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THE MONOPLANE AS A LIFTING VORTEX SURFACE[1]

By Hermann Blenk

In Prandtl's airfoil theory the monoplane was replaced by a single lifting vortex line (reference 1) and yielded fairly practical results. However, the theory remained restricted to the straight wing. Yawed wings and those curved in flight direction could not be computed with this first approximation; for these the chordwise lift distribution must be taken into consideration. For the two-dimensional problem the transition from the lifting line to the lifting surface has been explained by Birnbaum (reference 2). In the present report the transition to the three-dimensional problem is undertaken.

The first fundamental problem involves the prediction of flow, profile, and drag for prescribed circulation distribution on the straight rectangular wing (fig. 1(a)), the yawed wing for lateral boundaries parallel to the direction of flight (fig. 1(b)), the swept-back wing (fig. 1(c)), and the rectangular wing in slipping (fig. 1(d)), with the necessary series developments for carrying through the calculations, the practical range of convergence of which does not comprise the wing tips or the break point of the swept-back wing. The second problem concerns the calculation of the circulation distribution with given profile for a slipping rectangular monoplane with flat profile and aspect ratio 6, and a rectangular wing with cambered profile and variable aspect ratio - the latter serving as check of the so-called conversion formulas of the airfoil theory.

Nature of the Calculation

In the aim of explaining the type of calculations employed, the simplest case, the straight rectangular monoplane is to serve as model problem.

The wing is replaced by a continuous surface distribution of lifting vortices parallel to the wing center line. On the yawed wing these vortex lines are diagonal; on the wing with sweepback they are broken at the

[1]"Der Eindecker als tragende Wirbelfläche." Z.f.a.M.M., vol. 5, no. 1, Feb. 1925, pp. 36-47.

apex. The deviations of the wing profile from the $x - y$ plane are to be so small that - as in Birnbaum's report - the lifting vortices in the profile may be replaced by those in the $x - y$ plane (the x-axis in wing center line toward the right, the y-axis in flight direction rearward, and the z-axis at right angles to both downward being counted positive). This assumption is identical with the assumption of Prandtl's airfoil theory, according to which interference velocities are small relative to the flow velocity. Quantity $\gamma(x, y)$ is taken as the circulation per unit chord at the point $(x, y, 0)$; that is

$$\Gamma(x) = \int_{-t/2}^{t/2} \gamma(x, y)\, dy \tag{1}$$

is the circulation distribution over span b, if t is the chord of the rectangular wing. To simplify the calculations, the reduced coordinates $\xi = \frac{x}{b/2}$ and $\eta = \frac{y}{t/2}$ are used, so that

$$\Gamma(\xi) = \int_{-1}^{+1} \gamma(\xi, \eta)\, d\eta \tag{1'}$$

where $\gamma(\xi, \eta) = t/2\gamma(x, y)$ has the dimension of a circulation. For $\gamma(\xi, \eta)$ the ellipitical distribution over the span

$$\gamma(\xi, \eta) = \gamma_0(\eta)\sqrt{1 - \xi^2}$$

was chosen, and for the chordwise distribution the same functions used by Birnbaum in his treatment of the two-dimensional problem,

$$\text{(a)}\ \gamma_0(\eta) = a_0\sqrt{\frac{1 - \eta}{1 + \eta}} \qquad \text{(b)}\ \gamma_0(\eta) = b_0\sqrt{1 - \eta^2} \qquad \text{(c)}\ \gamma_0(\eta) = c_0\eta\sqrt{1-\eta^2}$$

According to Prandtl's theory, free vortex lines of intensity $\epsilon(\xi) = \frac{\partial\Gamma}{\partial\xi}$ pass from the lifting vortex surface rearward in flight direction. The downward velocity $w(\xi, \eta)$ is computed from $\gamma(\xi, \eta)$ and $\epsilon(\xi)$ by the Biot-Savart law, and from it the profile of the wing by the formula

$$z = \frac{1}{V} \int^{\eta} w(\xi, \eta) d\eta \tag{2}$$

where V is flow velocity. Lastly the drag is obtained by the formula

$$W = \frac{b}{2} \rho \int_{-1}^{+1} \int_{-1}^{+1} \gamma(\xi, \eta) w(\xi, \eta) d\xi d\eta \tag{3}$$

For the chordwise distribution by this formula the drag then gives the suction force S, which occurs at the leading edge of the wing; the drag itself is obtained by putting

$$\gamma_0(\eta) = \sqrt{\frac{1 - \eta}{1 + \eta + \alpha}}$$

and then passing from α to the boundary 0.

To obtain $w(\xi, \eta)$ the downward velocity

$$w = \frac{1}{4\pi} \int_{-b/2}^{+b/2} \int_{0}^{\infty} \frac{\partial\gamma}{\partial x'} \frac{(x - x')dx' dy}{\sqrt{(x - x')^2 + (y - y')^2}^3}$$

$$+ \frac{1}{4\pi} \int_{-b/2}^{+b/2} \gamma(x') \frac{y' dx'}{\sqrt{(x - x')^2 + y'^2}^3} \tag{4}$$

is computed for one lifting filament placed in the x-axis (fig. 2) at point A (with the coordinates x' and y') behind the filament. The double integral stems from the shedding vortex system (free vortices), the single integral from the lifting vortex itself (bound vortices). The integration with respect to y followed by the introduction of ξ and η gives

$$w = \frac{1}{2\pi b}\int_{-1}^{+1}\frac{\partial\gamma}{\partial\xi'}\frac{d\xi'}{\xi - \xi'} + \frac{\eta' t}{2\pi b^2}\int_{-1}^{+1}\frac{\partial\gamma}{\partial\xi'}\frac{d\xi'}{(\xi - \xi')\sqrt{(\xi - \xi')^2 + \eta'^2 t^2/b^2}}$$

$$+ \frac{\eta' t}{2\pi b^2}\int_{-1}^{+1}\frac{\gamma(\xi')d\xi'}{\sqrt{(\xi - \xi')^2 + \eta'^2 t^2/b^2}^3}$$

and after entering of the elliptic distribution $\gamma = \gamma_o\sqrt{1 - \xi^2}$:

$$w = \frac{\gamma_o}{2b} + \frac{\eta' t\gamma_o}{2\pi b^2}\int_{-1}^{+1}\frac{\xi' d\xi'}{(\xi - \xi')\sqrt{1 - \xi'^2}\sqrt{(\xi - \xi')^2 + \eta'^2 t^2/b^2}}$$

$$+ \frac{\eta' t\gamma_o}{2\pi b^2}\int_{-1}^{+1}\frac{\sqrt{1 - \xi'^2}\, d\xi'}{\sqrt{(\xi - \xi')^2 + \eta'^2 t^2/b^2}^3}$$

For $\eta' = 0$ there is obtained $w = \gamma_o/2b$ (formula (22a) of Prandtl's theory). The two additional cylinders result in elliptic integrals, the calculation of which is avoided by a special type of series development. The task is restricted to the central part of the wing, where $\sqrt{1 - \xi^2} >> \eta' t/b$, which obviously is so much better complied with, as the aspect ratio b/t is greater. The choice for the point A is an area $(\xi - \delta, \xi + \delta)$, such that

$$\sqrt{1 - \xi^2} >> \delta >> \eta' t/b \qquad (5)$$

Within the range $(\xi - \delta, \xi + \delta)$ quantities γ and $\frac{\partial\gamma}{\partial\xi}$ are developed from ξ in a Taylor series; outside the range where

$$(\eta' \, t/b)^2 << (\xi - \xi')^2$$

the roots in the denominator are developed binomially. Assuming uniform convergence of the series the integration is made, anticipating that in the correlation of the integrals with respect to the inner and outer region the quantity δ cancels. The series are broken off behind the squared terms, so that the terms progressing with

$$\frac{1}{\sqrt{1 - \xi^2}^{\,n}} \begin{pmatrix} n = 7, \ 9, \ . \ . \ . \text{ in the first integral} \\ n = 5, \ 7, \ . \ . \ . \text{ in the second integral} \end{pmatrix}$$

in the integrals with respect to the inner range cancel. As a result, all terms with $\frac{1}{\sqrt{1 - \xi^2}^{\,n}}$ from $n = 7$, and $n = 5$ are omitted in the respective integrals with respect to the outer range. In the binomial series all terms with $(\eta' \, t/b)^n$, from $n = 3$ on, drop out. Therefore the true values of the respective integrals with respect to the inner range up to the squared terms in $\eta' \, t/b$ are developed again. The result is:

$$w = \frac{\gamma_o}{2b} - \frac{\eta' \, t\gamma_o}{2\pi b^2}\left[\frac{2\xi^2}{\sqrt{1 - \xi^2}^3} - \frac{1}{\sqrt{1 - \xi^2}^3}\ln\frac{4(1 - \xi^2)^2}{\delta^2}\right.$$

$$+ \frac{1}{2}\eta'^2\frac{t^2}{b^2}\left(\frac{1}{\delta^2\sqrt{1 - \xi^2}^3} - \frac{1}{\sqrt{1 - \xi^2}^3}\left(\ln\frac{4\delta^2 b^2}{\eta'^2 t^2} \quad \frac{\eta'^2 t^2}{2\delta^2 b^2}\right)\right]$$

$$+ \frac{\eta' t \, \gamma_o}{2\pi b^2}\left[\frac{\sqrt{1 - \xi^2}}{\delta^4} - \frac{1 - 2\xi^2}{2\sqrt{1 - \xi^2}^3} - \frac{1}{2\sqrt{1 - \xi^2}^3}\ln\frac{4(1 - \xi^2)^2}{\delta^2}\right.$$

$$- \frac{3}{2}\eta'^2 \, t^2/b^2\left(\frac{\sqrt{1 - \xi^2}}{2\delta^4} - \frac{1}{2\delta^2\sqrt{1 - \xi^2}^3}\right)$$

$$+ \sqrt{1 - \xi^2}\left(\frac{2b^2}{\eta'^2 t^2} - \frac{1}{\delta^2} + \frac{3}{4}\frac{\eta'^2 t^2}{\delta^4 b^2}\right)$$

$$\left. - \frac{1}{2\sqrt{1 - \xi^2}^3}\left(- 2 + \frac{3}{2}\frac{\eta'^2 \, t^2}{\delta^2 b^2} + \ln\frac{4\delta^2 b^2}{\eta'^2 t^2}\right)\right]$$

After proving that δ actually drops out, the squared terms in $\eta' \, t/b$ are disregarded also, which leaves only the linear terms.

$$w = \gamma_o\left[A_o + \frac{B_o}{\eta'} + C_o\eta' + E_o \, \eta' \, \ln \eta'^2\right] \tag{6}$$

with

$$\left.\begin{aligned} A_o &= \frac{1}{2b}, \qquad B_o = \frac{1}{\pi t}\sqrt{1-\xi^2} \\ C_o &= \frac{t}{2\pi b^2}\frac{1}{\sqrt{1-\xi^2}}\left(\frac{1}{2}-\xi^2+\ln\frac{t}{4b(1-\xi^2)}\right) \\ E_o &= -\frac{t}{4\pi b^2}\frac{1}{\sqrt{1-\xi^2}^{\,3}} \end{aligned}\right\} \quad (7)$$

Substitution of $\eta - \eta'$ for η' in (6) followed by integration with respect to η' from -1 to +1, with one of the three basic functions inserted for γ_o gives $w(\xi, \eta)$ for the entire superficial wing. The formulas for induced velocity and profile (the new coefficients derived by (14) from the old ones) are as follows:

$$\left.\begin{aligned} &(a)\ w(\xi, \eta) = a_o\left[A_a{}^o + B_a{}^o\eta + C_a{}^o\eta^2\right] \\ &\qquad z = \frac{a_o}{V}\left[A_a{}^o\eta + \frac{1}{2}B_a{}^o\eta^2 + \frac{1}{3}C_a{}^o\eta^3\right] \\ &(b)\ w(\xi, \eta) = b_o\left[A_b{}^o + B_b{}^o\eta + D_b{}^o\eta^3\right] \\ &\qquad z = \frac{b_o}{V}\left[A_b{}^o\eta + \frac{1}{2}B_b{}^o\eta^2 + \frac{1}{4}D_b{}^o\eta^4\right] \\ &(c)\ w(\xi, \eta) = c_o\left[A_c{}^o + C_c{}^o\eta^2 + E_c{}^o\eta^4\right] \\ &\qquad z = \frac{c_o}{V}\left[A_c{}^o\eta + \frac{1}{3}C_c{}^o\eta^3 + \frac{1}{5}E_c{}^o\eta^5\right] \end{aligned}\right\} \quad (8)$$

The calculations for the yawed and the swept-back wing are made in the same way. The calculation for the wing in sweepback setting must be made, aside from (5), with the limitation $(2\xi \tan \beta + \eta' \, t/b) \leqq (\xi - \xi')$, where $\xi > 0$, $\xi' < 0$, and β angle of sweepback; as a result, there are no data for the middle of the wing.

Results of the Calculations

In place of (6) there is in each case an equation:

$$w = \gamma_o \left[A + \frac{B}{\eta'} + C\eta' + D \ln \eta'^2 + E\eta' \ln \eta'^2 \right] \tag{9}$$

The coefficients are different in each case; those for the straight rectangular wing carry the subscript o (cf. (7)), those of the yawed wing a cross bar, those for the swept-back wing an asterisk, and those for the slipping rectangular wing no subscript. Since the angle β of yaw and sweepback is small, everything is developed in powers of β:

$$\left.\begin{aligned}
\overline{A} &= \frac{1}{2\pi b}\left[\pi - 2\beta\left(\sin^{-1}\xi - \frac{\xi}{\sqrt{1-\xi^2}} \ln \frac{4b(1-\xi^2)}{t}\right)\right] \\
\overline{B} &= \frac{1}{\pi t}\sqrt{1-\xi^2} \\
\overline{C} &= \frac{t}{2\pi b^2}\frac{1}{\sqrt{1-\xi^2}^{\,3}}\left[\frac{1}{2} - \xi^2 + \ln \frac{4b(1-\xi^2)}{t} + \beta^2\left(1 + \xi^2 - \ln \frac{4b(1-\xi^2)}{t}\right)\right] \\
\overline{D} &= -\frac{1}{2\pi b}\frac{\xi}{\sqrt{1-\xi^2}}\beta, \qquad \overline{E} = -\frac{t}{4\pi b^2}\frac{1}{\sqrt{1-\xi^2}^{\,3}}(1-\beta^2)
\end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned}
A^* &= \frac{1}{2\pi b}\Bigg[\pi + \beta\Bigg(2\sin^{-1}\xi - \pi \\
&\quad - \frac{2}{\sqrt{1-\xi^2}}\ln\left(\frac{4b(1-\xi^2)}{t}\frac{\xi}{1+\sqrt{1-\xi^2}}\right) - \frac{2\xi}{1-\xi^2} \\
&\quad + \frac{1}{\xi(1-\xi^2)} + \frac{\xi}{\sqrt{1-\xi^2}^{\,3}}\ln\frac{1+\sqrt{1-\xi^2}}{\xi}\Bigg)\Bigg] \\
B^* &= \frac{1}{\pi t}\sqrt{1-\xi^2} \\
C^* &= \frac{t}{2\pi b^2}\frac{1}{\sqrt{1-\xi^2}}\Bigg[\frac{1}{2} - \xi^2 + \ln\frac{4b(1-\xi^2)}{t} \\
&\quad + \beta^2\left(1+\xi^2-\ln\frac{4b(1-\xi^2)}{t}\right)\Bigg] \\
D^* &= \frac{1}{2\pi b}\frac{\xi}{\sqrt{1-\xi^2}}\beta, \qquad E^* = -\frac{t}{4\pi b^2}\frac{1}{\sqrt{1-\xi^2}}(1-\beta^2)
\end{aligned}\right\} \quad (11)$$

$$\left.\begin{aligned}
A &= \frac{1}{2\pi b}\left[\pi - 2\beta\left(\sin^{-1}\xi - \frac{\xi}{\sqrt{1-\xi^2}}\ \ln\frac{4b(1-\xi^2)}{t}\right.\right. \\
&\quad \left.\left. + \frac{\xi}{\sqrt{1-\xi^2}}\right)\right] \qquad B = \frac{1}{\pi t}\sqrt{1-\xi^2} \\
C &= \frac{t}{2\pi b^2}\frac{1}{\sqrt{1-\xi^2}^{\,3}}\left[\frac{1}{2} - \xi^2 + \ln\frac{4b(1-\xi^2)}{t}\right. \\
&\quad \left. -\beta^2\left(2 + \xi^2 - \ln\frac{4b(1-\xi^2)}{t}\right)\right] \\
D &= -\frac{1}{2\pi b}\frac{\xi}{\sqrt{1-\xi^2}}\beta, \qquad E = -\frac{t}{4\pi b^2}\frac{1}{\sqrt{1-\xi^2}^{\,3}}(1+\beta^2)
\end{aligned}\right\} \qquad (12)$$

For $\beta = 0$ equations (10), (11), and (12) change to (7).

For the subsequent integration with respect to η' (see above) the same formulas are obtained in each case: namely,

$$\left.\begin{aligned}
&\text{(a)}\quad w(\xi,\eta) = a_o\left[A_a + B_a\eta + C_a\eta^2\right] \\
&\text{(b)}\quad w(\xi,\eta) = b_o\left[A_b + B_b\eta + C_b\eta^2 + D_b\eta^3\right] \\
&\text{(c)}\quad w(\xi,\eta) = c_o\left[A_c + B_c\eta + C_c\eta^2 + D_c\eta^3 + E_c\eta^4\right]
\end{aligned}\right\} \qquad (13)$$

The coefficients with the subscripts a, b, and c derive from the others by the formulas:

$$
\left.
\begin{aligned}
&\text{(a)}\;\begin{cases}
A_a = \pi\left[A + B + \frac{1}{2}C + D\ln\frac{1}{4} + E\left(\frac{1}{2} + \ln\frac{1}{2}\right)\right] \\
B_a = \pi\left[C + 2D + E\left(2 + \ln\frac{1}{4}\right)\right], \qquad C_a = \pi E
\end{cases} \\
&\text{(b)}\;\begin{cases}
A_b = \pi\left[\frac{1}{2}A + D\left(\ln\frac{1}{2} - \frac{1}{2}\right)\right] \\
B_b = \pi\left[B + \frac{1}{2}C + E\left(\frac{1}{2} + \ln\frac{1}{2}\right)\right] \\
C_b = \pi D, \qquad D_b = \pi\frac{1}{3}E
\end{cases} \\
&\text{(c)}\;\begin{cases}
A_c = \pi\left[-\frac{1}{2}B - \frac{1}{8}C + E\left(\frac{1}{4}\ln 2 - \frac{1}{16}\right)\right] \\
B_c = \pi[-D] \qquad C_c = \pi\left[B - \frac{1}{2}E\right] \\
D_c = \pi\frac{2}{3}D \qquad E_c = \pi\frac{1}{6}E
\end{cases}
\end{aligned}
\right\} \quad (14)
$$

From (13) the formulas for the profile z follow immediately. The drag is computed by (3); if A is the lift and $q = \frac{1}{2}\rho V^2$ the dynamic pressure, the formula in all cases reads $W = \frac{A^2}{\pi q b^2}$, although the formulas for w are no longer valid at the edge. The result was to be expected by Munk's stagger law, according to which a variation in the stagger of the lift elements is without effect on the drag provided the lift distribution of the individual parts is not changed thereby.

The same calculations were carried out for the slipping rectangular wing with the circulation distributions $\xi^n\sqrt{1 - \xi^2}$ (n - 1, 2, 3) over the span. The coefficients (in correspondence with n) carry an upper subscript.

For simplification the following abbreviations are used for the distributions and their derivatives:

$$f_1 = \xi\sqrt{1-\xi^2} \qquad f_1' = \frac{1-2\xi^2}{\sqrt{1-\xi^2}} \qquad f_1'' = \frac{2\xi^3-3\xi}{\sqrt{1-\xi^2}^{\,3}}$$

$$f_2 = \xi^2\sqrt{1-\xi^2} \qquad f_2' = \frac{2\xi-3\xi^3}{\sqrt{1-\xi^2}} \qquad f_2'' = \frac{2-9\xi^2+6\xi^4}{\sqrt{1-\xi^2}^{\,3}}$$

$$f_3 = \xi^3\sqrt{1-\xi^2} \qquad f_3' = \frac{3\xi^2-4\xi^4}{\sqrt{1-\xi^2}} \qquad f_3'' = \frac{6\xi-19\xi^3+12\xi^5}{\sqrt{1-\xi^2}^{\,3}}$$

Then:

$$\left.\begin{aligned}
A^{(1)} &= \frac{1}{2\pi b}\left[\pi 2\xi - \beta\left(4\xi\sin^{-1}\xi + 2f_1'\ln\frac{4b(1-\xi^2)}{t} - 2f_1'\right)\right] \\
B^{(1)} &= \frac{1}{\pi t}f_1, \qquad C^{(1)} = \frac{t}{2\pi b^2}\Bigg[-2\sin^{-1}\xi \\
&\quad - f_1''\left(\frac{1}{2} + \ln\frac{4b(1-\xi^2)}{t}\right) + \frac{\xi-2\xi^3}{\sqrt{1-\xi^2}^{\,3}} + \beta\pi 2 \\
&\quad + \beta^2\left\{-2\sin^{-1}\xi + f_1''\left(2-\ln\frac{4b(1-\xi^2)}{t}\right) + \frac{\xi-2\xi^3}{\sqrt{1-\xi^2}^{\,3}}\right\}\Bigg] \\
D^{(1)} &= \frac{1}{2\pi b}\beta f_1', \qquad E^{(1)} = \frac{t}{4\pi b^2}f_1''(1+\beta^2)
\end{aligned}\right\} \quad (15)$$

$$\left.\begin{aligned}
A^{(2)} &= \frac{1}{2\pi b}\left[\pi\left(3\xi^2 - \frac{1}{2}\right) - \beta\left((6\xi^2 - 1)\sin^{-1}\xi + 2f_2'\ln\frac{4b(1-\xi^2)}{t} - 2f_2'\right)\right] \\
B^{(2)} &= \frac{1}{\pi t} f_2, \qquad C^{(2)} = \frac{t}{2\pi b^2}\left[-6\xi\sin^{-1}\xi - f_2''\left(\frac{1}{2} + \ln\frac{4b(1-\xi^2)}{t}\right) + \frac{3 - 4\xi^2}{\sqrt{1-\xi^2}^{\,3}} + \beta\pi 6\xi \right. \\
&\left. + \beta^2\left\{-6\xi\sin^{-1}\xi + f_2''\left(2 - \ln\frac{4b(1-\xi^2)}{t}\right) + \frac{3-4\xi^2}{\sqrt{1-\xi^2}^{\,3}}\right\}\right], \qquad D^{(2)} = \frac{1}{2\pi b}\beta f_2' \\
E^{(2)} &= \frac{t}{4\pi b^2} f_2''(1 + \beta^2)
\end{aligned}\right\} \quad (16)$$

$$A^{(3)} = \frac{1}{2\pi b}\left[\pi(4\xi^3 - \xi) - \beta\left((8\xi^3 - 2\xi)\sin^{-1}\xi + 2f_3'\ln\frac{4b(1-\xi^2)}{t} - 2f_3'\right)\right]$$

$$B^{(3)} = \frac{1}{\pi t} f_3$$

$$C^{(3)} = \frac{t}{2\pi b^2}\left[-(12\xi^2 - 1)\sin^{-1}\xi - f_3''\left(\frac{1}{2} + \ln\frac{4b(1-\xi^2)}{t}\right) + \frac{10\xi - 17\xi^3 + 6\xi^5}{\sqrt{1 - {}^{2}}^{3}} + \beta\pi(12\xi^2 - 1)\right.$$
$$\left. + \beta^2\left\{-(12\xi^2 - 1)\sin^{-1}\xi + f_3''\left(2 - \ln\frac{4b(1-\xi^2)}{t}\right) + \frac{10\xi - 17\xi^3 + 6\xi^5}{\sqrt{1-\xi^2}^{3}}\right\}\right]$$

$$D^{(3)} = \frac{1}{2\pi b}\beta f_3', \qquad E^{(3)} = \frac{t}{4\pi b^2} f_3''(1 + \beta^2) \qquad (17)$$

The calculation gives the following values for lift A, moment about the x-axis M_x, moment about the y-axis M_y, drag W, and suction force S:

	γ (t) / (b)	$\sqrt{\frac{1-\eta}{1+\eta}}$	$\sqrt{1-\eta^2}$	$\eta\sqrt{1-\eta^2}$
A	$\sqrt{1-\xi^2}$	$\pi^2/4\ \rho b\ Va_o$	$\pi^2/8\ \rho b\ V\ b_o$	0
	$\xi\sqrt{1-\xi^2}$	0	0	0
	$\xi^2\sqrt{1-\xi^2}$	$\pi^2/16\ \rho b\ Va_2$	$\pi^2/32\ \rho b\ Vb_2$	0
	$\xi^3\sqrt{1-\xi^2}$	0	0	0
M_x	$\sqrt{1-\xi^2}$	$-\pi^2/16\ \rho bt\ Va_o$	0	$\pi^2/64\ \rho bt\ Vc_o$
	$\xi\sqrt{1-\xi^2}$	0	0	0
	$\xi^2\sqrt{1-\xi^2}$	$-\pi^2/64\ \rho bt\ Va_2$	0	$\pi^2/256\ \rho bt\ Vc_2$
	$\xi^3\sqrt{1-\xi^2}$	0	0	0
M_y	$\sqrt{1-\xi^2}$	0	0	0
	$\xi\sqrt{1-\xi^2}$	$\pi^2/32\ \rho b^2\ Va_1$	$\pi^2/64\ \rho b^2\ Vb_1$	0
	$\xi^2\sqrt{1-\xi^2}$	0	0	0
	$\xi^3\sqrt{1-\xi^2}$	$\pi^2/64\ \rho b^2\ Va_3$	$\pi^2/128\ \rho b^2\ Vb_3$	0
W	$\sqrt{1-\xi^2}$	$\pi^3/8\ \rho\ a_o^2$	$\pi^3/32\ \rho\ b_o^2$	0
	$\xi\sqrt{1-\xi^2}$	$\pi^3/16\ \rho\ a_1^2$	$\pi^3/64\ \rho\ b_1^2$	0
	$\xi^2\sqrt{1-\xi^2}$	$\pi^3/32\ \rho\ a_2^2$	$\pi^3/128\ \rho\ b_2^2$	0
	$\xi^3\sqrt{1-\xi^2}$	$3\ \pi^3/128\ \rho\ a_3^2$	$3\ \pi^3/512\ \rho\ b_3^2$	0
S	$\sqrt{1-\xi^2}$	$2\ \pi/3\ b/t\ \rho\ a_o^2$	0	0
	$\xi\sqrt{1-\xi^2}$	$2\ \pi/15\ b/t\ \rho\ a_1^2$	0	0
	$\xi^2\sqrt{1-\xi^2}$	$2\ \pi/35\ b/t\ \rho\ a_2^2$	0	0
	$\xi^3\sqrt{1-\xi^2}$	$2\ \pi/63\ b/t\ \rho\ a_3^2$	0	0

The values a_o, b_o, c_o, a_1, . . . b_3, c_3 are the circulations; the letter denotes the chordwise distribution, the subscript the spanwise distribution.

Discrepancies between the Theory of the Lifting Surface and the Theory of the Lifting Line for the Straight Monoplane

For the drag of the straight rectangular wing both theories give, as already mentioned, the same value. The argument here concerns the deviations in the flow or in the profile. For the theory of the lifting line also affords a profile of the wing, when the results of the two-dimensional problem of the lifting surface are used. The velocity is then $\frac{\Gamma_o}{2b}$, increased by the velocity valid for corresponding two-dimensional problem of the lifting surface, the latter elliptically distributed over the span (reference 1, pp. 20-21, and reference 2, p. 292).

$$\left.\begin{aligned}
&(a) \quad w = \frac{a_o \pi}{2b} + \sqrt{1 - \xi^2}\,\frac{a_o}{t} \\
&\qquad z = \frac{1}{V}\frac{a_o}{2\,b}\left[\pi + \frac{2\,b}{t}\sqrt{1 - \xi^2}\right]\eta \\
&(b) \quad w = \frac{b_o \pi}{4\,b} + \sqrt{1 - \xi^2}\,\frac{b_o}{t}\,\eta \\
&\qquad z = \frac{1}{V}\frac{b_o}{2\,b}\left[\frac{\pi}{2}\eta + \frac{2\,b}{t}\sqrt{1 - \xi^2}\,\frac{1}{2}\eta^2\right] \\
&(c) \quad w = 0 + \sqrt{1 - \xi^2}\,\frac{c_o}{t}\left(\eta^2 - \frac{1}{2}\right) \\
&\qquad z = \frac{1}{V}\frac{c_o}{t}\sqrt{1 - \xi^2}\left(\frac{1}{3}\eta^3 - \frac{1}{2}\eta\right)
\end{aligned}\right\} \quad (18)$$

These formulas agree in first approximation with those of the lifting surface theory; the deviations are of the next higher order in t/b, and which are (subscripts F and L referring to values of the lifting surface and the lifting line, respectively):

$$
\left.\begin{aligned}
&(a)\quad z_F - z_L = \frac{1}{V}\frac{a_o\ t}{4b^2\sqrt{1-\xi^2}^3}\left[\left(-\xi^2 + \ln\frac{8b(1-\xi^2)}{t}\right)(\eta + \eta^2) - \frac{1}{2}\eta^2 - \frac{1}{3}\eta^3\right]\\
&(b)\quad z_F - z_L = \frac{1}{V}\frac{b_o\ t}{8b^2\sqrt{1-\xi^2}^3}\left[\left(-\xi^2 + \ln\frac{8b(1-\xi^2)}{t}\right)\eta^2 - \frac{1}{6}\eta^4\right]\\
&(c)\quad z_F - z_L = \frac{1}{V}\frac{c_o\ t}{8b^2\sqrt{1-\xi^2}^3}\left[\left(-\frac{1}{4} + \xi^2 - \ln\frac{8b(1-\xi^2)}{t}\right)\frac{1}{2}\eta + \frac{1}{3}\eta^3 - \frac{1}{15}\eta^5\right]
\end{aligned}\right\}\quad(19)
$$

Figure 3 represents the profiles of both theories in the cross sections $\xi = 0$ and $\xi = \pm \frac{3}{4}$. The deviations of both curves increase toward the edge. The extent to which this agrees with reality, or perhaps is merely a result of the defective convergence of the series toward the edge, is not predictable without further study of this convergence.

For the practician, formula (19) affords the following empirical formulas for the additional angle of attack $\Delta\alpha$ and the additive curvature $\Delta\frac{1}{R}$, if $c_a = \frac{A}{\frac{1}{2}\rho V^2 bt}$ is the lift coefficient

$$\Delta\alpha = 0.059 \frac{t}{b} c_a, \qquad \Delta \frac{1}{R} = 0.056 \frac{t}{b} \frac{c_a}{t}$$

Twist of the Yawed Wing and the Swept-back Wing

From the foregoing it is seen that for elliptic lift distribution the wing situated at right angles to the flight direction must receive an elliptical spanwise distribution in first approximation. If the elliptic lift distribution is then to be retained for yawed or sweepback setting, a new corresponding twist of the wings must be effected. The extent of this twist is not immediately obtainable from the given formulas. From the profiles that were computed for the aspect ratio $\frac{b}{t} = 6$ and a number of angles it is seen that the twist increases proportionally to the angle β, for the yawed wing proportional to ξ and for the wing with sweepback approximately proportional to $\left(\frac{1}{4} - |\xi|\right)$. It also is apparent that the twist must be proportional to the angle of attack in first approximation. For on considering a flat plate with zero angle of attack it obviously need not be twisted for yawed or sweepback setting in order to maintain the same flow conditions and hence the elliptic lift distribution, which in this case is, to be sure, zero. The relationship between twist and angle of attack is inconvenient in practice, because the twist in a completed airplane cannot be varied with the angle of attack. So in a given case the twist must be effected for a medium angle of attack. The numerical data for aspect ratio 6, on the other hand, afford the following empirical formulas, which should be quite useful in practice; that is, for the ordinary profiles for which a_o and $b_o > 0$, $c_o < 0$, and b_o and c_o are small with respect to a_o. The angle of attack for the wing in yawed setting must be increased by about $1.3\beta\xi$ percent, for the wing in sweepback setting by $1.6\beta\left(\frac{1}{4} - |\xi|\right)$ percent of the geometrical angle of attack (computed from the zero lift direction) in the respective cross section without yawed or sweepback setting, if the positive course of the yawed wing $(\xi > 0)$ is pushed forward and β is the angle in degrees. The elliptic spanwise lift distribution is then retained.

The Slipping Rectangular Monoplane

An airplane moving with one corner leading rather than symmetrical to itself is said to slip. Consider this same wing as lying symmetrically in the coordinate system and the free vortices as traveling to the side to infinity (fig. 4). For the first, the calculations show little difference from the yawed wing for lateral boundary parallel to flight direction. Computing (as under Nature of the Calculation) the velocity w, the integral

$$\frac{\gamma_0}{2\pi b} J = \frac{\gamma_0}{2\pi b} \int_{-1}^{1} \frac{\xi' \, d\xi'}{\sqrt{1-\xi'^2}\left(\xi' - \xi - \eta' \frac{t}{b} \tan\beta\right)}$$

appears as an essential contribution.

$$J = \pi \qquad \text{for} \left| \xi + \eta' \frac{t}{b} \tan\beta \right| \leqq 1$$

and

$$= \pi \left[1 - \frac{\xi + \eta' \frac{t}{b} \tan\beta}{\sqrt{(\xi + \eta' \frac{t}{b} \tan\beta)^2 - 1}} \right] \qquad \text{for} \left| \xi + \eta' \frac{t}{b} \tan\beta \right| > 1$$

For the far greater part of the wing the first value applies at small β; but the second value must be allowed for in the shaded areas of figure 4. For in this area $\left| \xi + (\eta - \eta') \frac{t}{b} \tan\beta \right|$ is partly or wholly > 1, if η' runs through the integration range -1 to +1. Hence, if (ξ, η) lies in the shaded area, the portion due to J to $w(\xi, \eta)$ is:

$$
\left.
\begin{aligned}
&\frac{1}{2\pi b}\left\{\int_{-1}^{\eta+(\xi+1)b/t\cot\beta}\gamma_o\pi\,d\eta' + \int_{\eta+(\xi+1)b/t\cot\beta}^{1}\gamma_o\pi \right.\\
&\qquad\left.\times\left[1-\frac{\xi+(\eta-\eta')\frac{t}{b}\tan\beta}{\sqrt{\left(\xi+(\eta-\eta')\frac{t}{b}\tan\beta\right)^2-1}}\right]d\eta'\right\}\quad\text{for } \xi<0,\\
&\frac{1}{2\pi b}\left\{\int_{-1}^{\eta+(\xi-1)b/t\cot\beta}\gamma_o\pi\left[1-\frac{\xi+(\eta-\eta')\frac{t}{b}\tan\beta}{\sqrt{\left(\xi+(\eta-\eta')\frac{t}{b}\tan\beta\right)^2-1}}\right]d\eta'\right.\\
&\qquad\left.+\int_{\eta+(\xi-1)b/t\cot\beta}^{1}\gamma_o\pi\,d\eta'\right\}\quad\text{for } \xi>0
\end{aligned}
\right\}\qquad(20)
$$

For all bound vortices the shedding free vortices of which still cover the point (ξ, η) (under certain circumstances to be visualized as extended toward the negative η direction) induce on it the constant downward velocity $\frac{\gamma_o}{2b}$. Or simply expressed, if (ξ, η) lies in the shaded area of figure 4, the $w(\xi, \eta)$ calculated with $J = \pi$ must be supplemented by

$$
\left.
\begin{aligned}
&-\frac{1}{2b}\int_{\eta+(\xi+1)b/t\cot\beta}^{1}\gamma_o\frac{\xi+(\eta-\eta')\frac{t}{b}\tan\beta}{\sqrt{\left(\xi+(\eta-\eta')-\tan\beta\right)^2\frac{t}{b}1}}\,d\eta'\quad\text{for } \xi<0\\
&-\frac{1}{2b}\int_{-1}^{\eta+(\xi-1)b/t\cot\beta}\gamma_o\frac{\xi+(\eta-\eta')\frac{t}{b}\tan\beta}{\sqrt{\left(\xi+(\eta-\eta')\frac{t}{b}\tan\beta\right)^2-1}}\,d\eta'\quad\text{for } \xi>0
\end{aligned}
\right\}\qquad(21)
$$

These integrals are elliptic again. Their calculation is omitted since the correction obtained through it cannot be applied, because the simple formula for w itself becomes invalid at the wing edges. Presumably the corners exhibit very high velocities and hence sharp reversals in profiles.

The yawed wing experiences, for lateral boundary parallel to the direction of flight, if not twisted conformably to the yawed setting, a moment about the y-axis. In figure 5 the half with greater lift is indicated by +, that with smaller lift by -. The same applies to the slipping rectangular wing, as long as the two shaded corners are disregarded. The correction term for the downward velocity, which these corners furnish, then becomes negative for $\xi > 0$ and positive for $\xi < 0$; hence, to obtain elliptic lift distribution, the left corner must be set steeper, the right corner not so steep (perhaps even steep downward). If this is not done, the left corner gives less lift, but the right one more lift (fig. 5). Therefore it is impossible to say in what sense the moment about the y-axis of a symmetrical rectangular wing in yaw occurs, because without calculation it is not known which of the two effects predominates.

Second Problem for the Flat Monoplane of Aspect Ratio 6

For the practice the second fundamental problem, namely, the determination of a lift distribution for a given profile, will always be the more important one.

This problem is treated by assuming a circulation distribution with indeterminate coefficients, calculating the flow, profile, and drag for every single distribution as for the solution of the first fundamental problem, and then determine the indeterminate coefficients from the condition that the wing shall assume a prescribed profile. The wing copies the profile exactly at as many points as there are indeterminate coefficients in the lift distribution.

For the slipping rectangular flat wing the following circulation distribution with 12 coefficients is assumed:

$$\gamma = \sqrt{1-\xi^2}\left\{\left[a_0\sqrt{\frac{1-\eta}{1+\eta}} + b_0\sqrt{1-\eta^2} + c_0\eta\sqrt{1-\eta^2}\right]\right.$$

$$+\ \xi\left[a_1\sqrt{\frac{1-\eta}{1+\eta}} + b_1\sqrt{1-\eta^2} + c_1\eta\sqrt{1-\eta^2}\right]$$

$$+\ \xi^2\left[a_2\sqrt{\frac{1-\eta}{1+\eta}} + b_2\sqrt{1-\eta^2} + c^2\eta\sqrt{1-\eta^2}\right]$$

$$\left.+\ \xi^3\left[a_3\sqrt{\frac{1-\eta}{1+\eta}} + b_3\sqrt{1-\eta^2} + c_3\eta\sqrt{1-\eta^2}\right]\right\}$$

The results of the necessary calculations, which correspond to those of the first fundamental problem, have already been described under Results of the Calculations. Velocity, profile, lift, and moment are linearly composed of the individual parts; the drag is quadratic in the circulation and is therefore tedious to compute in certain cases. That the wing shall be flat simply implies that the velocity $w(\xi, \eta)$ over the entire wing shall be constant. Thus at 12 points of the wing the equation:

$$
\begin{aligned}
w(\xi\ ,\ \eta) = {} & a_o\,[A_a + B_a\eta + C_a\eta^2\,] + b_o\,[A_b + B_b\eta + C_b\eta^2 + D_b\eta^3\,] + c_o\,[\,A_c + B_c\eta + C_c\eta^2 + D_c\eta^3 + E_c\eta^4] \\
& + a_1\,[\,A_a^{(1)} + B_a^{(1)}\eta + C_a^{(1)}\eta^2\,] + b_1\,[\,A_b^{(1)} + \qquad\qquad] + c_1\,[\,A_c^{(1)} + \qquad\qquad\qquad] \\
& + a_2\,[A_a^{(2)} + \qquad\qquad\qquad\qquad] + b_2\,[A_b^{(2)} + \qquad\qquad] + c_2\,[\,A_c^{(2)} + \qquad\qquad\qquad] \\
& + a_3\,[A_a^{(3)} + \qquad\qquad\qquad\qquad] + b_3\,[A_b^{(3)} + \qquad\qquad] + c_3\,[A_c^{(3)} + \qquad\qquad\qquad] \\
& = \text{constant}
\end{aligned}
\tag{22}
$$

must be fulfilled.

The constant is proportional to the angle of attack α. The coefficients of equation (22) follow by means of (14) from (12), (15), (16), and (17). The intersection points of the straight line $\xi = \pm \frac{1}{4}, \pm \frac{3}{4}$ with the straight lines $\eta = 0, \pm \frac{1}{2}\sqrt{3}$ are chosen as the 12 points at which equation (22) is to be fulfilled. These values are chosen, because any contour is better approximated by one such third order with the same tangent angles in the points $\eta = 0, \pm \frac{1}{3}\sqrt{3}$ than otherwise by a curve of the third order (reference 2, p. 294). The solution of the 12 equations for different values of β $\left(\lambda = \frac{\text{[illegible]}\,10^3}{26\ V \tan \alpha}\right)$ affords the following values:

β	$a_0\lambda$	$b_0\lambda$	$c_0\lambda$	$a_1\lambda$	$b_1\lambda$	$c_1\lambda$	$a_2\lambda$	$b_2\lambda$	$c_2\lambda$	$a_3\lambda$	$b_3\lambda$	$c_3\lambda$
0	66.8	-1.46	-0.23	0	0	0	29.5	-20.5	7.35	0	0	0
.1	66.8	-1.46	-.17	-1.58	.09	1.07	29.4	-20.1	6.68	-1.77	3772	-1.95
.3	66.7	-1.14	-.25	-4.58	-1.01	1.43	30.5	-20.1	7.30	-5.40	11.4	-2.90
.5	66.5	-0.48	.29	-7.54	-1.76	1.68	31.5	-20.0	7.37	-9.02	19.0	-3.48

The spanwise circulation distribution is:

$$\begin{aligned}
\gamma &= 66.07\,\pi\sqrt{1-\xi^2}\,[1 \qquad\qquad\quad + 0.292\,\xi^2 \qquad\qquad\qquad\ ] \quad \text{for } \beta = 0 \\
&= 66.07\,\pi\sqrt{1-\xi^2}\,[1 - 0.023\,\xi + 0.293\,\xi^2 + 0.0014\,\xi^3] \quad \text{for } \beta = 0.1 \\
&= 66.13\,\pi\sqrt{1-\xi^2}\,[1 - 0.077\,\xi + 0.309\,\xi^2 + 0.0045\,\xi^3] \quad \text{for } \beta = 0.3 \\
&= 66.21\,\pi\sqrt{1-\xi^2}\,[1 - 0.127\,\xi + 0.325\,\xi^2 + 0.0076\,\xi^3] \quad \text{for } \beta = 0.5
\end{aligned}$$

(See fig. 6.)

The coefficient of ξ^2 for $\beta = 0$, according to Betz (reference 3), for a rectangular wing of identical profile and angle of attack is in round figures 0.320 (found by graphical interpolation for $L = 3.82$ corresponding to $b:t = 6$), which is in sufficient agreement with the value obtained here. The discrepancy is due to the fact that the series was broken off behind the term with ξ^2, whereas Betz completed the whole series.

The moment about the y-axis introduced by the yawed setting is:

$$M_y = -\frac{\pi^2}{16} \rho b^3 V^2 \tan\alpha \times 0 \quad \text{for } \beta = 0;$$

$$\times 1.50 \times 10^{-3} \quad \text{for } \beta = 0.1$$

$$\times 4.93 \times 10^{-3} \quad \text{for } \beta = 0.3$$

$$\times 8.17 \times 10^{-3} \quad \text{for } \beta = 0.5$$

M_y increases proportional to β; the greater lift is experienced by the trailing half of the wing. However, it should be noted that a test may well give the opposite result, because the two shaded corners (fig. 4) have been disregarded; they presumably have a considerable share in the formation of the moment.

The calculation of the total drag (with suction force) is considerably reduced for the wing in question, since $w = V \tan\alpha$ is constantly distributed over the span and chord of the wing. It simply gives

$$W + S = A \frac{w}{V} = A \tan\alpha = \frac{\pi^2}{2} \rho b^2 V^2 \tan^2\alpha \times 70.88 \times 10^{-3} \quad \text{for} \quad \beta = 0$$

$$\times 70.91 \times 10^{-3} \quad \text{for} \quad \beta = 0.1$$

$$\times 71.24 \times 10^{-3} \quad \text{for} \quad \beta = 0.3$$

$$\times 71.59 \times 10^{-3} \quad \text{for} \quad \beta = 0.5$$

The separation in drag and suction force is effected by computing the suction force alone; only the chord distribution (a) in equation (3) and the term of the speed arising from B are considered.

$$W = \frac{\pi^2}{2} \rho b^2 V^2 \tan^2\alpha \times 16.6 \times 10^{-3} \quad \text{for} \quad \beta = 0$$

$$\times 16.7 \times 10^{-3} \quad \text{for} \quad \beta = 0.1$$

$$\times 16.7 \times 10^{-3} \quad \text{for} \quad \beta = 0.3$$

$$\times 16.8 \times 10^{-3} \quad \text{for} \quad \beta = 0.5$$

For $\beta = 0$ the theory of the lifting line (reference 1, formula (23)) gives:

$$W = \frac{\pi \rho}{4}\left[\frac{1}{2}\left(a_o \pi + b_o \frac{\pi}{2}\right)^2 + \frac{1}{4}\left(a_o \pi + b_o \frac{\pi}{2}\right)\left(a_2 \pi + b_2 \frac{\pi}{2}\right)\right.$$

$$\left. + \frac{1}{8}\left(a_2 \pi + b_2 \frac{\pi}{2}\right)^2\right] = \frac{\pi^2}{2} \rho\, b^2 V^2 \tan^2 \alpha \times 16.00 \times 10^{-3}$$

in satisfactory agreement with the value computed here.

Conversion Formulas

The theory of the lifting line gives the formulas (reference 2, formulas (26a) and (27a)):

$$\alpha_1 - \alpha_2 = \frac{c_a}{\pi}\left(\frac{t_1}{b_1} - \frac{t_2}{b_2}\right), \qquad c_{w1} - c_{w2} = \frac{c_a^2}{\pi}\left(\frac{t_1}{b_1} - \frac{t_2}{b_2}\right) \tag{23}$$

with $c_a = \dfrac{A}{q\,b\,t}$ and $c_w = \dfrac{W}{q\,b\,t}$ $q = \dfrac{1}{2} \rho V^2$, dynamic pressure.

These formulas make it possible to convert the data from one case (1) to another (2) with different aspect ratio for fixed c_a; that is, to apply the test data obtained for one aspect ratio to other aspect ratios. These formulas were checked for a specific profile (Göttinger No. 389, fig. 7) up to $b/t = 7$ to 1, and found well confirmed up to $b/t = 3$; only those for $b/t = 2$ and 1 manifested considerable discrepancies (reference 4). It is suspected that these discrepancies are confirmed by the theory of the lifting surface.

The profile of the wing (that is, its backbone) is replaced by a curve which aerodynamically is as equivalent as possible to the profile. As such, a center line as symmetrical as possible is chosen. The calculation, reproduced below, did not give the desired results; the diagram $c_a(c_w)$ exhibited slight discrepancies with respect to the lifting line theory; the diagram $c_a(\alpha)$ showed, on the whole, substantially lower lift coefficients than the test. This result indicated that the backbone was not quite exactly chosen; it should perhaps be taken nearer to the suction side, the predominant effect of which is known.

In order to establish the ratio of the distance of the backbone from the upper and lower edge of the profile for the backbone to be aerodynamically equivalent to the profile, a number of circular crescent profiles that give equal lift for equal chord were analyzed by the Kármán and Trefftz method (reference 5). The looked-for ratio is dependent upon the peripheral angles of the two circular arcs of the crescent. The present profile was related to a circular crescent, and the ratio determined for these profiles was about 2/3. The backbone obtained with this value is represented in figure 7.

Now the lift distribution with the six coefficients a_o, b_o, c_o, a_2, b_2, c_2 for varying b/t is computed, as under 6, for a wing with this constant profile. To this end the angles of the tangent to the backbone are measured for

$$\eta = -\frac{\sqrt{3}}{2}, \qquad 0, \qquad \frac{\sqrt{3}}{2}$$

with respect to the chord drawn below the backbone.

$$\eta = -\frac{\sqrt{3}}{2} : \alpha_1 = -13^\circ\ 45'$$

$$\eta = 0 : \alpha_2 = 2^\circ\ 7'$$

$$\eta = \frac{\sqrt{3}}{2} : \alpha_3 = 6^\circ\ 23'\ 30''$$

At the points $\xi = \pm \frac{1}{4} \pm \frac{3}{4}$; $\eta = 0, \pm \frac{\sqrt{3}}{2}$ the equation

$$w(\xi, \eta) = a_o [A_a + B_a \eta + C_a \eta^2] + b_o [A_b + B_b \eta + C_b \eta^2 + D_b \eta^3]$$
$$+ c_o [A_c + B_c \eta + C_c \eta^2 + D_c \eta^3 + E_c \eta^4] + a_2 [A_a^{(2)} + B_a^{(2)} \eta$$
$$+ C_a^{(2)} \eta^2] + b_2 [A_b^{(2)} + \ldots] + c_2 [A_c^{(2)} + \ldots] = V(\alpha + \alpha_\nu) \quad (24)$$

must be fulfilled, with α_ν indicating one of the three values α_1, α_2, α_3 depending upon the value of η. The solution of the equations gives:

b/t	$a_0 \frac{10^3}{2\,t\,V}$	$b_0 \frac{10^3}{2\,t\,V}$	$c_0 \frac{10^3}{2\,t\,V}$	$a_2 \frac{10^3}{2\,t\,V}$	$b_2 \frac{10^3}{t\,V\,2}$	$c_2 \frac{10^3}{2\,t\,V}$
6	-21.8+400.8 α	100.0- 8.76 α	-66.6	-32.2+176.4 α	85.0-123.0 α	-65.4+ 37.7 α
4	-24.0+360.0 α	99.6- 22.92 α	-66.8- 1.36 α	-30.6+138.4 α	76.4-151.2 α	-67.6+ 74.8 α
2	-28.8+278.4 α	96.8- 66.0 α	-67.2+ 5.98 α	-25.6+ 92.4 α	56.0-137.6 α	-68.2+168.0 α
1	-31.8+211.8 α	90.6-120.8 α	-71.0+59.3 α	-19.1+ 34.4 α	33.5- 23.3 α	-62.2+210.3 α

From this c_a and c_w are computed as functions of α and compared with the test data, while it is borne in mind that α is referred to the chord below the profile; the difference is $1^o9'20''$. Regarding the values obtained here by means of the lifting surface theory like experimental data; that is, converting them by (23) to the aspect ratio ∞, produces no substantial discrepancies in the $c_a(c_w)$ diagram from the lifting line theory, or, in other words, the quoted conversion formula for c_w is confirmed. The $c_a(\alpha)$ diagram exhibits good agreement with the test, at least within the range of small angles of attack (fig. 8). The differences between test on conversion formula for α at $b/t = 2$ and 1 are largely confirmed. Obviously another effect disregarded so far as the theory is involved that causes the curvature of the curves $c_a(\alpha)$ and appears to be of significance especially for small aspect ratios. Its explanation with the available means of the airfoil theory is probably impossible, as the latter is essentially a linear theory.

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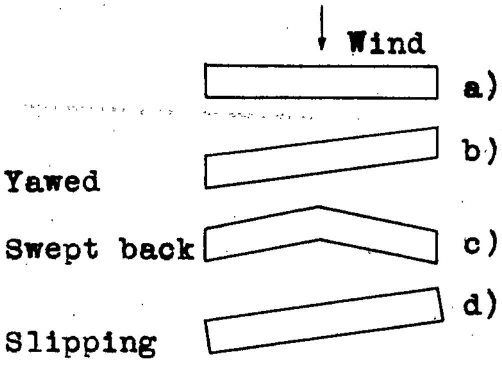

Figure 1.

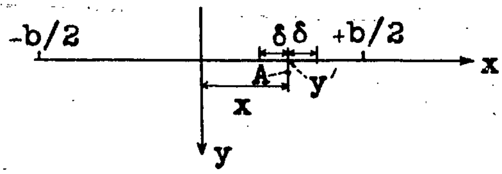

Figure 2.

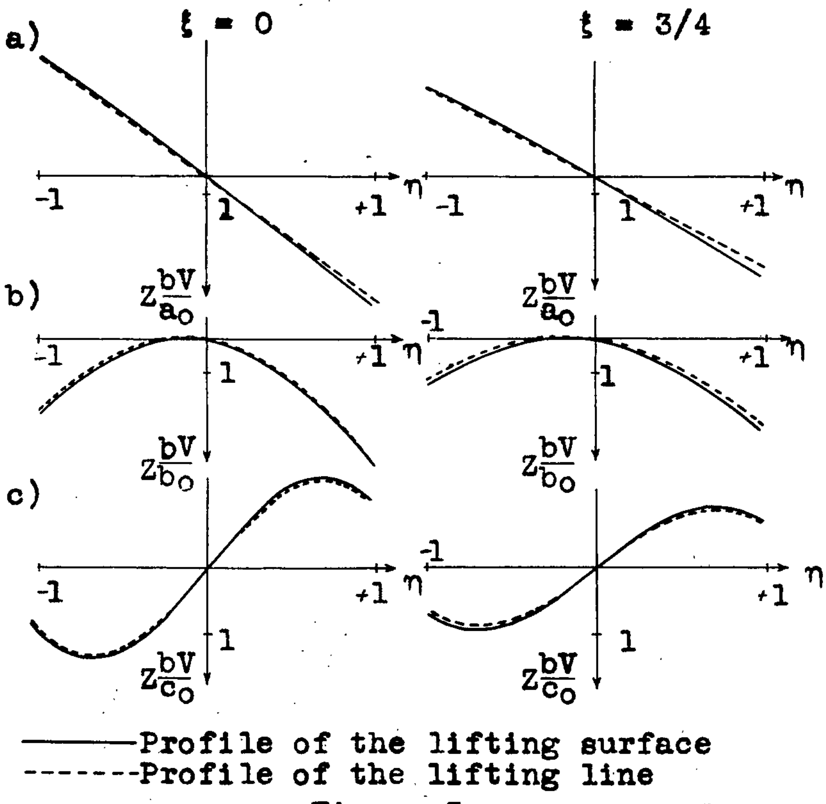

Figure 3.

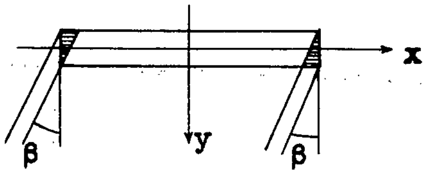

Figure 4.

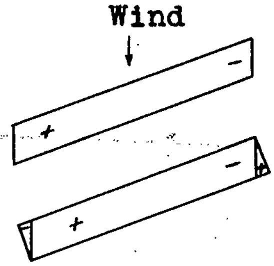

Figure 5.

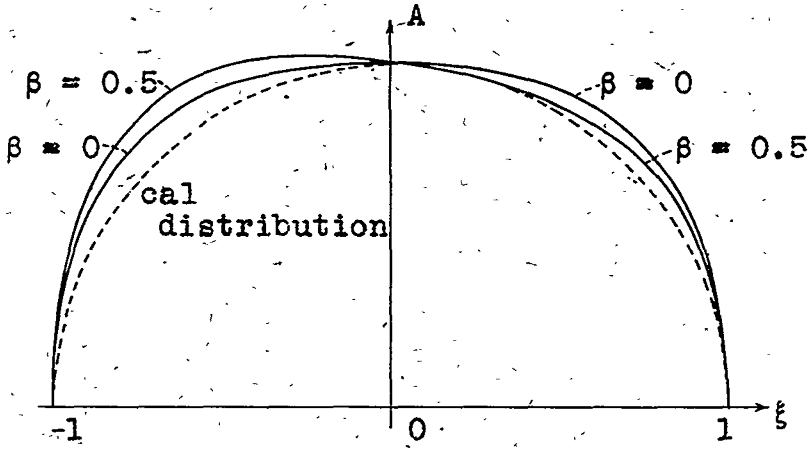

Figure 6.- Lift distribution of the flat rectangular wing of aspect ratio 6 at 0° and 28.6° yawed setting.

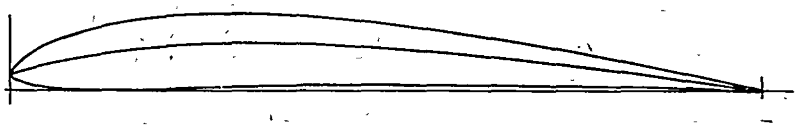

Figure 7.- Gottinger airfoil No. 389.

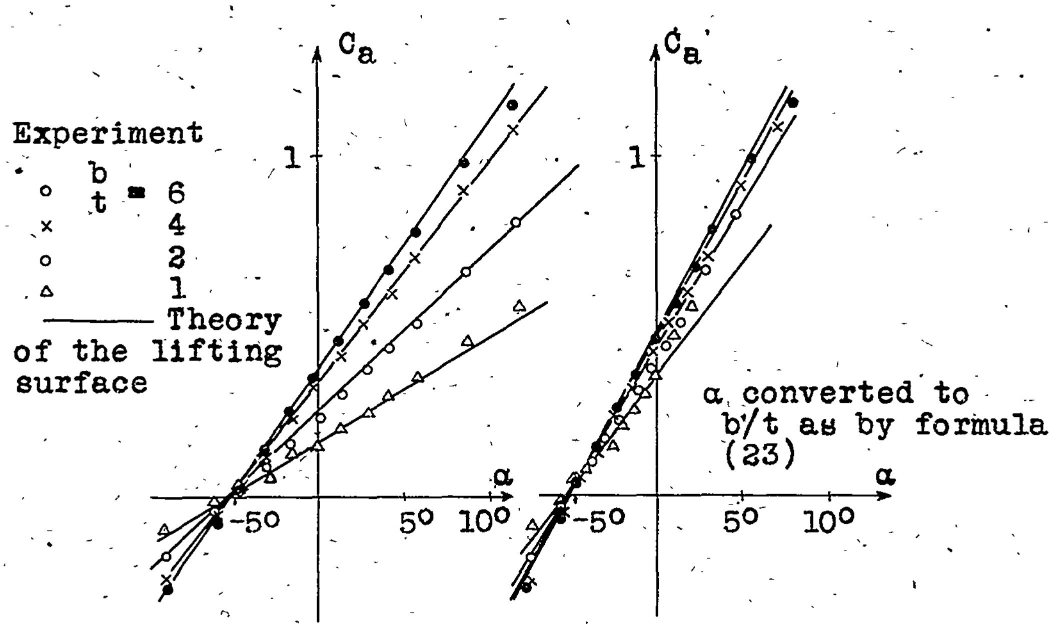

Figure 8.- Comparison of the theoretical and experimental C_a (α) curve.