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RESEARCH MEMORANDUM

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RESPONSE OF A ROTATING PROPELLER

TO AERODYNAMIC EXCITATION

By

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FOR REFERENCE

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NATIONAL ADVISORY COMMITTEE
FOR AERONAUTICS

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RESEARCH MEMORANDUM

RESPONSE OF A ROTATING PROPELLER TO AERODYNAMIC EXCITATION*

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SUMMARY

The flexural vibration of a rotating propeller blade with clamped shank is analyzed with the object of presenting, in matrix form, equations for the elastic bending moments in forced vibration resulting from aerodynamic forces applied at a fixed multiple of rotational speed. Matrix equations are also derived which define the critical speeds and mode shapes for any excitation order and the relation between critical speed and blade angle. Reference is given to standard works on the numerical solution of matrix equations of the forms derived.

The use of a segmented blade as an approximation to a continuous blade provides a simple means for obtaining the matrix solution from the integral equation of equilibrium, so that, in the numerical application of the method presented, the several matrix arrays of the basic physical characteristics of the propeller blade are of simple form, and their simplicity is preserved until, with the solution in sight, numerical manipulations well-known in matrix algebra yield the desired critical speeds and mode shapes from which the vibration at any operating condition may be synthesized.

A close correspondence between the familiar Stodola method and the matrix method is pointed out, indicating that any features of novelty are characteristic not of the analytical procedure but only of the abbreviation, condensation, and efficient organization of the numerical procedure made possible by the use of classical matrix theory.

INTRODUCTION

This report presents a theoretical analysis of the flexural vibration of an aircraft propeller blade subjected to harmonic aerodynamic exciting forces at a fixed multiple of propeller rotational frequency.

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*This report is a reproduction of Hamilton Standard Propellers' report No. HSP-613, of December 11, 1947, with some slight modifications to conform more nearly to NACA form.

Most direct practical application is found in the calculation of response to rotational frequency excitation below the first mode resonant frequency, but the analysis also provides a means for studying critical speeds at any frequency order. However, since current interest is centered on the former application, coupling of blade bending and torsion is neglected as being of little importance when operating far below the first torsional natural frequency, and damping is likewise ignored. Initial offset and sweep are also neglected, although their effects cannot be dismissed as a generality.

The propeller blade in this analysis is clamped at the shank, that is, constrained so that at some fixed location near the center of rotation only uniform rotational motion is possible. This condition of end fixity is chosen in accordance with the object of applying the results of the analysis to the particular case of vibration in response to air stream angularity, where a propeller of three or more blades vibrates in an unsymmetrical mode which fulfills this condition. The "reactionless modes" occurring in propellers of four or more blades are also covered by this boundary condition. The system is furthermore assumed to be linear, with small vibratory displacements, and simple bending theory is used, since the twist in conventional propeller blade designs is moderate.

The differential equations of equilibrium are first derived and transformed into integral equations, and they are then examined in the form of a matrix equation for a segmented blade, which permits the evaluation of vibratory response, critical speeds, and normal modes by simple classical methods which are particularly attractive because the solution in numerical form follows the symbolic form very closely.

PHILOSOPHY OF ANALYSIS

It must be recognized at the outset that the mathematical description of the vibration of a twisted, rotating, tapered beam, subjected to distributed vibratory loading, is essentially a complicated process, and that care must be taken to avoid dealing with expressions so cumbersome and involved in notation that the physical meanings of the various terms are completely hidden. The analysis here presented attempts to avoid such difficulties by using the concise abbreviations provided by simple matrix algebra, setting up matrix arrays of physical quantities wherever possible, and separating the operations of integration and differentiation from the physical quantities by the use of operational symbols in matrix form. It is thus possible to avoid writing large arrays of simultaneous equations, replete with multiple integrals, which might otherwise tend to disguise, through their complexity, the basic manipulations leading to a solution. By using concise matrix terminology, the problem is reduced to a classical form of equation which can easily be solved to yield such fundamental information as the critical speeds and normal modes of the vibrating system. As in simpler, nonrotating systems, a knowledge of the normal modes and critical frequencies is sufficient to synthesize the response of the system to an aerodynamic excitation at any operating condition.

LIST OF SYMBOLS AND UNITS

x	blade station radius, inches
l	blade tip radius, inches
y	blade vibratory displacement, scalar, also matrix column, inches
ω	circular frequency of vibration, radians per second
Ω	circular frequency of rotation, radians per second
ρ	mass per unit length of blade, also diagonal matrix, pound-seconds^2 per inch^2
E	Young's modulus, pounds per inch^2
I	blade section moment of inertia, scalar, also diagonal matrix, inches^4
θ	blade section angle from plane of rotation, radians
β	blade angle at reference station, radians
Θ	twist matrix of $\sin \theta$ and $\cos \theta$
Q	section shear force, pounds
M	section bending moment, scalar, also column matrix, pound-inches
T	centrifugal tension, scalar, also diagonal matrix, pounds
F	applied air force per unit length, scalar, also column matrix, pounds per inch
S	integral operator, scalar, also matrix as defined later, inches
D	derivative operator, scalar, also matrix as defined later, inches^{-1}
τ	altered centrifugal tension matrix, $\frac{1}{\Omega^2}T$, pound-seconds^2
φ	frequency matrix, seconds^{-2}

P frequency order, ω/Ω

φ_p frequency order matrix

C curvature, scalar, also column matrix, inches^{-1}

n number of blade segments

I unit matrix

λ scalar frequency parameter in matrix equations, seconds^{-2}

k modal column (moments), pound-inches

κ modal row (curvatures), inches^{-1}

u dynamic matrix, seconds^{-2}

J orthogonal unit matrix, $J^2 = -I$

Subscripts and matrix configurations are explained in the text and diagrams. Matrix notation follows the conventions generally adopted in reference 1, where possible.

EQUATIONS OF EQUILIBRIUM

The forces and moments acting upon a differential blade element, in the plane of rotation, are shown by figure 1. The differential equations of equilibrium are found by separately summing horizontal forces, vertical forces, and moments from this diagram. Equilibrium of forces in the direction of the x-axis is expressed by

$$dT + \Omega^2 \rho x \, dx = 0 \tag{1}$$

and in the y-direction, by

$$\omega^2 \rho y \, dx + dQ + F \, dx + \left(\frac{y}{x}\right)\Omega^2 \rho x \, dx + d\left(T\frac{dy}{dx}\right) = 0 \tag{2}$$

while the moments are summed by

$$Q \, dx + dM = 0 \tag{3}$$

from which may be written the three differential equations of equilibrium,

$$\frac{dT}{dx} + \Omega^2 \rho x = 0 \tag{4}$$

$$(\Omega^2 + \omega^2)\rho y + \frac{dQ}{dx} + F + \frac{d}{dx}\left(T\frac{dy}{dx}\right) = 0 \tag{5}$$

$$Q + \frac{dM}{dx} = 0 \tag{6}$$

Equation (4) may be integrated to obtain the centrifugal tension at any point along the blade

$$T = \Omega^2 \int_x^l \rho x \, dx \tag{7}$$

Integrating equation (6) and substituting the expression for Q, derived by integrating equation (5) gives

$$M = (\Omega^2 + \omega^2) \int_x^l dx \int_x^l \rho y \, dx + \int_x^l dx \int_x^l F \, dx - \int_x^l T\frac{dy}{dx} \, dx \tag{8}$$

For vibration normal to the plane of rotation, the equilibrium diagram of figure 1 would be altered only by the change in direction of the centrifugal force, $\Omega^2 \rho x \, dx$, which would then be parallel to the x-axis. This eliminates the fourth term of equation (2), for vibration normal to the plane of rotation. The equation of moments is then

$$M = \omega^2 \int_x^l dx \int_x^l \rho y \, dx + \int_x^l dx \int_x^l F \, dx - \int_x^l T\frac{dy}{dx} \, dx \tag{9}$$

where it must be understood that the M, y, and F symbols represent different quantities from those used in equation (8).

Now condense the notation by introducing the following operational symbols:

$$Sf(x) = \int_x^l f(x)\,dx$$

$$Df(x) = \frac{d}{dx}f(x)$$

Since equations (8) and (9) shall henceforth be used as simultaneous equations, introduce also subscripts p and r to denote quantities referred, respectively, to vibrations normal to the plane of rotation (parallel centrifugal field) and in the plane of rotation (radial centrifugal field). Equations (8) and (9), in reverse order, then become

$$\begin{aligned} M_p &= \omega^2 S^2 \rho y_p + S^2 F_p - STDy_p \\ M_r &= (\Omega^2 + \omega^2) S^2 \rho y_r + S^2 F_r - STDy_r \end{aligned} \qquad (10)$$

Further to condense these equations, it is convenient to apply a matrix notation, in which

$$M_{pr} = \begin{Bmatrix} M_p \\ M_r \end{Bmatrix},\ y_{pr} = \begin{Bmatrix} y_p \\ y_r \end{Bmatrix},\ F_{pr} = \begin{Bmatrix} F_p \\ F_r \end{Bmatrix},\ \varphi = \begin{bmatrix} \omega^2 & 0 \\ 0 & \Omega^2 + \omega^2 \end{bmatrix}$$

$$\rho = \begin{bmatrix} \rho & 0 \\ 0 & \rho \end{bmatrix},\ T = \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix},\ S = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix},\ D = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}$$

The symbols, ρ, T, S, and D may be used either as matrices or as matrix elements without ambiguity, since the equations in which they will be used will make their meanings evident. The matrix equation is thus

$$M_{pr} = \varphi S^2 \rho y_{pr} + S^2 F_{pr} - STDy_{pr} \tag{11}$$

In order to solve this equation for bending moments in terms of applied air loads, it is necessary first to eliminate the deflections, y_{pr}, by means of an additional moment-deflection relationship.

FLEXURAL RIGIDITY OF A TWISTED BEAM

Simple bending theory provides the scalar relation,

$$M = EIC$$

which applies to an untwisted beam, where bending takes place about one of the principal axes of inertia. This relation will be used in the twisted propeller blade, but it will be necessary to transform the moment and curvature from the twisted coordinates defined by the principal axes to the untwisted pr coordinates. Denoting moments about the minor axis of inertia by the subscript, f (for "flatwise" bending), and moments about the major axis by e (for "edgewise" bending), the coordinate transformation may be written as follows, referring to the orientation of axes shown in figure 2.

$$M_{fe} = \Theta M_{pr}$$

where

$$\Theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The curvature, expressed as a vector column matrix, may likewise be transformed in the same fashion —

$$C_{fe} = \Theta C_{pr}$$

and these coordinate transformations may be introduced into the moment-curvature relation,

$$M_{fe} = EI_{fe}C_{fe}$$

$$\Theta M_{pr} = EI_{fe}\Theta C_{pr}$$

whence

$$C_{pr} = \Theta^{-1}(EI_{fe})^{-1}\Theta M_{pr}$$

But the curvature, in untwisted coordinates, is given by the second derivative of the deflection.

$$C_{pr} = D^2 y_{pr}$$

Therefore

$$D^2 y_{pr} = \Theta^{-1}(EI_{fe})^{-1}\Theta M_{pr}$$

The curvatures may be integrated to obtain deflections, using the operator,

$$D^{-1}f(x) = \int_0^x f(x)\,dx$$

which, although an integral operator, differs from the S already defined, due to the necessity for integrating curvature and slope from the origin outward. The desired relation between moment and deflection is then

$$y_{pr} = D^{-2}\Theta^{-1}(EI_{fe})^{-1}\Theta M_{pr}$$

Substituting this into equation (11) yields

$$M_{pr} = \varphi S^2 \rho D^{-2} \Theta^{-1} (EI_{fe})^{-1} \Theta M_{pr} + S^2 F_{pr} - STDD^{-2} \Theta^{-1} (EI_{fe})^{-1} \Theta M_{pr}$$

or

$$M_{pr} = \left(\varphi S^2 \rho D^{-2} - STD^{-1}\right) \Theta^{-1} (EI_{fe})^{-1} \Theta M_{pr} + S^2 F_{pr} \tag{12}$$

This is an integral equation in operational matrix form, with x as an independent variable, relating elastic bending moments to applied aerodynamic forces. It can be solved by any of several methods, but this presentation will be confined to the discussion of a solution obtained by considering the propeller blade divided into a finite number of hinged segments. This treatment has the advantage of simplicity in directly reducing the matrix operators, S, D, and D^{-1} to convenient numerical form.

SEGMENTED BLADE

Consider the application of equation (12) to a propeller blade divided into n equal segments, having its distributed mass divided proportionately among the hinge points and having parallel springs across each hinge to represent the edgewise and flatwise flexibilities. The operators, S, D, and D^{-1}, can then be defined in series form, as follows:

$$Sf(x_i) = \int_{x_i}^{l} f(x)\,dx = \sum_{j=i}^{n} f(x_j)\,\Delta x$$

$$Df(x_i) = \frac{d}{dx} f(x) = \frac{f(x_i) - f(x_{i-1})}{\Delta x}$$

$$D^{-1} f(x_i) = \int_0^{x_i} f(x)\,dx = \sum_{j=1}^{i} f(x_j)\,\Delta x$$

The subscript nomenclature associated with these definitions is described by figure 3. Note that, because the moments in the segmented system are each one station removed from the forces which they equilibrate in accordance with equations (1), (2), and (3), the subscripts for moments and inertias differ from the subscripts for masses, deflections, air forces, and centrifugal tensions at each station.

There may be written n simultaneous equations, corresponding to the n hinge points, in place of each of the two simultaneous equations (10). Instead of writing them separately, they will be combined into a single matrix equation by using the following conventions.

$$M_p = \begin{Bmatrix} M_p(x_o) \\ M_p(x_1) \\ - \\ M_p(x_{n-1}) \end{Bmatrix}, \quad y_p = \begin{Bmatrix} y_p(x_1) \\ y_p(x_2) \\ - \\ y_p(x_n) \end{Bmatrix}, \quad F_p = \begin{Bmatrix} F_p(x_1) \\ F_p(x_2) \\ - \\ F_p(x_n) \end{Bmatrix},$$

$$\rho = \begin{bmatrix} \rho_1 & 0 & - & 0 \\ 0 & \rho_2 & - & 0 \\ - & - & - & - \\ 0 & 0 & - & \rho_n \end{bmatrix}, \quad T = \begin{bmatrix} T_1 & 0 & - & 0 \\ 0 & T_2 & - & 0 \\ - & - & - & - \\ 0 & 0 & - & T_n \end{bmatrix}, \quad S = \Delta x \begin{bmatrix} 1 & 1 & 1 & - & 1 \\ 0 & 1 & 1 & - & 1 \\ 0 & 0 & 1 & - & 1 \\ - & - & - & - & - \\ 0 & 0 & 0 & - & 1 \end{bmatrix},$$

$$D = \frac{1}{\Delta x} \begin{bmatrix} 1 & 0 & 0 & - & 0 & 0 \\ -1 & 1 & 0 & - & 0 & 0 \\ 0 & -1 & 1 & - & 0 & 0 \\ - & - & - & - & - & - \\ 0 & 0 & 0 & - & -1 & 1 \end{bmatrix}, \quad D^{-1} = \Delta x \begin{bmatrix} 1 & 0 & 0 & - & 0 \\ 1 & 1 & 0 & - & 0 \\ 1 & 1 & 1 & - & 0 \\ - & - & - & - & - \\ 1 & 1 & 1 & - & 1 \end{bmatrix}$$

Similar definitions apply to the r-coordinates. Note that D^{-1} is truly the matrix reciprocal of D. Equations (10) then are simultaneous matrix equations of n-order, and equation (12) will involve second-order matrices with n-order matrix elements, or, more concisely, equation (12) will become a 2n-order matrix equation. As an example, the twist matrix now becomes

$$
\Theta = \left[\begin{array}{ccccc|ccccc}
\cos\theta_1 & 0 & 0 & - & 0 & -\sin\theta_1 & 0 & - & - & 0 \\
0 & \cos\theta_2 & 0 & - & 0 & 0 & -\sin\theta_2 & - & - & 0 \\
0 & 0 & \cos\theta_3 & - & 0 & - & - & - & - & - \\
0 & 0 & 0 & - & \cos\theta_n & 0 & 0 & - & - & -\sin\theta_n \\
\hline
\sin\theta_1 & 0 & - & - & 0 & \cos\theta_1 & 0 & - & - & 0 \\
0 & \sin\theta_2 & - & - & 0 & 0 & \cos\theta_2 & - & - & 0 \\
- & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - \\
0 & 0 & - & - & \sin\theta_n & 0 & 0 & - & - & \cos\theta_n
\end{array}\right]
$$

and the stiffness matrix is

$$
I_{fe} = \left[\begin{array}{ccccc|ccccc}
I_f(x_o) & 0 & - & - & - & 0 & 0 & - & - & - \\
0 & I_f(x_1) & - & - & - & 0 & 0 & - & - & - \\
- & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - \\
- & - & - & - & I_f(x_{n-1}) & - & - & - & - & - \\
\hline
0 & 0 & - & - & - & I_e(x_o) & 0 & - & - & - \\
0 & 0 & - & - & - & 0 & I_e(x_1) & - & - & - \\
- & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & I_e(x_{n-1})
\end{array}\right]
$$

It is important to note that the reciprocals indicated in equation (12) are also of simple form, Θ^{-1} being the transposed of Θ, and $(I_{fe})^{-1}$ being a diagonal matrix of the reciprocals of the individual inertia elements. Furthermore, D^{-1} is the transposed of S.

Since, in propeller vibration studies, the bending moments referred to principal axes are directly related to measurable quantities, equation (12) shall be rewritten in the fe coordinates, whence

$$M_{fe} = \Theta(\varphi S^2 \rho D^{-2} - STD^{-1})\Theta^{-1}(EI_{fe})^{-1}M_{fe} + \Theta S^2 \Theta^{-1} F_{fe} \tag{13}$$

In solving this equation for resonant conditions it will be necessary either to choose a fixed rotational speed and find the natural frequencies or to choose a propeller vibration order (multiple of rotational speed) and find the critical speeds. Adopting the latter procedure and letting $P = \frac{\omega}{\Omega}$, write

$$\varphi = \Omega^2 \left[\begin{array}{cccc|cccc} P^2 & 0 & - & - & 0 & 0 & - & - \\ 0 & P^2 & - & - & 0 & 0 & - & - \\ - & - & - & - & - & - & - & - \\ - & - & - & P^2 & - & - & - & - \\ \hline 0 & 0 & - & - & (P^2+1) & 0 & - & - \\ 0 & 0 & - & - & 0 & (P^2+1) & - & - \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & (P^2+1) \end{array}\right] = \Omega^2 \varphi_p$$

$$T = \Omega^2 \tau$$

Also abbreviating the air moment column,

$$M_{fe}^{(a)} = \Theta S^2 \Theta^{-1} F_{fe}$$

gives

$$M_{fe} = \Omega^2 \Theta\left(\varphi_p S^2 \rho D^{-2} - S\tau D^{-1}\right)\Theta^{-1}\left(EI_{fe}\right)^{-1} M_{fe} + M_{fe}^{(a)}$$

Solving for M_{fe}

$$M_{fe} = \left(I - \Omega^2\theta\left(\varphi_p S^2 \rho D^{-2} - S\tau D^{-1}\right)\theta^{-1}\left(EI_{fe}\right)^{-1}\right)^{-1} M_{fe}^{(a)}$$

Letting $\lambda = \frac{1}{\Omega^2}$

and

$$u = \theta\left(\varphi_p S^2 \rho D^{-2} - S\tau D^{-1}\right)\theta^{-1}\left(EI_{fe}\right)^{-1} \tag{14}$$

$$M_{fe} = \lambda(\lambda I - u)^{-1} M_{fe}^{(a)} \tag{15}$$

Numerical quantities can be substituted into equation (15), and the bending moments caused by applied air moments at a predetermined rotational speed and frequency order may be directly calculated. Another form of solution, obtained by the use of Sylvester's Theorem, will often be more useful. This theorem, which provides a series expansion of a matrix polynomial, applied to equation (15) yields

$$M_{fe} = \sum_{r=1}^{2n} \frac{\lambda}{\lambda - \lambda_r} M^{(r)*} \tag{16}$$

where $M^{(r)}$ is a normal mode moment component of $M_{fe}^{(a)}$, obtained in terms of the modal row and column associated with a latent root, λ_r, from the relations,

$$(\lambda_r I - u)k_r = 0 \tag{17}$$

$$\kappa_r(\lambda_r I - u) = 0 \tag{18}$$

*The subscript r is used henceforth to denote any of the 2n modes of the segmented system. This preserves similarity with matrix notation in reference 1 and should cause no ambiguity, since the pr coordinates do not appear again in the report.

$$M^{(r)} = (\kappa_r k_r)^{-1} k_r \kappa_r M_{fe}{}^{(a)} \tag{19}$$

Equations (17) and (18) are known alternatively as the "characteristic equations" of a matrix, u, and are typical of problems in which there is an independent parameter, such as vibration frequency, as well as a group of coordinate dependent variables. The "latent roots," or values of the parameter for which there is a solution, and the coordinate values associated with each of these roots are frequently the solution desired, hence many classical methods are available for numerical solutions. In the case of the vibrating propeller blade represented by the characteristic equation (17), the latent roots of the matrix, u, are critical speeds (strictly, reciprocals of critical speeds squared) and the modal columns, k_r, are the bending-moment distributions corresponding to the natural modes for the critical speeds. The modal rows, κ_r, defined by equation (18), are the corresponding curvature distributions, as will be shown.

Numerical methods for determining the modal rows, columns, and roots are described in detail in reference 1. In the case at hand, the labor involved can be reduced by deriving a relation between the modal row and column, first forming the transposed of equation (17).

$$k_r'(\lambda_r I - u') = 0$$

Equation (14) stated,

$$u = \Theta(\varphi_p S^2 \rho D^{-2} - S\tau D^{-1})\Theta^{-1}(EI_{fe})^{-1}$$

Since

$$\Theta' = \Theta^{-1},\ S' = D^{-1},\ \varphi_p' = \varphi_p,\ \rho' = \rho,\ \tau' = \tau$$

and

$$\left((EI_{fe})^{-1}\right)' = (EI_{fe})^{-1}$$

it follows that

$$u' = (EI_{fe})^{-1}\Theta\left(\phi_p S^2 \rho D^{-2} - S\tau D^{-1}\right)\Theta^{-1}$$

$$= \left(EI_{fe}\right)^{-1} u \left(EI_{fe}\right)$$

whence

$$k_r'\left(\lambda_r I - \left(EI_{fe}\right)^{-1} u \left(EI_{fe}\right)\right) = 0$$

and

$$k_r'\left(EI_{fe}\right)^{-1}(\lambda_r I - u)(EI_{fe}) = 0$$

or

$$k_r'\left(EI_{fe}\right)^{-1}(\lambda_r I - u) = 0$$

Thus, the modal row is related to the modal column by the equation,

$$\kappa_r = k_r'\left(EI_{fe}\right)^{-1} \tag{20}$$

which shows that the modal row is a curvature distribution, since the modal column, from which it is here derived, satisfies the equation for the elastic moment.

EFFECT OF BLADE ANGLE ON CRITICAL SPEEDS

For small changes in blade angle, the changes in critical speeds can be found without recalculating u for a new blade angle. A partial derivative of the characteristic equation (17) may be formed as follows:

$$(\lambda_r I - u)k_r = 0$$

$$\left(\frac{\partial \lambda_r}{\partial \beta} I - \frac{\partial u}{\partial \beta}\right)k_r + (\lambda_r I - u)\frac{\partial k_r}{\partial \beta} = 0$$

Premultiplying by κ_r, the second term vanishes, leaving

$$\kappa_r\left(\frac{\partial\lambda_r}{\partial\beta}I - \frac{\partial u}{\partial\beta}\right)k_r = 0$$

whence

$$\frac{\partial\lambda_r}{\partial\beta} = \left(\kappa_r k_r\right)^{-1}\kappa_r \frac{\partial u}{\partial\beta}k_r \tag{21}$$

Now examine u and find its derivative.

$$u = \Theta\left(\varphi_p S^2\rho D^{-2} - S\tau D^{-1}\right)\Theta^{-1}\left(EI_{fe}\right)^{-1}$$

$$\frac{\partial u}{\partial\beta} = \frac{\partial\Theta}{\partial\beta}\left(\varphi_p S^2\rho D^{-2} - S\tau D^{-1}\right)\Theta^{-1}\left(EI_{fe}\right)^{-1}$$

$$+ \Theta\left(\varphi_p S^2\rho D^{-2} - S\tau D^{-1}\right)\frac{\partial\Theta^{-1}}{\partial\beta}\left(EI_{fe}\right)^{-1} \tag{22}$$

Since

$$\Theta = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}, \qquad \Theta^{-1} = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

and $d\theta = d\beta$

$$\frac{\partial\Theta}{\partial\beta} = \begin{bmatrix}-\sin\theta & -\cos\theta\\ \cos\theta & -\sin\theta\end{bmatrix}, \quad \frac{\partial\Theta^{-1}}{\partial\beta} = \begin{bmatrix}-\sin\theta & \cos\theta\\ -\cos\theta & -\sin\theta\end{bmatrix}$$

Let

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

then

$$\frac{\partial \Theta}{\partial \beta} = -J\Theta, \qquad \frac{\partial \Theta^{-1}}{\partial \beta} = \Theta^{-1}J$$

Substituting these derivatives into equation (22),

$$\frac{\partial u}{\partial \beta} = -Ju + u\left(EI_{fe}\right)J\left(EI_{fe}\right)^{-1}$$

Premultiply by κ_r and postmultiply by k_r, so that

$$\kappa_r \frac{\partial u}{\partial \beta} k_r = -\kappa_r J u k_r + \kappa_r u\left(EI_{fe}\right)J\left(EI_{fe}\right)^{-1} k_r$$

and by making use of the characteristic equation again, u is eliminated.

$$\kappa_r \frac{\partial u}{\partial \beta} k_r = -\kappa_r J \lambda_r k_r + \kappa_r \lambda_r \left(EI_{fe}\right)J\left(EI_{fe}\right)^{-1} k_r$$

The second term can be simplified by noting, from equation (20), that

$$\kappa_r' = \left(EI_{fe}\right)^{-1} k_r$$

and

$$k_r' = \kappa_r\left(EI_{fe}\right)$$

whence

$$\kappa_r \frac{\partial u}{\partial \beta} k_r = \lambda_r\left(-\kappa_r J k_r + k_r' J \kappa_r'\right)$$

But, since this is a scalar equation, the elements in any term may be transposed, so that

$$k_r' J \kappa_r' = -\kappa_r J k_r$$

and

$$\kappa_r \frac{\partial u}{\partial \beta} k_r = -2\lambda_r \kappa_r J k_r$$

whence, equation (21) becomes

$$\frac{\partial \lambda_r}{\partial \beta} = -2\lambda_r\left(\kappa_r k_r\right)^{-1} \kappa_r J k_r \tag{23}$$

As a last refinement, note that

$$\frac{\partial \lambda_r}{\partial \beta} = \frac{\partial {\Omega_r}^{-2}}{\partial \beta} = -\frac{1}{{\Omega_r}^4} \frac{\partial {\Omega_r}^2}{\partial \beta}$$

$$\frac{\partial {\Omega_r}^2}{\partial \beta} = -\frac{1}{{\lambda_r}^2} \frac{\partial \lambda_r}{\partial \Omega} = 2{\Omega_r}^2\left(\kappa_r k_r\right)^{-1} \kappa_r J k_r$$

$$\frac{1}{\Omega_r} \frac{\partial \Omega_r}{\partial \beta} = \left(\kappa_r k_r\right)^{-1} \kappa_r J k_r \tag{24}$$

This relationship makes it possible to make use of a solution, obtained for any arbitrary blade angle, to find the nature of the variation in critical speed with variations in blade angle, at least for a small range in angle. It is interesting to put this equation into an integral form for the distributed system, by noting the meanings of the operations indicated by equation (24). The scalar, $\kappa_r k_r$, is a sum of the products of each pair of curvatures and moments and hence represents twice the strain energy in a natural mode, so that, in terms of scalar moment elements, M_f and M_e, there may be written

$$\kappa_r k_r \equiv \int_0^l \frac{M_f^2}{EI_f}\,dx + \int_0^l \frac{M_e^2}{EI_e}\,dx$$

Furthermore, the operator, J, has the effect of interchanging flatwise and edgewise (f and e subscript) quantities, so that

$$\kappa_r J k_r \equiv \int_0^l \frac{M_f M_e}{EI_f}\,dx - \int_0^l \frac{M_e M_f}{EI_e}\,dx$$

and the changes in critical speed may therefore be expressed as a function of a small change in blade angle by the following scalar equation of definite integrals evaluated at the critical speed:

$$\frac{d\Omega_r}{\Omega_r} = \frac{\int_0^l \frac{M_f M_e}{EI_f}\,dx - \int_0^l \frac{M_e M_f}{EI_e}\,dx}{\int_0^l \frac{M_f^2}{EI_f}\,dx + \int_0^l \frac{M_e^2}{EI_e}\,dx}\,d\beta \tag{25}$$

This useful relationship might have been obtained by other methods, but the presentation here given has seemed the most straightforward to the writer.

NUMERICAL COMPUTATIONS

The foregoing analysis has provided several equations in matrix form which may be employed directly in numerical work. After forming the basic arrays of physical quantities and combining them to obtain the matrix, u, and the column, $M_{fe}^{(a)}$, equation (15) may be solved for the vibratory bending moments at the n blade stations by methods such as proposed by Aitken (reference 2) in order to find the nonresonant response of the propeller to the applied air forces at a given frequency order. If the critical speeds are desired, the characteristic equation (17), may be solved by repeated premultiplication of an arbitrary column, as described in reference 1, and by this means the bending-moment distribution of the normal mode is also obtained. The normal mode curvature distribution may be found by similar postmultiplication of an arbitrary row, according to the characteristic equation (18), or directly from the moment column, by equation (20). The critical speeds, normal mode columns and rows, and the applied air moments may be gathered together in equation (16) in order to express the resultant vibratory bending moments as functions of rotational speed.

For numerical purposes, it is often convenient to expand equation (16) in a form which will converge more rapidly, thus requiring the calculation of fewer mode shapes. It is easily shown that

$$M_{fe} = M_{fe}^{(a)} + \sum_{r=1}^{2n} \frac{\lambda_r}{\lambda - \lambda_r} M^{(r)} \tag{26}$$

which follows from observing that

$$\lim_{\substack{\Omega^2 \to 0 \\ (\lambda \to \infty)}} M_{fe} = \sum_{r=1}^{2n} M^{(r)} = M_{fe}^{(a)}$$

Since the matrix, u, includes an arbitrary blade angle, the numerical solution in terms of normal modes and frequencies yields complete information at a given frequency order only for a single blade angle; however, for small variations in blade angle, equation (24) may be used to determine the effect on critical speeds without going through a completely new solution.

In computing critical speeds for a fixed vibration order, it should be noted that some of the 2n latent roots, λ_r, may be negative. In the particular case of first-order vibrations, not more than one of these roots will, in general, be positive. The occurrence of negative roots implies imaginary critical speeds, which are difficult to picture as such, but may appear more logical if treated as the positive critical speeds which would exist were the centrifugal forces and inertia forces to be reversed. This condition is probably more easily understandable for the special case of $P = 0$, when the negative roots correspond to critical speeds for several modes of buckling under reversed centrifugal forces. Figure 4 presents a sketch of the natural frequency spectrum of a propeller as a function of rotational speed, using frequency squared along each axis, so that the negative roots appear as intersections in the third quadrant.

STODOLA'S METHOD

In the calculation of natural frequencies in problems involving the vibration of beams, the Stodola process is often employed. This is normally a method by which a deflection curve is assumed and used to calculate the inertia loading, which is then successively integrated along the length of the beam in order to obtain the shear, moment, slope, and a new deflection shape. The new deflection curve is then carried through the same process again, and repeated iterations finally converge on the true mode shape, with an increase in amplitude through each iteration in proportion to the lowest natural frequency. After solving for the lowest mode and frequency, higher modes can also be found, but since there is a tendency for the fundamental mode to become more prominent through each iteration, it is necessary to employ the orthogonality relations in order to eliminate the unwanted mode or modes in each higher mode solution.

The Stodola process could also be used with an assumed moment distribution, carrying through each series of integrations to obtain an improved moment mode shape, and it is interesting to note that the solution of the matrix equation (17) for its latent roots and modal columns is very closely related to this type of Stodola process. Premultiplication of an arbitrary moment column by u, upon examination of the definition of u, equation (14), includes in a lumped form the several steps of successively integrating the curvature, slope, loading, and shear, including centrifugal tension effects, to obtain an improved moment distribution, and the comparison of successive mode shapes, after convergence has been attained, yields the lowest natural frequency in the form of the dominant latent root. The process of modifying the u-matrix in order to eliminate the dominant mode and permit solution for the subdominant root, when carefully examined is found to provide a new matrix which represents the same series of operations during each iteration, with the addition of an "orthogonalizing" step included in each iteration. Use of the Stodola process in this manner has long since been successfully accomplished at least in the somewhat simplified case of a nonrotating untwisted beam (reference 3). The rotating, twisted propeller blade requires numerical calculations of larger volume and hence greater difficulty, but no new principles are involved.

The matrix method of calculation, therefore, is not basically novel, but represents a technique for organizing in an efficient manner the multitude of operations involved in a large-scale numerical calculation. One of its strongest attractions is that it permits a proper perspective to be maintained not only in the analytical phases of the problem but also throughout the numerical work.

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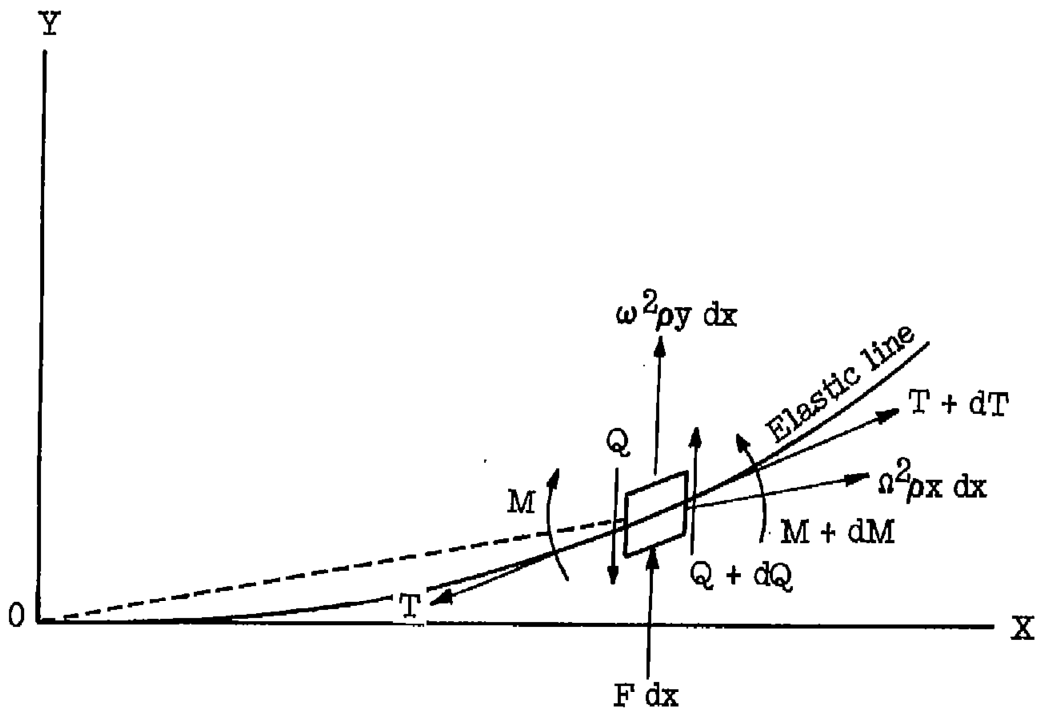

Figure 1.- Equilibrium of forces and moments on a differential element--in plane of rotation (radial centrifugal field).

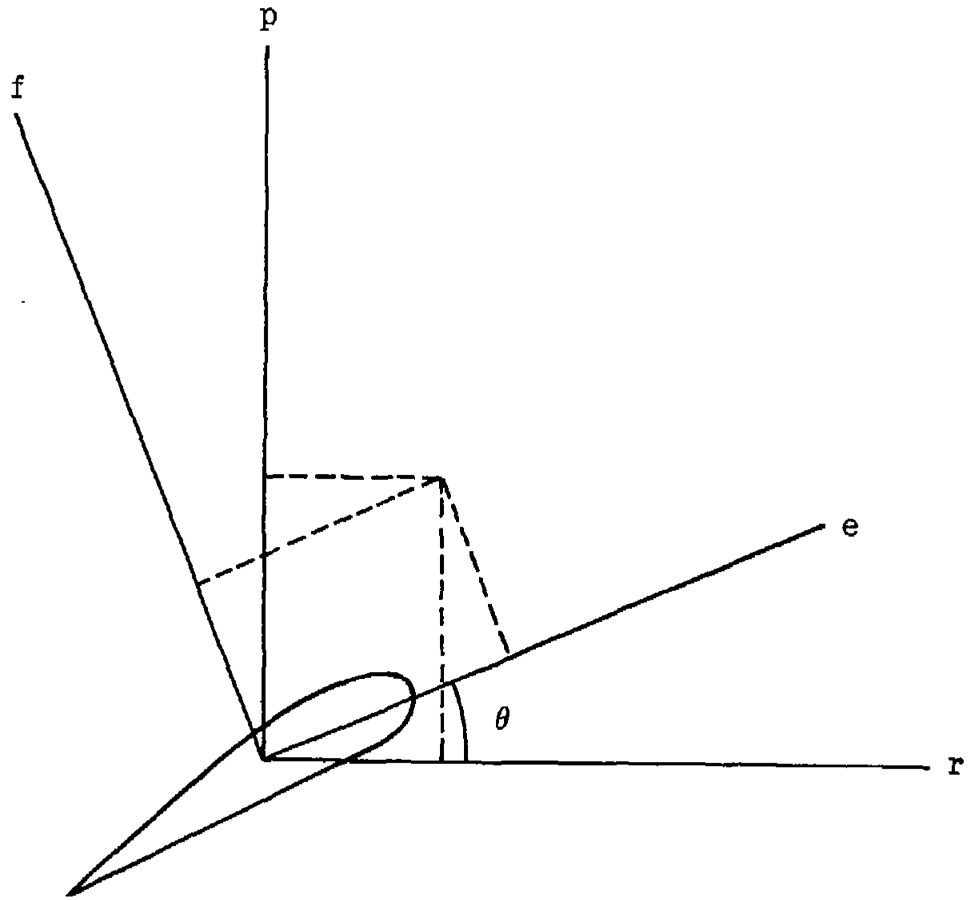

Figure 2.- Orientation of twisted (fe) coordinates in relation to untwisted (pr) coordinates.

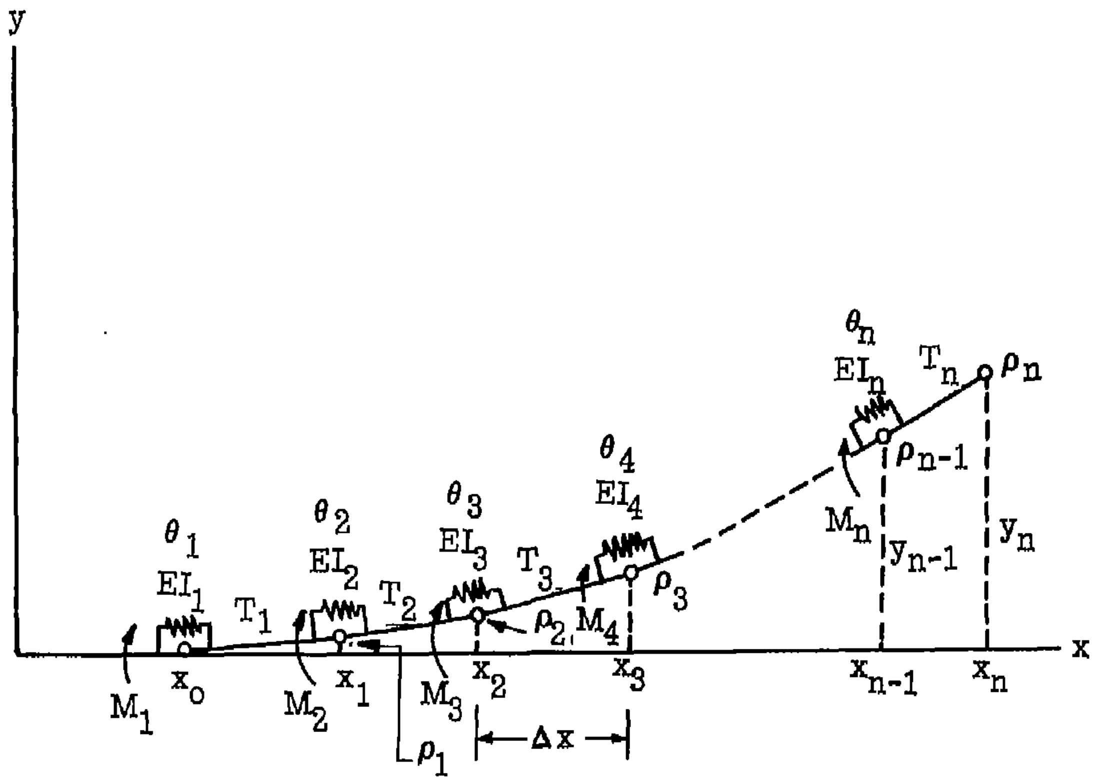

Figure 3.- Nomenclature of segmented blade.

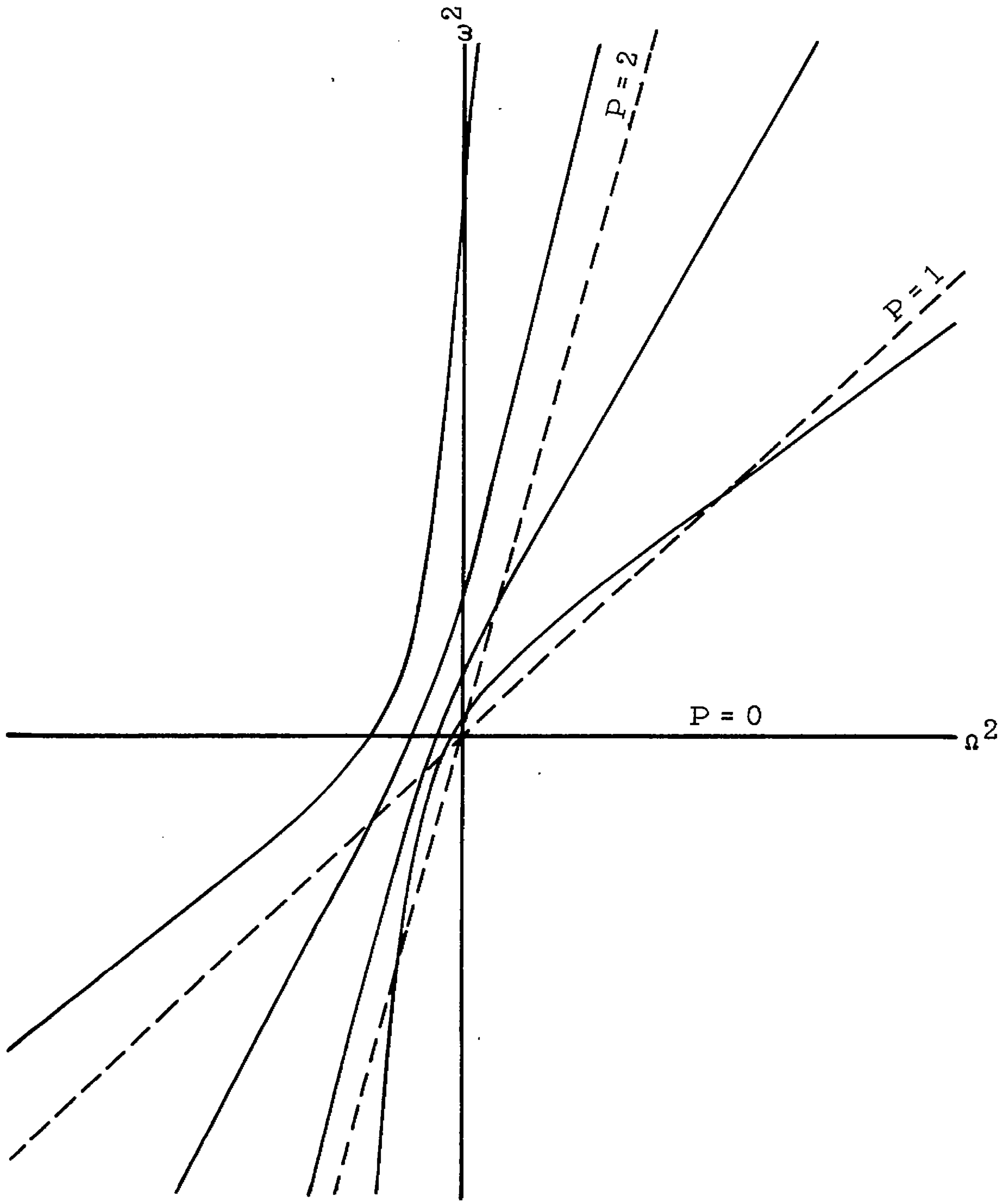

Figure 4.- Frequency spectrum of rotating propeller.