Exact Solutions to Category 1 Problem 3

Thomas Hagstrom[1] **and Igor Nazarov**
Dept. of Mathematics and Statistics
The University of New Mexico, Albuquerque, NM 87131
email: hagstrom@math.unm.edu, FAX: (505) 277-5505.

The goal of this problem was to provide a detailed study of the accuracy of boundary treatments with a range of incidence angles including shear and a sonic point.

There are three parts. In each we solve the linearized Euler equations on a prescibed domain: $(-2,2)\times(0,1)$ with initial conditions consisting of a pressure dipole, entropy and vorticity disturbances. Here $x_1 = \pm 2$ are the artificial boundaries, the speed of sound is scaled to 1, and we solve up to $t = 64$.

PART 1

For part 1 the base flow is a uniform subsonic flow skew to the boundaries:

$$U_1 = 0.3, \quad U_2 = 0.4. \tag{1}$$

In addition, periodic boundary conditions are prescribed x_2.

The exact solution is given by the following formulas:

$$p = P(x_1 - U_1 t, x_2 - U_2 t, t), \quad \rho = D(x_1 - U_1 t, x_2 - U_2 t, t),$$
$$u = U(x_1 - U_1 t, x_2 - U_2 t, t), \quad v = V(x_1 - U_1 t, x_2 - U_2 t, t),$$

where

$$P(x_1, x_2, t) = \sum_{i=1}^{2} B_i \sum_{k=-\infty}^{\infty} \int_{-\infty}^{t-r_{ik}} \frac{e^{-\mu_i (s-\tau_i)^2}}{\sqrt{(t-s)^2 - r_{ik}^2}} ds,$$

$$D(x_1, x_2, t) = P(x_1, x_2, t) + S \sum_{k=-\infty}^{\infty} e^{-\mu_S r_{Sk}^2},$$

$$U(x_1, x_2, t) = -\int_0^t \frac{\partial P}{\partial x_1}(x_1, x_2, s) ds + U_0(x_1, x_2),$$

$$V(x_1, x_2, t) = -\int_0^t \frac{\partial P}{\partial x_2}(x_1, x_2, s) ds + V_0(x_1, x_2),$$

$$U_0(x_1, x_2) = f_1(x_1) \int_{-2}^{2} \frac{\partial P}{\partial t}(z, x_2, 0) dz - \int_{-2}^{x_1} \frac{\partial P}{\partial t}(z, x_2, 0) dz,$$

$$V_0(x_1, x_2) = -f_1'(x_1) \int_0^{x_2} \int_{-2}^{2} \frac{\partial P}{\partial t}(z, w, 0) dz dw.$$

[1]Supported, in part, by ARO Grant DAAD19-03-1-0146, NSF Grant DMS-0306285, NASA Contract NAG3-2692, and BSF Grant 2002019. Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the author and do not necessarily reflect the views of ARO, NSF, or NASA.

and

$$r_{ik}^2 = (x_1 - x_{1,i})^2 + (x_2 - x_{2,ik})^2,$$
$$r_{Sk}^2 = (x_1 - x_{1,S})^2 + (x_2 - x_{2,Sk})^2,$$
$$f_1(x_1) = \begin{cases} 0, & x_1 < -1.9 \\ 1 - e^{-((x_1+1.9)/2.5)^8}, & |x_1| \leq 1.9 \\ 1, & x_1 > 1.9. \end{cases}$$

The parameters B_i, μ_i, $x_{1,i}$, $x_{2,ik}$, S, μ_S, $x_{1,S}$, $x_{2,Sk}$ are chosen so that, to a high degree of accuracy (11 digits), the initial data is supported on $(-2, 2)$ and the boundary conditions are satisfied. The integrals are evaluated using a combination of Gaussian quadrature and endpoint corrected trapezoid rules, again to high accuracy. The infinite sums are truncated after the point where their contributions are below machine precision. We also note that the jump in f_1 is approximately 4×10^{-13}.

Precisely we chose a dipole-like initial configuration for the pressure pulse:

$$\tau_1 = \tau_2 = -.95, \quad \mu_1 = \mu_2 = 30, \quad B_2 = -B_1 = 1,$$

$$x_{1,1} = -x_{1,2} = 0.1, \quad x_{2,10} = x_{2,20} = 1/2.$$

and for the entropy pulse:

$$\mu_S = 12, \quad S = 1, \quad x_{1,S} = 0, \quad x_{2,S0} = 1/2.$$

To guarantee periodicity we have:

$$x_{2,ik}, x_{2,Sk} = \frac{1}{2} + k, \quad -\infty < k < \infty.$$

We note that similar solutions have been used to test boundary conditions for the linearized Euler equations in [2] and for the scalar wave equation in [1].

PARTS 2 AND 3

In part 2 the base flow is given by the subsonic Couette flow:

$$U_1 = Mx_2, \quad M = 0.9, \quad U_2 = 0, \tag{2}$$

and in part 3 by the transonic Couette flow:

$$U_1 = Mx_2, \quad M = 1.2, \quad U_2 = 0. \tag{3}$$

For these problems we replace the periodic boundary conditions by the wall boundary condition, $v = 0$. The initial conditions are defined by the same functions and parameters as part 1 except that the image source locations $x_{2,ik}$ are determined to guarantee compatibility with the wall conditions. For $k \geq 0$:

$$x_{2,i,k+1} = 2 - x_{2,i,-k}, \quad x_{2,i,-(k+1)} = -x_{2,ik},$$

$$x_{2,S,k+1} = 2 - x_{2,S,-k}, \quad x_{2,S,-(k+1)} = -x_{2,Sk}.$$

In this case we don't have a code which evaluates an exact solution. Instead we use a well-resolved numerical solution on a sufficiently long domain to eliminate the influence of the boundaries. Tha basic numerical scheme is identical to the one we used to solve these and other benchmark problems, and is described in more detail elsewhere in the proceedings. In time we use a standard 4th order Runge-Kutta method with time step $dt = 1/2000$: $128,000$ steps for the entire solution. Space derivatives are calculated using an 8th order difference scheme on a square grid with an extra point near the boundaries (added for stability). Thus the mesh in the domain $[-L, L] \times [0, 1]$ has $(nx + 3) * (ny + 3)$ points $(nx = 128 * 2 * L,\ ny = 128;\ h_x = h_y = 1/128)$

The length of the domain is chosen so that reflection from the left and right boundaries causing possible errors would not come before time $t = 64.0$

$$\frac{L-2}{M+1} + (L-2) > 64.0$$

Hence, $L = 44$ for Problem 2 ($M = .9$), and $L = 47$ for Problem 3 ($M = 1.2$). We note that this required $385,120$ points in the transonic case. We have not fully assessed the accuracy of this solution, but preliminary comparisons with coarser mesh solutions suggests that it is accurate to more than three digits.

References

[1] B. Alpert, L. Greengard, and T. Hagstrom. Nonreflecting boundary conditions for the time-dependent wave equation. *J. Comput. Phys.*, 180:270–296, 2002.

[2] T. Hagstrom and J. Goodrich. Accurate radiation boundary conditions for the linearized Euler equations in Cartesian domains. *SIAM J. Sci. Comput.*, 24:770–795, 2002.