NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM

No. 1116

STABILITY OF PLATES AND SHELLS BEYOND THE PROPORTIONAL LIMIT

By. A. A. Ilyushin

TRANSLATION

"Ustoichivost Plastinok i Obolochek za Predelom Uprugosti"
Prikladnaya Matematika i Mekhanika, N. S. 8, No. 5, 1944

Washington
October 1947

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STABILITY OF PLATES AND SHELLS BEYOND THE PROPORTIONAL LIMIT*

By A. A. Ilyushin

The problem of elastic equilibrium of plates, shells, and, in general, of thin-walled elements of metal structures and machines has been solved in the basic works of Bryan (reference 1), Timoshenko (reference 2), and many of their followers (reference 3). But essential difficulties were met in connection with the experimental verification of Euler's formula that could not be resolved within the bounds of the theory of elasticity. In practical applications, rods, plates, tubes, and other forms of shells, usually having only relatively thin walls, under the action of compressive forces often go beyond the proportional limit before reaching the critical (elastic) point, and therefore become unstable at appreciably lower loads.

This circumstance of the diminishing value of the formulas giving the critical loads under elastic deformations was completely cleared up in the fundamental works of Engesser (reference 4), and von Kármán (reference 5), who showed the limits of applicability of Euler's formula and extended these limits to the whole range of elastico-plastic deformations of rods under compression. The widely known result that Euler's formula remains valid even for plastic deformations if Young's modulus E is replaced by von Kármán's modulus K received such exact experimental verification that it was transferred automatically to many other cases of instability.

Thus, for rectangular plates compressed in the x-direction by a force P, Bleich (reference 6) without proof proposes to generalize the equation of Bryan to the case of plastic deformations by the introduction of a corrective factor

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*"Ustoichivost Plastinok i Obolochek za Predelom Uprugosti." Prikladnaya Matematika i Mekhanika, N.S. 8, No. 5, 1944, pp. 337-360.

$$D\left(\frac{K}{E}\frac{\partial^4 w}{\partial x^4} + 2\sqrt{\frac{K}{E}}\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + P\frac{\partial^2 w}{\partial x^2} = 0 \qquad (1)$$

Geckeler (reference 7) just as formally generalizes the equation of longitudinal bending of thin-walled tubes that buckle into axially symmetrical waves

$$D\frac{K}{E}\frac{d^4 w}{dx^4} + P\frac{d^2 w}{dx^2} + \frac{Eh}{R^2}w = 0 \qquad (2)$$

and shows that an analogous consideration of plasticity with the aid of von Kármán's modulus can be made in many other cases.

Among engineers the idea has very wide circulation that if the material of a plate or shell has a pronounced yield-plateau at a stress equal to the yield limit the plate will completely loose its load-carrying ability.

Let us note at the very beginning that all these conjectures turn out not only to be groundless but also, generally speaking, incorrect. They lead to an incorrect estimate of the load-bearing ability of structures and to too great a weight.

To Brylaard (reference 8) belongs the attempt to formulate the problem of the stability of compressed plates beyond the elastic limit on the basis of the Hencky-Mises theory of plasticity. But he was not able to cope with the difficulties of analysis of the stresses and strains in the elastico-plastic region, and after obtaining his formulas for calculation he had to introduce empirical corrections. As for other works it is difficult to point out any in which the reasoning and deductions were more basic or, better, less baseless than in the mentioned works of Bleich and Geckeler.

In the present paper is examined the method of investigating the stability of plates and shells beyond the elastic limit, that proceeds from the generalization of the Hencky-Mises theory of elastico-plastic deformations given in the works of Smirnov-Alyayev (reference 9), Schmidt (reference 10), and in our papers (reference 11).

1. LAW OF PLASTICITY FOR STATE OF PLANE STRESS IN VARIATIONAL FORM

Let us define the coordinates x, y, z, such that the xy-plane is tangent to the middle surface of the shell; take the x- and y-axes directed along the tangents to the arbitrarily chosen orthogonal curvilinear coordinates α, β, and the z-axis normal to the surface.

The basic thesis of the theory of shells is that at each surface z = constant there exists a state of plane stress varying with z. The connection between the stresses X_x, Y_y, X_y, and strains e_{xx}, e_{yy}, e_{xy}, can be written in the form

$$\left.\begin{aligned} X_x &= 4G\left[1-\omega(\epsilon_i)\right]\left(e_{xx}+\frac{1}{2}e_{yy}\right) \\ Y_y &= 4G\left[1-\omega(\epsilon_i)\right]\left(e_{yy}+\frac{1}{2}e_{xx}\right) \\ X_y &= G\left[1-\omega(\epsilon_i)\right]e_{xy} \end{aligned}\right\} \tag{1.1}$$

The "intensity of stress" σ_i

$$\sigma_i = \sqrt{X_x^2 + Y_y^2 - X_xY_y + 3X_y^2} \tag{1.2}$$

and "intensity of strain" ϵ_i

$$\epsilon_i = \frac{2}{\sqrt{3}}\sqrt{e_{xx}^2 + e_{yy}^2 + e_{xx}e_{yy} + \frac{1}{4}e_{xy}^2} \tag{1.3}$$

are connected by the relation

$$\sigma_i = 3\mu\epsilon_i = 3G(1-\omega)\epsilon_i \tag{1.4}$$

so that the function ω is defined by the physical properties of the material.

Let us find the variation of the intensity of strain

$$\delta\epsilon_i = \frac{\partial\epsilon_i}{\partial e_{xx}}\delta e_{xx} + \frac{\partial\epsilon_i}{\partial e_{yy}}\delta e_{yy} + \frac{\partial\epsilon_i}{\partial e_{xy}}\delta e_{xy}$$

$$= \frac{1}{3\mu\epsilon_i}\left(X_x\delta e_{xx} + Y_y\delta e_{yy} + X_y\delta e_{xy}\right)$$

From this we get

$$\sigma_i\delta\epsilon_i = X_x\delta e_{xx} + Y_y\delta e_{yy} + X_y\delta e_{xy} \tag{1.5}$$

Under the action of given external forces let the shell have the definite stress condition X_x, Y_y, X_y. Instability sets in when, at certain values of the external forces in addition to the indicated equilibrium state of stress, another state $X_x + \delta X_x$, $Y_y + \delta Y_y$, $X_y + \delta X_y$ is possible, where the transition from the first to the second takes place under constant forces.

Assume that we know the strain differences between the second and first equilibrium states δe_{xx}, δe_{yy}, δe_{xy}.

Before instability one region of a shell II[1] may be in a plastic state and another region Y in an elastic state. After instability the region II will divide into two parts; II → II and II → Y; in the former, loading will continue, that is, the transition from the first state to the second produces the further plastic deformations

$$\sigma_i\delta\epsilon_i > 0$$

while in the latter, unloading occurs; that is, this transition produces again elastic deformations

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[1]Translator's note: II – plasticheskii (plastic); Y–uprugii (elastic).

$$\sigma_i \delta \epsilon_i < 0$$

Thus the boundary between the two regions $II \rightarrow II$ and $II \rightarrow Y$ has the equation

$$\sigma_i \delta \epsilon_i = X_x \delta e_{xx} + Y_y \delta e_{yy} + X_y \delta e_{xy} = 0 \tag{1.6}$$

and represents a certain surface $z = z_0\ (\alpha,\beta)$.

It is not necessary to investigate specially the region of purely elastic deformations, since the function $\omega(\epsilon_i)$ for usual materials has everywhere a continuous derivative and therefore at instability the region Y will remain elastic; to it are applied the results of the theory of elasticity.

Let us investigate the relation between the variations of stress and strain for the region $II \longrightarrow II$. Varying (1.1) we find

$$\delta X_x = 4G\left[(1-\omega)\left(\delta e_{xx} + \frac{1}{2}\delta e_{yy}\right) - \left(e_{xx} + \frac{1}{2}e_{yy}\right)\frac{d\omega}{d\epsilon_i}\delta\epsilon_i\right]$$

$$\delta X_y = G\left[(1-\omega)\delta e_{xy} - e_{xy}\frac{d\omega}{d\epsilon_i}\delta\epsilon_i\right]$$

Let us introduce the symbols

$$\left.\begin{aligned} a &= \frac{X_x}{\sigma_i}\sqrt{\frac{\omega'}{1-\omega}} \\ b &= \frac{Y_y}{\sigma_i}\sqrt{\frac{\omega'}{1-\omega}} \\ c &= \frac{X_y}{\sigma_i}\sqrt{\frac{\omega'}{1-\omega}} \\ \omega' &= \frac{d\omega}{d\epsilon_i} \end{aligned}\right\} \tag{1.7}$$

$$\bar{\mu} = \frac{\mu}{\sigma_i} = \frac{G}{\sigma_i}(1 - \omega) \tag{1.8}$$

Using these in equations (1.1) and (1.5) we get finally

$$\left.\begin{aligned}
\frac{\delta X_x}{\sigma_i} &= (4\bar{\mu} - a^2)\ \delta e_{xx} + (2\bar{\mu} - ab)\ \delta e_{yy} - ac\delta e_{xy} \\
\frac{\delta Y_y}{\sigma_i} &= (2\bar{\mu} - ab)\ \delta e_{xx} + (4\bar{\mu} - b^2)\ \delta e_{yy} - bc\delta e_{xy} \\
\frac{\delta X_y}{\sigma_i} &= -ac\delta e_{xx} - bc\delta e_{yy} + (\bar{\mu} - c^2)\ \delta e_{xy}
\end{aligned}\right\} \tag{1.9}$$

For the region $II \to Y$ it is necessary to put in (1.9) $\omega = 0$, $\bar{\mu} = \bar{\mu}_o = \dfrac{G}{\sigma_i}$. Thus

$$\left.\begin{aligned}
\frac{\delta X_x}{\sigma_i} &= 4\bar{\mu}_o\delta e_{xx} + 2\bar{\mu}_o\delta e_{yy} \\
\frac{\delta Y_y}{\sigma_i} &= 2\bar{\mu}_o\delta e_{xx} + 4\bar{\mu}_o\delta e_{yy} \\
\frac{\delta X_y}{\sigma_i} &= \bar{\mu}_o\delta e_{xy}
\end{aligned}\right\} \tag{1.10}$$

The expression (1.9) shows that, in the region $II \to II$, as the result of instability, there appears the characteristic aspect of anisotropy when the additional normal stresses depend also on the variations of the shear strains and the additional shear stresses on the variations of the normal strains.

2. LAW OF PLASTICITY IN VARIATIONAL FORM

FOR PLATES AND SHELLS

Instability is attended by changes in the first and second quadratic forms of the middle surface. Let ϵ_1, ϵ_2 be the direct strain in the x and y directions and $\gamma = 2\epsilon_3$ the shear strain in the xy-plane appropriate to the change in the first quadratic form, and χ_1, χ_2, τ, the curvatures and twist appropriate to the change in the second quadratic form. The quantities considered represent the differences between the strains and curvatures after and before buckling, for which reason the quantities χ_1, χ_2, τ are called also distortions.

The six quantities ϵ_1, ϵ_2, ϵ_3, χ_1, χ_2, τ can be expressed in terms of the displacements u, v, w, by means of partial derivatives of not higher than the second order in the curvilinear coordinates α, β in the most general case. These expressions we shall consider known, inasmuch as they are derived in many treatments on the theory of elasticity and elastic stability.

The strains in a surface at a distance z from the middle surface are given by linear functions of z:

$$\left.\begin{aligned} \delta e_{xx} &= \epsilon_1 - z\chi_1 \\ \delta e_{yy} &= \epsilon_2 - z\chi_2 \\ \delta e_{xy} &= 2\epsilon_3 - 2z\tau \end{aligned}\right\} \tag{2.1}$$

On the basis of equation (1.6), we find the boundary of the regions II → II and II → Y, denoting the appropriate ordinate by z_0

$$z_0 = \frac{X_x\epsilon_1 + Y_y\epsilon_2 + 2X_y\epsilon_3}{X_x\chi_1 + Y_y\chi_2 + 2X_y\tau} \tag{2.2}$$

For definiteness, we assume that the region II → II adjoins the surface of the shell $z = \frac{1}{2}h$ and the region II → Y adjoins the surface $z = \frac{1}{2}h$, where h is the thickness of the shell.

Then from equations (1.9), (1.10), and (2.1) we have for

$$
\left.
\begin{aligned}
&\tfrac{1}{2} h \geqslant z \geqslant z_o \\
&\frac{\delta X_x}{\sigma_i} = (4\bar{\mu} - a^2)\epsilon_1 + (2\bar{\mu} - ab)\epsilon_2 - 2ac\epsilon_3 - z\Big[(4\bar{\mu} - a^2)\chi_1 \\
&\qquad + (2\bar{\mu} - ab)\chi_2 - 2ac\tau\Big] \\
&\frac{\delta Y_y}{\sigma_i} = (2\bar{\mu} - ab)\epsilon_1 + (4\bar{\mu} - b^2)\epsilon_2 - 2bc\epsilon_3 - z\Big[(2\bar{\mu} - ab)\chi_1 \\
&\qquad + (4\bar{\mu} - b^2)\chi_2 - 2bc\tau\Big] \\
&\frac{\delta X_y}{\sigma_i} = -ac\epsilon_1 - bc\epsilon_2 + 2(\bar{\mu} - c^2)\epsilon_3 - z\Big[-ac\chi_1 - bc\chi_2 + 2(\bar{\mu} - c^2)\tau\Big]
\end{aligned}
\right\} \quad (2.3)
$$

for $z_o \geqslant z \geqslant -\frac{1}{2} h$

$$
\left.
\begin{aligned}
\frac{\delta X_x}{\sigma_i} &= 4\bar{\mu}_o\epsilon_1 + 2\bar{\mu}_o\epsilon_2 - z(4\bar{\mu}_o\chi_1 + 2\bar{\mu}_o\chi_2) \\
\frac{\delta Y_y}{\sigma_i} &= 2\bar{\mu}_o\epsilon_1 + 4\bar{\mu}_o\epsilon_2 - z(2\bar{\mu}_o\chi_1 + 4\bar{\mu}_o\chi_2) \\
\frac{\delta X_y}{\sigma_i} &= 2\bar{\mu}_o\epsilon_3 - z2\bar{\mu}_o\tau
\end{aligned}
\right\} \quad (2.4)
$$

Now we may write the variations of the resultant middle-surface forces

$$
\left.
\begin{aligned}
\delta T_1 &= \int_{-\frac{1}{2}h}^{\frac{1}{2}h} \delta X_x \, dz \\
\delta T_2 &= \int_{-\frac{1}{2}h}^{\frac{1}{2}h} \delta Y_y \, dz \\
\delta S &= \int_{-\frac{1}{2}h}^{\frac{1}{2}h} \delta X_y \, dz
\end{aligned}
\right\} \quad (2.5)
$$

Employing the following dimensionless quantities for the distortions and the ordinate z

$$\bar{\chi}_1 = \frac{1}{2} h\chi_1, \quad \bar{\chi}_2 = \frac{1}{2} h\chi_2, \quad \bar{\tau} = \frac{1}{2} h\tau = \bar{\chi}_3, \quad \bar{z} = \frac{2}{h} z, \quad (1 \geqslant \bar{z} \geqslant -1) \qquad (2.6)$$

and taking into account the different expressions for δX_x, δY_y, δX_y for $1 \geqslant \bar{z} \geqslant \bar{z}_o$ and $\bar{z}_o \geqslant \bar{z} \geqslant -1$, we find

$$\left.\begin{aligned}
\frac{2\delta T_1}{h\sigma_i} &= \int_{-1}^{\bar{z}_o} \frac{\delta X_x}{\sigma_i} d\bar{z} + \int_{\bar{z}_o}^{1} \frac{\delta X_x}{\sigma_i} d\bar{z} = \sum_{i=1}^{3} \left[\epsilon_i t_1^{(i)} + \bar{\chi}_i \tau_1^{(i)}\right] \\
\frac{2\delta T_2}{h\sigma_i} &= \int_{-1}^{\bar{z}_o} \frac{\delta Y_y}{\sigma_i} d\bar{z} + \int_{\bar{z}_o}^{1} \frac{\delta Y_y}{\sigma_i} d\bar{z} = \sum_{i=1}^{3} \left[\epsilon_i t_2^{(i)} + \bar{\chi}_i \tau_2^{(i)}\right] \\
\frac{2\delta S}{h\sigma_i} &= \int_{-1}^{\bar{z}_o} \frac{\delta X_y}{\sigma_i} d\bar{z} + \int_{\bar{z}_o}^{1} \frac{\delta X_y}{\sigma_i} d\bar{z} = \sum_{i=1}^{3} \left[\epsilon_i t_3^{(i)} + \bar{\chi}_i \tau_3^{(i)}\right]
\end{aligned}\right\} \qquad (2.7)$$

where $t_k^{(i)}$, $\tau_k^{(i)}$ are defined by the formulas

$$t_1' = 4(\bar{\mu}_o + \bar{\mu}) + 4(\bar{\mu}_o - \bar{\mu})\bar{z}_o - a^2(1 - \bar{z}_o)$$

$$t_1'' = 2(\bar{\mu}_o + \bar{\mu}) + 2(\bar{\mu}_o - \bar{\mu})\bar{z}_o - ab(1 - \bar{z}_o)$$

$$t_1''' = -2ac(1 - \bar{z}_o)$$

$$t_2' = 2(\bar{\mu}_o + \bar{\mu}) + 2(\bar{\mu}_o - \bar{\mu})z_o - ab(1 - \bar{z}_o)$$

$$t_2'' = 4(\bar{\mu}_o + \bar{\mu}) + 4(\bar{\mu}_o - \bar{\mu})\bar{z}_o - b^2(1 - \bar{z}_o)$$

$$t_2''' = -2bc(1 - \bar{z}_o)$$

$$t_3' = -ac(1 - \bar{z}_o)$$

$$t_3'' = -bc(1 - \bar{z}_o)$$

$$t_3''' = 2(\bar{\mu}_o + \bar{\mu}) + 2(\bar{\mu}_o - \bar{\mu})\bar{z}_o - 2c^2(1 - \bar{z}_o)$$

$$\tau_1' = \left[2(\bar{\mu}_o - \bar{\mu}) + \frac{1}{2} a^2\right](1 - \bar{z}_o^2)$$

$$\tau_1'' = \left(\bar{\mu}_o - \bar{\mu} + \frac{1}{2} ab\right)(1 - \bar{z}_o^2)$$

$$\tau_1''' = ac(1 - \bar{z}_o^2)$$

$$\tau_2' = \left(\bar{\mu}_o - \bar{\mu} + \frac{1}{2} ab\right)(1 - \bar{z}_o^2)$$

$$\tau_2'' = \left[2(\bar{\mu}_o - \bar{\mu}) + \frac{1}{2} b^2\right](1 - \bar{z}_o^2)$$

$$\tau_2''' = bc(1 - \bar{z}_o^2)$$

$$\tau_3' = \frac{1}{2} ac(1 - \bar{z}_o^2)$$

$$\tau_3'' = \frac{1}{2} bc(1 - \bar{z}_o^2)$$

$$\tau_3''' = (\bar{\mu}_o - \bar{\mu} + c^2)(1 - \bar{z}_o^2)$$

In order to simplify equations (2.7), let us consider the expressions (2.3) and (2.4). With the introduction of the symbols

$$\left.\begin{aligned} \epsilon &= \frac{a}{s_1}\epsilon_1 + \frac{b}{s_1}\epsilon_2 + \frac{2c}{s_1}\epsilon_3 \\ \bar{X} &= \frac{a}{s_1}\bar{X}_1 + \frac{b}{s_1}\bar{X}_2 + \frac{2c}{s_1}\bar{\tau} \\ s_1^2 &= a^2 + b^2 - ab + 3c^2 \end{aligned}\right\} \tag{2.8}$$

these expressions, for $1 \geqslant \bar{z} \geqslant \bar{z}_o$ can be transformed into

$$\left.\begin{aligned}
\frac{1}{\sigma_i}\left(\delta X_x - \frac{1}{2}\,\delta Y_y\right) &= 3\bar{\mu}\left(\epsilon_1 - \bar{z}\bar{X}_1\right) - s_i\left(a - \frac{1}{2}\,b\right)\left(\epsilon - \bar{z}\bar{X}\right) \\
\frac{1}{\sigma_i}\left(\delta Y_y - \frac{1}{2}\,\delta X_x\right) &= 3\bar{\mu}\left(\epsilon_2 - \bar{z}\bar{X}_2\right) - s_i\left(b - \frac{1}{2}\,a\right)\left(\epsilon - \bar{z}\bar{X}\right) \\
\frac{1}{\sigma_i}\,\delta X_y &= 2\bar{\mu}\left(\epsilon_3 - \bar{z}\bar{\tau}\right) - s_i c\left(\epsilon - \bar{z}\bar{X}\right)
\end{aligned}\right\} \qquad (2.9)$$

Multiply the numerator and denominator of the expression (2.2) by

$$\frac{1}{s_i\sigma_i}\sqrt{\frac{\omega'}{1 - \omega}}$$

and divide both sides of the equation by $\frac{1}{2}\,h$; as is easily seen, we get

$$\bar{z}_o = \frac{\epsilon}{\bar{X}} \qquad (2.10)$$

Introducing $\epsilon = \bar{z}_o\bar{X}$ into equation (2.9), we get finally:
for $1 \geqslant \bar{z} \geqslant \bar{z}_o$

$$\left.\begin{aligned}
\frac{1}{\sigma_i}\left(\delta X_x - \frac{1}{2}\,\delta Y_y\right) &= 3\bar{\mu}\epsilon_1 - 3\bar{\mu}\bar{X}_1\bar{z} + s_i\left(a - \frac{1}{2}\,b\right)\left(\bar{z} - \bar{z}_o\right)\bar{X} \\
\frac{1}{\sigma_i}\left(\delta Y_y - \frac{1}{2}\,\delta X_x\right) &= 3\bar{\mu}\epsilon_2 - 3\bar{\mu}\bar{X}_2\bar{z} + s_i\left(b - \frac{1}{2}\,a\right)\left(\bar{z} - \bar{z}_o\right)\bar{X} \\
\frac{1}{\sigma_i}\,\delta X_y &= 2\bar{\mu}\epsilon_3 - 3\bar{\mu}\bar{\tau}\bar{z} + s_i c\left(\bar{z} - \bar{z}_o\right)\bar{X}
\end{aligned}\right\} \qquad (2.11)$$

for $\bar{z}_o \geqslant \bar{z} \geqslant -1$

$$\left.\begin{aligned}
\frac{1}{\sigma_i}\left(\delta X_x - \frac{1}{2}\,\delta Y_y\right) &= 3\bar{\mu}_o\epsilon_1 - 3\bar{\mu}_o\bar{X}_1\bar{z} \\
\frac{1}{\sigma_i}\left(\delta Y_y - \frac{1}{2}\,\delta X_x\right) &= 3\bar{\mu}_o\epsilon_2 - 3\bar{\mu}_o\bar{X}_2\bar{z} \\
\frac{1}{\sigma_i}\,\delta X_y &= 2\bar{\mu}_o\epsilon_3 - 2\bar{\mu}_o\bar{\tau}\bar{z}
\end{aligned}\right\} \qquad (2.12)$$

From these equalities it is easily seen that the system of equations (2.7) can be written in the form

$$\left.\begin{aligned}
\frac{2}{h\sigma_i}\left(\delta T_1 - \frac{1}{2}\,\delta T_2\right) &= 3\left[\bar{\mu}_o + \bar{\mu} + (\bar{\mu}_o - \bar{\mu})\bar{z}_o\right]\epsilon_1 + \frac{3}{2}(\bar{\mu}_o - \bar{\mu})\left(1 - \bar{z}_o^{\,2}\right)\bar{X}_1 \\
&\quad + \frac{1}{2}\,s_i\left(a - \frac{b}{2}\right)\left(1 - \bar{z}_o\right)^2\,\bar{X} \\
\frac{2}{h\sigma_i}\left(\delta T_2 - \frac{1}{2}\,\delta T_1\right) &= 3\left[\bar{\mu}_o + \bar{\mu} + (\bar{\mu}_o - \bar{\mu})\bar{z}_o\right]\epsilon_2 + \frac{3}{2}(\bar{\mu}_o - \bar{\mu})\left(1 - \bar{z}_o^{\,2}\right)\bar{X}_2 \\
&\quad + \frac{1}{2}\,s_i\left(b - \frac{1}{2}\,a\right)\left(1 - \bar{z}_o\right)^2\bar{X} \\
\frac{2}{h\sigma_i}\,\delta S &= 2\left[\bar{\mu}_o + \bar{\mu} + (\bar{\mu}_o - \bar{\mu})\bar{z}_o\right]\epsilon_3 + (\bar{\mu}_o - \bar{\mu})\left(1 - \bar{z}_o^{\,2}\right)\bar{\tau} + \frac{1}{2}\,s_i c\left(1 - \bar{z}_o\right)^2\bar{X}
\end{aligned}\right\} \qquad (2\ 13)$$

The relations (2.13) permit the derivation of a result extremely important for the theory of stability of shells: multiplying the first by a/s_i, second by b/s_i, third by $3c/s_i$, and adding we obtain

$$\left\{s_i^{\,2}\left(1 - \bar{z}_o\right)^2 + 3(\bar{\mu}_o - \bar{\mu})\left(1 - \bar{z}_o^{\,2}\right) + 6\bar{z}_o\left[\bar{\mu}_o + \bar{\mu} + (\bar{\mu}_o - \bar{\mu})\bar{z}_o\right]\right\}\bar{X}$$
$$= \frac{4}{h\sigma_i s_i}\left[\left(a - \frac{b}{2}\right)\delta T_1 + \left(b - \frac{1}{2}\,a\right)\delta T_2 + 3c\,\delta S\right] \qquad (2.14)$$

From equations (1.7) and (2.8) we have

$$s_i = \sqrt{\frac{\omega'}{1 - \omega}} \tag{2.15}$$

Differentiating equation (1.4) we get

$$\frac{\omega'}{1 - \omega} = s_i^2 = 3\bar{\mu} - \frac{1}{\sigma_i}\frac{d\sigma_i}{d\epsilon_i} = \frac{1}{\sigma_i}\left(\frac{\sigma_i}{\epsilon_i} - \frac{d\sigma_i}{d\epsilon_i}\right) \tag{2.16}$$

Returning to the initial expressions for the forces and strains before instability ($T_1 = hX_x$, $S = hX_y$ . . .) we transform equation (2.14) into

$$(1 - \bar{z}_o)^2 - \frac{4E}{E - \frac{d\sigma_i}{d\epsilon_i}}(1 - \bar{z}_o)$$
$$+ \frac{4E}{E - \frac{d\sigma_i}{d\epsilon_i}}\left[1 - (1 - \omega)\frac{e_{xx}\delta T_1 + e_{yy}\delta T_2 + e_{xy}\delta S}{\bar{X}_1 T_1 + \bar{X}_2 T_2 + 2\bar{\tau} S}\right] = 0 \tag{2.17}$$

where $E = 3G$.

Solving this quadratic equation for $\bar{z}_o = 2z_o/h$ we find the thickness of the region II→II relativo to the thickness of the shell

$$\frac{1 - \bar{z}_o}{2} = \frac{1}{h}\left(\frac{h}{2} - z_o\right) = \frac{E - \sqrt{E\sigma_i'(1 + \varphi)}}{E - \sigma_i'} \tag{2.18}$$

Here the quantity φ is defined by

$$\varphi = \frac{E - \sigma_i'}{\sigma_i'}(1 - \omega)\frac{e_{xx}\delta T_1 + e_{yy}\delta T_2 + e_{xy}\delta S}{T_1\bar{X}_1 + T_2\bar{X}_2 + 2S\bar{\tau}} \tag{2.19}$$

and is invariant with respect to a rotation of axes x, y about the z-axis, because the bilinear forms in the numerator and denominator are invariants.

We have obtained the following result: The regions $II \rightarrow II$ and $II \rightarrow Y$ resulting after buckling of a shell are divided by a surface given by equation (2.18); the position of this surface within the shell depends, in general, both on its state of stress and strain before instability and on the form of buckling, namely, the ratio of the work done by the additional middle-surface forces on the strains in the base state $e_{xx}\delta T_1 + e_{yy}\delta T_2 + e_{xy}\delta S$ to work, arising at buckling from the distortion of the middle surface, done by the projections on the normal of the forces T_1, T_2, S . . . $T_1\bar{X}_1 + T_2\bar{X}_2 + 2S\bar{\tau}$.

Let us pass on to the calculation of the bending and twisting moments resulting from the system of stresses (equations (2.11) and (2.12)). By definition we have

$$\left.\begin{aligned} \delta M_1 &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \delta X_x z\, dz \\ \delta M_2 &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \delta Y_y z\, dz \\ \delta H &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \delta X_y z\, dz \end{aligned}\right\} \tag{2.20}$$

Calculation gives

$$\left.\begin{aligned}
\frac{24}{h^2\sigma_i}\left(\delta M_1 - \frac{1}{2}\,\delta M_2\right) &= -9(\bar{\mu}_0 - \bar{\mu})\left(1 - \bar{z}_0{}^2\right)\epsilon_1 - 6\left[\bar{\mu}_0 + \bar{\mu} + (\bar{\mu}_0 - \bar{\mu})\bar{z}_0{}^3\right]\bar{\chi}_1 \\
&\quad + s_i\left(a - \frac{1}{2}\,b\right)\left(\bar{z}_0{}^3 - 3\bar{z}_0 + 2\right)\bar{\chi} \\
\frac{24}{h^2\sigma_i}\left(\delta M_2 - \frac{1}{2}\,\delta M_1\right) &= -9(\bar{\mu}_0 - \bar{\mu})\left(1 - \bar{z}_0{}^2\right)\epsilon_2 - 6\left[\bar{\mu}_0 + \bar{\mu} + (\bar{\mu}_0 - \bar{\mu})z_0{}^3\right]\bar{\chi}_2 \\
&\quad + s_i\left(b - \frac{1}{2}\,a\right)\left(\bar{z}_0{}^3 - 3\bar{z}_0 + 2\right)\bar{\chi} \\
\frac{24}{h^2\sigma_i}\,\delta H &= -6(\bar{\mu}_0 - \bar{\mu})\left(1 - \bar{z}_0{}^2\right)\epsilon_3 - 4\left[\bar{\mu}_0 + \bar{\mu} + \left(\bar{\mu}_0 - \bar{\mu}\right)\bar{z}_0{}^3\right]\bar{\tau} \\
&\quad + s_i c\left(\bar{z}_0{}^3 - 3\bar{z}_0 + 2\right)\bar{\chi}
\end{aligned}\right\} \qquad (2.21)$$

From this too can be derived a relationship analogous to equation (2.17) but cubic in $\bar{z}_0$ and having in place of φ a new quantity containing the ratio

$$\frac{e_{xx}\delta M_1 + e_{yy}\delta M_2 + e_{xy}\delta H}{T_1\bar{\chi}_1 + T_2\bar{\chi}_2 + 2S\bar{\tau}}$$

Thus, for the general case of instability, the connections between the variations of the resultant forces and moments and the variations of strains, curvatures, and twist or, in the final analysis, the variations of the three displacements are given by the relations (2.13) and (2.21) where in place of $\bar{z}_0$ its expression (2.10) must be introduced.

The five quantities: the variations of the three components of displacement u, v, w and the variations of the two transverse forces N_1, N_2, must satisfy five differential equations of equilibrium of the shell.

All the expressions obtained above remain valid both in the case of elastico-plastic strains and in the case of purely elastic strains; in the latter case it is merely necessary to put

$$\omega(\epsilon_i) \equiv 0$$

Let us list the interesting features of the expressions (2.13) and (2.21).

(1) If at the elastic limit the variations of the forces δT_1, δT_2, δS depend only on the strains ϵ_1, ϵ_2, ϵ_3 and the variations of the moments δM_1, δM_2, δH only on the distortions χ_1, χ_2, τ then beyond the elastic limit both depend on the strains and the distortions.

(2) If at the elastic limit the stiffnesses of the shell in extension and bending depend simply on the thickness of the wall and the modulus of elasticity, then beyond the elastic limit they depend both on the construction of the shell and the forces acting.

3. CASE WHERE INSTABILITY IS NOT ACCOMPANIED BY A CHANGE OF MIDDLE-SURFACE FORCES

Placing

$$\delta T_1 = \delta T_2 = \delta S = 0 \tag{3.1}$$

we find accordingly from equations (2.19) and (2.18)

$$\varphi = 0, \ \frac{1}{2}(1 - \bar{z}_o) = \frac{1}{h}\left(\frac{h}{2} - z_o\right) = \frac{\sqrt{E}}{\sqrt{E} + \sqrt{\frac{d\sigma_i}{d\epsilon_i}}} \tag{3.2}$$

In this case, as is seen, the surface separating the regions $II \rightarrow II$ and $II \rightarrow Y$ does not depend on the form of buckling and is defined only by the state of stress before buckling, namely, the invariant $E'' = d\sigma_i/d\epsilon_i$, that is, the slope of the curve $\sigma_i = \sigma_i(\epsilon_i)$ at each point of the shell; if the intensity of stress σ_i before buckling is the same at every point, then the boundary between the regions $II \rightarrow II$ and $II \rightarrow Y$ is a surface parallel to the middle surface.

Taking into account relations (1.7), (2.8), and (2.15), we have

$$\left.\begin{aligned}
&S_i\left(a - \frac{1}{2}b\right) = \frac{1}{\sigma_i}\left(\frac{\sigma_i}{\epsilon_i} - \frac{d\sigma_i}{d\epsilon_i}\right)\left(\bar{X}_x - \frac{1}{2}\bar{Y}_y\right), \quad S_i c = \frac{1}{\sigma_i}\left(\frac{\sigma_i}{\epsilon_i} - \frac{d\sigma_i}{d\epsilon_i}\right)\bar{X}_y \\
&S_i\left(b - \frac{1}{2}a\right) = \frac{1}{\sigma_i}\left(\frac{\sigma_i}{\epsilon_i} - \frac{d\sigma_i}{d\epsilon_i}\right)\left(\bar{Y}_y - \frac{1}{2}\bar{X}_x\right), \quad \tilde{X} = \tilde{X}_1\bar{X}_x + \tilde{X}_2\bar{Y}_y + 2\tilde{\tau}\bar{X}_y
\end{aligned}\right\} \tag{3.3}$$

where the quantities

$$\bar{X}_x = \frac{X_x}{\sigma_i} \qquad \bar{Y}_y = \frac{Y_y}{\sigma_i} \qquad \bar{X}_y = \frac{X_y}{\sigma_i}$$

represent stresses referred to their intensity before instability, so that

$$\bar{X}_x^2 + \bar{Y}_y^2 - \bar{X}_x\bar{Y}_y + 3\bar{X}_y^2 = 1$$

On the basis of equation (3.1) we find from equation (2.13) expressions for the strains in the middle surface of the shell

$$\left.\begin{aligned}
-\epsilon_1 &= \frac{(\mu_o - \mu)\left(1 - \bar{z}_o^2\right)}{2\left[\mu_o + \mu + (\mu_o - \mu)\bar{z}_o\right]}\tilde{X}_1 + \left(\frac{\sigma_i}{\epsilon_i} - \frac{d\sigma_i}{d\epsilon_i}\right)\frac{\left(1 - \bar{z}_o\right)^2\left(\bar{X}_x - \frac{1}{2}\bar{Y}_y\right)}{6\left[\mu_o + \mu + (\mu_o - \mu)\bar{z}_o\right]}\tilde{X} \\
-\epsilon_2 &= \frac{(\mu_o - \mu)\left(1 - \bar{z}_o^2\right)}{2\left[\mu_o + \mu + (\mu_o - \mu)\bar{z}_o\right]}\tilde{X}_2 + \left(\frac{\sigma_i}{\epsilon_i} - \frac{d\sigma_i}{d\epsilon_i}\right)\frac{\left(1 - \bar{z}_o\right)^2\left(\bar{Y}_y - \frac{1}{2}\bar{X}_x\right)}{6\left[\mu_o + \mu + (\mu_o - \mu)\bar{z}_o\right]}\tilde{X} \\
-\epsilon_3 &= \frac{(\mu_o - \mu)\left(1 - \bar{z}_o^2\right)}{2\left[\mu_o + \mu + (\mu_o - \mu)\bar{z}_o\right]}\tilde{\tau} + \left(\frac{\sigma_i}{\epsilon_i} - \frac{d\sigma_i}{d\epsilon_i}\right)\frac{\left(1 - \bar{z}_o\right)^2\bar{X}_y}{4\left[\mu_o + \mu + (\mu_o - \mu)\bar{z}_o\right]}\tilde{X}
\end{aligned}\right\} \tag{3.4}$$

In contrast to the case of elastic instability here the following phenomenon is observed; if buckling of the shell beyond the elastic limit takes place under constant middle-surface forces, then it is accompanied by middle-surface strains proportional to the ensuing distortions.

Entering these values of the strains into the expressions for the bending moments (equation (2.21)) and collecting terms, we obtain

$$\left.\begin{aligned}
\frac{24}{h^2}\left(\delta M_1 - \frac{1}{2}\,\delta M_2\right) &= -12\mu_o(1-\psi)\bar{\chi}_1 + 12\mu_o(1-\chi)\left(\bar{X}_x - \frac{1}{2}\,\bar{Y}_y\right)\bar{\chi} \\
\frac{24}{h^2}\left(\delta M_2 - \frac{1}{2}\,\delta M_1\right) &= -12\mu_o(1-\psi)\bar{\chi}_2 + 12\mu_o(1-\chi)\left(\bar{Y}_y - \frac{1}{2}\,\bar{X}_x\right)\bar{\chi} \\
\frac{24}{h^2}\,\delta H &= -8\mu_o(1-\psi)\bar{\tau} + 12\mu_o(1-\chi)\bar{X}_y\bar{\chi}
\end{aligned}\right\} \quad (3.5)$$

where

$$\left.\begin{aligned}
8\mu_o\psi &= \frac{3\left(\mu_o-\mu\right)^2\left(1-\bar{z}_o^{\,2}\right)^2}{\mu_o+\mu+(\mu_o-\mu)\bar{z}_o} + 4(\mu_o-\mu)\left(1-\bar{z}_o^{\,3}\right) \\
12\mu_o\chi &= 12\mu_o - \frac{1}{2}\left(\frac{\sigma_i}{\epsilon_i} - \frac{d\sigma_i}{d\epsilon_i}\right)\left[\frac{3(\mu_o-\mu)\left(1-\bar{z}_o^{\,2}\right)\left(1-\bar{z}_o\right)^2}{\mu_o+\mu+(\mu_o-\mu)\bar{z}_o} + 2\left(\bar{z}_o^{\,3} - 3\bar{z}_o + 2\right)\right]
\end{aligned}\right\} \quad (3.6)$$

Let us now introduce the quantities

$$K = \frac{4E\,\dfrac{d\sigma_i}{d\epsilon_i}}{\left(\sqrt{E} + \sqrt{\dfrac{d\sigma_i}{d\epsilon_i}}\right)^2}\,; \quad k = \frac{K}{E} \quad (3.7)$$

The first represents a generalized von Kármán modulus for the case of a complex state of stress; the second, the relative modulus.

On the basis of equations (1.4) and (3.2) we have

$$\mu_0 - \mu = \mu_0\omega \qquad \bar{z}_0 = -1 + \sqrt{k}$$

and, consequently, the functions ψ and χ can be expressed in terms of the rigidity decrease function ω and the relative modulus k. After rather cumbersome transformations we obtain

$$\psi = \omega\left(1 - \frac{1}{2}\sqrt{k}\right)\left[\left(1 - \frac{1}{2}\sqrt{k}\right)^2 + \frac{3}{4}\frac{k}{1 - \left(1 - \frac{1}{2}\sqrt{k}\right)\omega}\right] \qquad \chi = k + \psi \qquad (3.8)$$

Solving the system (3.5) for the bending moments we obtain finally the basic formulas

$$\left.\begin{aligned}
\frac{\delta M_1}{D} &= -(1 - \psi)\left(\chi_1 + \frac{1}{2}\chi_2\right) + \frac{3}{4}(1 - \psi - k)\bar{X}_x\left(\bar{X}_x\chi_1 + \bar{Y}_y\chi_2 + 2\bar{X}_y\tau\right) \\
\frac{\delta M_2}{D} &= -(1 - \psi)\left(\chi_2 + \frac{1}{2}\chi_1\right) + \frac{3}{4}(1 - \psi - k)\bar{Y}_y\left(\bar{X}_x\chi_1 + \bar{Y}_y\chi_2 + 2\bar{X}_y\tau\right) \\
\frac{\delta H}{D} &= -\frac{1}{2}(1 - \psi)\tau + \frac{3}{4}(1 - \psi - k)\bar{X}_y\left(\bar{X}_x\chi_1 + \bar{Y}_y\chi_2 + 2\bar{X}_y\tau\right)
\end{aligned}\right\} \qquad (3.9)$$

where D denotes the usual bending stiffness of the shell

$$D = \frac{Eh^3}{12(1 - \nu^2)}$$

for the value of Poisson's ratio $\nu = 0.5$.

Let us point out some consequences of the relations obtained:

(1) If the characteristic of the material of the shell $\sigma_i = \sigma_i(\epsilon_i)$ has a pronounced yield-plateau, so that at some point the intensity of stress $\sigma_i = \sigma_s$ and $d\sigma_i/d\epsilon_i = 0$, then the generalized von Kármán modulus at that point becomes zero, but, in the general case, the shell,

nevertheless, does not lose its stiffness completely, since, besides $k = 0$, $\psi = \omega < 1$. The smallest stiffness will correspond to the end of the yield-plateau where the rigidity decrease is greater than any other point on the plateau. The only exceptions are several particular cases to which belong von Kármán's problem of the stability of a strut.

(2) The bending moments resulting from buckling are linear homogeneous functions of the distortions and have the potential

$$W = \frac{D}{2}\left[(1-\psi)\left(\chi_1^2 + \chi_1\chi_2 + \chi_2^2 + \tau^2\right) - \frac{3}{4}(1-\psi-k)\left(\bar{X}_x\chi_1 + \bar{Y}_y\chi_2 + 2\bar{X}_y\tau\right)^2\right]$$

$$\left.\begin{aligned} \delta M_1 &= -\frac{\partial W}{\partial \chi_1} \\ \delta M_2 &= -\frac{\partial W}{\partial \chi_2} \\ \delta H &= -\frac{1}{2}\frac{\partial W}{\partial \tau} \end{aligned}\right\} \qquad (3.10)$$

The function W represents the work of the bending moments on the distortions of the middle surface resulting from buckling referred to unit area of the middle surface, so that the total work will equal

$$\iint W\, df$$

This makes possible the application of the method of Timoshenko to the analysis of elastico-plastic stability of shells.

(3) As the two terms on the right-hand sides of the relations (3.9) show, the stiffness of the shell at instability beyond the elastic limit depends on both the plastic deformations before instability and the relations between the acting stresses X_x, Y_y, and X_y. Exceptions are those cases where before buckling there exists the relation

$$\frac{d\sigma_i}{d\epsilon_i} = \frac{\sigma_i}{\epsilon_i} \qquad \sigma_i = A\epsilon_i$$

true, generally speaking, only in the elastic region where $A = E$. However, for some materials the characteristic $\sigma_1 = \sigma_1(\epsilon_1)$ may have points where the tangent goes through the origin of coordinates so that $A < E$; at these points

$$\omega = \frac{4(1 - \sqrt{k})}{(2 - \sqrt{k})^2} \qquad 1 - \psi = k = \frac{4A}{(\sqrt{E} + \sqrt{A})^2}$$

The formulas (3.9) in this case show that the stiffness of the shell is proportional to the modulus K, and the critical forces are obtained from their elastic values by replacement of the modulus.

(4) The forces T_1, T_2, S perform on the middle surface strains ϵ_1, ϵ_2, $\gamma = 2\epsilon_3$ work which on the basis of equation (3.4) or equation (2.2) equals

$$T_1\epsilon_1 + T_2\epsilon_2 + 2S\epsilon_3 = z_o\left(T_1\chi_1 + T_2\chi_2 + 2S\tau\right)$$

If stability is investigated by the method of Timoshenko, this quantity may be omitted from the expression for the work of the inner forces, if also there is omitted the equal work of the peripheral (external) forces on the displacements corresponding to the strains ϵ_1, ϵ_2, ϵ_3; that is, the work of the external forces should be calculated only from the displacements arising from the distortions of the middle surface.

4. STABILITY OF PLATES

Let x,y be Cartesian coordinates in the middle surface of the plate and $w(x,y)$, its deflection under buckling. The curvatures, as is well known, will be expressed by the formulas

$$\chi_1 = \frac{\partial^2 w}{\partial x^2} \qquad \chi_2 = \frac{\partial^2 w}{\partial y^2} \qquad \tau = \frac{\partial^2 w}{\partial x \, \partial y} \tag{4.1}$$

The work of the peripheral forces on the displacements arising from the distortions of the middle surface is equal to

$$-\frac{1}{2}\iint\left[T_1\left(\frac{\partial w}{\partial x}\right)^2 + T_2\left(\frac{\partial w}{\partial y}\right)^2 + 2S\,\frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y}\right]dx\,dy = -\frac{h}{2}\iint\sigma_i\left(\bar{X}_x w_x{}^2 + \bar{Y}_y w_y{}^2 + 2\bar{X}_y\,w_x w_y\right)dx\,dy \tag{4.2}$$

and, consequently, the increment (variation) of total energy of the shell due to buckling is

$$J = \iint\left[W + \frac{h}{2}\,\sigma_i\left(\bar{X}_x w_x{}^2 + \bar{Y}_y w_y{}^2 + 2\bar{X}_y w_x w_y\right)\right]dx\,dy \tag{4.3}$$

The condition that this quantity equal zero

$$J = 0 \tag{4.4}$$

represents the generalized equation of Timoshenko for the case of elastico-plastic deformations.

In the first approximation the equilibrium of the shell after instability is neutral; that is

$$\delta J = 0$$

In order to prove this statement, it is sufficient to show that the differential equation of equilibrium of the shell after buckling and the boundary conditions proceed from the condition (4.4). Varying equation (4.3), we obtain on the basis of equation (3.10)

$$\begin{aligned}
\delta J = &\iint\Big[-\delta M_1\delta\chi_1 - \delta M_2\delta\chi_2 - 2\delta H\delta\tau + h\sigma_i\left(\bar{X}_x w_x + \bar{X}_y w_y\right)\delta w_x \\
&+ h\sigma_i\left(\bar{Y}_y w_y + \bar{X}_y w_x\right)\delta w_y\Big]\,dx\,dy \\
= &-\iint\left[\frac{\partial^2\delta M_1}{\partial x^2} + 2\,\frac{\partial^2\delta H}{\partial x\,\partial y} + \frac{\partial^2\delta M_2}{\partial y^2} + h\sigma_i\left(\bar{X}_x\chi_1 + \bar{Y}_y\chi_2 + 2\bar{X}_y\tau\right)\right]\delta w\,dx\,dy \\
&-\int\Big\{(\delta M_1 l + \delta H m)\,\delta w_x + (\delta M_2 m + \delta H l)\,\delta w_y \\
&- \left[h\sigma_i\left(\bar{X}_x w_x + \bar{Y}_y w_y\right) + \left(\frac{\partial\delta M_1}{\partial x} + \frac{\partial\delta H}{\partial y}\right)l + \left(\frac{\partial\delta M_2}{\partial y} + \frac{\partial\delta H}{\partial x}\right)m\right]\delta w\Big\}\,ds = 0
\end{aligned} \tag{4.5}$$

Here, l, m are the direction cosines of the normal to the boundary of the plate and $\bar{X}_\nu$, $\bar{Y}_\nu$ are the stresses on the boundary

$$\bar{X}_\nu = \bar{X}_x l + \bar{X}_y m$$

$$\bar{Y}_\nu = \bar{Y}_y m + \bar{X}_y l$$

and the quantities

$$\frac{\partial \delta M_1}{\partial x} + \frac{\partial \delta H}{\partial y} = \delta N_1$$

$$\frac{\partial \delta M_2}{\partial y} + \frac{\partial \delta H}{\partial x} = \delta N_2$$

represent the transverse forces.

Because of the arbitrariness of the variations δw and $\frac{\partial \delta w}{\partial \nu} = \delta w_x l + \delta w_y m$, we find from equation (4.5) both the differential equation of equilibrium

$$\frac{\partial^2 \delta M_1}{\partial x^2} + 2\frac{\partial^2 \delta H}{\partial x\, \partial y} + \frac{\partial^2 \delta M_2}{\partial y^2} + h\sigma_1\left(\bar{X}_x \chi_1 + \bar{Y}_y \chi_2 + 2\bar{X}_y \tau\right) = 0 \qquad (4.6)$$

and the kinematic boundary conditions, either combined or in the form of Kirchoff's conditions; that is, the usual conditions for plates.

In the general case the forces T_1, T_2, S appear as known functions of the coordinates x, y and hence the stresses

$$X_x = \frac{T_1}{h} \quad Y_y = \frac{T_2}{h} \quad X_y = \frac{S}{h}$$

and the quantities ψ and k are variable. Taking into account equations (3.9) and (4.1), we conclude that the equation of equilibrium (4.6) will be a homogeneous linear differential equation of the fourth order with variable coefficients containing all possible derivatives in x and y from the fourth to the second order.

Let us write this equation in explicit form for the case where the forces acting in the middle surface of the plate are constant. For axes of coordinates x, y, we shall take the principal axes of stress, since the latter will be straight lines.

In this case the shear stress X_y may be put equal to zero without loss of generality. From equation (3.9) we have

$$\left.\begin{aligned}
\frac{\delta M_1}{D} &= -(1-\psi)\left(\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\frac{\partial^2 w}{\partial y^2}\right) + \frac{3}{4}(1-\psi-k)\bar{X}_x\left(\bar{X}_x\frac{\partial^2 w}{\partial x^2} + \bar{Y}_y\frac{\partial^2 w}{\partial y^2}\right)\\
\frac{\delta M_2}{D} &= -(1-\psi)\left(\frac{\partial^2 w}{\partial y^2} + \frac{1}{2}\frac{\partial^2 w}{\partial x^2}\right) + \frac{3}{4}(1-\psi-k)\bar{Y}_y\left(\bar{X}_x\frac{\partial^2 w}{\partial x^2} + \bar{Y}_y\frac{\partial^2 w}{\partial y^2}\right)\\
\frac{\delta H}{D} &= -\frac{1}{2}(1-\psi)\frac{\partial^2 w}{\partial x\,\partial y}
\end{aligned}\right\} \tag{4.7}$$

Introducing these expressions into equation (4.6) we find the basic differential equation of stability (generalization of Bryan's equation)

$$\begin{aligned}
&\left(1 - \frac{3}{4}\frac{1-\psi-k}{1-\psi}\bar{X}_x^2\right)\frac{\partial^4 w}{\partial x^4} + 2\left(1 - \frac{3}{4}\frac{1-\psi-k}{1-\psi}\bar{Y}_y\bar{X}_x\right)\frac{\partial^4 w}{\partial x^2\,\partial y^2}\\
&\quad + \left(1 - \frac{3}{4}\frac{1-\psi-k}{1-\psi}\bar{Y}_y^2\right)\frac{\partial^4 w}{\partial y^4} - \frac{h\sigma_i}{(1-\psi)D}\left(\bar{X}_x\frac{\partial^2 w}{\partial x^2} + \bar{Y}_y\frac{\partial^2 w}{\partial y^2}\right) = 0
\end{aligned} \tag{4.8}$$

The quantities ψ and k are functions of the intensity of stress and can be evaluated for given intensity σ_i by formulas (3.7) and (3.8) if the characteristic of the material of the shell is given $\sigma_i = \sigma_i(\epsilon_i)$.

If, under the action of the given forces, the shell does not go too far beyond the elastic limit, so that σ_i differs little from the yield limit of the material σ_s the function ψ will be an extremely small quantity and can be neglected; on the other hand, the generalized modulus of von Kármán k equal to one at the elastic limit may diminish greatly in accordance with the great change in the modulus $d\sigma_i/d\epsilon_i$ in the zone of transition to the yield limit.

The generalized equation of Timoshenko for a uniformly stressed plate takes the form

$$(1 - \psi)\iint\left(\chi_1{}^2 + \chi_2{}^2 + \chi_1\chi_2 + \tau^2\right) dx\, dy - \frac{3}{4}(1 - \psi - k)\iint\left(\bar{X}_x\chi_1 + \bar{Y}_y\chi_2\right)^2 dx\, dy + \frac{h\sigma_i}{D}\iint\left(\bar{X}_x w_x{}^2 + \bar{Y}_y w_y{}^2\right) dx\, dy = 0 \qquad (4.9)$$

It is easily seen that the difficulty of finding the critical forces by equation (4.8) or equation (4.9) is not much greater than in the elastic case. The difficulty lies in that the sought critical stress (for example, σ_i) enters implicitly in the coefficient of every term of equation (4.8) or equation (4.9); whereas in elastic problems it appears as a coefficient of only one term. This difficulty is easily avoided if instead of seeking the critical stress σ_{icr} for given dimensions of the shell we seek the critical value of some characteristic dimension for a given value of σ_i

Let l be the characteristic dimension of a plate. In analogy to a strut, let us designate the flexibility of the plate in relation to the dimension l by

$$\lambda = \frac{l}{h}\sqrt{12(1 - \nu^2)} \qquad (4.10)$$

that is, the flexibility of a flat strip of length l and width 1.

With the use of the dimensionless coordinates of points in the plate

$$\bar{x} = \frac{x}{l} \qquad \bar{y} = \frac{y}{l}$$

equation (4.8) may be put into the form

$$\frac{(1-\psi)E}{\sigma_i}\left(\Phi_{11}\frac{\partial^4 w}{\partial\bar{x}^4}+2\Phi_{12}\frac{\partial^4 w}{\partial\bar{x}^2\,\partial\bar{y}^2}+\Phi_{22}\frac{\partial^4 w}{\partial\bar{y}^4}\right)-\lambda^2\left(\bar{X}_x\frac{\partial^2 w}{\partial\bar{x}^2}+\bar{Y}_y\frac{\partial^2 w}{\partial\bar{y}^2}\right)=0 \tag{4.11}$$

and equation (4.9) solved for λ^2 becomes

$$\lambda^2=\frac{\bar{\psi}E}{\sigma_i}\frac{\iint\left[\Phi_{11}\left(\frac{\partial^2 w}{\partial\bar{x}^2}\right)^2+2\Phi_{12}\frac{\partial^2 w}{\partial\bar{x}^2}\frac{\partial^2 w}{\partial\bar{y}^2}+\Phi_{22}\left(\frac{\partial^2 w}{\partial\bar{y}^2}\right)^2+\left(\frac{\partial^2 w}{\partial\bar{x}\,\partial\bar{y}}\right)^2-\frac{\partial^2 w}{\partial\bar{x}^2}\frac{\partial^2 w}{\partial\bar{y}^2}\right]dx\,dy}{-\bar{X}_x\iint\left(\frac{\partial w}{\partial\bar{x}}\right)^2dx\,dy-\bar{Y}_y\iint\left(\frac{\partial w}{\partial\bar{y}}\right)^2dx\,dy} \tag{4.12}$$

where

$$\Phi_{11}=1-\frac{3}{4}\,\frac{1-\psi-k}{1-\psi}\,\bar{X}_x^{\,2}\qquad \Phi_{22}=1-\frac{3}{4}\,\frac{1-\psi-k}{1-\psi}\,\bar{Y}_y^{\,2}$$

$$\Phi_{12}=1-\frac{3}{4}\,\frac{1-\psi-k}{1-\psi}\,\bar{X}_x\bar{Y}_y\qquad \bar{\psi}=1-\psi$$

The problem comes down to finding the characteristic value of the parameter λ.

Let us examine some concrete problems:

1. Stability of a compressed strip.- A strip, whose length l is considerably greater than its width b is compressed in the longitudinal direction with the stress $-X_x$, with the other stresses equal to zero $Y_y = X_y = 0$; the long edges are free of forces. We have

$$-\bar{X}_x = 1 \qquad X_x = \sigma_i$$

Since the bending moment δM_2 and the twisting moment δH are zero along the long edges $(y = 0,\ y = b)$, then because of the small width they may be taken equal to zero everywhere.

From equation (4.7) we find

$$\tau = 0 \qquad \chi_2 = -\frac{1}{2}\chi_1 \qquad \delta M_1 = -\frac{3}{4}kD\chi_1 \tag{4.13}$$

From equation (4.6) we obtain

$$\frac{\partial^4 w}{\partial \bar{x}^4} + \frac{4\sigma_i}{3Ek}\lambda^2 \frac{\partial^2 w}{\partial \bar{x}^2} = 0 \tag{4.14}$$

If the characteristic value of the parameter

$$\gamma^2 = \frac{4\sigma_i}{3Ek}\lambda^2$$

is known for given edge conditions at the ends $\bar{x} = 0$, $\bar{x} = 1$, the critical flexibility becomes known:

$$\frac{2}{\sqrt{3}}\lambda_{cr} = \gamma_{cr}\sqrt{\frac{E}{\sigma_i}k} \tag{4.15}$$

Thus, for a freely supported strip $\gamma_{cr} = \pi$ and the formula (4.15) coincides with the results of von Kármán–Engesser.

2. Cylindrical form of buckling.– If the transverse dimension b of a rectangular plate is sufficiently great in comparison with the thickness h and the edges $y = 0$, $y = b$ are free, or if the dimension b is considerably greater than the length l, so that by St. Venant's principle the boundary conditions at the sides $y = 0$, $y = b$ do not influence the doformations of the plate, the form of buckling will be cylindrical. Assuming that in equation (4.11)

$$\bar{\bar{X}}_x = -1 \qquad \bar{Y}_y = 0 \qquad w = w(\bar{x})$$

we obtain

$$\frac{E(1 - \psi + 3k)}{4\sigma_i}\frac{d^4 w}{d\bar{x}^4} + \lambda^2 \frac{d^2 w}{d\bar{x}^2} = 0 \tag{4.16}$$

Finding the critical value of the parameter

$$\gamma = \frac{4\sigma_i}{E(1 - \psi + 3k)} \lambda^2$$

for given edge conditions at $\bar{x} = 0$, $\bar{x} = 1$, we obtain the critical flexibility

$$\lambda_{cr} = \gamma_{cr}\sqrt{\frac{E(1 - \psi + 3k)}{4\sigma_i}} \qquad (4.17)$$

For freely supported edges $\gamma_{cr} = \pi$.

3. Stability of a uniformly compressed plate of arbitrary plan form.- Taking as the condition of the problem

$$X_x = Y_y = -\sigma_i \qquad \bar{X}_x = \bar{Y}_y = -1$$

we transform the differential equation (4.11) into

$$\nabla^4 w + \gamma^2 \nabla^2 w = 0 \qquad \gamma^2 = \frac{4\sigma_i}{E(1 - \psi + 3k)} \lambda^2 \qquad (4.18)$$

This equation differs from the well-known equation for elastic buckling only in the expression for the parameter γ, and therefore the characteristic numbers γ_{cr} will be the same as in elastic problems.

Hence, the general solution of the posed problem is given by the formula

$$\lambda_{cr} = \gamma_{cr}\sqrt{\frac{E(1 - \psi + 3k)}{4\sigma_i}} \qquad (4.19)$$

where γ_{cr} must be taken from the solution of elastic problems. For example, for a circular plate of radius l, clamped at the edge, we have $\gamma_{cr} = 3.8317$.

4. Stability of a freely supported rectangular plate compressed in one direction.— A plate with the boundary

$$x(x - a)\ y(y - b) = 0$$

is uniformly compressed in the x-direction. We have

$$\bar{Y}_y = 0 \qquad \bar{X}_x = -1 \qquad \sigma_i = -X_x$$

Assuming, in accordance with the boundary conditions,

$$w = B \sin \frac{\pi y}{b} \sin \frac{m\pi x}{a}$$

we find from the generalized equation of Timoshenko (4.12)

$$\lambda = \frac{\pi l}{a}\sqrt{\frac{E(1 - \psi)}{\sigma_i}\left[\frac{1 - \psi + 3k}{4(1 - \psi)} m^2 + \frac{a^4}{m^2 b^2} + \frac{2a^2}{b^2}\right]} \tag{4.20}$$

Let us investigate first a plate infinitely long in the direction of loading $a = \infty$. For the characteristic dimension l we may take the width b. The number of half waves m will be infinite and the half wave length a' will be

$$a' = \left(\frac{a}{m}\right)_{\substack{a=\infty \\ m=\infty}} \qquad l = b$$

We have

$$\lambda = \pi\sqrt{\frac{E(1 - \psi)}{\sigma_i}\left[\frac{1 - \psi + 3k}{4(1 - \psi)}\left(\frac{b}{a'}\right)^2 + \left(\frac{a'}{b}\right)^2 + 2\right]}$$

and, consequently, the smallest flexibility corresponds to the wave length

$$a' = b\sqrt[4]{\frac{1 - \psi + 3k}{4(1 - \psi)}} \tag{4.21}$$

and is equal to

$$\lambda_{cr} = \pi\sqrt{\frac{2E(1-\psi)}{\sigma_1}\left[1+\sqrt{\frac{1-\psi+3k}{4(1-\psi)}}\right]} \qquad (4.22)$$

It is seen from equation (4.21) that each critical stress σ_1 has its own wave length and, for example, at the beginning of the flow plateau of the material the wave length may be 30 to 40 percent less than at the limit of proportionality. As is seen from equation (4.20), for a plate with arbitrary ratio of sides a and b, the critical number of half-waves is an integer satisfying the inequality

$$m + 1 > \frac{a'a}{b^2} > m > 0$$

For a square plate $a = b = l$ we have $m = 1$ and

$$\lambda_{cr} = \frac{\pi}{2}\sqrt{\frac{E}{\sigma_1}\left[13(1-\psi) + 3k\right]} \qquad (4.23)$$

There is no need to extend the number of examples of stability of compressed plates, inasmuch as the method of solution of the problems remains general and differs little from the solution of corresponding questions of elastic stability, not only in the course of the calculations but also in the possible forms of buckling, in the sense of the form of the function $w(x,y)$. We always obtain the exact or, in the general case, approximate value of the critical flexibility or critical forces if we put the deflection $w(x,y)$ obtained for the elastic problem into the generalized equation of Timoshenko (4.12). Thus, we will obtain the exact expression for the critical forces and flexibility for a rectangular supported plate compressed in two directions, if we place

$$w = B \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

and select the appropriate values of m and n.

5. STABILITY OF CYLINDRICAL SHELLS LOADED BY EXTERNAL PRESSURE AND AXIAL COMPRESSION

Let us investigate the cylindrical form of buckling of a cylindrical shell under the action of uniform external pressure p and uniformly distributed axial force. The x-axis is in the direction of the generators of the cylinder, and the y-axis is tangential. The stresses X_x, Y_y are taken compressive (negative).

From the conditions of the problem we have

$$\chi_1 = \tau = 0$$

$$X_y = 0$$

$$\bar{X}_x = \frac{X_x}{\sqrt{X_x^2 - X_x Y_y + Y_y^2}}$$

$$\bar{Y}_y = \frac{Y_y}{\sqrt{X_x^2 - X_x Y_y + Y_y^2}}$$

From equations (3.9) we obtain expressions for the moments in terms of the nonzero distortion χ_2,

$$\left.\begin{aligned} \delta M_1 &= -\frac{D}{2}\left[1 - \psi - \frac{3}{2}(1 - \psi - k)\bar{X}_x\bar{Y}_y\right]\chi_2 \\ \delta M_2 &= -D\left[1 - \psi - \frac{3}{4}(1 - \psi - k)\bar{Y}_y^{\,2}\right]\chi_2 \\ \delta H &= 0 \end{aligned}\right\} \qquad (5.1)$$

The stiffness

$$D'' = D\left[1 - \psi - \frac{3}{4}(1 - \psi - k)\bar{Y}_y^{\,2}\right] \qquad (5.2)$$

depends not only on the degree of plastic deformation (σ_i, k, ψ), but also on the ratio of the tangential to the axial stresses. The smallest stiffness occurs when the quantity $\bar{Y}_y^2$ is a maximum $\left(X_x = \frac{1}{2} Y_y\right)$.

For a circular cylinder this condition means that the resulting axial force is equal to the lateral pressure multiplied by the area of an end of the cylinder; it is realized in the case of cylinders with closed ends, subjected to external pressure on all surfaces. The smallest stiffness will be

$$D'' = kD$$

that is, obtained by replacing Young's modulus by the generalized von Kármán modulus K.

By way of an example let us investigate a complete circular cylinder of radius R, subjected to a constant external pressure p, and axial compressive force Q, so that

$$Y_y = -p\frac{R}{h} \qquad X_x = -\frac{Q}{2\pi Rh}$$

The corresponding elastic problem was solved first by M. Levy (reference 12). Referring to the well-known book of Timoshenko (reference 13) we shall write, without derivation, the expression for the curvature χ_2 in terms of the normal deflection w, and the moment δM_2

$$\left.\begin{aligned} \chi_2 &= \frac{1}{R^2}\left(\frac{d^2w}{d\theta^2} + w\right) \\ \delta M_2 &= Rpw = -h\sigma_i\bar{Y}_y w \end{aligned}\right\} \qquad (5.3)$$

On the basis of equations (5.1) the differential equation of stability will have the form

$$\frac{d^2w}{d\theta^2} + \left(1 - \frac{\sigma_i R^2 h}{D''}\right)w = 0$$

and the critical stress is found from the condition of periodicity of its solution

$$\frac{\sigma_1 R^2 h}{D''} \bar{Y}_y = -3 \tag{5.4}$$

Using expression (4.10) for the flexibility λ and choosing the circumference $l = 2\pi R$ as the characteristic dimension, we obtain the critical flexibility

$$\lambda_{cr} = \pi \sqrt{\frac{3E}{\sigma_1} \frac{4(1 - \psi) - 3(1 - \psi - k)\bar{Y}_y^{\,2}}{-\bar{Y}_y}} \tag{5.5}$$

or, in the case of uniform pressure on all the surfaces of the cylinder $\left(X_x = \frac{1}{2} Y_y\right)$

$$\lambda_{cr} = \pi \sqrt{\frac{6\sqrt{3}\,Ek}{\sigma_1}} \tag{5.6}$$

and in the case of absence of axial force $(X_x = 0)$

$$\lambda_{cr} = \pi \sqrt{\frac{3E}{\sigma_1}(1 - \psi + 3k)}$$

6. LONGITUDINAL BUCKLING OF A CYLINDRICAL TUBE UNDER AXIAL FORCE AND UNIFORM LATERAL PRESSURE

Let us investigate the axial symmetric form of longitudinal buckling of a cylindrical tube of radius R. For the elastic case this problem has been studied from various points of view by Lorenz, (reference 14), Tzell, Timoshenko (reference 15), and others. The complication beyond the elastic limit consists in the fact that buckling is accompanied by changes in the middle surface forces, and hence, in place of the simple system of equations (3.9), the general expressions for forces and moments (equations (2.13), (2.18), (2.19), and (2.21)) must be used; while the boundary of the plastic region z_0 will be a function of the form of buckling, and therefore, in general, the differential equation of equilibrium becomes considerably complicated and nonlinear.

Let us retain the coordinate system in "5. Stability of Cylindrical Shells Loaded by External Pressure and Axial Compression." We shall take the base state of the shell before buckling such that the axial compressive stress $-X_x$ is twice as great as the tangential stress $-Y_y$ caused by the uniform lateral pressure

$$X_x = 2Y_y \qquad T_1 = 2T_2 \qquad S = 0 \tag{6.1}$$

As is well known, at the elastic limit the choice of the quantity Y_y does not influence the critical value of the axial stress X_x, and beyond the elastic limit our condition introduces a basic simplification into the solution of the problem. In fact, the condition (6.1) is equivalent to the assumption that before buckling the tangential strain $e_{yy} = 0$ as follows from equations (1.1); moreover, after buckling we have

$$\delta T_1 = 0 \qquad \delta S = 0 \qquad \delta T_2 \neq 0 \tag{6.2}$$

From equations (2.19) we now obtain

$$\varphi = 0$$

consequently, the boundary z_o between the elastic and plastic regions according to equation (2.18) turns out to be constant:

$$\bar{z}_o = -\frac{\sqrt{E} - \sqrt{\dfrac{d\sigma_i}{d\epsilon_i}}}{\sqrt{E} + \sqrt{\dfrac{d\sigma_i}{d\epsilon_i}}} = -1 + \sqrt{k} \tag{6.3}$$

From equation (2.13) we have

$$\left.\begin{aligned}
-\frac{\delta T_2}{h\sigma_i} &= 3\left[\bar{\mu}_o + \bar{\mu} + (\bar{\mu}_o - \bar{\mu})\bar{z}_o\right]\epsilon_1 + \frac{3}{2}(\bar{\mu}_o - \bar{\mu})(1 - \bar{z}_o^{\,2})\bar{X}_1 \\
&\quad + \frac{3}{8}a^2(1 - \bar{z}_o)^2\left(\bar{X}_1 + \frac{1}{2}\bar{X}_2\right) \\
\frac{2\delta T_2}{h\sigma_i} &= 3\left[\bar{\mu}_o + \bar{\mu} + (\bar{\mu}_o - \bar{\mu})\bar{z}_o\right]\epsilon_2 + \frac{3}{2}(\bar{\mu}_o - \bar{\mu})(1 - \bar{z}_o^{\,2})\bar{X}_2 \\
\delta S &= 0 \qquad \epsilon_3 = 0
\end{aligned}\right\} \tag{6.4}$$

The distortions χ_1, χ_2 and the strain ϵ_2 are expressed in terms of the normal deflection w (positive inwards) by the formulas

$$\chi_1 = \frac{2}{h}\bar{\chi}_1 = \frac{d^2w}{dx^2} \qquad \chi_2 = \frac{2}{h}\bar{\chi}_2 = \frac{w}{R^2} \qquad \epsilon_2 = -\frac{w}{R} \tag{6.5}$$

and, consequently, equations (6.4) give, firstly,

$$\delta T_2 = -k'Eh\frac{w}{R}$$

$$k' = 1 - \frac{1}{2}\omega\left[1 - \bar{z}_o + \frac{h}{8R}(1 - \bar{z}_o^2)\right] = 1 - \omega\left(1 - \frac{1}{2}\sqrt{k}\right) \tag{6.6}$$

where k, as before, is the relative generalized von Kármán modulus and, secondly,

$$-\left(\epsilon_1 + \frac{1}{2}\epsilon_2\right) = \frac{4(\bar{\mu}_o - \bar{\mu})(1 - \bar{z}_o^2) + a^2(1 - \bar{z}_o)^2}{8\left[\bar{\mu}_o + \bar{\mu} + (\bar{\mu}_o - \bar{\mu})\bar{z}_o\right]}\left(\bar{\chi}_1 + \frac{1}{2}\bar{\chi}_2\right) \tag{6.7}$$

Formulas (2.21) give the value of the moment δM_1

$$\delta M_1 = -k''D\left(\chi_1 + \frac{1}{2}\chi_2\right)$$

where the coefficient k'' is given by

$$k'' = 1 - \frac{1}{2}\omega(1 - \bar{z}_o^3) - \frac{1}{4}\left(1 - \omega - \frac{1}{E}\frac{d\sigma_i}{d\epsilon_i}\right)(2 - 3\bar{z}_o + \bar{z}_o^3)$$
$$- \frac{3}{8}\frac{\omega\left(1 - \bar{z}_o^2\right)}{2 - \omega(1 - \bar{z}_o)}\left[\omega\left(1 - \bar{z}_o^2\right) + \left(1 - \omega - \frac{1}{E}\frac{d\sigma_i}{d\epsilon_i}\right)(1 - \bar{z}_o)^2\right]$$

Taking into account that on the basis of equation (6.3)

$$\frac{1}{E}\frac{d\sigma_i}{d\epsilon_i} = \left(\frac{1 + \bar{z}_o}{1 - \bar{z}_o}\right)^2$$

we obtain after simple transformations

$$K'' = (1 + \bar{z}_o)^2 = k$$

Thus,

$$\delta M_1 = -kD\left(\frac{d^2w}{dx^2} + \frac{w}{2R^2}\right) \qquad (6.8)$$

The differential equation of equilibrium of an element of the shell after buckling will be

$$\frac{d^2\delta M_1}{dx^2} + T_1 \frac{d^2w}{dx^2} + \frac{\delta T_2}{R} = 0 \qquad (6.9)$$

Taking as the characteristic dimension of the shell the radius R and using the expression (4.10) for the flexibility λ we have

$$\left.\begin{aligned} & k\frac{d^4w}{d\bar{x}^4} - \lambda^2\frac{X_x}{E}\frac{d^2w}{d\bar{x}^2} + k'\lambda^2 w = 0 \\ & \lambda = \frac{R}{h}\sqrt{12(1-\nu^2)} \approx \frac{3R}{h} \\ & \bar{x} = \frac{x}{R} \end{aligned}\right\} \qquad (6.10)$$

where we have discarded small quantities of the order of h/R in comparison with 1.

In our case

$$\sigma_i = -\frac{\sqrt{3}}{2}X_x \qquad (6.11)$$

and since the coefficients k, k' are functions of the intensity of stress the convenient expression (6.10) may be written finally in the form

$$k \frac{d^4 w}{d\bar{x}^4} + \frac{2\sigma_1}{E\sqrt{3}} \lambda^2 \frac{d^2 w}{d\bar{x}^2} + k' \lambda^2 w = 0 \qquad (6.12)$$

Satisfying the conditions of support at the ends of the tube

$$w = 0 \qquad \frac{d^2 w}{d\bar{x}^2} = 0$$

we find the critical flexibility

$$\lambda_{cr} = \frac{E}{\sigma_1} \sqrt{3kk'}$$

$$= \frac{E}{\sigma_1} \sqrt{3k\left(1 - \frac{2 - \sqrt{k}}{2} \omega\right)} \qquad (6.13)$$

The result obtained is very simple and may easily be subjected to experimental verification, but it is necessary to remember that it is strictly correct only in the case where the tube is subjected to axial force and lateral pressure such that the ratio of axial to tangential stress is equal to two.

7. NUMERICAL RESULTS FOR THE MILD STEEL USED IN VON KÁRMÁN'S EXPERIMENTS

In this paper, giving the method of investigating stability, we cannot dwell at length upon the analysis of the multitudinous technically important problems of stability and give formulas for calculation, tables, and graphs for all the various steels used in technology. In endeavoring to show that the established viewpoint concerning the possibility of a wide application in practice or formulas for elastic critical loads corrected according to the von Kármán modulus is, in general, completely wrong, we shall present numerical results in accordance with the above given formulas only for the steel used in the experiments of von Kármán.

The mechanical properties of the steel are obtained from the tensile stress-strain curve for which Young's modulus, tensile strength, proportional limit, and yield limit are

$$E = 2.17 \times 10^6, \ \sigma_b = 6800, \ \sigma_p = 2600, \ \sigma_s = 3250 \ (\text{in kg/cm}^2)$$

The first three columns of table I were computed from the stress-strain curve, and then by the use of formulas (1.4), (3.7), (3.8) the values of rigidity-decrease ω, generalized modulus of von Kármán, and the parameter ψ were determined and put in the last three columns of table I.

In table II are given values of the critical flexibility

$$\lambda = \frac{l}{h}\sqrt{12(1 - \nu^2)}$$

for all of the above investigated cases of instability of plates and shells. For each value of the critical stress σ_i are given three values of flexibility, in the first line λ' – exact by our formulas, in the second λ'' – approximated by correction according to von Kármán's modulus, in the third λ''' – computed by the formulas of the theory of elasticity.

The last value λ''' is obtained if in our formulas we place $\omega = 0$ ($k = 1$, $\psi = 0$).

Thus in table II are presented numerical values of the stiffnesses computed from the data of table I for the following cases:

(1) Von Kármán strut and narrow strip ($l = a$, $\sigma_i = -X_x$, $Y_y = 0$):

$$\lambda_1' = \lambda_1'' = \pi\sqrt{\frac{Ek}{\sigma_i}} \qquad \lambda_1''' = \pi\sqrt{\frac{E}{\sigma_i}}$$

(2) Wide strip with two free edges (cylindrical bending; $l = a$, $\sigma_i = -X_x$, $Y_y = 0$):

$$\lambda_2' = \pi\sqrt{\frac{E(1 - \psi + 3k)}{4\sigma_i}} \qquad \lambda_2'' = \pi\sqrt{\frac{Ek}{\sigma_i}} \qquad \lambda_2''' = \pi\sqrt{\frac{E}{\sigma_i}}$$

(3) Circular plate clamped at the edge under uniform compression ($l = R$, $X_x = Y_y = -\sigma_1$):

$$\lambda_3{}' = 3.84\sqrt{\frac{E(1 - \psi + 3k)}{4\sigma_1}} \qquad \lambda_3{}'' = 3.84\sqrt{\frac{Ek}{\sigma_1}} \qquad \lambda_3{}''' = 3.84\sqrt{\frac{E}{\sigma_1}}$$

(4) Long narrow plate, freely supported on all edges and compressed in the direction of the long edges (width $b = l$, half-wave length designated by a'; $X_x = -\sigma_1$, $Y_y = 0$):

$$\lambda_4{}' = \pi\sqrt{\frac{2E(1 - \psi)}{\sigma_1}\left[1 + \frac{1 - \psi + 3k}{4(1 - \psi)}\right]}$$

$$\left(\frac{a'}{b}\right)' = \sqrt[4]{\frac{1 - \psi + 3k}{4(1 - \psi)}}$$

$$\lambda_4{}'' = \pi\sqrt{\frac{4E}{\sigma_1}\sqrt{K}} \; ; \quad \left(\frac{a'}{b}\right)'' = \sqrt[4]{k}$$

$$\lambda_4{}''' = \pi\sqrt{\frac{4E}{\sigma_1}} \; ; \quad \left(\frac{a'}{b}\right)''' = 1$$

(5) Freely supported square plate compressed in one direction ($l = a = b$, $X_x = -\sigma_1$, $Y_y = 0$):

$$\lambda_5{}' = \pi\sqrt{\frac{E}{4\sigma_1}\left[13(1 - \psi) + 3k\right]}$$

$$\lambda_5{}'' = \pi\sqrt{\frac{E}{\sigma_1}(1 + 2\sqrt{k} + k)},$$

$$\lambda_5{}''' = \pi\sqrt{\frac{4E}{\sigma_1}}$$

(6) Tube under external pressure in the absence of axial forces $\left(l = 2\pi R,\ X_x = 0,\ Y_y = -\sigma_i = -\frac{pR}{h}\right)$:

$$\lambda_6' = \pi\sqrt{\frac{3E}{\sigma_i}(1 - \psi + 3k)} \qquad \lambda_6'' = \pi\sqrt{\frac{12Ek}{\sigma_i}} \qquad \lambda_6''' = \sqrt{\frac{12E}{\sigma_i}}$$

(7) Tube with external pressure on all surfaces, that is, with axial force $Q = \pi R^2 p \left(l = 2\pi R,\ X_x = \frac{1}{2} Y_y = -\frac{pR}{2h} = -\frac{\sigma_i}{\sqrt{3}}\right)$:

$$\lambda_7' = \lambda_7'' = \pi\sqrt{\frac{12Ek}{\sigma_i}} \qquad \lambda_7''' = \pi\sqrt{\frac{12E}{\sigma_i}}$$

(8) Longitudinal buckling of a tube under axial load and lateral pressure $\left(l = R,\ X_x = 2\ Y_y = -\frac{2}{\sqrt{3}}\sigma_i\right)$:

$$\lambda_8' = \frac{E}{\sigma_i}\sqrt{3k\left(1 - \frac{2 - \sqrt{k}}{2}\omega\right)}\ ; \quad \lambda_8'' = \frac{E}{\sigma_i}\sqrt{3k}\ ; \quad \lambda_8''' = \frac{E}{\sigma_i}\sqrt{3}$$

If the curves $\sigma_i = \sigma_i(\lambda)$ are drawn, they will all have a point of inflection at the yield plateau of the steel $\sigma_i = \sigma_s = 3250$ kilograms per centimeter^2.

As a rule, the approximate theory with von Kármán's correction factor gives appreciably lowered values of the critical flexibility and, on the other hand, the elastic theory gives appreciably raised values. It is interesting to note that in almost all cases (except for the square plate, where $\lambda_5'' = 81.3$) the values of the flexibility λ'' became zero at the inflection points, which means complete loss of load bearing ability at the yield limit according to the approximate theory; whereas, in fact, in the majority of cases the flexibilities have finite values at the inflection points

$$\lambda_2' = 30 \sim 34 \qquad \lambda_3' = 37 \sim 42 \qquad \lambda_4' = 105 \sim 119$$

$$\lambda_5' = 119 \sim 123 \qquad \lambda_6' = 104 \sim 118$$

and, consequently, load bearing ability is maintained not only at the yield limit but appreciably beyond. Thus for the flexibilities

$$\lambda_2 = 30 \quad \lambda_3 = 36 \quad \lambda_4 = 85 \quad \lambda_5 = 87 \quad \lambda_6 = 104$$

stability obtains at all stresses up to $\sigma_i = 4000$ kilograms per centimeter^2, inclusive.

Table II shows the complete failure of the approximate theory of stability of plates and shells in which plastic deformations are taken into account with the aid of the corrective modulus of von Kármán. This conclusion can be established definitely only by experiment, but it is to be remembered that: (1) the experiments of von Kármán gave complete agreement with his theory of the stability of struts and; (2) the theory of stability of plates and shells proposed here contains no arbitrary hypotheses and, being based on classical laws of plasticity and laws of mechanics, can already now give this conclusion with a high degree of authenticity

Translated by E. B. Schwartz
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TABLE I

σ_i	$\epsilon_i \times 10^3$	$\frac{d\sigma_i}{d\epsilon_i} \times 10^{-6}$	ω	k	ψ
2600	1.20	2.17	0.	1.000	0.
2800	1.31	1.98	.014	.940	.007
3000	1.43	1.54	.034	.825	.017
3100	1.51	1.12	.054	.685	.026
3240	1.92	.06	.212	.0805	.149
3250	2.1 to 2.7	0	.285 to .445	0	.285 to .445
3260	2.9	.042	.482	.056	.360
3300	3.3	.117	.540	.141	.369
3500	4.7	.140	.657	.163	.456
4000	8.8	.115	.790	.139	.605

TABLE II

σ_1	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
2600	90.8	90.8	111	181.6	181.6	314	314	1440
	90.8	90.8	111	181.6	181.6	314	314	1440
	90.8	90.8	111	181.6	181.6	314	314	1440
2800	84.8	85.4	104	173	173	296	296	1290
	84.8	84.8	103.6	173	175	296	296	1300
	88.5	88.5	108	177	177	306	306	1340
3000	76.8	78.7	96.1	166	166	273	266	1120
	76.8	76.8	91.3	161	161	266	266	1130
	84.5	84.5	103	169	169	293	293	1250
3100	69.0	72.4	88.4	162	160	251	239	984
	69.0	69.0	84.2	151	152	239	239	1000
	83.2	83.2	101.5	166.4	166.4	289	289	1210
3240	23.0	42.2	51.6	132	136	146	80.5	296
	23.0	23.0	28.0	46.2	105	80.5	80.5	328
	81.5	81.5	99.5	163	163	283	283	1155
3250	0	34 to 30	41.5 to 36.6	119 to 105	123 to 119	118 to 104	0	0
	0	0	0	0	81.3	0	0	0
	81.3	81.3	99.4	162.6	162.6	282	282	1150
3260	19.3	36.5	44.5	114	118	126	66.5	206
	19.3	19.3	23.6	38.5	100	66.5	66.5	271
	81.1	81.1	99.3	162.2	162.2	281	281	1145
3300	30.3	41.6	50.9	116.5	119	144	105	318
	30.3	30.3	37.0	60.5	110.5	105	105	425
	80.5	80.5	98.3	161.0	161.0	279	279	1130
3500	31.6	39.7	48.5	106	108	138	108	298
	31.6	31.6	38.6	63.2	109.5	109	108	432
	78.3	78.3	95.5	156.6	156.6	271	271	1070
4000	27.3	33.0	40.2	85.5	87	114	95	210
	27.3	27.3	33.4	54.6	100	95	95	350
	73.2	73.2	89.5	146.4	146.4	254	254	935