G_3{}^2\,J_3{}^2\,U_{12}\,U_8 + 6\,G_3{}^2\,H_3{}^2\,U_{12}\,U_8 +$$
$$6\,G_3{}^2\,H_3\,K_3\,U_{11}\,U_8 - 2\,G_3\,H_3\,J_3{}^2\,U_{11}\,U_8 + 6\,G_3\,H_3{}^3\,U_{11}\,U_8 + G_3{}^2\,H_3\,J_3\,U_{10}\,U_8 +$$
$$6\,G_3\,H_3\,K_3{}^2\,U_1\,U_8 - 2\,H_3\,J_3{}^2\,K_3\,U_1\,U_8 + 6\,H_3{}^3\,K_3\,U_1\,U_8 - G_3{}^2\,K_3{}^2\,U_7{}^2 -$$
$$4\,G_3\,H_3{}^2\,K_3\,U_7{}^2 - H_3{}^4\,U_7{}^2 - 4\,G_3\,H_3\,K_3{}^2\,U_6\,U_7 - 4\,H_3{}^3\,K_3\,U_6\,U_7 +$$
$$2\,G_3\,H_3\,J_3\,K_3\,U_5\,U_7 + H_3{}^3\,J_3\,U_5\,U_7 + G_3\,J_3\,K_3{}^2\,U_4\,U_7 + 2\,H_3{}^2\,J_3\,K_3\,U_4\,U_7 +$$
$$6\,G_3\,H_3{}^2\,K_3\,U_3\,U_7 - H_3{}^2\,J_3{}^2\,U_3\,U_7 + 2\,H_3{}^4\,U_3\,U_7 + 6\,G_3\,H_3\,K_3{}^2\,U_2\,U_7 -$$
$$2\,H_3\,J_3{}^2\,K_3\,U_2\,U_7 + 6\,H_3{}^3\,K_3\,U_2\,U_7 + 2\,G_3{}^3\,K_3\,U_{13}\,U_7 - G_3{}^2\,J_3{}^2\,U_{13}\,U_7 +$$
$$6\,G_3{}^2\,H_3{}^2\,U_{13}\,U_7 + 6\,G_3{}^2\,H_3\,K_3\,U_{12}\,U_7 - 2\,G_3\,H_3\,J_3{}^2\,U_{12}\,U_7 + 6\,G_3\,H_3{}^3\,U_{12}\,U_7 +$$
$$6\,G_3\,H_3{}^2\,K_3\,U_{11}\,U_7 - H_3{}^2\,J_3{}^2\,U_{11}\,U_7 + 2\,H_3{}^4\,U_{11}\,U_7 + G_3{}^2\,J_3\,K_3\,U_{10}\,U_7 +$$
$$2\,G_3\,H_3{}^2\,J_3\,U_{10}\,U_7 + 2\,G_3\,K_3{}^3\,U_1\,U_7 - J_3{}^2\,K_3{}^2\,U_1\,U_7 + 6\,H_3{}^2\,K_3{}^2\,U_1\,U_7 -$$

$$H_3{}^2\,K_3{}^2\,U_6{}^2 + G_3\,J_3\,K_3{}^2\,U_5\,U_6 + 2\,H_3{}^2\,J_3\,K_3\,U_5\,U_6 + H_3\,J_3\,K_3{}^2\,U_4\,U_6 +$$
$$6\,G_3\,H_3\,K_3{}^2\,U_3\,U_6 - 2\,H_3\,J_3{}^2\,K_3\,U_3\,U_6 + 6\,H_3{}^3\,K_3\,U_3\,U_6 + 2\,G_3\,K_3{}^3\,U_2\,U_6 -$$
$$J_3{}^2\,K_3{}^2\,U_2\,U_6 + 6\,H_3{}^2\,K_3{}^2\,U_2\,U_6 + 6\,G_3{}^2\,H_3\,K_3\,U_{13}\,U_6 - 2\,G_3\,H_3\,J_3{}^2\,U_{13}\,U_6 +$$
$$6\,G_3\,H_3{}^3\,U_{13}\,U_6 + 6\,G_3\,H_3{}^2\,K_3\,U_{12}\,U_6 - H_3{}^2\,J_3{}^2\,U_{12}\,U_6 + 2\,H_3{}^4\,U_{12}\,U_6 +$$
$$2\,H_3{}^3\,K_3\,U_{11}\,U_6 + 2\,G_3\,H_3\,J_3\,K_3\,U_{10}\,U_6 + H_3{}^3\,J_3\,U_{10}\,U_6 + 2\,H_3\,K_3{}^3\,U_1\,U_6 -$$
$$3\,G_3\,H_3{}^2\,K_3\,U_5{}^2 - H_3{}^4\,U_5{}^2 - 6\,G_3\,H_3\,K_3{}^2\,U_4\,U_5 - 6\,H_3{}^3\,K_3\,U_4\,U_5 +$$
$$H_3{}^3\,J_3\,U_3\,U_5 + 3\,H_3{}^2\,J_3\,K_3\,U_2\,U_5 - 3\,G_3{}^2\,H_3\,J_3\,U_{13}\,U_5 - 3\,G_3{}^2\,J_3\,K_3\,U_{12}\,U_5 +$$

$$G_3\,J_3^3\,U_{12}\,U_5 - 6\,G_3\,H_3^2\,J_3\,U_{12}\,U_5 - 6\,G_3\,H_3\,J_3\,K_3\,U_{11}\,U_5 + H_3\,J_3^3\,U_{11}\,U_5 -$$
$$3\,H_3^3\,J_3\,U_{11}\,U_5 + 4\,G_3^2\,H_3\,K_3\,U_{10}\,U_5 - G_3\,H_3\,J_3^2\,U_{10}\,U_5 + 4\,G_3\,H_3^3\,U_{10}\,U_5 +$$
$$3\,H_3\,J_3\,K_3^2\,U_1\,U_5 - G_3\,K_3^3\,U_4^2 - 3\,H_3^2\,K_3^2\,U_4^2 + 3\,H_3^2\,J_3\,K_3\,U_3\,U_4 +$$
$$3\,H_3\,J_3\,K_3^2\,U_2\,U_4 - 3\,G_3^2\,J_3\,K_3\,U_{13}\,U_4 + G_3\,J_3^3\,U_{13}\,U_4 - 6\,G_3\,H_3^2\,J_3\,U_{13}\,U_4 -$$
$$6\,G_3\,H_3\,J_3\,K_3\,U_{12}\,U_4 + H_3\,J_3^3\,U_{12}\,U_4 - 3\,H_3^3\,J_3\,U_{12}\,U_4 - 3\,H_3^2\,J_3\,K_3\,U_{11}\,U_4 +$$
$$2\,G_3^2\,K_3^2\,U_{10}\,U_4 - G_3\,J_3^2\,K_3\,U_{10}\,U_4 + 8\,G_3\,H_3^2\,K_3\,U_{10}\,U_4 - H_3^2\,J_3^2\,U_{10}\,U_4 +$$
$$2\,H_3^4\,U_{10}\,U_4 + J_3\,K_3^3\,U_1\,U_4 - H_3^4\,U_3^2 - 8\,H_3^3\,K_3\,U_2\,U_3 -$$
$$2\,G_3^2\,H_3^2\,U_{13}\,U_3 - 4\,G_3^2\,H_3\,K_3\,U_{12}\,U_3 + 4\,G_3\,H_3\,J_3^2\,U_{12}\,U_3 - 4\,G_3\,H_3^3\,U_{12}\,U_3 -$$
$$2\,G_3^2\,K_3^2\,U_{11}\,U_3 + 4\,G_3\,J_3^2\,K_3\,U_{11}\,U_3 - 8\,G_3\,H_3^2\,K_3\,U_{11}\,U_3 - J_3^4\,U_{11}\,U_3 +$$
$$4\,H_3^2\,J_3^2\,U_{11}\,U_3 - 2\,H_3^4\,U_{11}\,U_3 - 3\,G_3\,H_3^2\,J_3\,U_{10}\,U_3 - 12\,H_3^2\,K_3^2\,U_1\,U_3 -$$
$$6\,H_3^2\,K_3^2\,U_2^2 - 4\,G_3^2\,H_3\,K_3\,U_{13}\,U_2 + 4\,G_3\,H_3\,J_3^2\,U_{13}\,U_2 - 4\,G_3\,H_3^3\,U_{13}\,U_2 -$$
$$2\,G_3^2\,K_3^2\,U_{12}\,U_2 + 4\,G_3\,J_3^2\,K_3\,U_{12}\,U_2 - 8\,G_3\,H_3^2\,K_3\,U_{12}\,U_2 - J_3^4\,U_{12}\,U_2 +$$
$$4\,H_3^2\,J_3^2\,U_{12}\,U_2 - 2\,H_3^4\,U_{12}\,U_2 - 4\,G_3\,H_3\,K_3^2\,U_{11}\,U_2 + 4\,H_3\,J_3^2\,K_3\,U_{11}\,U_2 -$$
$$4\,H_3^3\,K_3\,U_{11}\,U_2 - 6\,G_3\,H_3\,J_3\,K_3\,U_{10}\,U_2 + H_3\,J_3^3\,U_{10}\,U_2 - 3\,H_3^3\,J_3\,U_{10}\,U_2 -$$
$$8\,H_3\,K_3^3\,U_1\,U_2 - G_3^4\,U_{13}^2 - 8\,G_3^3\,H_3\,U_{12}\,U_{13} - 12\,G_3^2\,H_3^2\,U_{11}\,U_{13} +$$
$$G_3^3\,J_3\,U_{10}\,U_{13} - 2\,G_3^2\,K_3^2\,U_1\,U_{13} + 4\,G_3\,J_3^2\,K_3\,U_1\,U_{13} - 8\,G_3\,H_3^2\,K_3\,U_1\,U_{13} -$$
$$J_3^4\,U_1\,U_{13} + 4\,H_3^2\,J_3^2\,U_1\,U_{13} - 2\,H_3^4\,U_1\,U_{13} - 6\,G_3^2\,H_3^2\,U_{12}^2 -$$
$$8\,G_3\,H_3^3\,U_{11}\,U_{12} + 3\,G_3^2\,H_3\,J_3\,U_{10}\,U_{12} - 4\,G_3\,H_3\,K_3^2\,U_1\,U_{12} + 4\,H_3\,J_3^2\,K_3\,U_1\,U_{12} -$$
$$4\,H_3^3\,K_3\,U_1\,U_{12} - H_3^4\,U_{11}^2 + 3\,G_3\,H_3^2\,J_3\,U_{10}\,U_{11} - 2\,H_3^2\,K_3^2\,U_1\,U_{11} -$$
$$G_3^3\,K_3\,U_{10}^2 - 3\,G_3^2\,H_3^2\,U_{10}^2 - 3\,G_3\,J_3\,K_3^2\,U_1\,U_{10} + J_3^3\,K_3\,U_1\,U_{10} -$$
$$6\,H_3^2\,J_3\,K_3\,U_1\,U_{10} - K_3^4\,U_1^2$$

$$\Xi_6 = -H_3^3\,K_3\,U_9^2 + 2\,G_3\,H_3\,J_3\,K_3\,U_8\,U_9 + H_3^3\,J_3\,U_8\,U_9 + H_3^2\,J_3\,K_3\,U_7\,U_9 +$$
$$4\,G_3\,H_3\,K_3^2\,U_5\,U_9 - H_3\,J_3^2\,K_3\,U_5\,U_9 + 4\,H_3^3\,K_3\,U_5\,U_9 + 2\,H_3^2\,K_3^2\,U_4\,U_9 -$$
$$3\,G_3\,J_3\,K_3^2\,U_3\,U_9 + J_3^3\,K_3\,U_3\,U_9 - 6\,H_3^2\,J_3\,K_3\,U_3\,U_9 - 3\,H_3\,J_3\,K_3^2\,U_2\,U_9 +$$
$$3\,G_3\,H_3^2\,J_3\,U_{13}\,U_9 + H_3^3\,J_3\,U_{12}\,U_9 - 6\,G_3\,H_3^2\,K_3\,U_{10}\,U_9 - 2\,H_3^4\,U_{10}\,U_9 -$$
$$2\,G_3^2\,H_3\,K_3\,U_8^2 - 2\,G_3\,H_3^3\,U_8^2 - 2\,G_3^2\,K_3^2\,U_7\,U_8 - 8\,G_3\,H_3^2\,K_3\,U_7\,U_8 -$$
$$2\,H_3^4\,U_7\,U_8 - 4\,G_3\,H_3\,K_3^2\,U_6\,U_8 - 4\,H_3^3\,K_3\,U_6\,U_8 + 2\,G_3\,H_3\,J_3\,K_3\,U_5\,U_8 +$$
$$H_3^3\,J_3\,U_5\,U_8 + G_3\,J_3\,K_3^2\,U_4\,U_8 + 2\,H_3^2\,J_3\,K_3\,U_4\,U_8 + 6\,G_3\,H_3^2\,K_3\,U_3\,U_8 -$$
$$H_3^2\,J_3^2\,U_3\,U_8 + 2\,H_3^4\,U_3\,U_8 + 6\,G_3\,H_3\,K_3^2\,U_2\,U_8 - 2\,H_3\,J_3^2\,K_3\,U_2\,U_8 +$$
$$6\,H_3^3\,K_3\,U_2\,U_8 + 2\,G_3^3\,K_3\,U_{13}\,U_8 - G_3^2\,J_3^2\,U_{13}\,U_8 + 6\,G_3^2\,H_3^2\,U_{13}\,U_8 +$$
$$6\,G_3^2\,H_3\,K_3\,U_{12}\,U_8 - 2\,G_3\,H_3\,J_3^2\,U_{12}\,U_8 + 6\,G_3\,H_3^3\,U_{12}\,U_8 + 6\,G_3\,H_3^2\,K_3\,U_{11}\,U_8 -$$
$$H_3^2\,J_3^2\,U_{11}\,U_8 + 2\,H_3^4\,U_{11}\,U_8 + G_3^2\,J_3\,K_3\,U_{10}\,U_8 + 2\,G_3\,H_3^2\,J_3\,U_{10}\,U_8 +$$
$$2\,G_3\,K_3^3\,U_1\,U_8 - J_3^2\,K_3^2\,U_1\,U_8 + 6\,H_3^2\,K_3^2\,U_1\,U_8 - 2\,G_3\,H_3\,K_3^2\,U_7^2 -$$
$$2\,H_3^3\,K_3\,U_7^2 - 2\,H_3^2\,K_3^2\,U_6\,U_7 + G_3\,J_3\,K_3^2\,U_5\,U_7 + 2\,H_3^2\,J_3\,K_3\,U_5\,U_7 +$$
$$H_3\,J_3\,K_3^2\,U_4\,U_7 + 6\,G_3\,H_3\,K_3^2\,U_3\,U_7 - 2\,H_3\,J_3^2\,K_3\,U_3\,U_7 + 6\,H_3^3\,K_3\,U_3\,U_7 +$$
$$2\,G_3\,K_3^3\,U_2\,U_7 - J_3^2\,K_3^2\,U_2\,U_7 + 6\,H_3^2\,K_3^2\,U_2\,U_7 + 6\,G_3^2\,H_3\,K_3\,U_{13}\,U_7 -$$
$$2\,G_3\,H_3\,J_3^2\,U_{13}\,U_7 + 6\,G_3\,H_3^3\,U_{13}\,U_7 + 6\,G_3\,H_3^2\,K_3\,U_{12}\,U_7 - H_3^2\,J_3^2\,U_{12}\,U_7 +$$

$$2\,H_3{}^4\,U_{12}\,U_7+2\,H_3{}^3\,K_3\,U_{11}\,U_7+2\,G_3\,H_3\,J_3\,K_3\,U_{10}\,U_7+H_3{}^3\,J_3\,U_{10}\,U_7+$$
$$2\,H_3\,K_3{}^3\,U_1\,U_7+H_3\,J_3\,K_3{}^2\,U_5\,U_6+2\,G_3\,K_3{}^3\,U_3\,U_6-J_3{}^2\,K_3{}^2\,U_3\,U_6+$$
$$6\,H_3{}^2\,K_3{}^2\,U_3\,U_6+2\,H_3\,K_3{}^3\,U_2\,U_6+6\,G_3\,H_3{}^2\,K_3\,U_{13}\,U_6-H_3{}^2\,J_3{}^2\,U_{13}\,U_6+$$
$$2\,H_3{}^4\,U_{13}\,U_6+2\,H_3{}^3\,K_3\,U_{12}\,U_6+H_3{}^2\,J_3\,K_3\,U_{10}\,U_6-3\,G_3\,H_3\,K_3{}^2\,U_5{}^2-$$

$$3\,H_3{}^3\,K_3\,U_5{}^2-2\,G_3\,K_3{}^3\,U_4\,U_5-6\,H_3{}^2\,K_3{}^2\,U_4\,U_5+3\,H_3{}^2\,J_3\,K_3\,U_3\,U_5+$$
$$3\,H_3\,J_3\,K_3{}^2\,U_2\,U_5-3\,G_3{}^2\,J_3\,K_3\,U_{13}\,U_5+G_3\,J_3{}^3\,U_{13}\,U_5-6\,G_3\,H_3{}^2\,J_3\,U_{13}\,U_5-$$
$$6\,G_3\,H_3\,J_3\,K_3\,U_{12}\,U_5+H_3\,J_3{}^3\,U_{12}\,U_5-3\,H_3{}^3\,J_3\,U_{12}\,U_5-3\,H_3{}^2\,J_3\,K_3\,U_{11}\,U_5+$$
$$2\,G_3{}^2\,K_3{}^2\,U_{10}\,U_5-G_3\,J_3{}^2\,K_3\,U_{10}\,U_5+8\,G_3\,H_3{}^2\,K_3\,U_{10}\,U_5-H_3{}^2\,J_3{}^2\,U_{10}\,U_5+$$
$$2\,H_3{}^4\,U_{10}\,U_5+J_3\,K_3{}^3\,U_1\,U_5-H_3\,K_3{}^3\,U_4{}^2+3\,H_3\,J_3\,K_3{}^2\,U_3\,U_4+$$
$$J_3\,K_3{}^3\,U_2\,U_4-6\,G_3\,H_3\,J_3\,K_3\,U_{13}\,U_4+H_3\,J_3{}^3\,U_{13}\,U_4-3\,H_3{}^3\,J_3\,U_{13}\,U_4-$$
$$3\,H_3{}^2\,J_3\,K_3\,U_{12}\,U_4+4\,G_3\,H_3\,K_3{}^2\,U_{10}\,U_4-H_3\,J_3{}^2\,K_3\,U_{10}\,U_4+4\,H_3{}^3\,K_3\,U_{10}\,U_4-$$
$$4\,H_3{}^3\,K_3\,U_3{}^2-12\,H_3{}^2\,K_3{}^2\,U_2\,U_3-4\,G_3{}^2\,H_3\,K_3\,U_{13}\,U_3+4\,G_3\,H_3\,J_3{}^2\,U_{13}\,U_3-$$
$$4\,G_3\,H_3{}^3\,U_{13}\,U_3-2\,G_3{}^2\,K_3{}^2\,U_{12}\,U_3+4\,G_3\,J_3{}^2\,K_3\,U_{12}\,U_3-8\,G_3\,H_3{}^2\,K_3\,U_{12}\,U_3-$$
$$J_3{}^4\,U_{12}\,U_3+4\,H_3{}^2\,J_3{}^2\,U_{12}\,U_3-2\,H_3{}^4\,U_{12}\,U_3-4\,G_3\,H_3\,K_3{}^2\,U_{11}\,U_3+$$

$$4\,H_3\,J_3{}^2\,K_3\,U_{11}\,U_3-4\,H_3{}^3\,K_3\,U_{11}\,U_3-6\,G_3\,H_3\,J_3\,K_3\,U_{10}\,U_3+H_3\,J_3{}^3\,U_{10}\,U_3-$$
$$3\,H_3{}^3\,J_3\,U_{10}\,U_3-8\,H_3\,K_3{}^3\,U_1\,U_3-4\,H_3\,K_3{}^3\,U_2{}^2-2\,G_3{}^2\,K_3{}^2\,U_{13}\,U_2+$$
$$4\,G_3\,J_3{}^2\,K_3\,U_{13}\,U_2-8\,G_3\,H_3{}^2\,K_3\,U_{13}\,U_2-J_3{}^4\,U_{13}\,U_2+4\,H_3{}^2\,J_3{}^2\,U_{13}\,U_2-$$
$$2\,H_3{}^4\,U_{13}\,U_2-4\,G_3\,H_3\,K_3{}^2\,U_{12}\,U_2+4\,H_3\,J_3{}^2\,K_3\,U_{12}\,U_2-4\,H_3{}^3\,K_3\,U_{12}\,U_2-$$
$$2\,H_3{}^2\,K_3{}^2\,U_{11}\,U_2-3\,G_3\,J_3\,K_3{}^2\,U_{10}\,U_2+J_3{}^3\,K_3\,U_{10}\,U_2-6\,H_3{}^2\,J_3\,K_3\,U_{10}\,U_2-$$
$$2\,K_3{}^4\,U_1\,U_2-4\,G_3{}^3\,H_3\,U_{13}{}^2-12\,G_3{}^2\,H_3{}^2\,U_{12}\,U_{13}-8\,G_3\,H_3{}^3\,U_{11}\,U_{13}+$$
$$3\,G_3{}^2\,H_3\,J_3\,U_{10}\,U_{13}-4\,G_3\,H_3\,K_3{}^2\,U_1\,U_{13}+4\,H_3\,J_3{}^2\,K_3\,U_1\,U_{13}-4\,H_3{}^3\,K_3\,U_1\,U_{13}-$$
$$4\,G_3\,H_3{}^3\,U_{12}{}^2-2\,H_3{}^4\,U_{11}\,U_{12}+3\,G_3\,H_3{}^2\,J_3\,U_{10}\,U_{12}-2\,H_3{}^2\,K_3{}^2\,U_1\,U_{12}+$$
$$H_3{}^3\,J_3\,U_{10}\,U_{11}-3\,G_3{}^2\,H_3\,K_3\,U_{10}{}^2-3\,G_3\,H_3{}^3\,U_{10}{}^2-3\,H_3\,J_3\,K_3{}^2\,U_1\,U_{10}$$

$$\Xi_7=H_3{}^2\,J_3\,K_3\,U_8\,U_9+2\,H_3{}^2\,K_3{}^2\,U_5\,U_9-3\,H_3\,J_3\,K_3{}^2\,U_3\,U_9+H_3{}^3\,J_3\,U_{13}\,U_9-$$
$$2\,H_3{}^3\,K_3\,U_{10}\,U_9-G_3{}^2\,K_3{}^2\,U_8{}^2-4\,G_3\,H_3{}^2\,K_3\,U_8{}^2-H_3{}^4\,U_8{}^2-$$
$$4\,G_3\,H_3\,K_3{}^2\,U_7\,U_8-4\,H_3{}^3\,K_3\,U_7\,U_8-2\,H_3{}^2\,K_3{}^2\,U_6\,U_8+G_3\,J_3\,K_3{}^2\,U_5\,U_8+$$
$$2\,H_3{}^2\,J_3\,K_3\,U_5\,U_8+H_3\,J_3\,K_3{}^2\,U_4\,U_8+6\,G_3\,H_3\,K_3{}^2\,U_3\,U_8-2\,H_3\,J_3{}^2\,K_3\,U_3\,U_8+$$
$$6\,H_3{}^3\,K_3\,U_3\,U_8+2\,G_3\,K_3{}^3\,U_2\,U_8-J_3{}^2\,K_3{}^2\,U_2\,U_8+6\,H_3{}^2\,K_3{}^2\,U_2\,U_8+$$
$$6\,G_3{}^2\,H_3\,K_3\,U_{13}\,U_8-2\,G_3\,H_3\,J_3{}^2\,U_{13}\,U_8+6\,G_3\,H_3{}^3\,U_{13}\,U_8+6\,G_3\,H_3{}^2\,K_3\,U_{12}\,U_8-$$
$$H_3{}^2\,J_3{}^2\,U_{12}\,U_8+2\,H_3{}^4\,U_{12}\,U_8+2\,H_3{}^3\,K_3\,U_{11}\,U_8+2\,G_3\,H_3\,J_3\,K_3\,U_{10}\,U_8+$$
$$H_3{}^3\,J_3\,U_{10}\,U_8+2\,H_3\,K_3{}^3\,U_1\,U_8-H_3{}^2\,K_3{}^2\,U_7{}^2+H_3\,J_3\,K_3{}^2\,U_5\,U_7+$$
$$2\,G_3\,K_3{}^3\,U_3\,U_7-J_3{}^2\,K_3{}^2\,U_3\,U_7+6\,H_3{}^2\,K_3{}^2\,U_3\,U_7+2\,H_3\,K_3{}^3\,U_2\,U_7+$$
$$6\,G_3\,H_3{}^2\,K_3\,U_{13}\,U_7-H_3{}^2\,J_3{}^2\,U_{13}\,U_7+2\,H_3{}^4\,U_{13}\,U_7+2\,H_3{}^3\,K_3\,U_{12}\,U_7+$$

$$H_3{}^2\,J_3\,K_3\,U_{10}\,U_7+2\,H_3\,K_3{}^3\,U_3\,U_6+2\,H_3{}^3\,K_3\,U_{13}\,U_6-G_3\,K_3{}^3\,U_5{}^2-$$
$$3\,H_3{}^2\,K_3{}^2\,U_5{}^2-2\,H_3\,K_3{}^3\,U_4\,U_5+3\,H_3\,J_3\,K_3{}^2\,U_3\,U_5+J_3\,K_3{}^3\,U_2\,U_5-$$
$$6\,G_3\,H_3\,J_3\,K_3\,U_{13}\,U_5+H_3\,J_3{}^3\,U_{13}\,U_5-3\,H_3{}^3\,J_3\,U_{13}\,U_5-3\,H_3{}^2\,J_3\,K_3\,U_{12}\,U_5+$$

$$4\,G_3\,H_3\,K_3{}^2\,U_{10}\,U_5 - H_3\,J_3{}^2\,K_3\,U_{10}\,U_5 + 4\,H_3{}^3\,K_3\,U_{10}\,U_5 + J_3\,K_3{}^3\,U_3\,U_4 -$$
$$3\,H_3{}^2\,J_3\,K_3\,U_{13}\,U_4 + 2\,H_3{}^2\,K_3{}^2\,U_{10}\,U_4 - 6\,H_3{}^2\,K_3{}^2\,U_3{}^2 - 8\,H_3\,K_3{}^3\,U_2\,U_3 -$$
$$2\,G_3{}^2\,K_3{}^2\,U_{13}\,U_3 + 4\,G_3\,J_3{}^2\,K_3\,U_{13}\,U_3 - 8\,G_3\,H_3{}^2\,K_3\,U_{13}\,U_3 - J_3{}^4\,U_{13}\,U_3 +$$
$$4\,H_3{}^2\,J_3{}^2\,U_{13}\,U_3 - 2\,H_3{}^4\,U_{13}\,U_3 - 4\,G_3\,H_3\,K_3{}^2\,U_{12}\,U_3 + 4\,H_3\,J_3{}^2\,K_3\,U_{12}\,U_3 -$$
$$4\,H_3{}^3\,K_3\,U_{12}\,U_3 - 2\,H_3{}^2\,K_3{}^2\,U_{11}\,U_3 - 3\,G_3\,J_3\,K_3{}^2\,U_{10}\,U_3 + J_3{}^3\,K_3\,U_{10}\,U_3 -$$
$$6\,H_3{}^2\,J_3\,K_3\,U_{10}\,U_3 - 2\,K_3{}^4\,U_1\,U_3 - K_3{}^4\,U_2{}^2 - 4\,G_3\,H_3\,K_3{}^2\,U_{13}\,U_2 +$$
$$4\,H_3\,J_3{}^2\,K_3\,U_{13}\,U_2 - 4\,H_3{}^3\,K_3\,U_{13}\,U_2 - 2\,H_3{}^2\,K_3{}^2\,U_{12}\,U_2 - 3\,H_3\,J_3\,K_3{}^2\,U_{10}\,U_2 -$$
$$6\,G_3{}^2\,H_3{}^2\,U_{13}{}^2 - 8\,G_3\,H_3{}^3\,U_{12}\,U_{13} - 2\,H_3{}^4\,U_{11}\,U_{13} + 3\,G_3\,H_3{}^2\,J_3\,U_{10}\,U_{13} -$$
$$2\,H_3{}^2\,K_3{}^2\,U_1\,U_{13} - H_3{}^4\,U_{12}{}^2 + H_3{}^3\,J_3\,U_{10}\,U_{12} - 3\,G_3\,H_3{}^2\,K_3\,U_{10}{}^2 -$$
$$H_3{}^4\,U_{10}{}^2$$

$$\Xi_8 = -2\,G_3\,H_3\,K_3{}^2\,U_8{}^2 - 2\,H_3{}^3\,K_3\,U_8{}^2 - 2\,H_3{}^2\,K_3{}^2\,U_7\,U_8 + H_3\,J_3\,K_3{}^2\,U_5\,U_8 +$$
$$2\,G_3\,K_3{}^3\,U_3\,U_8 - J_3{}^2\,K_3{}^2\,U_3\,U_8 + 6\,H_3{}^2\,K_3{}^2\,U_3\,U_8 + 2\,H_3\,K_3{}^3\,U_2\,U_8 +$$
$$6\,G_3\,H_3{}^2\,K_3\,U_{13}\,U_8 - H_3{}^2\,J_3{}^2\,U_{13}\,U_8 + 2\,H_3{}^4\,U_{13}\,U_8 + 2\,H_3{}^3\,K_3\,U_{12}\,U_8 +$$
$$H_3{}^2\,J_3\,K_3\,U_{10}\,U_8 + 2\,H_3\,K_3{}^3\,U_3\,U_7 + 2\,H_3{}^3\,K_3\,U_{13}\,U_7 - H_3\,K_3{}^3\,U_5{}^2 +$$
$$J_3\,K_3{}^3\,U_3\,U_5 - 3\,H_3{}^2\,J_3\,K_3\,U_{13}\,U_5 + 2\,H_3{}^2\,K_3{}^2\,U_{10}\,U_5 - 4\,H_3\,K_3{}^3\,U_3{}^2 -$$
$$2\,K_3{}^4\,U_2\,U_3 - 4\,G_3\,H_3\,K_3{}^2\,U_{13}\,U_3 + 4\,H_3\,J_3{}^2\,K_3\,U_{13}\,U_3 - 4\,H_3{}^3\,K_3\,U_{13}\,U_3 -$$
$$2\,H_3{}^2\,K_3{}^2\,U_{12}\,U_3 - 3\,H_3\,J_3\,K_3{}^2\,U_{10}\,U_3 - 2\,H_3{}^2\,K_3{}^2\,U_{13}\,U_2 - 4\,G_3\,H_3{}^3\,U_{13}{}^2 -$$
$$2\,H_3{}^4\,U_{12}\,U_{13} + H_3{}^3\,J_3\,U_{10}\,U_{13} - H_3{}^3\,K_3\,U_{10}{}^2$$

$$\Xi_9 = -H_3{}^2\,K_3{}^2\,U_8{}^2 + 2\,H_3\,K_3{}^3\,U_3\,U_8 + 2\,H_3{}^3\,K_3\,U_{13}\,U_8 - K_3{}^4\,U_3{}^2 -$$
$$2\,H_3{}^2\,K_3{}^2\,U_{13}\,U_3 - H_3{}^4\,U_{13}{}^2 = 0$$

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