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THE THEORY OF A FREE JET OF A COMPRESSIBLE GAS

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SUMMARY

In the present report the theory of free turbulence propagation and the boundary layer theory are developed for a plane-parallel free stream of a compressible fluid. In constructing the theory use was made of the turbulence hypothesis by Taylor (transport of vorticity) which gives best agreement with test results for problems involving heat transfer in free jets.

The theory developed here considers two kinds of flow:

1. The boundary layer of a jet with temperature different from that of the surroundings and velocities that are small by comparison with the velocity of sound (Bairstow number $Ba \lesssim 0.5$).

2. The boundary layer of a jet of high velocity and at a temperature equal to that of the surroundings.

The first deals with compressibility effect arising from the difference in temperatures inside and outside the jet; the second with the compressibility effect arising from the high flow velocities.

The results indicated that the compressibility had only a slight effect on the properties of the free jet. Furthermore it was found that a drop in the jet temperature had approximately the same effect on the properties of the jet regardless of whether the reduction was due to artificial cooling of the jet or to the conversion of thermal energy to kinetic at high flow velocities.

Compressibility factors (r and S) are introduced with the aid of which it is possible to reduce the variations in all fundamental properties of the boundary layer under the

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influence of compressibility to simple linear relations. The results obtained are valid for flow velocities up to the velocity of sound and for considerable temperature differences (up to $\pm 100 - 150^{\circ}$ C).

INTRODUCTION

In 1926 Tollmien's paper was published (reference 1) in which the author, on the basis of the semi-empirical general turbulence theory of Prandtl, developed the theory of so-called free turbulence — that is, turbulence in a free, heated stream. In the same paper, making use of his proposed theory, Tollmien solved three problems on the propagation of free, heated jets:

(a) The boundary layer of an infinite plane-parallel jet;

(b) Plane-parallel jet escaping from a very narrow opening;

(c) Axially symmetric jet escaping from very narrow opening.

About 3 or 4 years later (1929–1930) Swain's paper (reference 2) and Schlichting's paper (reference 3) extending the theory of free turbulence to the case of the wake behind a body and developing the laws of flow in axially symmetric and plane wakes appeared. These laws are applicable to flows not too near the body.

The work of these authors was supplemented by experimental investigations of the velocity fields of flow. The use of one empirical constant enabled the above-mentioned theories to be brought into excellent agreement with test results. In fact this agreement determined the success of the Prandtl-Tollmien free-turbulence theory and assured it wide theoretical and practical application. In 1935 the article by Kuethe (reference 4) appeared in which an approximate method is worked out for the computation of the velocity profile in the initial part of a round jet.

In 1935, 1936, and 1938 four papers by the present author were published in which Tollmien's theory was extended to the case of plane-parallel flow and axially symmetric jets escaping from openings of finite diameter (an approximate theory of the initial part of the jet was proposed); formulas were worked out for the aerodynamic computation of the plane-parallel jet; axially symmetric jet flow in the open working part of a wind tunnel with round and elliptic sections, hot and cold air jets, and moreover, methods were proposed for computing the air resistance of railway cars (in tunnels or on the open track), pipe systems and heat interchangers (references 5, 6, 7). These proposals and the flow theory itself were satisfactorily confirmed by test results and each year find wide application to engineering practice.

The rapid development of the mechanics of turbulent flow has prompted the application of the physical model of the phenomena as conceived by Prandtl and Tollmien to the solution of heat problems, those of the temperature distribution along the jet axis and over its cross-sections of heat diffusion from the jet to the surrounding space, and so forth. It is interesting to note that as a direct consequence of the Prandtl theory, in the case of a free jet and the wake behind a body, complete similarity is obtained between the temperature and velocity fields. In order to check this extremely important result Fage and Falkner (reference 8), in 1932, made measurements of the velocity and temperature fields in the wake behind a long cylinder of elliptic cross section. They showed that the theoretical velocity fields of Prandtl-Schlichting were very well confirmed by test results while there was no similarity of the velocity and temperature fields, and the heat transfer from the wake to the undisturbed flow is of greater intensity than follows from the Prandtl theory.

Taylor (reference 9) was the first to note the contradiction revealed in the free turbulence theory of Prandtl and presented a hypothesis according to which the tangential turbulent stresses in the flow were to be determined by the transverse transport of vorticity and not by the momentum as proposed by Prandtl. The imperfection in the Prandtl theory was also pointed out in that it took no account of the local pressure gradients which have an appreciable effect on the momentum interchange but not on the vorticity transport. The above hypothesis, with regard to the turbulence, was first proposed as far back as 1915 (reference 10). On the basis of this hypothesis using only one empirical constant, as Prandtl did, Taylor obtained the velocity and temperature profiles in the wake behind a long cylinder as experimentally determined by Fage and Falkner.

The Taylor theory of free turbulence gave velocity profiles accurately, the same as those given by the Prandtl theory, and at the same time removed the imperfection of the latter theory as regards application to heat problems. This permits all the solutions of the problems in the field of flow mechanics that were based on the Prandtl theory to retain their validity. It made it necessary, however, to give preference to the Taylor theory for the further development of the problems of free flow and the wake behind a body, particularly in those cases when the problems are concerned with temperature profiles and heat transfer.

In the present report devoted to the further development of the theory of the free jet, a theory of free turbulence in a compressible gas is worked out and solutions are given of boundary layer problems of a free flow for the following two cases;

(a) The boundary layer of a plane-parallel jet at small flow velocities with a temperature different from that of the surroundings, that is, a nonisothermal layer;

(b) Isothermal boundary layer at large (up to $Ba_0=1$) flow velocities.

It may be noted in conclusion that the free turbulence problems, in addition to being of interest in themselves, also possess a general interest since free turbulence represents the simplest case of turbulence free from the effect of viscosity. The study of free turbulence is a necessary preliminary stage in the study of turbulent flows in general. It is, therefore, hoped by the author that the solution proposed in his present paper of the problems of free turbulence in a compressible gas possesses a certain usefulness for the study of turbulence in other cases of compressible flow.

I. EQUATION OF MOTION FOR FREE TURBULENCE

The problem will be restricted to two-dimensional flow. In this case the differential equation of motion in the direction of the axis of abscissas assumes the following form:

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho V \frac{\partial u}{\partial y} = - \mu \frac{\partial^2 u}{\partial y^2} \frac{\partial P}{\partial x} \qquad (1)$$

where

u, V instantaneous velocity components

ρ, P, μ instantaneous values of the density, pressure, and viscosity

For the flow of a liquid of small viscosity about solid bodies the flow, as was shown by Prandtl as far back as 1904, may be divided into two regions; namely, a relatively thin layer of fluid lying close to the solid walls – the boundary layer – in which the effect of the viscosity cannot be neglected, however small its value may be, and the remaining part of the flow, in which the viscosity plays no part and which is therefore subject to the laws of flow of ideal fluids. The boundary layer, in turn, is assumed to consist of a very thin sublayer of purely laminar flow in direct contact with the wall (no transverse turbulent fluctuations can be developed since they are dissipated by the wall) and a remaining turbulent portion of the boundary layer in which the effect of the viscosity may be neglected. Thus the study of the flow about solid bodies, the motion through pipes and, in general, of all fluid flows in the presence of rigid boundaries does not, in principle, permit neglecting entirely the effect of the viscosity. This circumstance constitutes the great obstacle in the development of the theory of turbulent flows.

The distinguishing property of free turbulent jets is the absence of rigid flow boundaries and hence of a laminar sublayer. This makes it possible to neglect entirely the effect of viscosity in all cases of free turbulence and explains the dynamic similarity of the jet flows – the nondependence on the Reynolds number – over a very wide range of Reynolds number.

The differential equation of motion for two-dimensional free turbulence may thus be written in the following form:

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho V \frac{\partial u}{\partial y} = - \frac{\partial P}{\partial x} \tag{2}$$

Due to the quasi-stationary state of the turbulent motion all its characteristics may be broken up into mean and fluctuating components:

$$\left.\begin{aligned} u &= \bar{u} + u' \\ V &= \bar{V} + V' \\ \rho &= \bar{\rho} + \rho' \\ P &= \bar{P} + P' \end{aligned}\right\} \qquad (2a)$$

On the average, over a certain finite time interval, the fluctuating component is evidently equal to zero:

$$\overline{V'} = \overline{u'} = \overline{\rho'} = \overline{P'} = 0 \qquad (2b)$$

In the general case, however, this is not true of the squares of the fluctuations and their products.

In equation (2) the mean values and fluctuations are substituted for the instantaneous magnitudes. To average over the time, take account of conditions (equation (2b)) and neglect moments of the third order:

$$\left(\rho' u' \frac{\partial u'}{\partial x}; \ \rho' V' \frac{\partial u'}{\partial y}\right)$$

there is obtained the differential equation for the average turbulent flow of a compressible fluid:

$$\left[\bar{\rho}\,\bar{u}\,\frac{\partial \bar{u}}{\partial x} + \bar{\rho}\,\bar{V}\,\frac{\partial \bar{u}}{\partial y}\right] + \left[\overline{\rho' u'}\,\frac{\partial \bar{u}}{\partial x} + u\,\overline{\rho'\,\frac{\partial u'}{\partial x}} + \bar{\rho}\,\overline{u'\,\frac{\partial u'}{\partial x}}\right]$$

$$+ \left[\overline{\rho' V'}\,\frac{\partial \bar{u}}{\partial y} + \bar{V}\,\overline{\rho'\,\frac{\partial u'}{\partial y}} + \bar{\rho}\,\overline{V'\,\frac{\partial u'}{\partial y}}\right] = -\frac{\partial \bar{P}}{\partial x} \qquad (3)$$

To estimate the order of magnitude of the individual terms that enter the above equation it is not difficult to see that in the case of free turbulence all terms in the second brackets (with derivatives in respect to x) are very small by comparison with the corresponding terms of the third brackets (with derivatives in respect to y) if the principal direction of the flow coincides with the axis of abscissas. Similarly one of the terms of the third brackets

$$\overline{V}\ \overline{\rho' \frac{\partial u'}{\partial u}}$$

is negligibly small. By neglecting the above small terms the following form of the differential equation of motion for the case of two-dimensional purely turbulent flow is:

$$\overline{\rho}\ \overline{u}\ \frac{\partial \overline{u}}{\partial x} + \overline{\rho\ V}\ \frac{\partial \overline{u}}{\partial y} + \overline{\rho' V'}\ \frac{\partial \overline{u}}{\partial y} + \overline{\rho}\ \overline{V'\ \frac{\partial u'}{\partial y}} = -\ \frac{\partial \overline{P}}{\partial x} \quad (3a)$$

Free jets propagated in infinite space filled with liquid at rest, and wakes behind a body surrounded by infinite undisturbed flow, possess such small pressure gradients that they may be neglected. With this in mind the differential equation of motion for free turbulence in a compressible gas is:

$$\overline{\rho}\ \overline{u}\ \frac{\partial \overline{u}}{\partial x} + \overline{\rho\ V}\ \frac{\partial \overline{u}}{\partial y} + \overline{V' \left[\overline{\rho}\ \frac{\partial u'}{\partial y} + p'\ \frac{\partial \overline{u}}{\partial y}\right]} = 0 \qquad (4)$$

The further steps in the solution of the problem depend on the choice of physical model for the turbulent flow. At the present time there are two models of interest in their application to free turbulence, namely, those of Prandtl and Taylor. With the aid of the Prandtl model results agreeing with experiment are obtained for problems in the field of jet mechanics (velocity fields, frictional stress, and so forth) but strong disagreement is obtained for heat problem solutions (temperature field, heat transfer). The Taylor physical model gives the same solutions as the Prandtl model for the mechanical problems and furthermore leads to solutions of the heat problems that are in good agreement with experiments. In what follows, therefore, use is made of the Taylor model. The latter is based on the assumption that the turbulent tangential stresses in the flow arise from the transverse transport of vorticity, that is, from the correlation between the vortex fluctuations and the transverse velocity components. In two-dimensional flow directed along the axis of abscissas the vorticity is given by

$$\overline{\omega} = \frac{1}{2}\left[\frac{\partial \overline{u}}{\partial y} - \frac{\partial \overline{V}}{\partial x}\right] \tag{4a}$$

where the magnitude of $\partial\overline{V}/\partial x$ in free turbulence is negligibly small by comparison with $\partial\overline{u}/\partial y$ so that

$$\overline{\omega} = \frac{1}{2}\frac{\partial \overline{u}}{\partial y} \tag{4b}$$

In its transverse transport, immediately before the loss of its individuality, the particle encounters a layer where the value of the vorticity differs from that in the layer from which it arrived by the amount

$$\Delta\overline{\omega} = l_T \frac{\partial\overline{\omega}}{\partial y} = \frac{1}{2} l_T \frac{\partial^2\overline{u}}{\partial y^2}$$

where l_T is the mean free path of the fluid particle in the turbulent flow. The loss of individuality of the fluid particle should be accompanied by a discontinuous change (fluctuation) of vorticity of amount

$$\omega' = \Delta\overline{\omega}$$

From equation (4b) it is clear, however, that

$$\omega' = \frac{1}{2}\frac{\partial u'}{\partial y}$$

hence

$$\frac{\partial u'}{\partial y} = l_T \frac{\partial^2\overline{u}}{\partial y^2} \tag{4c}$$

With the loss of individuality of a given particle there are naturally associated fluctuations of the flow velocity:

$$u' = l_T \frac{\partial\overline{u}}{\partial y} \tag{4d}$$

and of the fluid density:

$$\rho' = l_T \frac{\partial \overline{\rho}}{\partial y} \tag{4e}$$

By making use of equations (4c) and (4e) equation (4) was reduced to the form

$$\overline{\rho u} \frac{\partial \overline{u}}{\partial x} + \overline{\rho V} \frac{\partial \overline{u}}{\partial y} + \overline{V' l_T} \left[\overline{\rho} \frac{\partial^2 \overline{u}}{\partial y^2} + \frac{\partial \overline{\rho}}{\partial y} \frac{\partial \overline{u}}{\partial y}\right] = 0 \tag{5}$$

or

$$\overline{\rho u} \frac{\partial \overline{u}}{\partial x} + \overline{\rho V} \frac{\partial \overline{u}}{\partial y} + \overline{V' l_T} \frac{\partial}{\partial y} \left[\overline{\rho} \frac{\partial \overline{u}}{\partial y}\right] = 0 \tag{6}$$

which is the differential equation obtained on the basis of the Taylor turbulence model.

With respect to the magnitude $\overline{V' l_T}$ it is necessary to make some assumptions by which it is associated with the velocity of motion and with the coordinates of the system. It is possible, for example, to make use of the generally accepted idea of Prandtl, namely, that the transverse velocity fluctuations are of the same order of magnitude as the longitudinal fluctuations:

$$V' \sim u'$$

that is,

$$V' \sim l_T \frac{\partial \overline{u}}{\partial y} \tag{6a}$$

Including the proportionality constant in the magnitude of the free path of the particle – the mixing length (l_T) – (equation (6)) is reduced to the more simple form*

$$\rho u \frac{\partial u}{\partial x} + \rho V \frac{\partial u}{\partial y} = l_T^2 \frac{\partial u}{\partial y} \frac{\partial}{\partial y} \left[\rho \frac{\partial u}{\partial y}\right] \tag{7}$$

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*With a view toward simplicity of notation the averaging bars over the letters are omitted in what follows so that ρ, u, V, and l_T are to stand for their mean values in time ($\overline{\rho}$, $\overline{u}$, $\overline{V}$, and $\overline{l_T}$).

In the special case of an incompressible fluid (ρ = constant) the following is obtained

$$u \frac{\partial u}{\partial x} + V \frac{\partial u}{\partial y} = l_T^2 \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y^2} \tag{7a}$$

The corresponding equation derived from the turbulence model of Prandtl for the free turbulence in an incompressible fluid was obtained by Tollmien in 1926 in the following form (reference 1):

$$u \frac{\partial u}{\partial x} + V \frac{\partial u}{\partial y} = 2l^2 \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y^2} \tag{7b}$$

To take into account the fact that the constant of proportionality is determined from experimental data, it is seen that in the case of free turbulence the Prandtl and Taylor models give rise to the same equation of motion.

The value of the mixing lengths, as given by Prandtl and Taylor respectively, differ by the constant magnitude

$$l_T = \sqrt{2l} \tag{7c}$$

For the purpose of retaining the form of computation adopted by Tollmien and others the Prandtl value of the mixing length is assumed. The differential equation of motion for free turbulence in a compressible gas then becomes

$$\rho u \frac{\partial u}{\partial x} + \rho V \frac{\partial u}{\partial y} = 2l^2 \frac{\partial u}{\partial y} \frac{\partial}{\partial y} \left[\rho \frac{\partial u}{\partial y}\right] \tag{8}$$

Comparison of the above equation with the generally known equation for stationary flow

$$\rho u \frac{\partial u}{\partial x} + \rho V \frac{\partial u}{\partial y} = \frac{\partial \tau_{xy}}{\partial y} \tag{8a}$$

reveals the presence of "apparent" tangential stresses the magnitude of which is given by the equation

$$\tau_{xy} = \int 2l^2 \frac{\partial u}{\partial y} \frac{\partial}{\partial y}\left[\rho \frac{\partial u}{\partial y}\right] dy \tag{8b}$$

In the case of an incompressible fluid equation (8b) reduces to the generally familiar Prandtl law of turbulent friction:

$$\tau_{xy} = \rho l^2 \left(\frac{\partial u}{\partial y}\right)^2 \tag{8c}$$

To solve equation (8) it is necessary to know a relation between the mixing length l and the coordinates of the system. The absence in the case of free flow, of rigid boundaries that damp the fluctuating motions of the particles led Prandtl to the assumption of constancy of the mixing length in the transverse direction of flow:

$$l(y) = \text{constant} \tag{9}$$

It thus remains to establish the law of variation of the mixing length along the axis of abscissas:

$$l = l(x)$$

The available experimental investigations of free flows make it possible without any particular difficulty to determine the form of the function $l(x)$. A sufficient basis for this is the experimentally established fact of similarity of the boundary layers in various cross sections of a given free flow (jets or wakes, reference 1). This similarity was revealed in a large number of experimental papers (Tollmien, Förthmann, Ruden, Schlichting, and others) by constructing velocity profiles in nondimensional coordinates, for example, in the form of the relation

$$\frac{u}{u_m} = f\left(\frac{y}{b}\right) \tag{9a}$$

where

u velocity at a point with ordinate y

u_m velocity on the axis of the jet

b width of the jet (or of its boundary layer) in the given cross section

The nondimensional velocity profiles (equation (9a)) were found to agree for the various cross sections.

The similarity of the boundary layers at any two cross-sections of a given free flow must also be obtained with regard to geometric factors. In other words equality is to be expected between the nondimensional mixing lengths for the various flow cross sections:

$$\frac{l_1}{b_1} = \frac{l_2}{b_2} \ldots\ldots = \text{constant} \qquad (9b)$$

It is thus sufficient to establish the law of increase of width of jet along the axis of abscissas in order that the law of increase of the mixing length be known. A very interesting consideration of Prandtl permits the solution of the relation $b = b(x)$. It is shown by Prandtl (reference 11) that the widening of the jet (or of the boundary layer of the jet) arises from the transverse velocity fluctuations V', that is,

$$\frac{db}{dt} \sim V' \sim l \frac{\partial u}{\partial y} \qquad (10a)$$

Because of the similarity of the velocity profiles at the different jet cross-sections the following equation may be written

$$\frac{\partial u}{\partial y} \sim \frac{u_m}{b}$$

and further

$$\frac{db}{dt} \approx \frac{l}{b} u_m \sim u_m \qquad (10b)$$

On the other hand, the rate of expansion of the jet

$$\frac{db}{dt} = \frac{db}{dx}\frac{dx}{dt} \qquad (10c)$$

that is,

$$\frac{db}{dt} \cong u_m \frac{db}{dx} \qquad)$$

Comparison of expressions (10b) and (10c) leads to the solution of the problem of the law of increase in width of the free jet and of the mixing length in the flow direction:

$$\left.\begin{aligned} \frac{db}{dx} &= \text{constant} \\ b &= x \text{ constant} \\ l &= c\,x \end{aligned}\right\} \qquad (11)$$

The law obtained for the increase in the mixing length along the flow

$$l = c\,x \qquad (12a)$$

is valid for free jets of various shapes: for the boundary layer of an infinite two-dimensional flow, for a plane-parallel stream, axially symmetric stream and, in general, for those cases of free streams for which the flow profiles are similar. In the same manner, as previously described, the law of variation of the mixing length in a plane-parallel wake, axially symmetric wake, and so forth, may be obtained. In the present paper, which is devoted to the free jet only, no consideration will be given to wakes. By making use of the relation obtained for the mixing length along the jet the differential equation for free turbulence in a compressible fluid is reduced to the new form

$$\rho u \frac{\partial u}{\partial x} + \rho V \frac{\partial u}{\partial y} = 2c^2x^2 \frac{\partial u}{\partial y}\frac{\partial}{\partial y}\left[\rho \frac{\partial u}{\partial y}\right] \qquad (12)$$

This is the general equation satisfying any case of a free jet of a compressible gas. The magnitude c is the only empirical constant in the theory of free turbulence. In the special case of an incompressible fluid equation (12) assumes the familiar form given in Tollmien's paper:

$$u \frac{\partial u}{\partial x} + V \frac{\partial u}{\partial y} = 2c^2x^2 \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y^2} \tag{12b}$$

II. DIFFERENTIAL EQUATION OF THE TURBULENT BOUNDARY LAYER IN A FREE JET OF A COMPRESSIBLE GAS (PLANE-PARALLEL PROBLEM)

Assume a plane-parallel flow of compressible gas extending to infinity in OY direction (fig. 1) with undisturbed velocity u_0, density ρ_0, and starting from the point O, mix with the surrounding gas at rest. In accord with the law previously derived of the linear increase in the width of the boundary layer b together with the condition of similarity of velocity profiles the velocity along any line $O\varphi$ drawn from the origin of coordinates O (the latter coincided with the point where the boundary layer thickness $b = 0$) remains constant as will be shown. From the similarity of the velocity profiles it follows that the velocity at corresponding points of the flow are equal, that is, for

$$\frac{y_1}{b_1} = \frac{y_2}{b_2} = \frac{y_3}{b_3} = \cdots = \text{constant}$$

there is

$$\frac{u_1}{u_0} = \frac{u_2}{u_0} = \frac{u_3}{u_0} = \cdots = \text{constant}$$

But from equation (11)

$$b = x \text{ constant}$$

hence for

$$y/x = \text{constant} \tag{13a}$$

there is the condition

$$\frac{u}{u_o} = \text{constant} \tag{13b}$$

which proves what was required, since equation (13a) is the equation of a straight line through O. Thus in the turbulent boundary layer of a free flow the rays from a correspondingly chosen origin of coordinates are "isotachs."

The result obtained indicates that if the problem of the free plane boundary layer is solved in coordinates x and $\eta = y/x$ the velocity will depend only on η:

$$u = u_o f(\eta) \tag{14}$$

In order to eliminate the empirical constant from equation (12) the following equation is set

$$2c^2 = \alpha^3 \tag{15a}$$

and the following system of coordinates is chosen

$$x;\ \varphi = \frac{y}{\alpha x} \tag{15b}$$

Then

$$u = u_o f(\varphi) \tag{15c}$$

There is introduced, as is usually done for compressible flow cases, the stream function for the product of the mean velocity by the mean density and there is obtained

$$\rho u = \frac{\partial \psi}{\partial y} \text{ and } \rho V = - \frac{\partial \psi}{\partial x} \tag{15d}$$

The density of the fluid in the case of free turbulence, for which the pressure gradients may be neglected, depends only on the temperatures.

The temperature fields in free flows, as shown by the tests of Fage and Falkner, Ruden, Olsen, and others, are similar as the velocity fields. Otherwise expressed, the investigations of the temperatures show that in the free boundary layer the isotherms, like the isotachs are straight lines from the origin. Thus the temperatures, and hence also the densities, depend only on the nondimensional coordinates (φ):

$$\left.\begin{aligned} t &= t_o\ \theta(\varphi) \\ \rho &= \rho_o\ \kappa(\varphi) \end{aligned}\right\} \tag{16}$$

On the basis of the foregoing a certain function $F(\varphi)$, the first derivative of which is equal to the principal component of the momentum, is introduced

$$\rho\ u = \rho_o\ u_o\ F' \tag{17a}$$

where ρ_o and u_o are the density and velocity, respectively, in undisturbed flow,

Hence

$$\psi = \int \rho u dy = \rho_o\ u_o\ ax \int F' d\varphi$$

that is, the formula for the stream function is

$$\psi = ax\rho_o u_o F \tag{17b}$$

and for the transverse momentum:

$$\rho V = -\frac{\partial\psi}{\partial x} = -a\rho_o u_o \left[F - xF' \frac{\partial\varphi}{\partial x}\right]$$

or

$$\rho V = a\rho_o u_o \, [\varphi F' - F] \tag{17c}$$

Substituting expressions (16), (17a), and (17c) in the differential equation (12) and making certain elementary transformations of the latter, using the expressions

$$a = \sqrt[3]{2c^2}; \quad \varphi = \frac{y}{ax}; \quad \frac{\partial\varphi}{\partial x} = -\frac{\varphi}{x}; \quad \frac{\partial\varphi}{\partial y} = \frac{1}{ax} \tag{17d}$$

the differential equation of the turbulent boundary layer in a plane-parallel free flow of a compressible gas in the form is

$$-F \frac{\partial\left(\frac{F'}{\kappa}\right)}{\partial\varphi} = \frac{\partial\left(\frac{F'}{\kappa}\right)}{\partial\varphi} \frac{\partial}{\partial\varphi}\left[\kappa \frac{\partial\left(\frac{F'}{\kappa}\right)}{\partial\varphi}\right] \tag{18}$$

At all points of the boundary layer except its outer (φ_2) and inner (φ_1) boundaries the derivative of the velocity is not equal to zero, that is,

$$\frac{\partial\left(\frac{F'}{\kappa}\right)}{\partial\varphi} \neq 0$$

Therefore the following is obtained

$$-F = \frac{\partial}{\partial x} \kappa\left[\frac{\partial\left(\frac{F'}{\kappa}\right)}{\partial\varphi}\right] \tag{19a}$$

and further

$$-F = \frac{\partial}{\partial\varphi}\left[F'' - \frac{\kappa'}{\kappa} F'\right] \tag{19b}$$

Thus the differential equation of the turbulent boundary layer assumes the following form

$$F''' = -F + \frac{\partial}{\partial\varphi}\left[\frac{\kappa'}{\kappa} F'\right] \qquad (20)$$

To solve the above equation it is necessary to know the density function $\kappa = \kappa(\varphi)$.

In the case of an incompressible fluid for which $\kappa(\varphi)$ = constant = 1 and $\kappa' = \partial\kappa/\partial\varphi = 0$ equation (20) reduces to the well-known Tollmien equation:

$$F''' = -F \qquad (20a)$$

corresponding to the case of the turbulent boundary layer in a free plane-parallel flow of an incompressible gas.

In what follows equation (20) corresponding to the compressible gas will be solved for two cases:

1. Free flow at small velocities (up to Ba ≅ 0.5) with a temperature differing from that of the surrounding space, that is, a nonisothermal flow.

2. Isothermal flow at large velocities (up to Ba = 1).

III. BOUNDARY LAYER OF NONISOTHERMAL PLANE-PARALLEL JET OF COMPRESSIBLE GAS AT MODERATE FLOW VELOCITIES

1. Heat Balance in the Turbulent Stream

In the present section the flow of a compressible fluid (gas) in the boundary layer of a plane-parallel stream at moderate velocities but at temperatures differing from those of the surrounding fluid at rest shall be considered. In the later sections it will be shown that the effect of the compressibility of the gas arising from the high flow velocities is not large. Up to values of the Bairstow number of the order Ba ≑ 0.5 – 0.6 the effect of the compressibility is barely appreciable. For this reason the equations and results which will be obtained in the present section devoted to the nonisothermal jet of small velocity will maintain their validity up to velocities of the order of 0.5 – 0.6 of the velocity of sound.

To obtain the law of temperature distribution in the boundary layer of a free jet use is made of the differential equation of heat balance, wherein the molecular heat conduction and the conversion of the energy of the viscous forces into heat is neglected with respect to the turbulent heat transfer in the same manner as the friction due to the viscosity in the dynamic equation (2) was disregarded with respect to the turbulent friction. Then

$$\rho \frac{\partial T}{\partial t} + \rho u \frac{\partial T}{\partial x} + \rho V \frac{\partial T}{\partial y} = 0 \qquad (21)$$

where

T temperature of the fluid

t time, seconds

It is convenient to break up all the characteristics of the turbulent flow into their mean values and fluctuations about the mean values:

$$u = \overline{u} + u', \quad V = \overline{V} + V', \quad \rho = \overline{\rho} + \rho', \quad T = \overline{T} + T' \qquad (22a)$$

so that on the average for a finite time interval the fluctuating components are reduced to zero:

$$\overline{V'} = \overline{u'} = \overline{\rho'} = \overline{T'} = 0$$

Averaging with respect to time, while taking account of equations (22a) and (22b) and neglecting moments of the third order:

$$\left(\rho' u' \frac{\partial T'}{\partial x}; \quad \rho' V' \frac{\partial T'}{\partial y}\right)$$

equation (21) is transformed into the differential equation of heat balance for the turbulent quasi-stationary $\left(\overline{\rho \frac{\partial T}{\partial t}} = 0\right)$ flow:

$$\left[\bar{\rho}\,\bar{u}\,\frac{\partial \bar{T}}{\partial x} + \bar{\rho}\,\bar{V}\,\frac{\partial \bar{T}}{\partial y}\right] + \left[\overline{\rho' u'}\,\frac{\partial \bar{T}}{\partial x} + \bar{\rho}\,\overline{u'\,\frac{\partial T'}{\partial x}} + \bar{u}\,\overline{\rho'\,\frac{\partial T'}{\partial x}}\right]$$

$$+ \left[\overline{\rho' V'}\,\frac{\partial \bar{T}}{\partial y} + \bar{\rho}\,\overline{V'\,\frac{\partial T'}{\partial y}} + V\,\overline{\rho'\,\frac{\partial T'}{\partial y}}\right] = 0 \qquad (22c)$$

Assuming as in the case of equation (4d) and density equation (4e) variations that the temperature change is the discontinuity at the instant of loss of individuality of the fluid particle transported by the flow over a distance equal to the mean value of the mixing length (l_T) resulted in

$$T' = l_T\,\frac{\partial \bar{T}}{\partial y} \qquad (22d)$$

Because of the presence in equation (22c) of fluctuations of the temperature gradients its further transformation becomes impossible. In order to eliminate this difficulty the equation of continuity of the flow is here resorted to:

$$\left[\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial \rho'}{\partial t}\right] + \left[\frac{\partial(\overline{\rho u})}{\partial x} + \frac{\partial(\bar{\rho} u')}{\partial x} + \frac{\partial(\rho' \bar{u})}{\partial x} + \frac{\partial(\rho' u')}{\partial x}\right]$$

$$+ \left[\frac{\partial(\overline{\rho V})}{\partial y} + \frac{\partial(\bar{\rho} V')}{\partial y} + \frac{\partial(\rho' \bar{V})}{\partial y} + \frac{\partial(\rho' V')}{\partial y}\right] = 0 \qquad (22e)$$

which, after averaging, assumes the following form

$$\frac{\overline{\partial \rho}}{\partial t} + \frac{\partial(\overline{\rho u})}{\partial x} + \frac{\partial(\overline{\rho' u'})}{\partial x} + \frac{\partial(\overline{\rho V})}{\partial y} + \frac{\partial(\overline{\rho' V'})}{\partial y} = 0 \qquad (22f)$$

Subtracting the averaged equation of continuity from the instantaneous:

$$\frac{\partial \rho'}{\partial t} + \frac{\partial(\bar{\rho} u')}{\partial x} + \frac{\partial(\rho' \bar{u})}{\partial x} + \frac{\partial(\bar{\rho} V')}{\partial y} + \frac{\partial(\rho' \bar{V})}{\partial y} = 0$$

and multiplying by T' gives the following:

$$T'\frac{\partial\rho'}{\partial t}+T'\frac{\partial(\bar{\rho}u')}{\partial x}+T'\frac{\partial(\rho'\bar{u})}{\partial x}+T'\frac{\partial(\bar{\rho}V')}{\partial y}+T'\frac{\partial(\rho'\bar{V})}{\partial y}=0$$

whence

$$\frac{\partial(\rho'T')}{\partial t}+\frac{\partial(\bar{\rho}\bar{u}'T')}{\partial x}+\frac{\partial(\bar{u}\rho'T')}{\partial x}+\frac{\partial(\bar{\rho}V'T')}{\partial y}+\frac{\partial(\bar{V}\rho'T')}{\partial y}$$

$$=\rho'\frac{\partial T'}{\partial t}+\bar{\rho}u'\frac{\partial T'}{\partial x}+\bar{u}\rho'\frac{\partial T'}{\partial x}+\bar{\rho}V'\frac{\partial T'}{\partial y}+\bar{V}\rho'\frac{\partial T'}{\partial y}$$

By averaging the latter expression and taking into account the quasi-stationary state $\left(\frac{\partial(\overline{\rho'T'})}{\partial t}=\overline{\rho'\frac{\partial T'}{\partial t}}=0\right)$ the following is obtained:

$$\left[\bar{\rho}\overline{u'\frac{\partial T'}{\partial x}}+\bar{u}\overline{\rho'\frac{\partial T'}{\partial x}}\right]+\left[\bar{\rho}\overline{V'\frac{\partial T'}{\partial y}}+\bar{V}\overline{\rho'\frac{\partial T'}{\partial y}}\right]=$$

$$=\left[\frac{\partial(\bar{\rho}\overline{u'T'})}{\partial x}+\frac{\partial(\bar{u}\overline{\rho'T'})}{\partial x}\right]+\left[\frac{\partial(\bar{\rho}\,\overline{V'T'})}{\partial y}+\frac{\partial(\bar{V}\overline{\rho'T'})}{\partial x}\right]$$

whence

$$\left[\bar{\rho}\bar{u}\frac{\partial\bar{T}}{\partial x}+\bar{\rho}\bar{V}\frac{\partial\bar{T}}{\partial y}\right]+\left[\overline{\rho'u'}\frac{\partial\bar{T}}{\partial x}+\frac{\partial(\bar{\rho}\overline{u'T'})}{\partial x}+\frac{\partial(\bar{u}\overline{\rho'T'})}{\partial x}\right]+$$
$$+\left[\overline{\rho'V'}\frac{\partial\bar{T}}{\partial y}+\frac{\partial(\bar{\rho}\,\overline{V'T'})}{\partial y}+\frac{\partial(\bar{V}\overline{\rho'T'})}{\partial y}\right]=0. \qquad (23)$$

Neglecting, in analogy to what was done with differential equation (3), the small terms entering the second brackets and the term $\partial(\bar{V}\overline{\rho'T'})/\partial y$ in the third brackets the differential equation of heat balance in the turbulent flow is obtained in the sufficiently simple form:

$$\bar{\rho}\bar{u}\frac{\partial\bar{T}}{\partial x}+\bar{\rho}\bar{V}\frac{\partial\bar{T}}{\partial y}+\overline{\rho'V'}\frac{\partial\bar{T}}{\partial y}+\frac{\partial(\bar{\rho}\,\overline{V'T'})}{\partial y}=0. \qquad (24)$$

For the purpose of further converting this equation the relations shall be taken into account

$$\rho' = l_T \frac{\partial \bar{\rho}}{\partial y}, \qquad V' = l_T \frac{\partial \bar{u}}{\partial y}, \qquad T' = l_T \frac{\partial \bar{T}}{\partial y} \tag{25}$$

and the bars dropped from the notation, that is, $\bar{\rho} = \rho$, $\bar{u} = u$, $\bar{T} = T$, $\bar{V} = V$ shall be set. Then

$$\rho u \frac{\partial T}{\partial x} + \rho V \frac{\partial T}{\partial y} + + \overline{V' l_T} \frac{\partial \rho}{\partial y} \frac{\partial T}{\partial y} + \frac{\partial}{\partial y}\left[\overline{V' l_T}\, \rho \frac{\partial T}{\partial y}\right] = 0 \tag{26}$$

or

$$\rho u \frac{\partial T}{\partial x} + \rho V \frac{\partial T}{\partial y} = l_T^2 \cdot \frac{\partial \rho}{\partial y} \cdot \frac{\partial u}{\partial y} \cdot \frac{\partial T}{\partial y} + \left[l_T^2 \cdot \rho \cdot \frac{\partial u}{\partial y} \cdot \frac{\partial T}{\partial y} \right]. \tag{26b}$$

The right hand side of the above equation gives the transverse gradient of the turbulent heat transfer

$$\frac{\partial W_T}{\partial y} = l_T^2 \frac{\partial \rho}{\partial y} \cdot \frac{\partial u}{\partial y} \cdot \frac{\partial T}{\partial y} + \frac{\partial}{\partial y}\left[l_T^2 \rho \frac{\partial u}{\partial y} \frac{\partial T}{\partial y} \right]. \tag{27a}$$

From this the expression for the heat transfer in the turbulent flow of a compressible gas is obtained:

$$W_T = \int l_T^2 \frac{\partial \rho}{\partial y} \frac{\partial u}{\partial y} \frac{\partial T}{\partial y} dy + l_T^2 \rho \frac{\partial u}{\partial y} \frac{\partial T}{\partial y} + \text{const}, \tag{27b}$$

which, in the particular case of an incompressible gas, $\partial\rho/\partial y = 0$ assumes the following form that

$$W_T = l_T^2 \rho \frac{\partial u}{\partial y} \cdot \frac{\partial T}{\partial y}. \tag{27c}$$

In the case of free turbulence in a compressible gas, according to Taylor model, it is assumed

$$l_T = \sqrt{2 \cdot c \cdot x}, \tag{27d}$$

the differential equation of heat balance is written thus:

$$\rho \cdot u \cdot \frac{\partial T}{\partial x} + \rho V \frac{\partial T}{\partial y} = = 2 c^2 x^2 \left[\frac{\partial \rho}{\partial y} \frac{\partial u}{\partial y} \cdot \frac{\partial T}{\partial y} + \frac{\partial}{\partial y}\left(\rho \frac{\partial u}{\partial y} \cdot \frac{\partial T}{\partial y} \right) \right] = 0. \tag{28}$$

For the turbulence model of Prandtl

$$l = c\ x$$

and therefore

$$\rho \cdot u \frac{\partial T}{\partial x} + \rho V \frac{\partial T}{\partial y} = c^2 x^2 \left| \frac{\partial \rho}{\partial y} \cdot \frac{\partial u}{\partial y} \frac{\partial T}{\partial y} + \frac{\partial}{\partial y}\left(\rho \frac{\partial u}{\partial y} \cdot \frac{\partial T}{\partial y}\right) \right|. \qquad (28a)$$

Comparison of equations (28) and (28a) shows that the Prandtl model gives a heat transfer half as great as that given by the Taylor model. Moreover, the Prandtl model, as shown by Taylor, leads to similarity between the temperature and velocity fields; a result which is not obtained from the Taylor model. In view of the fact that the results of Fage's and Ruden's tests confirm Taylor's free turbulence model and refute the Prandtl model, equation (28) is used as a basis for this discussion.

For an incompressible fluid (ρ = constant) the equation of heat balance reduces to the following form:

$$u \frac{\partial T}{\partial x} + V \frac{\partial T}{\partial y} = 2c^2 x^2 \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y} \frac{\partial T}{\partial y}\right] \qquad (29)$$

2. Temperature and Density Distribution Laws

According to the results obtained in section II of this report

$$\left.\begin{array}{l} \rho u = \rho_0 u_0 F'(\varphi); \qquad \rho V = \rho_0 \cdot u_0 \cdot d(\varphi F' - F); \qquad \rho = \rho_0 \varkappa(\varphi); \\ \varphi = \dfrac{y}{ax}; \qquad a = \sqrt[3]{2c^2}; \\ -F = \dfrac{\partial}{\partial \varphi}\left[\varkappa \cdot \dfrac{\partial\left(\dfrac{F'}{\varkappa}\right)}{\partial \varphi}\right] \end{array}\right\} \qquad (30)$$

It is assumed that the excess temperature fields (difference between the temperatures of the stream and those of the surrounding fluid) in the various cross-sections of the boundary layer of the free jet are similar

$$\frac{\Delta T}{\Delta T_0} = \frac{T - T_{ata}}{T_0 - T_{ata}} = \theta(\varphi), \qquad (31)$$

where

T temperature along a ray φ drawn from the origin of coordinates (start of boundary layer of jet)

T_0 temperature in the region of undisturbed flow

T_{ata} temperature of the fluid at rest in the space surrounding the jet

Then

$$\Delta T = \Delta T_0 \Theta; \qquad \frac{\partial T}{\partial x} = \Delta T_0 \frac{\partial \Theta}{\partial x}, \qquad \frac{\partial T}{\partial y} = \Delta T_0 \frac{\partial \Theta}{\partial y}. \tag{32a}$$

Making use of expressions (30) and (32a) to transform equation (28) it is found that

$$-F\Theta' = \varkappa' \cdot \frac{\partial\left(\frac{F'}{\varkappa}\right)}{\partial\varphi} \cdot \Theta' + \frac{\partial}{\partial\varphi}\left[\varkappa \cdot \frac{\partial\left(\frac{F'}{\varkappa}\right)}{\partial\varphi} \cdot \Theta'\right] \tag{32c}$$

and further

$$-F = \varkappa' \cdot \frac{\partial\left(\frac{F'}{\varkappa}\right)}{\partial\varphi} + \frac{\partial}{\partial\varphi}\left[\varkappa \cdot \frac{\partial\left(\frac{F'}{\varkappa}\right)}{\partial\varphi}\right] + \varkappa \frac{\partial\left(\frac{F'}{\varkappa}\right)}{\partial\varphi} \frac{\Theta''}{\Theta'}. \tag{32d}$$

The above equation on comparison with equation (30) leads to the differential equation

$$\frac{\varkappa'}{\varkappa} + \frac{\Theta''}{\Theta'} = 0. \tag{33}$$

In the case of an incompressible fluid (κ = constant; $\kappa' = 0$):

$$\ln \Theta' = \text{const}; \qquad \Theta' = \text{const}; \qquad \Theta = c_1\varphi + c_2. \tag{34}$$

With $\Theta(\varphi_1) = 1$ at the inner boundary of the flow (in the region of constant velocity) and $\Theta(\varphi_2) = 0$ at the outer boundary, the constants of integration are:

$$c_1\varphi_2 + c_2 = 0; \qquad c_1\varphi_1 + c_2 = 1;$$

whence

$$c_1(\varphi_1 - \varphi_2) = 1;$$

$$c_1 = \frac{1}{\varphi_1 - \varphi_2};$$

$$c_2 = -\frac{\varphi_2}{\varphi_1 - \varphi_2}.$$

From this the law of temperature distribution in the boundary layer of a plane-parallel flow of an incompressible fluid is found to be:

$$\Theta = \frac{\varphi - \varphi_2}{\varphi_1 - \varphi_2} \tag{35}$$

The obtained linear law of temperature distribution is satisfactorily confirmed by Ruden's tests (fig. 2, reference 12).

A certain amount of disagreement with the tests occurs only at the boundaries where the temperature profile departs from a straight line and passes smoothly over to the boundary values of the temperatures. The thermal boundary layer is found to be somewhat thicker than the dynamic boundary layer. This fact is explained by the following reasoning. The linearity of the temperature law is obtained on the assumption of purely turbulent heat transfer with the molecular heat conduction neglected. This assumption was based on the analogy with the dynamic problem where the neglecting of the molecular viscosity led to a velocity profile which was excellently confirmed by tests. A specific characteristic of the velocity profile was that near the boundaries of the layer the velocity gradients and also the frictional stresses were so small that allowance for the viscosity could have no appreciable effect on the deformation of the velocity profile. In contrast to this the temperature field was obtained with large gradients near the boundaries. This indicates that the molecular heat conduction at the limits of the dynamic boundary layer is of appreciable magnitude so that the temperature field departs from the straight line law and the thermal boundary layer will be thicker than the dynamic. Subsequently it is attempted to perfect the temperature distribution law in the free jet by taking the molecular heat conduction into account. For the computation, however, of the density and velocity fields in the boundary layer of a free jet such refinement of the temperature law is not justified since the accuracy of the density field will not thereby be appreciably increased while the mathematical labor will be considerably completed.

On the basis of the foregoing it was preferred to investigate the laws of flow in a compressible fluid without allowance for the effect of the molecular heat conduction and to restrict the problem to the solution of the previous differential equation:

$$\frac{\kappa'}{\kappa} + \frac{\Theta''}{\Theta'} = 0 \tag{36a}$$

whence

$$\ln (\kappa \ \theta') = \text{constant}$$

and

$$\kappa \ \theta' = c_1 \tag{36b}$$

The further solution of equation (36b) is predicated upon a relation between the density and temperature functions. With this in mind Clapeyron's equation is used:

$$\left.\begin{aligned} P &= g \ R \ \rho \ T \\ P_o &= g \ R \ \rho_o \ T_o \end{aligned}\right\} \tag{36c}$$

which in the case of a free jet with constant pressure along and at right angles to the flow ($P = P_o$ = constant) leads to inverse proportionality between the absolute temperature and the densities:

$$\frac{\rho}{\rho_o} = \frac{T_o}{T} = \frac{T_{ata} + \Delta T_o}{T_{ata} + \Delta T} \tag{36d}$$

whence

$$\kappa = \frac{\rho}{\rho_o} = \frac{T_{ata} + \Delta T_o}{T_{ata} + \theta \Delta T_o} \tag{36e}$$

Nondimensional parameters characterizing the degree of heating (or cooling) of the jet are introduced:

$$t = \frac{\Delta T_o}{T_{ata}} \tag{37a}$$

where

ΔT_0 excess temperature in the region of constant velocity $(u = u_0)$

T_{ata} absolute temperature in the gas at rest surrounding the jet

This affords, in final form, the relation between the density and temperature functions in the boundary layer of a plane-parallel turbulent flow:

$$\kappa = \frac{1 + t}{1 + \theta t} \tag{37b}$$

Substitution of the above expression in differential equation (36b) gives

$$\frac{\theta'}{1 + t\theta} = \frac{c_1}{1 + t} \tag{37c}$$

or

$$\frac{d(1 + t\theta)}{1 + t\theta} = \frac{c_1 t}{1 + t} \, d\varphi \tag{37d}$$

Equation (37d) is easily integrated:

$$\ln (1 + t\theta) = D_1\varphi + D_2 \tag{38a}$$

The constants of integration (D_1, D_2) are determined from the boundary conditions given above:

$$\theta(\varphi_1) = 1$$

$$\theta(\varphi_2) = 0$$

resulting in

$$1 + t\theta = (1 + t)^{\frac{\varphi - \varphi_2}{\varphi_1 - \varphi_2}} \tag{38b}$$

and thus in the temperature distribution formula:

$$\theta = \frac{\Delta T}{\Delta T_0} = \frac{(1+t)^{\frac{\varphi-\varphi_2}{\varphi_1-\varphi_2}}-1}{t} \tag{39}$$

The above expression substituted in equation (37b) gives the law of density distribution in the boundary layer:

$$\kappa = \frac{\rho}{\rho_0} = (1+t)^{1-\frac{\varphi-\varphi_2}{\varphi_1-\varphi_2}} \tag{40a}$$

or

$$\kappa = \frac{\rho}{\rho_0} = (1+t)^{\frac{\varphi_1-\varphi}{\varphi_1-\varphi_2}} \tag{40b}$$

3. Development of the Differential Equation

Consider the problem of the flow of nonisothermal jet at velocities that are small compared to the velocity of sound (up to Ba = 0.5). For this condition the density profiles may be considered practically independent of the velocity profiles. The density will depend only on the temperature, the character of the dependence having been established in the foregoing as

$$\kappa = \frac{\rho}{\rho_0} = (1+t)^{\frac{\varphi_1-\varphi}{\varphi_1-\varphi_2}} \tag{41}$$

The derivative of the density at a given point will then be

$$\kappa' = \frac{\partial x}{\partial \varphi} = -(1+t)^{\frac{\varphi_1-\varphi}{\varphi_1-\varphi_2}} \frac{\ln(1+t)}{\varphi_1-\varphi_2} \tag{42}$$

The ratio of the derivative of the density to the value of the latter will be constant for a given value of the jet temperature

$$\frac{\kappa'}{\kappa} = -\frac{\ln(1 + t)}{\varphi_1 - \varphi_2} \tag{43}$$

Returning to the general differential equation (20) of the boundary layer in a compressible gas stream:

$$F''' = -F + \frac{\partial}{\partial\varphi}\left[\frac{\kappa'}{\kappa} F'\right] \tag{44}$$

its special form for a nonisothermal jet of moderate velocities is obtained as:

$$F''' = -F - \frac{\ln(1 + t)}{\varphi_1 - \varphi_2} F'' \tag{45}$$

After introduction of a special notation for the parameter which depends only on the temperature of the stream:

$$S = \frac{\ln(1 + t)}{\varphi_1 - \varphi_2} \tag{46a}$$

The differential equation will then have the following form

$$F''' = -F - SF'' \tag{46b}$$

The above equation is a common linear differential equation of the third order whose general integral is of the form

$$F = C_1 e^{k_1\varphi} + C_2 e^{k_2\varphi} + C_3 e^{k_3\varphi} \tag{46c}$$

where

C_1, C_2, C_3 constants of integration

k_1, k_2, k_3 roots of the characteristic equation

$$k^3 + \alpha k^2 + 1 = 0 \tag{46d}$$

which in this instance reduces to the equation by Cardan. As is usual for jet boundary layers equation (46c) has five boundary conditions.

1. At the inner boundary of the layer where $\varphi = \varphi_1$

   (a) The gradient of the momentum is equal to zero:

$$\frac{\partial(pu)}{\partial\rho} = 0 \text{ — that is, } F''(\varphi_1) = 0 \tag{46$_1$}$$

   (b) The momentum is equal to the momentum of the undisturbed flow:

$$\rho u = \rho_0 u_0 \text{ — that is, } F'(\varphi_1) = 1 \tag{46$_2$}$$

   (c) The transverse component of the velocity vanishes:

$$\rho_0 v_0 = 0 \text{ — that is, } F(\varphi_1) = \varphi_1 \tag{46$_3$}$$

2. On the outer boundary of the layer where $\varphi = \varphi_2$

   (d) The velocity gradient is equal to zero:

$$\frac{\partial(\rho u)}{\partial\varphi} = 0 \text{ — that is, } F''(\varphi_2) = 0 \tag{46$_4$}$$

   (e) The velocity is equal to zero:

$$\rho u = 0 \text{ — that is, } F'(\varphi_2) = 0 \tag{46$_5$}$$

The five conditions (46_{1-5}) are used to ascertain the three constants of integration C_1, C_2, C_3 and the values of the nondimensional coordinates of the outer and inner limits of the boundary layer φ_1 and φ_2. To each value of the compressibility parameter S there corresponds certain values of the constants of integration and the nondimensional coordinates.

4. Integration of the Differential Equation of the Nonisothermal Jet

According to the foregoing the boundary layer of the nonisothermal jet is characterized by the differential equation

$$F''' + SF'' + F = 0 \tag{47}$$

the integral of which is

$$F = C_1 e^{k_1\varphi} + C_2 e^{k_2\varphi} + C_3 e^{k_3\varphi} \tag{48}$$

The values of k_1, k_2, k_3 entering this integral are the roots of the characteristic equation

$$k^3 + Sk^2 + 1 = 0 \tag{49a}$$

By means of the substitution

$$k = \frac{x - S}{3} \tag{49b}$$

the given cubic equation is reduced to the Cardan solution:

$$X^3 + 3PX + 2q = 0 \tag{49c}$$

in which

$$P = - S^2$$

$$q = S^3 + \frac{27}{2} \tag{49d}$$

The roots of this equation are determined by the Cardan formula:

$$X_1 = u + v, \quad X_2 = w_1 u + w_2 v, \quad X_3 = w_2 u + w_1 v \tag{49e}$$

the terms u and v being given by

$$u = \sqrt[3]{-q + \sqrt{q^2 + P^3}}; \quad v = \sqrt{-q - \sqrt{q^2 + P^3}}$$

The coefficients w_1 and w_2 are the conjugate imaginary cube roots of unity:

$$w_1 = -\frac{1}{2} + i\,\frac{\sqrt{3}}{2}; \quad w_2 = -\frac{1}{2} - i\,\frac{\sqrt{3}}{2} \tag{49f}$$

In the present case:

$$\left.\begin{aligned} u &= \sqrt[3]{-\left[S^3 + \frac{27}{2}\right] + \sqrt{\left[S^3 + \frac{27}{2}\right]^2 - S^6}} \\ v &= \sqrt[3]{-\left[S^3 + \frac{27}{2}\right] - \sqrt{\left[S^3 + \frac{27}{2}\right]^2 - S^6}} \end{aligned}\right\} \tag{50a}$$

Since u and v are real numbers, X_1 is the real root of the Cardan equation (49c) and X_2 and X_3 are the conjugate imaginary roots. Correspondingly k_1 is the real root of the characteristic equation (49a) and k_2 and k_3 the conjugate imaginary roots.

Setting:

$$k_1 = \alpha_1, \quad k_2 = \alpha_2 + \beta_2 i, \quad k_3 = \alpha_2 - \beta_2 i \tag{50b}$$

integral (48) is transformed into

$$F(\varphi) = C_1 e^{\alpha_1 \varphi} + C_2 e^{(\alpha_2 + \beta_2 i)\varphi} + C_3 e^{(\alpha_2 - \beta_2 i)\varphi} \tag{50c}$$

As is known a pair of conjugate imaginary solutions of a linear differential equation of the third order defines a pair of real solutions expressed in terms of trigonometric functions:

$$\left.\begin{aligned} C_2 e^{(\alpha_2 + \beta_2 i)\varphi} &\rightarrow C_2 e^{\alpha_2 \varphi} \cos(\beta_2 \varphi) \\ C_3 e^{(\alpha_2 - \beta_2 i)\varphi} &\rightarrow C_3 e^{\alpha_2 \varphi} \sin(\beta_2 \varphi) \end{aligned}\right\}: \tag{50d}$$

In its final form the differential equation of the boundary layer may be given in the following form:

$$F(\varphi)=C_1e^{\alpha_1\varphi}+C_2e^{\alpha_2\varphi}\cos(\beta_2\varphi)+C_3e^{\alpha_2\varphi}\sin(\beta_2\varphi) \tag{51a}$$

The magnitudes α_1, α_2, and β_2 are readily expressed in terms of the "thermal compressibility factor" of the jet:

$$S=\frac{\ln(1+t)}{\varphi_1-\varphi_2}$$

For this purpose the expressions (49b) and (50a, b) are resorted to, while taking into account the fact that S is a small magnitude of the order of 0.05 - 0.15. Then

$$\sqrt{\left[S^3+\frac{27}{2}\right]^2-S^6}\cong\frac{27}{2}+S^3;$$

$$u=\sqrt[3]{-\left[S^3+\frac{27}{2}\right]+\left[S^3+\frac{27}{2}\right]}=0;$$

$$v=\sqrt[3]{-2\left[S^3+\frac{27}{2}\right]}=-\left[3+\frac{2S^3}{27}\right];$$

$$w_1=\frac{-1+i\sqrt{3}}{2};\qquad w_2=\frac{-1-i\sqrt{3}}{2}$$

whence

$$\left.\begin{aligned}X_1&=u+v=-\left[3+\frac{2S^3}{27}\right];\\X_2&=w_1u+w_2v=\frac{1+i\sqrt{3}}{2}\left[3+\frac{2S^3}{27}\right]\\X_3&=w_2u+w_1v=\frac{1-i\sqrt{3}}{2}\left[3+\frac{2S^3}{27}\right]\end{aligned}\right\} \tag{51b}$$

and further

$$\left.\begin{aligned}k_1&=\frac{X_1-S}{3}=1-\frac{S}{3}-\frac{2S^3}{81};\\k_2&=\frac{X_2-S}{3}=\left[\frac{1}{2}-\frac{S}{6}+\frac{S^3}{81}\right]+i\frac{\sqrt{3}}{2}\left[1+\frac{2S^3}{81}\right];\\k_3&=\frac{X_3-S}{3}=\left[\frac{1}{2}-\frac{S}{3}+\frac{S^3}{81}\right]-i\frac{\sqrt{3}}{2}\left[1+\frac{2S^3}{81}\right].\end{aligned}\right\} \tag{51c}$$

A comparison of equations (51a) and (51b) yields the formulas for computing α_1, α_2, and β_2 from equation (50c) for given values of the compressibility factor:

$$\left.\begin{aligned}\alpha_1 &= -\left[1+\frac{S}{3}+\frac{2S^3}{81}\right];\\ \alpha_2 &= \left[\frac{1}{2}-\frac{S}{3}+\frac{S^3}{81}\right]; \quad \beta_2=\frac{\sqrt{3}}{2}\left[1+\frac{2S^3}{81}\right]\end{aligned}\right\} \tag{51d}$$

In the particular case of an incompressible fluid, $S \sim 0$, it is

$$\alpha_1 = -1, \quad \alpha_2 = \frac{1}{2}, \quad \beta_2 = \frac{\sqrt{3}}{2},$$

which is in complete agreement with the corresponding Tollmien solution:

$$F = e^{-\varphi} + C_2 e^{\frac{\varphi}{2}} \cos\left[\frac{\sqrt{3}}{2}\varphi\right] + C_3 e^{\frac{\varphi}{2}} \sin\left[\frac{\sqrt{3}}{2}\varphi\right] \tag{51e}$$

Estimating the order of magnitude of the individual terms in equation (51d) it is readily apparent that the terms containing (S^3) can be neglected since S is usually considerably less than unity. Thus, for example, if the temperature of the jet is 100°C higher than the surrounding temperature, S will have the value 0.1 and $S^3 \cong 0.001$. Thus without appreciable impairment of the accuracy

$$\alpha_1 = -1 - \frac{S}{3}, \quad \alpha_2 = \frac{1}{2} - \frac{S}{3}, \quad \beta_2 = \frac{\sqrt{3}}{2} \tag{52a}$$

The basic function of the boundary layer then assumes the following form:

$$F(\varphi) = e^{-\frac{S}{3}\varphi}\left\{C_1 e^{-\varphi} + C_2 e^{\frac{\varphi}{2}} \cos\left[\frac{\sqrt{3}}{2}\varphi\right] + C_3 e^{\frac{\varphi}{2}} \sin\left[\frac{\sqrt{3}}{2}\varphi\right]\right\} \tag{52b}$$

The expression in braces corresponds exactly to the Tollmien solution for an incompressible fluid:

$$F_0(\varphi) = C_1 e^{-\varphi} + C_2 e^{\frac{\varphi}{2}} \cos\left[\frac{\sqrt{3}}{2}\varphi\right] + C_3 e^{\frac{\varphi}{2}} \sin\left[\frac{\sqrt{3}}{2}\varphi\right] \tag{52c}$$

The first derivative of $F(\varphi)$, equal to the nondimensional momentum, is given by

$$\frac{\rho u}{\rho_0 u_0} = F'(\varphi) = e^{-\frac{S}{3}\varphi}\left[-C_1 e^{-\varphi} + \frac{1}{2}(C_2 + C_3\sqrt{3})\, e^{\frac{\varphi}{2}} \cos\left[\frac{\sqrt{3}}{2}\varphi\right] - \right.$$
$$\left. - \frac{1}{2}(C_2\sqrt{3} - C_3)\, e^{\frac{\varphi}{2}} \sin\left[\frac{\sqrt{3}}{2}\varphi\right]\right] -$$
$$- \frac{S}{3} e^{-\frac{S}{3}\varphi}\left[C_1 e^{-\varphi} + C_2 e^{\frac{\varphi}{2}} \cos\left[\frac{\sqrt{3}}{2}\varphi\right] + C_3 e^{\frac{\varphi}{2}} \sin\left[\frac{\sqrt{3}}{2}\varphi\right]\right] \tag{52d}$$

The second derivative of $F(\varphi)$, which is the nondimensional velocity gradient, is

$$F''(\varphi) = e^{-\frac{S}{3}\varphi}\left[C_1 e^{-\varphi} + \frac{1}{2}(C_3\sqrt{3} - C_2)\, e^{\frac{\varphi}{2}} \cos\left[\frac{\sqrt{3}}{2}\varphi\right] - \right.$$
$$\left. - \frac{1}{2}(C_3 + C_2\sqrt{3})\, e^{\frac{\varphi}{2}} \sin\left[\frac{\sqrt{3}}{2}\varphi\right]\right] - \frac{2S}{3} e^{-\frac{S}{3}\varphi}\left[-C_1 e^{-\varphi} + \frac{1}{2}(C_2 + \right.$$
$$\left. + C_3\sqrt{3})\, e^{\frac{\varphi}{2}} \cos\left[\frac{\sqrt{3}}{2}\varphi\right] - \frac{1}{2}(C_2\sqrt{3} - C_3)\, e^{\frac{\varphi}{2}} \sin\left[\frac{\sqrt{3}}{2}\varphi\right]\right] +$$
$$+ \frac{S^2}{9} e^{-\frac{S}{3}\varphi}\left[C_1 e^{-\varphi} + C_2 e^{\frac{\varphi}{2}} \cos\left[\frac{\sqrt{3}}{2}\varphi\right] + C_3 e^{\frac{\varphi}{2}} \sin\left[\frac{\sqrt{3}}{2}\varphi\right]\right] \tag{52e}$$

For these considerations the factor $S^2/9$ in the foregoing equation are neglected.

There remains, on the basis of the five boundary conditions (46_{1-5}), the determination of the constants of integration C_1, C_2, C_3 and the values of the nondimensional coordinates of the outer and inner limits (φ_1) and (φ_2) of the boundary layer:

$$F(\varphi_1) = \varphi_1, \qquad F'(\varphi_1) = 1, \qquad F''(\varphi_1) = 0,$$
$$F'(\varphi_2) = 0, \qquad F''(\varphi_2) = 0$$

The problem of predicting the basic function $F(\varphi)$ is then solved.

The five boundary conditions lead to five transcendental equations with five unknowns $(C_1, C_2, C_3, \varphi_1,$ and $\varphi_2)$ solvable for any particular value of the compressibility factor. By applying the transformation of variables proposed by Tollmien

$$\bar{\varphi} = \varphi - \varphi_1, \tag{53a}$$

these equations can be considerably simplified. The substitution is made so that

$$F(\bar{\varphi})=F(\varphi), \quad F'(\bar{\varphi})=F'(\varphi), \quad F''(\bar{\varphi})=F''(\varphi), \tag{53b}$$

Then

$$F(\bar{\varphi})=e^{-\frac{S}{3}\varphi}\left[D_1e^{-\bar{\varphi}}+D_2e^{\frac{\varphi}{2}}\cos\left[\frac{\sqrt{3}}{2}\varphi\right]+D_3e^{\frac{\varphi}{2}}\sin\left[\frac{\sqrt{3}}{2}\varphi\right]\right], \tag{53c}$$

and hence, while bearing in mind that,

$$\bar{\varphi}_1=0_1,\bar{\varphi}_2=\varphi_2-\varphi_1, F_0=D_1e^{-\bar{\varphi}}+D_2e^{\frac{\bar{\varphi}}{2}}\cos\left[\frac{\sqrt{3}}{2}\varphi\right]+D_3e^{\frac{\bar{\varphi}}{2}}\sin\left[\frac{\sqrt{3}}{2}\bar{\varphi}\right]$$

five equations with five unknowns:

$$\left.\begin{aligned} &F_{01}=\varphi_1; \quad F_{01}'-\frac{S}{3}F_{01}=1; \quad F_{01}''-\frac{2S}{3}F'_{01}=0;\\ &F_{02}'-\frac{S}{3}F_{01}=0; \quad F_{02}''-\frac{2S}{3}F'_{02}=0, \end{aligned}\right\} \tag{54a}$$

which are simplified to

$$F_{01}=\varphi_1; \quad F_{01}'\cong 1+\frac{S}{3}; \quad F_{01}''\simeq\frac{2S}{3}; \quad F_{02}'\simeq-0{,}13S; \quad F_{02}''\simeq 0. \tag{54b}$$

With the first three equations of the above system the coefficients D_1, D_2, and D_3 of equation (53c) can be expressed in terms of φ_1:

$$D_1+D_2=\varphi_1$$

$$-D_1+\frac{1}{2}(D_2+D_3\sqrt{3})=1+\frac{S}{3}; \tag{54c}$$

$$D_1+\frac{1}{2}(D_3\sqrt{3}-D_2)=\frac{2S}{3}. \tag{54d}$$

from which

$$D_1=\frac{\varphi_1-1}{3}+\frac{S}{9}; \quad D_2=\frac{\varphi_1+0{,}5}{1{,}5}-\frac{S}{9}; \quad D_3=\frac{1}{\sqrt{3}}+\frac{S}{1{,}7}. \tag{54e}$$

With equations (54c) the five equations of (54b) (with account taken of equations (52) and (52e)) can be reduced to a system of two equations with two unknowns:

$$-D_1e^{-(\varphi_2-\varphi_1)}+\frac{1}{2}(D_2+D_3\sqrt{3})e^{\frac{\varphi_2-\varphi_1}{2}}\cos\left[\frac{\sqrt{3}}{2}(\varphi_2-\varphi_1)\right]-$$

$$-\frac{1}{2}[D_2\sqrt{3}-D_3]e^{\frac{(\varphi_2-\varphi_1)}{2}}\sin\left[\frac{\sqrt{3}}{2}(\varphi_2-\varphi_1)\right]=-0{,}13\,S; \tag{54f}$$

$$D_1 e^{-(\varphi_2-\varphi_1)} + \frac{1}{2}\left[D_3\sqrt{3} - D_2\right] e^{\frac{\varphi_2-\varphi_1}{2}} \cos\left[\frac{\sqrt{3}}{2}(\varphi_2-\varphi_1)\right] -$$
$$- \frac{1}{2}\left[D_2\sqrt{3} + D_3\right] e^{\frac{\varphi_2-\varphi_1}{2}} \sin\left[\frac{\sqrt{3}}{2}(\varphi_2-\varphi_1)\right] \cong 0, \qquad (54g)$$

where D_1, D_2, and D_3 are taken from equations (54c). Addition and subtraction of equations (54f) and (54g) give two new equations of a somewhat simpler form:

$$\left.\begin{aligned} &D_3\sqrt{3}\, e^{\frac{\varphi_2-\varphi_1}{2}} \cos\left[\frac{\sqrt{3}}{2}(\varphi_2-\varphi_1)\right] - \\ &\quad - D_2\sqrt{3}\, e^{\frac{\varphi_2-\varphi_1}{2}} \sin\left[\frac{\sqrt{3}}{2}(\varphi_2-\varphi_1)\right] = -0{,}13\,S; \\ &2D_1 e^{-(\varphi_2-\varphi_1)} - D_2 e^{\frac{\varphi_2-\varphi_1}{2}} \cos\left[\frac{\sqrt{3}}{2}(\varphi_2-\varphi_1)\right] - \\ &\quad - D_3 e^{\frac{\varphi_2-\varphi_1}{2}} \sin\left[\frac{\sqrt{3}}{2}(\varphi_2-\varphi_1)\right] = 0{,}13\,S \end{aligned}\right\} \qquad (55)$$

Next it is attempted to determine the functional relation between the constants D_1, D_2, D_3, φ_1, and φ_2 and the compressibility factor S making use of the fact, as will be shown later, that the compressibility of the air is of only slight effect on the free jet. Putting

$$\left.\begin{aligned} D_1 = D_{10} + \Delta D_1; \quad D_2 = D_{20} + \Delta D_2; \quad D_3 = D_{30} + \Delta D_3; \\ \varphi_1 = \varphi_{10} + \Delta\varphi_1; \quad \varphi_2 = \varphi_{20} + \Delta\varphi_2, \end{aligned}\right\} \qquad (56a)$$

where D_{10}, D_{20}, D_{30}, φ_{10}, and φ_{20} are known values of the constants for the incompressible jet:

$$\left.\begin{aligned} D_{10} = \frac{\varphi_{10} - 1}{3} = -0{,}0062, \qquad D_{20} = \frac{\varphi_{10} + 0{,}5}{1{,}5} = 0{,}987, \\ D_{30} = \frac{1}{\sqrt{3}} = 0{,}578, \qquad \varphi_{10} = 0{,}981;\ \varphi_{20} = -2{,}04, \end{aligned}\right\} \qquad (56b)$$

and the small increments (compare equations (86a) and (86c) with (84b)):

$$\left.\begin{aligned} \Delta D_1 &= \frac{3\Delta\varphi_1 + S}{9}; \qquad \Delta D_3 = \frac{S}{1.7} \\ \Delta D_2 &= \frac{6\Delta\varphi_1 - S}{9}; \qquad \Delta\varphi_1 = \varphi_1 - \varphi_{10}, \quad \Delta\varphi_2 = \varphi_2 - \varphi_{20} \end{aligned}\right\} \tag{56c}$$

Reverting to equations (55) expanding into series while neglecting all terms of higher degree than the second and the products of the small increments and using the values

$$e^{1.51} = 4.51 \qquad e^{3.02} = 20.4$$

$$\sin(-2.62) = -\,0.5, \quad \cos(-2.62) = -\,0.865$$

results in

$$0.082S = 0.380\,\Delta\varphi_2 - 0.255\Delta\varphi_1$$

$$4.634S = 0.130\Delta\varphi_2 - 13.60\Delta\varphi_1$$

The solution of these equations leads to a functional relation between the deformation of the boundary layer and the compressibility factor:

$$\Delta\varphi_1 \cong -\,0.34S; \qquad \Delta\varphi_2 \cong 0 \tag{57a}$$

which yield the corrections for the integration constants:

$$\Delta D_1 \cong 0; \qquad \Delta D_2 = \frac{S}{9}; \qquad \Delta D_3 = \frac{S}{1.7} \tag{57b}$$

The constants of the auxiliary function $F(\overline{\varphi})$ are then equal to

$$D_1 = -\,0.0062, \quad D_2 = 0.987 + 0.11S; \quad D_3 = 0.587 + 0.59S \tag{57c}$$

and the ordinates of the outer and inner limits of the boundary layer

$$\varphi_1 = 0.98 - 0.34S, \quad \varphi_2 = -\,2.04 \tag{58}$$

Equations (57c) and (58) give the integration constants of the fundamental function $F(\varphi)$. The first three boundary conditions of the given problem are used:

$$F(\varphi_1) = \varphi_1, \quad F'(\varphi_1) = 1, \quad F''(\varphi_1) = 0 \tag{59a}$$

After substitution of the values F, F' and F'' from equations (52b), (52d), and (52e) the first boundary condition gives

$$\varphi_1 = e^{-\frac{S}{3}\varphi_1}\left\{C_1 e^{-\varphi_1} + C_2 e^{\frac{\varphi_1}{2}} \cos\left[\frac{\sqrt{3}}{2}\varphi_1\right] + C_3 e^{\frac{\varphi_1}{2}} \sin\left[\frac{\sqrt{3}}{2}\varphi_1\right]\right\} \tag{59b}$$

The constants of the compressible gas are again expressed in the form

$$C_1 = C_{1o} + \Delta C_1, \quad C_2 = C_{2o} + \Delta C_2, \quad C_3 = C_{3o} + \Delta C_3 \tag{59c}$$

where

$$C_{1o} = -0.0176, \quad C_{2o} = 0.1337, \quad C_{3o} = 0.6876 \tag{59d}$$

are the corresponding values of the constants obtained by Tollmien for the particular case of an incompressible flow. In the same manner, according to (58) the coordinate of the inner boundary of the layer may be written:

$$\varphi_1 = \varphi_{1o} - 0.34S \tag{59e}$$

where $\varphi_{1o} = 0.981$ is the coordinate of the inner boundary for the incompressible fluid. Substitution of (59c) and (59e) in (59b) and use of the series up to the second power term (due to the smallness of S) of the exponential and trigonometric expressions gives

$$\left.\begin{aligned} &e^{-\frac{S}{2}\varphi_1} = 1 - \frac{S}{3}\varphi_{10}; \quad e^{-\varphi_1} = e^{-\varphi_{10}}\,[1 + 0.34S]; \\ &e^{\frac{\varphi_1}{2}} = e^{\frac{\varphi_{10}}{2}}\,[1 - 0.17S] \\ &\cos\left[\frac{\sqrt{3}}{2}\varphi_1\right] = \cos\left[\frac{\sqrt{3}}{2}\varphi_{10}\right] + 0.3S\,\sin\left[\frac{\sqrt{3}}{2}\varphi_{10}\right] \\ &\sin\left[\frac{\sqrt{3}}{2}\varphi_1\right] = \sin\left[\frac{\sqrt{3}}{2}\varphi_{10}\right] - 0.3S\,\cos\left[\frac{\sqrt{3}}{2}\varphi_{10}\right] \end{aligned}\right\} \qquad (59f)$$

The products and squares of small terms are disregarded and the following relation is taken from the boundary conditions for the incompressible gas:

$$\varphi_{10} = C_{10}e^{-\varphi_{10}} + C_{20}e^{\frac{\varphi_{10}}{2}}\cos\left[\frac{\sqrt{3}}{2}\varphi_{10}\right] + C_{30}e^{\frac{\varphi_{10}}{2}}\sin\left[\frac{\sqrt{3}}{2}\varphi_{10}\right]$$

whence the equation connecting the increase in the constants due to the compressibility with the compressibility factor S:

$$0.375\ \Delta C_1 + 1.08\Delta C_2 + 1.225\ \Delta C_3 = 0.340\,S \qquad (60)$$

The second boundary condition $F'(\varphi_1) = 1$ in combination with equations (52d) and (59b) gives

$$1 = e^{-\frac{S}{3}\varphi_1}\left\{-C_1e^{\varphi_1} + \frac{1}{2}\,(C_2 + C_3\,\sqrt{3})e^{\frac{\varphi_1}{2}}\cos\left[\frac{\sqrt{3}}{2}\varphi_1\right]\right.$$

$$\left. - \frac{1}{2}\,(C_2\sqrt{3} - C_3)e^{\frac{\varphi_1}{2}}\sin\left[\frac{\sqrt{3}}{2}\varphi_1\right]\right\} - \frac{S}{3}\,\varphi_1 \qquad (61)$$

Substitution of expressions (59c), (59e), and (59) in equation (61a), while neglecting products of small quantities, and making use of the particular form of this equation obtained for incompressible gas yields:

$$1 = -C_1 e^{-\varphi_{10}} + \frac{1}{2}(C_{20} + C_{30}\sqrt{3}) e^{\frac{\varphi_{10}}{2}} \cos\left[\frac{\sqrt{3}}{2}\varphi_{10}\right]$$

$$-\frac{1}{2}(C_{20}\sqrt{3} - C_{30}) e^{\frac{\varphi_{10}}{2}} \sin\left[-\frac{\sqrt{3}}{2}\varphi_{10}\right]$$

a second equation for the relation between the increments in the constants of the function $F(\varphi)$ and the compressibility factor:

$$-0.375\Delta C_1 - 0.52\Delta C_2 + 1.55\Delta C_3 = 0.654\ S \qquad (62)$$

The third boundary condition $F''(\varphi_1) = 0$, together with equations (52e) and (61), gives

$$0 = e^{-\frac{S}{3}\varphi_1}\left\{-C_1 e^{-\varphi_1} + \frac{1}{2}(C_3\sqrt{3} - C_2) e^{\frac{\varphi_1}{2}} \cos\left[\frac{\sqrt{3}}{2}\varphi_1\right]\right.$$

$$\left.-\frac{1}{2}(C_3 + C_2\sqrt{3}) e^{\frac{\varphi_1}{2}} \sin\left[\frac{\sqrt{3}}{2}\varphi_1\right]\right\} - \frac{2S}{3}\left[1 + \frac{S}{2}\varphi_1\right] \qquad (63)$$

By substitution of expressions (59c), (59e), and (59f) in equation (63), while neglecting the products and squares of small terms and taking account of the fact that in the case of incompressible gas equation (63) assumes the form:

$$0 = C_{10} e^{-\varphi_{10}} + \frac{1}{2}(C_{30}\sqrt{3} - C_{20}) e^{\frac{\varphi_{10}}{2}} \cos\left[\frac{\sqrt{3}}{2}\varphi_{10}\right]$$

$$-\frac{1}{2}(C_{30} + C_{20}\sqrt{3}) e^{\frac{\varphi_{10}}{2}} \sin\left[\frac{\sqrt{3}}{2}\varphi_{10}\right]$$

the third relation between the increments of the constants of the function $F(\varphi)$ and the compressibility factor is:

$$0.375\ \Delta C_1 - 1.6\ \Delta C_2 + 0.32\ \Delta C_3 = 0.332\ S \qquad (64)$$

The solution of the system of three simultaneous equations:

$$\left.\begin{aligned} 0.375\ \Delta C_1 - 1.60\ \Delta C_2 + 0.32\ \Delta C_3 &= 0.372\ S, \\ -0.375\ \Delta C_1 - 0.52\ \Delta C_2 + 1.55\ \Delta C_3 &= 0.654\ S \\ 0.375\ \Delta C_1 + 1.08\ \Delta C_2 + 1.225\ \Delta C_3 &= 0.340\ S \end{aligned}\right\} \qquad (65)$$

gives the laws of variation of the constants of integration of equation (52b) under the effect of compressibility:

$$\Delta C_1 = 0, \quad \Delta C_2 = -0.14\ S, \quad \Delta C_3 = 0.385\ S \qquad (65a)$$

hence the integration constants:

$$C_1 = -\ 0.0176; \quad C_2 = 0.1337 - 0.140\ S;$$

$$C_3 = 0.6776 + 0.385\ S \qquad (66)$$

Substitution of expressions (52b) and (52e) in (66) while neglecting the products of small quantities, the function F and its derivatives are obtained in the final form

$$F = F_o + \Delta F; \quad F' = F'_o + \Delta F'; \quad F'' = F''_o + \Delta F'' \qquad (67)$$

where F_o, F_o', and F_o'' are the values of the functions and its derivatives for incompressible gas (Tollmien's solution):

$$\left.\begin{aligned} F_o(\varphi) &= -\ 0.0176e^{-\varphi} + 0.1377e^{\frac{\varphi}{2}} \cos\left[\frac{\sqrt{3}}{2}\varphi + 0.6876e^{\frac{\varphi}{2}} \sin\frac{\sqrt{3}}{2}\varphi\right] \\ F_o'(\varphi) &= 0.0176e^{-\varphi} + 0.6623e^{\frac{\varphi}{2}} \cos\left[\frac{\sqrt{3}}{2}\varphi + 0.228\ e^{\frac{\varphi}{2}} \sin\frac{\sqrt{3}}{2}\varphi\right] \\ F_o''(\varphi) &= -0.0176e^{-\varphi} + 0.528e^{\frac{\varphi}{2}} \cos\left[\frac{\sqrt{3}}{2}\varphi - 0.930e^{\frac{\varphi}{2}} \sin\frac{\sqrt{3}}{2}\varphi\right] \end{aligned}\right\} \qquad (67$$

and ΔF, $\Delta F'$, and $\Delta F''$ are the increments of the function and its derivatives under the influence of compressibility. The increment of the function is

$$\Delta F = -\frac{S}{3}\left[\varphi F_o + 0.42e^{\frac{\varphi}{2}}\cos\left[\frac{\sqrt{3}}{2}\varphi\right] - 1.16e^{\frac{\varphi}{2}}\cos\left[\frac{\sqrt{3}}{2}\varphi\right]\right] \quad (67b)$$

that of the first derivative

$$\Delta F' = -\frac{S}{3}\left[\varphi F_o' + F_o - 0.80e^{\frac{\varphi}{2}}\cos\left[\frac{\sqrt{3}}{2}\varphi\right] - 0.93e^{\frac{\varphi}{2}}\sin\left[\frac{\sqrt{3}}{2}\varphi\right]\right] (67c)$$

and that of the second derivative

$$\Delta F'' = -\frac{S}{3}\left[\varphi F_o'' + 2F_o' - 1.2e^{\frac{\varphi}{2}}\cos\left[\frac{\sqrt{3}}{2}\varphi\right] + 0.22e^{\frac{\varphi}{2}}\sin\left[\frac{\sqrt{3}}{2}\varphi\right]\right] (67d)$$

In conclusion it should be noted that the author's carefully conducted numerical solution, based directly on equations (55), showed complete agreement with the functional results, (67a) and (67d), approximately obtained, notwithstanding the fact that for the numerical solution a very large value was chosen for the compressibility factor (S = 0.135) corresponding to the case where the jet has a temperature 150°C above the surrounding temperature.

It will be recalled that the first derivative of the function F is the momentum of the flow in the direction of the X-axis:

$$\frac{\rho u}{\rho_o u_o} = F' \quad (68a)$$

To obtain the velocity u/u_o it is necessary to apply to the law of density distribution in the boundary layer:

$$\kappa = \frac{\rho}{\rho_o} = (1+t)^{\frac{\varphi_1 - \varphi}{\varphi_1 - \varphi_2}} \quad (68b)$$

then

$$\frac{u}{u_o} = \frac{F'}{\kappa} \quad (68c)$$

The momentum velocity ratios in the direction of the y-axis are

$$\frac{\rho V}{\rho_0 V_0} = a\,(\varphi F' - F); \qquad \frac{V}{V_0} = a\,\frac{\varphi F' - F}{\kappa} \tag{63d}$$

To facilitate the computation of the components of the velocity and other characteristic magnitudes of the nonisothermal layer, table I is appended with the computed values:

$$F_0,\ F_0',\ F_0'',\ u\,\frac{3}{S}\,\Delta F,\ \frac{3}{S}\,\Delta F';\ \frac{3}{S}\,\Delta F''$$

The tables 2, 3, and 4 contain the values:

$$F,\ F',\ F'',\ \kappa,\ \frac{F'}{\kappa};\ \ \frac{\varphi F' - F}{\kappa};\ \varphi F' - F\,\frac{F'^2}{\kappa'},\ \theta$$

from which the velocity profiles, velocity heads, densities, and temperatures may be computed for the following values of the compressibility factor:

$$S = -\,0.074; \qquad S = 0.0605; \qquad S = 0.1115;$$

the given values of S corresponding to the jet temperatures of -60°, 60°, and 120° C, respectively, above the surrounding temperature.

5. Fundamental Properties of Boundary Layer of Nonisothermal Jet

(a) Geometry of the Jet

The nonisothermal jet, that is, having a temperature other than the surrounding space has, as explained in the foregoing, the interesting property that its outer boundary ($u = 0$) remains constant for variation within very wide limits of the temperature increment ($\Delta t = \pm\,150^\circ$ C):

$$\varphi_2 = \text{constant} = -\,2.04 \tag{69}$$

The inner boundary ($u = u_0$) expands somewhat when the jet is cooled and compresses when heated:

$$\varphi_1 = 0.981 - 0.34\,S \qquad (70)$$

where

$S = \dfrac{\ln(1+t)}{\varphi_1 - \varphi_2}$ compressibility factor of the jet

$t = \Delta t_0/T_{ata}$ ratio of the temperature increment (Δt in °C) to absolute temperature ($T_{ata} = 273 + t_{ata}$ °C) of the surrounding fluid.

According to formula (70) the following results are obtained: $\Delta t = -60°$; $\varphi_1 = 1.005$; $\Delta t = +60°$; $\varphi_1 = 0.960$; $\Delta t = +120°$; $\varphi_1 = 0.942$; $\Delta t = 0° - \varphi_1 = 0.981$.

The nondimensional width of the jet boundary layer depends on the temperature difference above the surrounding temperature in the following manner:

$$\bar{b} = \frac{b}{ax} = 3.02 - 0.34\,S \qquad (71)$$

The boundary that separates the initial mass of the jet from the entrained mass is determined, as is known, from the condition that at the partition surface (φ_3) the stream function $\psi_3 = 0$, or, what amounts to the same thing, $F(\varphi_3) = 0$. The relationship between the boundary of the core of constant mass flow of the jet and the compressibility factor was obtained as follows:

In the case of incompressible fluid it is

$$F_0(\varphi_{30}) = 0$$

For a compressible fluid

$$F = F_0 + \Delta F, \quad \varphi_3 = \varphi_{30} + \Delta\varphi_3$$

then

$$F(\varphi_3) = F_0(\varphi_3) + \Delta F(\varphi_3) = F_0(\varphi_{30} + \Delta\varphi_3) + \Delta F(\varphi_{30} + \Delta\varphi_3) \cong F_0(\varphi_{30})$$
$$+ \Delta\varphi_3 F_0'(\varphi_{30}) + \Delta F(\varphi_{30}) + \Delta\varphi_3 \Delta F'(\varphi_{30}) = 0$$

Omission of the small magnitudes of the second order ($\Delta\varphi$, $\Delta F'$) while bearing in mind that

$$F_0(\varphi_{30}) = 0$$

gives

$$\Delta F(\varphi_{30}) + \Delta\varphi_3 F_0'(\varphi_{30}) = 0$$

hence

$$\Delta\varphi_3 = -\frac{\Delta F(\varphi_{30})}{F_0'(\varphi_{30})} \cong \frac{\frac{S}{3}\ 0.52}{0.59} \cong 0.29\ S$$

The result is the formula for the boundary of the constant mass flow core:

$$\varphi_3 = -\ 0.185 + 0.29\ S \tag{72}$$

The same result is obtained by direct interpolation from tables 2, 3, and 4.

(b) Velocity, Temperature, and Density Profiles

In the representation of the velocity profiles the magnitude

$$\bar{\varphi} = \frac{\varphi - \varphi_2}{\varphi_1 - \varphi_2}$$

serves as ordinate, the effects of heating and cooling of the jet can be compared from the plotted velocity distribution curves. The curves shown in figure 3 are for $\Delta t = +\ 60^\circ$ C, -60° C, and $+\ 120^\circ$ C.

On figures 4 and 5 are shown the density and temperature fields in the jet boundary layer for $\Delta t = -\ 60^\circ$ and $+\ 60^\circ$ C.

$$\frac{\rho}{\rho_0} = \kappa(\bar{\varphi}),\ \frac{\Delta t}{\Delta t_0} = \Theta(\bar{\varphi})$$

Figure 6 gives the field of velocity heads (F'^2/k) for $\Delta t = -60^\circ$ C:

$$\frac{\rho u^2}{\rho_o u_o^2} = \frac{F'^2}{\kappa}(\overline{\varphi})$$

Figure 7 finally gives the fields of transverse velocity fields for $\Delta t = -60^\circ$ C:

$$\frac{V}{V_o} = \frac{\varphi F' - F}{\kappa}\ (\overline{\varphi})$$

The corresponding fields obtained by Tollmien for the incompressible fluid ($\Delta t_o = 0$) are shown in figures 3 to 7 for comparison. (See tables II, III, and IV.)

(c) Rate of Mass Flow

The quantity discharged per second in the turbulent compressible jet does not entirely correspond to the values of the stream function on account of the fluctuations in the density. The mass flow per second is

$$m = \int \overline{\rho\ u}dy = \int \rho\ udy + \int \overline{\rho' u'}dy = \psi + \int \overline{\rho' u'}dy; \qquad (73)$$

But the stream function is determined by the expression

$$\psi = ax\rho_o u_o F$$

while

$$\overline{\rho' u'} = l^2 \frac{\partial\rho}{\partial y}\frac{\partial u}{\partial y} = \alpha\rho_o u_o \frac{\partial\kappa}{\partial\varphi}\frac{\partial\left(\frac{F'}{\kappa}\right)}{\partial\varphi}$$

so that the mass flow per second is

$$m = ax\rho_o u_o \left[F + a\int \kappa' \frac{\partial\left(\frac{F'}{\kappa}\right)}{\partial\varphi}d\varphi + \text{constant}\right] \qquad (74)$$

or in nondimensional form:

$$\bar{m} = \frac{m}{ax\rho_o u_o} = F + a\int \kappa' \frac{\partial\left(\frac{F'}{\kappa}\right)}{\partial\varphi} d\varphi + \text{constant} \qquad (75)$$

In particular the nondimensional value of the entrained mass of fluid sucked into the free jet from the surrounding space is*

$$\bar{m}_2 = -\left[F_2 + a\int_3^2 \kappa' \frac{\partial\left(\frac{F'}{\kappa}\right)}{\partial\varphi} d\varphi\right] \qquad (76)$$

The nondimensional magnitude of the retarded mass of fluid in the initial jet core is

$$\bar{m}_1 = F_1 + a\int_1^3 \kappa' \frac{\partial\left(\frac{F'}{\kappa}\right)}{\partial\varphi} d\varphi \qquad (77)$$

The values of F_2 and F_1 may be obtained from tables of integrals and by approximate integration (for example, by Simpson's method) the values of the integrals:

$$A_2 = \int_3^2 \kappa' \frac{\partial\left(\frac{F'}{\kappa}\right)}{\partial\varphi} d\varphi; \quad A_1 = \int_1^3 \kappa' \frac{\partial\left(\frac{F'}{\kappa}\right)}{\partial\varphi} d\varphi \qquad (78)$$

for various values of the temperature of the flow. Furthermore, assuming a few values of the turbulence factor a, for example, taking $a = 0.0845$, according to the tests of Tollmien, it is not difficult to determine the values of the mass of the initial jet and of the entrained mass for various values of the temperature.

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*The subscript 1 hereinafter refers to the boundary of the region of constant velocities; subscript 2 to the region of fluid at rest; subscript 3 to the core of constant mass flow.

The integral in expressions (76) and (77) may be integrated by parts:

$$\int_3^2 \kappa' \frac{\partial\left(\frac{F'}{\kappa}\right)}{\partial\varphi} d\varphi = \frac{\kappa'}{\kappa} F' \Big|_3^2 - \int_3^2 \frac{\kappa'}{\kappa} F' d\varphi$$

From the boundary conditions it is known that

$$F'_2 = 0$$

Moreover, according to equations (41) and (46a):

$$\kappa = (1+t)^{\frac{\varphi_1 - \varphi}{\varphi_1 - \varphi_2}}; \qquad S = \frac{\ln(1+t)}{\varphi_1 - \varphi_2}$$

hence

$$\kappa = e^{(\varphi_1 - \varphi) S}; \quad \kappa' = - S e^{(\varphi_1 - \varphi) S}; \kappa'' = S^2 e^{(\varphi_1 - \varphi) S} \tag{79}$$

Omission, as in all parentheses of the preceding section, of the terms with factor S^2 leaves

$$\int_3^2 \kappa' \frac{\partial\left(\frac{F'}{\kappa}\right)}{\partial\varphi} d\varphi = - \frac{\kappa'}{\kappa} F' \Big|_3^2 = - SF'_3 \tag{80}$$

whence the expression for the entrained mass:

$$m_2 = - \left[F(\varphi_2) - aSF'(\varphi_3)\right] \tag{81}$$

A glance at tables II, III, and IV corresponding to particular cases of the nonisothermal jet:

$$S = - 0.074 \; (\Delta t_o = - 60^\circ \text{ C})$$
$$S = 0.0605 \; (\Delta t_o = + 60^\circ \text{ C})$$
$$S = 0.1115 \; (\Delta t_o = + 120^\circ \text{ C})$$

shows the values of $F'(\varphi_3)$ corresponding to the particular values of the compressibility factor S on the assumption of a = 0.0845 for the coefficient of jet structure (turbulence) - according to the data of Tollmien and CAHI Therefore

$$a\,S\,F'(\varphi_3) \cong 0.05\,S \tag{82}$$

The values of $F(\varphi_2)$ for $\varphi_2 = -2.04$ is computed according to formulas (67) and (67b):

$$F(\varphi_2) = -(0.388 - 0.27\,S) \tag{83}$$

Lastly equations (82) and (83) added together give the final expression for the nondimensional value of the entrained mass of the jet:

$$\overline{m}_2 = 0.388 + 0.22\,S \tag{84}$$

The initial mass of the jet is

$$\frac{m_1}{ax\rho_0 u_0} = \overline{m}_1 = F_1 + a\int_1^3 \varkappa' \cdot \frac{\partial\left(\frac{F'}{\varkappa}\right)}{\partial\varphi} \cdot d\varphi =$$

$$= F' + a \cdot \frac{\varkappa'}{\varkappa} \cdot F' \Big|_1^3 - a\int_1^3 \frac{\varkappa''}{\varkappa} \cdot F' \cdot d\varphi \tag{85}$$

According to (79)

$$\frac{\varkappa''}{\varkappa} = S^2 \simeq 0; \qquad \frac{\varkappa'}{\varkappa} = -S,$$

hence

$$\overline{m}_1 = F_1 + aS\,[F'_1 - F'_3] \simeq F_1 - 0{,}034\,S. \tag{86a}$$

Further, equations (67) and (67c) give

$$F_1 = 0{,}981 + \Delta F(\varphi_1) \simeq 0{,}98 + 0{,}02\,S. \tag{86b}$$

Substitution of (86b) in (86a) yields the final expression for the nondimensional mass of the initial part of the jet:

$$\overline{m}_1 = 0{,}981 - 0{,}014\,S. \tag{87}$$

The total (nondimensional) mass flow in the boundary layer is equal to

$$\overline{m}_1 + \overline{m}_2 = 1{,}369 + 0{,}2\,S. \qquad (88)$$

The smallness of the value of S indicates that the mass flow per second for the nonisothermal boundary layer differs very little from that for the incompressible jet.

(d) Frictional Stress

In section I an expression was derived for the frictional stresses in the turbulent boundary layer of a compressible gas, which according to equation (12a) can be written

$$\tau_{xy} = 2\,c^2 x^2 \int \frac{\partial u}{\partial y}\,\frac{\partial}{\partial y}\left(\rho\,\frac{\partial u}{\partial y}\right) dy + \text{const}, \qquad (89)$$

But

$$u = u_0\,\frac{F'}{\varkappa};\quad \rho = \rho_0 \varkappa;\quad 2\,c^2 = a^3;\quad \frac{y}{ax} = \varphi,$$

hence

$$\tau_{xy} = a\rho_0 u_0^2 \int \frac{\partial\left(\frac{F'}{\varkappa}\right)}{\partial\varphi}\cdot\frac{\partial}{\partial\varphi}\left[\varkappa\cdot\frac{\partial\left(\frac{F'}{\varkappa}\right)}{\partial\varphi}\right] d\varphi + \text{const}.$$

The nondimensional value of the frictional stress is

$$\overline{\tau} = \frac{\tau_{xy}}{a\rho_0\,\frac{u_0^2}{2}} = 2\int \frac{\partial\left(\frac{F'}{\varkappa}\right)}{\partial\varphi}\cdot\frac{\partial}{\partial\varphi}\left[\varkappa\cdot\frac{\partial\left(\frac{F'}{\varkappa}\right)}{\partial\varphi}\right] d\varphi + \text{const}. \qquad (90)$$

After corresponding transformations partial integration yields

$$\overline{\tau} = \varkappa\cdot\left[\frac{\partial\left(\frac{F'}{\varkappa}\right)}{\partial\varphi}\right]^2 + \int \varkappa'\cdot\left[\frac{\partial\left(\frac{F'}{\varkappa}\right)}{\partial\varphi}\right]^2 d\varphi + \text{const}. \qquad (91)$$

The values of $\overline{\tau}$ of greatest interest is that at the boundary of the core of constant flow $\overline{\tau} = \overline{\tau}_3$ since it determined the energy loss due to suction of the entrained mass. This value of $\overline{\tau}$ is evidently given by

$$\overline{\tau}_3 = \varkappa\cdot\left[\frac{\partial\left(\frac{F'}{\varkappa}\right)}{\partial\varphi}\right]_3^2 + \int_3^1 \varkappa'\cdot\left[\frac{\partial\left(\frac{F'}{\varkappa}\right)}{\partial\varphi}\right]^2\cdot d\varphi. \qquad (92)$$

At the inner limit of the boundary layer

$$\frac{\partial\left(\frac{F'}{\varkappa}\right)}{\partial\varphi}=0,$$

hence

$$\bar{\tau}_2=-\varkappa_3\left[\frac{\partial\left(\frac{F'_3}{\varkappa_3}\right)}{\partial\varphi}\right]+\int_3^{\varphi}\varkappa'\cdot\left[\frac{\partial\left(\frac{F'}{\varkappa}\right)}{\partial\varphi}\right]^2\cdot d\varphi. \tag{93}$$

In addition

$$\frac{\partial\left(\frac{F'}{\varkappa}\right)}{\partial\varphi}=\frac{\varkappa F''-F'\varkappa'}{\varkappa^2},$$

where according to (79)

$$\varkappa=e^{S(\varphi_1-\varphi)}; \qquad \varkappa'=-S\cdot e^{S(\varphi_1-\varphi)},$$

whence

$$\frac{\partial\left(\frac{F'}{\varkappa}\right)}{\partial\varphi}=\frac{F''-SF'}{e^{S(\varphi_1-\varphi)}}.$$

Since the values of S are small

$$e^{S(\varphi_1-\varphi)}=1+S(\varphi_1-\varphi).$$

it results in

$$\frac{\partial\left(\frac{F'}{\varkappa}\right)}{\partial\varphi}=F''+S(\varphi-\varphi_1)F''+SF',$$

$$\left[\frac{\partial\left(\frac{F'}{\varkappa}\right)}{\partial\varphi}\right]^2=F''^2+2S(\varphi-\varphi_1)F''^2+2SF'F'',$$

$$=1-S(\varphi-\varphi_1)$$

$$\varkappa\left[\frac{\partial\left(\frac{F'}{\varkappa}\right)}{\partial\varphi}\right]^2=F''^2+S(\varphi-\varphi_1)F''^2+2SF'F'';$$

$$\varkappa'\cong S$$

$$\varkappa'\left[\frac{\partial\left(\frac{F'}{\varkappa}\right)}{\partial\varphi}\right]^2=SF''^2\sim SF_0''^2;$$

$$\int_3^1\varkappa'\left[\frac{\partial\left(\frac{F'}{\varkappa}\right)}{\partial\varphi}\right]^2 d\varphi=S\int_3^1 F_0''^2 d\varphi.$$

With the aid of the approximate integration method and table I the integral becomes

$$\int_3^1 F_0''^2 d\varphi = 0,4$$

within the limits $\varphi_3 = -0.185$ to $\varphi_1 = 0.981$ (the given values of φ_3 and φ_1 correspond to the incompressible jet). Moreover, at the point φ_3 (for various values of S):

$$F''_3 \cong 0,52, \qquad F'_3 \cong 0,6.$$

whence

$$\bar{\tau}_3 = 0,27 - S\,0,32 + 0,62\,S + 0,4\,S.$$

The final expression for the nondimensional value of the frictional stress at the boundary between the initial and entrained masses of the jet

$$\bar{\tau}_3 = \frac{\tau_3}{a\rho_0 \frac{u_0^2}{2}} = 0,27 + 0,7\,S. \tag{94}$$

(e) Heat Transfer

In section VI (par. 1) the differential equation was obtained for the transverse heat transfer due to the turbulent fluctuations in the free jet of the compressible gas:

$$\frac{1}{C_p \cdot g} \cdot \frac{\partial W_T}{\partial y} = 2c^2 x^2 \left[\frac{\partial \rho}{\partial y} \cdot \frac{\partial u}{\partial y} \cdot \frac{\partial T}{\partial y} + \frac{\partial}{\partial y} \left(\rho \cdot \frac{\partial u}{\partial y} \frac{\partial T}{\partial y} \right) \right] \tag{95}$$

The heat transfer from the initial mass to the entrained mass across 1 m² of partition surface φ_3 is studied next:

$$W_T = C_p \cdot g \cdot 2 \cdot c^2 x^2 \left[\int_1^3 \frac{\partial \rho}{\partial y} \cdot \frac{\partial u}{\partial y} \cdot \frac{\partial T}{\partial y} \cdot dy + \rho \cdot \frac{\partial u}{\partial y} \cdot \frac{\partial T}{\partial y} \Bigg|_1^3 \right]. \tag{95a}$$

From previous results

$$y = ax\varphi; \quad 2c^2 = a^3; \quad \rho = \rho \cdot e^{(\varphi_1 - \varphi)S};$$

$$\frac{\partial \rho}{\partial \varphi} = -\rho_0 \cdot e^{(\varphi_1 - \varphi)S}; \quad \frac{\partial T}{\partial \varphi} = \Delta T_0 \frac{1+t}{t} \cdot S \cdot e^{(\varphi - \varphi_1)S};$$

$$\frac{\partial u}{\partial \varphi} = u_0 \cdot e^{(\varphi - \varphi_1)S} \, [F'' + SF'],$$

whence

$$W_T = - C_p\ g\ \rho_o u_o \Delta T_o a\ \frac{1+t}{t} S\left[S\int_1^3 (F''+SF')\ e^{(\varphi-\varphi_1)S} d\varphi - (F'' + SF')e^{(\varphi-\varphi_1)S}\Big|_1^3\right] \qquad (96)$$

Use is also made of the known relation

$$\frac{1 + t}{t} = \frac{e^{(\varphi_1-\varphi_2)S}}{e^{(\varphi_1-\varphi_2)S}-1} = \frac{1}{(\varphi_1 - \varphi_2)S} + 1$$

After certain simplifications, while neglecting very small terms, equation (96) assumes the form:

$$\frac{W_T}{C_p\ g\ \rho_o\ u_o\ \Delta T_o a} \cong - \frac{SF'}{\varphi_1 - \varphi_2}\Big|_1^3 + \left[\frac{1}{\varphi_1 - \varphi_2} + S\right] F'' + F''\ (\varphi - \varphi_1)S + SF'\Big|_1^3 \qquad (96b)$$

Further transformations give

$$\frac{W_T}{aC_p\ g\ \rho_o\ u_o \Delta T_o} \cong \left[\frac{1}{\varphi_1 - \varphi_2} + S\ \frac{\varphi_3 - \varphi_2}{\varphi_1 - \varphi_2}\right] F''(\varphi_3) \qquad (97)$$

But*

$$F''(\varphi_3) = F_o''(\varphi_3) + \Delta F''(\varphi_3) \cong F_o''(\varphi_{30}) + \Delta\varphi_3 F'''\ (\varphi_{30}) + \Delta F''\ (\varphi_{30}) \cong F_o''\ (\varphi_{30})$$

and, further,

$$\varphi_1 = \varphi_{1o} - 0.34S$$

$$\varphi_3 = - 0.185 + 0.29S$$

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*In region φ_3 the magnitudes F''' and $\Delta F'''$ assume values of the order of zero.

The foregoing relations yield

$$\frac{W_T}{agC_p\rho_o u_o \Delta T_o} = \frac{F_o''(\varphi_{3o})}{\varphi_{1o} - \varphi_{2o}} + 0.34S$$

where

$$F_o''(\varphi_{3o}) = 0.52, \quad \varphi_{1o} = 0.981, \quad \varphi_{2o} = -2.040$$

Thus the nondimensional magnitude characterizing the heat transfer through the boundary of the intial mass of the nonisothermal jet is equal to

$$\overline{W}_T = \frac{W_T}{agC_p\rho_o u_o \Delta T_o} = 0.172 + 0.34S \qquad (99a)$$

The coefficient of heat transfer from the initial mass to the entrained mass equal to the amount of heat transferred per hour through 1 square meter per degree difference in temperature is

$$a_T = \frac{W_T}{\Delta T_o} = 3600\ a\ g\ C_p\ \rho_o\ u_o\ [0.172 + 0.34S] \frac{\text{cal}}{\text{m}^2\ \text{hr}\ {}^\circ\text{C}} \qquad (99b)$$

For a turbulence coefficient $a = 0.0845$ and $g = 9.81$, it yields

$$\alpha_T = 3000\ C_p\ \rho_o\ u_o(0.172 + 0.34\ S) \qquad (100a)$$

For air with a specific heat of the order of

$$C_p = 0.24$$

it is

$$a_T = 720\ \rho_o\ u_o\ (0.172 + 0.34\ S) \frac{\text{cal}}{\text{m}^2\ \text{hr}\ {}^\circ\text{C}} \qquad (100b)$$

The effect of the compressibility on the heat transfer in the nonisothermal jet is appreciable. For example, for a temperature difference $\Delta T_o = 60^\circ$ C ($S \cong 0.07$) the heat transfer in the compressible jet differs from that of the incompressible fluid by about 15 percent.

(e) Concluding Remarks on the Nonisothermal Jet

As is seen by the previous discussion the effect of the compressibility of the fluid (gas) on the fundamental properties of the boundary layer of a nonisothermal flow is insignificant. In particular, on lowering the jet temperature 60° C below that of the surrounding medium (this corresponds to an increase in the velocity to Ba = 1.0 in a jet of high velocities) the angle of divergence of the boundary layer increases by 0.7 percent; the angle of dissolution of the core of constant mass flow increases by 11 percent; the nondimensional value of the entrained mass by 3.7 percent; the mass of the initial jet remains practically constant; the frictional stresses decrease by 16.8 percent; the heat diffusion is reduced by 15 percent.

The results obtained are evidence of the maintenance of the dynamic similarity of the jet for appreciable changes in its temperature.* It has further been shown that if $S = \ln(1 + t)/(\varphi_1 - \varphi_2)$ is taken as the compressibility factor of the nonisothermal jet, where $t = \Delta t_0/T_{a t a}$ is the ratio of the temperature increment of the jet to the absolute temperature of the surrounding medium, the change in the fundamental properties of the jet with S is linear except for the nondimensional coordinate giving the dissolution of the outer boundary of the jet which remains unchanged. It will be shown later that the cooling of the jet at small velocities has the same effect on its fundamental properties (velocity profile and friction) as an increase in the Bairstow number.

IV. BOUNDARY LAYER OF A PLANE-PARALLEL JET AT LARGE VELOCITIES

1. General Considerations

The investigation so far involved the case of a free turbulent jet having a temperature different from that of the surroundings and a velocity small by comparison with the velocity of sound; that is, the effect

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*The similarity of the jet for a wide variation in the Reynolds number was discussed in a previous report (reference 6).

of compressibility arising from the difference in temperature alone was studied. The following deals with jets of large velocities (up to the velocity of sound) under the condition that the temperature in the reservoir from which the jet escapes is equal to that of the surroundings, or otherwise expressed, the effect of compressibility due to high flow velocities will be investigated.

2. Derivation of the Density Function

The air temperature ahead of the nozzle (in the region of small velocities) is equal to the temperature of the surrounding medium. In this case there will be no heat transfer between the jet and the surrounding space so that the heat content of the air will be uniquely associated with the flow velocities (the energies of the pulsating and transverse motions are neglected):

$$c_p(T - T_o) = \frac{A}{2g}(u_o^2 - u^2) \qquad (101)$$

where

c_p specific heat at constant pressure

$A = \frac{1}{427}$ heat equivalent of mechanical work

g acceleration of gravity

The relation (101) expressed in nondimensional form gives

$$\frac{T}{T_o} = 1 + \frac{Au_o^2}{2gc_pT_o}\left[1 - \left(\frac{u}{u_o}\right)^2\right] \qquad (101a)$$

The velocity of sound in the region of undisturbed flow reads the value

$$c_o = \sqrt{\frac{c_p}{c_v} g R T_o} \qquad (101b)$$

where

c_v specific heat at constant volume

$c_p/c_v = k$ adiabatic coefficient

R gas constant

The Bairstow number in the undisturbed region is

$$Ba_o = u_o/c_o \quad \text{and} \quad AR = c_p - c_v$$

hence

$$\frac{T}{T_o} = 1 + \frac{k-1}{2} Ba_o^2 \left[1 - \left(\frac{u}{u_o}\right)^2\right] \tag{102}$$

The foregoing expression gives the relation between the temperature of the flow and the velocity. Particularly in the outer boundary of the flow (region of air at rest) where $u = 0$:

$$\frac{T_{ata}}{T_o} = 1 + \frac{k-1}{2} Ba_o^2$$

At the inner boundary of the boundary layer (undisturbed flow with velocity u_o):

$$\frac{T}{T_o} = 1$$

A thermometer, however, mounted at any stationary point of the flow will show the same temperature T_o – the temperature of the air at rest – since the velocity of the flow drops to zero directly at the wall of the thermometer. Thus a stationary thermometer in the flow shows not the actual temperature of the flow but the temperature of the retarded air, the stagnation temperature.

The above flow considered from the point of view of the stagnation temperature is isothermal as a result of which it was possible to assume that no heat transfer exists and to apply the heat equation in form (101).

The foregoing problems of free turbulence involve isobars (the pressure gradients are negligible and not taken into account); hence the density is inversely proportional to the absolute temperature:

$$\frac{\rho_0}{\rho} = \frac{T}{T_0} = 1 + \frac{k-1}{2} Ba_0^2 \left[1 - \left(\frac{u}{u_0}\right)^2\right] \tag{102a}$$

Therefore the density function is:

$$\kappa(\varphi) = \frac{1}{1 + \frac{k-1}{2} Ba_0^2 \left[1 - \left(\frac{F'}{\kappa(\varphi)}\right)^2\right]} \tag{102b}$$

The solution of the above equation for κ gives the density function in the form

$$\kappa = \frac{1 + \sqrt{1 + 2(k-1) Ba_0^2 \left(1 + \frac{k-1}{2} Ba_0^2\right) F'^2}}{2\left(1 + \frac{k-1}{2} Ba_0^2\right)} \tag{103}$$

and the introduction of the special function of the Bairstow number:

$$r = 2(k-1)\ Ba_0^2 \left[1 + \frac{k-1}{2} Ba_0^2\right] \tag{103a}$$

whence the calculation of the derivative of the density function affords

$$\frac{\kappa'}{\kappa} = \frac{rF'F''}{\sqrt{1 + rF'^2}\left[1 + \sqrt{1 + rF'^2}\right]} \tag{104}$$

3. Derivation of the Fundamental Differential Equation

With equation (104) the general differential equation (20) is reduced to the following special form which satisfies the problem in question.

$$F''' = -F + \frac{\partial}{\partial\varphi}\left[\frac{rF'^2F''}{\sqrt{1 + rF'^2}\,[1 + \sqrt{1 + rF'^2}]}\right] \qquad (105)$$

whence

$$F''' = -F + \frac{\partial}{\partial\varphi}\left[F'' - \frac{F''}{\sqrt{1 + rF'^2}}\right]$$

and

$$-F = \frac{\partial}{\partial\varphi}\left[\frac{F''}{\sqrt{1 + rF'^2}}\right] \qquad (105a)$$

Further transformations yield a differential equation of the form

$$F''' = -F\sqrt{1 + rF'^2} + \frac{rF'F''^2}{1 + rF'^2} \qquad (106)$$

In the special case of incompressible flow when $r = 0$ (equation (106)) reduces, as expected, to the known Tollmien equation:

$$F''' = -F \qquad (106a)$$

Differential equation (106) as well as the Tollmien equation (106a) contains the five boundary conditions (46_{1-5}) which are applied to obtain the three constants of integration and the values of the nondimensional coordinates of the outer and inner limits φ_1 and φ_2 of the boundary layer. To each value of the compressibility parameter (r) there corresponds the values of the integration constants and nondimensional coordinates.

Since the functional solution of equation (106) is impossible, it is necessary to apply the method of numerical integration to each particular value of the compressibility parameter (r) or, what amounts to the same thing, the Bairstow number (Ba). The most suitable method for solving the given equation appears to be the Adams method which gives good agreement for the given case.

4. Numerical Integration of Differential Equations by the Adams Method

If the values of a function and its derivatives at a certain point are known, the values at a neighboring point (a + h) can be obtained with the aid of the Taylor series. After the values of the functions and its derivatives at the point (a + h) are determined, the values at other neighboring points (a + 2h), and so forth, can then be found. In general if $Y_k = Y(a + kh)$ is known, then

$$Y_{k+1} = Y_k + hY'_k + \frac{h^2}{2!} Y''_k + \frac{h^3}{3!} Y'''_k + \ldots$$

Thus, passing successively from point to point, it is possible to compute a table of values of the required integral, that is, of the required function $Y(x)$ over the entire integration range. The smaller the size of the interval h the more accurately are the values Y_k determined although, on account of the large number of intervals, the accumulation of errors may become considerable unless a sufficient number of terms of the Taylor series is employed. The fundamental disadvantage of this method, which was proposed by Euler, is that it is necessary to compute the higher derivatives which, for arbitrary form of the function Y may yield very complicated expressions. In such cases a considerably less complicated method is that by Adams in which the increment in function at a certain interval is expressed by the first differences in the so-called linear increments of the function, the second differences, in the neighboring intervals, and so forth. The Adams method does not require the higher derivatives and gives good accuracy even for large integration ranges. Particularly, as will be shown later, in solving the differential equation of motion for the compressible gas jet (equation (106)) - the Adams method gives very good agreement.

The linear increment in a certain interval (a, a + h) is given by the product

$$\eta_n = h\, Y'_n\, (a)$$

The first difference of the linear increments of the function in neighboring intervals is given by

$$\Delta\eta_{n-1} = \eta_n - \eta_{n-1}$$

The second difference by

$$\Delta^2\eta_{n-2} = \Delta\eta_{n-1} - \Delta\eta_{n-2}$$

the third difference by

$$\Delta^3\eta_{n-3} = \Delta^2\eta_{n-2} - \Delta^2\eta_{n-3}$$

and so forth.

In general if there is a table of values of the required function at $n + 1$ points it is possible to draw up a table of linear increments at $n + 1$ points, first differences at n points, second differences at $(n - 1)$ points, third differences at $(n - 2)$ points, and so forth. The Adams method makes it possible to compute the value of the function at the $(n + 2)$th point, the value of the linear increment at the $(n + 1)$th point, the value of the first difference at the nth point, the value of the second difference at the $(n - 1)$th point, and so forth. By the same method it is possible to proceed to the $(n + 3)$th point, and so on up to the end of the entire integration range.

The extension of the integration table from one interval to the next by the Adams method is effected on the basis of the following considerations:

1. The Taylor series affords the increment in the function:

$$\Delta Y_n = Y_{n+1} - Y_n = hY'_n + \frac{h^2}{2} Y'_n + \frac{h^3}{6} Y'''_n + \ldots$$

2. According to the definition of the linear increments

$$\eta_{n-1} = h\, Y'_{n-1},\ \eta_{n-2} = \eta\, Y'_{n-2}$$

3. Application of the Taylor series results in the values of the derivatives in the foregoing intervals:

$$Y'_{n-1} = Y'_n - h\,Y''_n + \frac{h^2}{2}\,Y'''_n - \frac{h^3}{6}\,Y_n^{IV} + \frac{h^4}{24}\,Y^V - \ldots$$

$$Y'_{n-2} = Y'_n - 2hY''_n + \frac{4}{2}\,h^2\,Y'''_n - \frac{8}{6}\,h^3Y_n^{IV} + \frac{16}{24}\,h^4Y^V - \ldots$$

$$Y'_{n-3} = Y'_n - 3h\,Y''_n + \frac{9}{2}\,h^2Y'''_n - \frac{27}{6}\,h^3Y_n^{IV} + \frac{81}{24}\,h^4Y^V - \ldots$$

4. Substitution of the previous values in the expressions for the linear increments gives

$$\eta_{n-1} = h\,Y'_n - h^2Y''_n + \frac{h^3}{2}\,Y'''_n - \frac{h^4}{6}\,Y_n^{IV} + \frac{h^5}{24}\,Y_n^V - \ldots$$

$$\eta_{n-2} = h\,Y'_n - 2h^2\,Y''_n + \frac{4}{2}\,h^3Y'''_n - \frac{8}{6}\,h^4Y_n^{IV} + \frac{16}{24}\,h^5Y_n^V - \ldots$$

$$\eta_{n-3} = h\,Y'_n - 3h^2\,Y''_n + \frac{9}{2}\,h^3Y'''_n - \frac{27}{6}h^4\,Y_n^{IV} + \frac{81}{24}\,h^5Y_n^V$$

5. By definition of the first and higher differences

$$\Delta\eta_{n-1} = \eta_n - \eta_{n-1}$$

$$\Delta^2\eta_{n-2} = \Delta\eta_{n-1} - \Delta\eta_{n-2} = \eta_n - 2\eta_{n-1} + \eta_{n-2}$$

$$\Delta^3\eta_{-3} = \Delta^2\eta_{n-2} - \Delta^2\eta_{n-3} = \eta_{n-1} + 3\eta_{n-2} - \eta_{n-3}$$

6. Substitution of the expressions obtained for the linear increments in the differences yields

$$\Delta\eta_{n-1} = h^2 Y''_n - \frac{h^3}{2} Y'''_n + \frac{h^4}{6} Y^{IV}_n - \frac{h^5}{24} Y^V_n,$$

$$\Delta^2\eta_{n-2} = h^3 Y'''_n - h^4 Y^{IV}_n + \frac{14}{24} h^5 Y^V_n$$

$$\Delta^2\eta_{n-3} = h^4 Y^{IV}_n - \frac{36}{24} h^5 Y^V_n$$

7. Next it is shown that the true increment in the function may be expressed in terms of its linear increments and differences:

$$\Delta Y_n = \eta_n + \alpha \Delta \eta_{n-1} + \beta \Delta^2\eta_{n-2} + \gamma \Delta^3\eta_{n-3} \tag{107}$$

For this purpose the coefficients α, β, and γ are computed by comparison of the Adams series (107) with the Taylor series:

$$\Delta Y_n = \eta_n + \alpha \left[h^2 Y''_n - \frac{h^3}{2} Y'''_n + \frac{h^4}{6} Y^{IV}_n - \frac{h^5}{24} Y^V_n\right] +$$

$$+ \beta \left[h^3 h'''_n - h^4 Y^{IV}_n + \frac{14}{24} h^5 Y^V_n\right] + \gamma \left[h^4 Y^{IV}_n - \frac{36}{24} h^5 Y^V_n\right]$$

$$\Delta Y_n = h Y'_n + \frac{h^2}{2} Y''_n + \frac{h^3}{6} Y'''_n + \frac{h^4}{24} Y^{IV}_n + \frac{h^5}{120} Y^V_n$$

This may be done by equating the coefficients in the two series of terms of the same degree in h, which then yields the following system of equations:

$$\alpha = \frac{1}{2}$$

$$-\frac{\alpha}{2} + \beta = \frac{1}{6}$$

$$\frac{\alpha}{6} - \beta + \gamma = \frac{1}{24}$$

$$-\frac{\alpha}{24} + \frac{14}{24}\beta - \frac{36}{24}\gamma = \frac{1}{120}$$

To determine the coefficients the first three equations are employed; then

$$\alpha = \frac{1}{2}; \ \beta = \frac{5}{12}; \ \gamma = \frac{3}{8}$$

From the fourth equation the order of the computation error may be ascertained. In the given case the error will be of the order

$$\epsilon = \left[\frac{1}{120} - \left(-\frac{\alpha}{24} + \frac{14}{24}\beta - \frac{36}{24}\gamma\right)\right] h^5 \ Y^V{}_n = \frac{49}{144} h^5 \ Y^V{}_n$$

By taking, in order to increase the integration accuracy, a term with fourth order difference $\delta\Delta^4\eta_{n-4}$, the Adams equation, after the corresponding computations are carried out, takes the following form:

$$\Delta\eta_n = \eta_n + \frac{1}{2}\Delta\eta_{n-1} + \frac{5}{12}\Delta^2\eta_{n-2} + \frac{3}{8}\Delta^3\eta_{n-3} + \frac{251}{720}\Delta^4\eta_{n-4} \tag{108}$$

With the aid of the Adams series in the previous form the increment in the function (increment of the required integral) from point to point can be accurately computed up to the terms of the fifth order. Trial computation tables have shown that the Adams series in form (108) is sufficiently accurate for integrating the differential equation of motion in the boundary layer of the free compressible jet.

It is seen, from series (108), that before starting the computations it is necessary to have prepared a table of values of the function, its first derivative, linear increment of the function and first, second, third, and fourth differences, respectively, at the fifth, fourth, third, second, and first points. In other words, in order to start the numerical integration of differential equation (106) by the Adams method, with the aid of series (108), it is first necessary to compute the integral for the above five points. This initial computation is then carried out with the aid of the Taylor series on the basis of the fact that at the initial point the values of the function and its derivatives are known from the boundary conditions.

5. Integration of the Differential Equation of the Boundary Layer of the Compressible Jet

The differential equation to be integrated is

$$F''' = -F\sqrt{1 + rF'^2} + \frac{rF'F''^2}{1+rF'^2} \tag{106}$$

with the boundary conditions

$$F(\varphi_1) = \varphi_1; \quad F'(\varphi_1) = 1; \quad F''(\varphi_1) = 0;$$

$$F'(\varphi_2) = 0; \quad F''(\varphi_2) = 0$$

As has been shown in the foregoing the first five rows for the numerical integration table are to be computed by Taylor series. The computation is made in sequence starting with point φ_1 (the inner limit of the boundary layer). Since the argument φ decreases from φ_1 to φ_2, the integration interval should be taken with negative sign. Let:

$$\Delta\varphi = h = -0.05$$

With subscript o denoting the values of the function at F and its derivatives at F', F'', F''', and F^{IV} at point φ_1 and with subscript n the corresponding values at the point $\varphi_n - nh$, the Taylor series affords

$$\left.\begin{aligned} F_{n+1} &= F_n - hF'_n + \frac{h^2}{2}F_n'' - \frac{h^3}{6}F_n''' \\ F'_{n+1} &= F_n' - hF_n'' + \frac{h^2}{2}F_n''' - \frac{h^3}{6}F_n^{IV} \\ F''_{n+1} &= F_n'' - hF'''_n + \frac{h^2}{2}F_n^{IV} \end{aligned}\right\} \tag{109$_{1-3}$}$$

Moreover the differential equation gives

$$F'''_{n+1} = -F_{n+1}\sqrt{1 + rF'^2_{n+1}} + \frac{rF'_{n+1}F''^2_{n+1}}{1 + rF'^2_{n+1}} \qquad (109_4)$$

and by further differentiating,

$$F^{IV}_{n+1} = \frac{rF''^3_{n+1} + 2rF'_{n+1}F''_{n+1}F'''_{n+1}}{1 + rF'^2_{n+1}} - \frac{2r^2F'^2_{n+1}F''^3_{n+1}}{(1 + rF'^2_{n+1})^2}$$

$$- \frac{rF_{n+1}F'_{n+1}F''_{n+1}}{\sqrt{1 + rF'^2_{n+1}}} - F'_{n+1}\sqrt{1 + rF'^2_{n+1}} \qquad (109_5)$$

The differentiation is limited to the fourth derivative in order to avoid the very complicated expression for the fifth derivative. Computations show that the accuracy thus obtained is entirely satisfactory.

The procedure of the preliminary computations is as follows. The boundary conditions give:

$$F(\varphi_1) = \varphi_1; \quad F'(\varphi_1) = 1; \quad F''(\varphi_1) = 0$$

With a given value of φ_1 for given value of the parameter r

$$F'''(\varphi_1) \text{ and } F^{IV}(\varphi_1)$$

are computed. Then by means of (107_{1-3}) and (109_{1-5}) the values of F, F', F'', F''', and F^{IV} at a neighboring (second) point $(\varphi_1 - h)$ are obtained by (107) and (109). The same procedure is followed for all five initial points up to $(\varphi_1 - 4h)$. The values of F^{IV} are required only up to point 3. In the remaining rows of the table F^{IV} is not required (for the Adams method). By denoting the linear increment of the function F and its first two derivatives F', F'' by $\eta = hF'_n$; $\zeta_n = hF''_n$; $\mu_n = hF'''_n$ tables are constructed of the linear increments of the functions and their differences for the computed five points. The tables of differences are computed by the formulas given in section 4:

$$\eta_n = h\,F'_n;$$

$$\Delta\,\eta_{n-1} = \eta_n - \eta_{n-1};$$

$$\Delta^2\eta_{n-2} = \eta_n - 2\eta_{n-1} + \eta_{n-2};$$

$$\Delta^3\eta_{n-3} = \eta_n - 3\eta_{n-1} + 3\eta_{n-2} - \eta_{n-3};$$

$$\Delta^4\eta_{n-4} = \eta_n - 4\eta_{n-1} + 6\eta_{n-2} - 4\eta_{n-3} + \eta_{n-4}$$

These formulas are also used for the differences ζ, μ, where, however

$$\zeta_n = hF''_n; \quad \mu_n = hF'''_n$$

from the Adams series furthermore:

$$\Delta\,Y_n = \eta_n + \frac{1}{2}\,\Delta\,\eta_{n-1} + \frac{5}{12}\,\Delta^2\eta_{n-2} + \frac{3}{8}\,\Delta^3\eta_{n-3} + \frac{251}{720}\,\Delta^4\eta_{n-4}$$

follow the increments in the function and its derivatives at the sixth, seventh, and so forth, points:

$$\Delta F_n = \eta_n + \frac{1}{2}\,\Delta\eta_{n-1} + \frac{5}{12}\,\Delta^2\eta_{n-2} + \frac{3}{8}\,\Delta^3\eta_{n-3} + \frac{251}{720}\,\Delta^4\eta_{n-4};$$

$$\Delta F'_n = \zeta_n + \frac{1}{2}\,\Delta\zeta_{n-1} + \frac{5}{12}\,\Delta^2\zeta_{n-2} + \frac{3}{8}\,\Delta^3\zeta_{n-3} + \frac{251}{720}\,\Delta\zeta^4{}_{n-4}$$

$$\Delta F''_n = \mu_n + \frac{1}{2}\,\Delta\mu_{n-1} + \frac{5}{12}\,\Delta^2\mu_{n-2} + \frac{3}{8}\,\Delta^3\mu_{n-3} + \frac{251}{720}\,\Delta^4\mu_{n-4}$$

each time increasing the computation table by one row. The calculation is broken off when for a given initial value of φ_1 there is obtained at the end of the table a point φ_2 satisfying simultaneously the two boundary conditions at the outer limit of the boundary layer:

$$F'(\varphi_2) = 0; \quad F''(\varphi_2) = 0$$

If one of the derivatives becomes zero ahead of the other it indicates that the initial value of φ_1 was not well chosen. In this case it is necessary to recompute the table, assuming a new value of φ_1, and so forth, until a successful result is obtained. The computations carried out show that usually three to four approximations are sufficient to obtain an accurate integral table for a given value of Ba.

In the present paper the computations were carried out for air ($K = 1.41$) first for $Ba = 1$. Subsequently, for the other values of Ba, it was possible to obtain a good result in a smaller number of trials by interpolation between the known results for $Ba = 1$ and $Ba \sim 0$ (the latter corresponds to the solution of Tollmien which gives the extreme values $\varphi_1 = 0.981$ and $\varphi_2 = -2.04$).

By this method of integration the final tables V, VI, and VII were obtained for the values of the function F and its derivatives F', F'', and F''', corresponding to various values of the argument φ. Table V was obtained for $Ba_0 = 1.0$ ($\varphi_1 = 0.923$; $\varphi_2 = -2.04$); table VI for $Ba_0 = 0.9$ ($\varphi_1 = 0.935$; $\varphi_2 = -2.04$); table VII for $Ba_0 = 0.5$ ($\varphi_1 = 0.968$; $\varphi_2 = -2.04$). Moreover, for comparison there is presented table VIII in which the same magnitudes are given for $Ba_0 \sim 0$ (Tollmien solution), that is, for the case of an incompressible fluid.

Tables V to VIII include also a number of auxiliary magnitudes

$$\left(\frac{F'}{K};\ \varphi F' - F;\ \frac{\varphi F' - F}{K};\ \frac{F'^2}{K}\right)$$

by which the longitudinal and transverse velocity components, velocity heads, and so forth, can be computed:

$$\frac{\rho v}{\rho_o v_o} = \varphi F' - F, \quad \frac{u}{u_o} = \frac{F'}{\kappa}$$

$$\frac{v}{v_o} = \frac{\varphi F' - F}{\kappa}, \quad \frac{\rho u^2}{\rho_o u_o^2} = \frac{F'^2}{\kappa}$$

6. Fundamental Properties of the Turbulent Boundary Layer in the Plane-Parallel Flow of a Compressible Gas

The foregoing solution of the differential equation of the boundary layer of a compressible gas jet makes it possible to estimate the effect of the compressibility on a turbulent jet of high velocity

(a) Geometry of the High-Velocity Jet

As is seen from the tables of numerical integration the effect of the compressibility is first of all to decrease the thickness of the boundary layer (region of mixing of the jet with the surrounding fluid) with increasing flow velocity; that is, the coordinate φ_1 of the boundary of the constant velocity core $(u = u_o)$ decreases with increase in Ba. In particular for air $(K = 1.41)$ the following values of φ_1 for various values of Ba are obtained:

$Ba_o = 1$	$\varphi_1 = 0.923$
$Ba_o = 0.9$	$\varphi_1 = 0.934$
$Ba_o = 0.5$	$\varphi_1 = 0.966$
$Ba \sim 0$ (incompressible fluid)	$\varphi_1 = 0.981$

The relation between φ_1 and Ba in terms of the factor of compressibility:

$$r = 2(K - 1)\, Ba_o^2 \left[1 + \frac{K - 1}{2} Ba_o^2\right] \qquad (110)$$

can be approximated by the linear expression

$$\varphi_1 = 0.981 \, [1 - 0.06 \, r] \tag{111}$$

In contrast to φ_1 the coordinate of the outer limit φ_2 $(u = 0)$ of the boundary layer φ_2 remains constant with change in the flow velocity:

$$\varphi_2 = \text{constant} = -2.04$$

In view of the foregoing the nondimensional width of the boundary layer is connected with the compressibility factor by the relation

$$\bar{b} = \frac{b}{a \, x} = 3.02 \, [1 - 0.02 r] \tag{112}$$

A very important geometric characteristic of the jet is the surface separating the initial mass flowing under the plane OA, figure 1, from the associated mass consisting of the particles entrained from the surrounding flow of the gas at rest. This partition surface should also be a flow surface since it is the boundary of the initial stream of constant mass flow per second. The value of the stream function $a\psi_3$ on the surface should be equal to zero:

$$\psi_3 = 0 \tag{113}$$

But according to equation (17b)

$$\psi = ax\rho_0 u_0 F$$

hence it is seen that on the partition surface at the boundary of the constant flow core

$$F(\varphi_3) = 0 \tag{113b}$$

The foregoing expression is thus the fundamental condition for determining the partition surface of the jet. It is seen that the boundary of the core of constant flow is a plane the position of which is determined by the value of the nondimensional coordinate

$$\varphi_3 [F] = \varphi (0) \tag{113c}$$

The integration tables in which the values of $F(\varphi)$ are given afford, by interpolation, the corresponding values:

Ba = 1	$\varphi_3 = - 0.205$
Ba = 0.9	$\varphi_3 = - 0.200$
Ba = 0.5	$\varphi_3 = - 0.192$
Ba → 0 (incompressible fluid)	$\varphi_3 = - 0.185$

It is interesting to note that the relation between φ_3 and r is also linear:

$$\varphi_3 = - 0.185 [1 - 0.11 r] \tag{114}$$

Moreover, it has been shown that in contrast to the boundary layer width which decreases with increase in the flow velocity, the width of the core of constant mass flow increases with increase in velocity. The latter effect is explained by the deformation of the profile of velocity heads in the boundary layer due to the effect of the compressibility of the gas at high flow velocities.

(b) Velocity Profiles

For greater clearness in comparing the velocity fields obtained for various values of Ba it is necessary to choose an absolute (independent of Ba) system of coordinates. Lay off on the axis of ordinates:

$$\bar{\varphi} = \frac{\varphi - \varphi_2}{\varphi_1 - \varphi_2}$$

and on the axis of abscissas

$$F' = \frac{\rho u}{\rho_0 u_0}; \quad \frac{F'}{\kappa} = \frac{u}{u_0}$$

Then for all values of Ba the values of the coordinates on the inner ($\bar{\varphi}_1 = 1$) and outer ($\bar{\varphi}_2 = 0$) limits of the boundary layer are the same, regardless of the fact that the true coordinate of the inner limit φ_1 changes with change in Ba.

Figure 8 gives the curves F', F'/κ for Ba = 1 along with the curve F' corresponding to the incompressible fluid ($\kappa = 1$) for comparison. As may be seen the velocity profile of the boundary layer becomes fuller with increase in the velocity in the undisturbed region (Ba = 1) while the F' profile becomes less full. In general, however, a weak effect of the compressibility on the velocity profile of the jet is observed up to Ba = 1.

Figure 9 shows the same curves for Ba = 0.5. The character of the curves is the same but the effect of the compressibility is now so weak that it may be practically neglected.

Figure 10 gives the density curves in the boundary layer for Ba = 0.5 and Ba = 1.

Figure 11 gives the velocity heads F'^2/κ for Ba = 1 and Ba ~ 0.

Figure 12 shows a comparison of the transverse velocity component distribution:

$$\frac{v}{v_0} = \frac{\varphi F' - F}{\kappa}$$

for air at Ba = 1 and for the incompressible fluid (Tollmien result).

(c) Rate of Flow Discharge

As established in a previous part of this paper the nondimensional rates of gas flow are subject to formula (75) which, applied to the entrained mass, gives

$$\bar{m}_2 = -\left[F_2 + a \int_3^2 \kappa' \frac{\partial\left(\frac{F'}{\kappa}\right)}{\partial\varphi} d\varphi\right] \qquad (115)$$

The nondimensional magnitude of the initial mass in the jet is

$$\bar{m}_1 = F_1 + a \int_1^3 \kappa' \frac{\partial\left(\frac{F'}{\kappa}\right)}{\partial\varphi} d\varphi \tag{116}$$

The integral tables give the values $F_2(Ba_0)$ and $F_1(Ba_0)$ and approximate integration of the values of the integrals

$$A_2 = \int_3^2 \kappa' \frac{\partial\left(\frac{F'}{\kappa}\right)}{\partial\varphi} d\varphi$$

$$A_1 = \int_1^3 \kappa' \frac{\partial\left(\frac{F'}{\kappa}\right)}{\partial\varphi} d\varphi$$

for various values of Ba_0. Further, assuming with Tollmien $a = 0.0845$, it is necessary to determine the magnitudes aA_2, aA_1 and find the values of the entrained and initial masses. (Computation shows that the values of aA_2 and aA_1 are relatively very small.)

It is interesting to note that initial mass, entrained mass, and total mass flow through the boundary in the boundary layer of the plane-parallel flow of the high-velocity jet are linear functions of the compressibility factor:

(a) Initial mass:

$$\bar{m}_1 = 0.981 \, [1 - 0.06r] \tag{117}$$

(b) Entrained mass:

$$\bar{m}_2 = 0.388 \, [1 - 0.10r] \tag{118}$$

(c) Total mass rate of flow:

$$\overline{m}_1 + \overline{m}_2 = 1.369\,[1 - 0.07\ r] \qquad (119)$$

In the particular case of an incompressible gas the conventional Tollmien values are:

$$\overline{m}_1 = 0.981, \quad \overline{m}_2 = 0.388, \quad \overline{m}_1 + \overline{m}_2 = 1.369$$

(d) Frictional Stresses

According to formula (93) the nondimensional frictional stresses at the boundary of the core of constant mass flow is:

$$\overline{\tau}_3 = -\kappa_3 \frac{\partial\left(\frac{F'}{\kappa}\right)_3}{\partial\varphi} + \int_3^1 \kappa' \left[\frac{\partial\left(\frac{F'}{\kappa}\right)}{\partial\varphi}\ d\varphi\right]^2 \qquad (120)$$

With the use of the available integral tables the magnitudes corresponding to the various values of the compressibility factor r are obtained by the trapesoidal method. It is interesting to note that for the stresses also a linear relation obtains:

$$\frac{\tau_3}{a\rho_0 \frac{u_0^2}{2}} = \overline{\overline{\tau}}_3 = 0.27\ [1 - 0.167\ r] \qquad (121)$$

where $0.27 = \overline{\overline{\tau}}_{30}$ is the nondimensional value of the frictional stress in the incompressible fluid.

(e) Conclusions with Regard to the High Velocity Jet

The laws of variation of the fundamental properties of the turbulent boundary layer of a plane-parallel stream of compressible gas (air) at high flow velocities have been obtained in the foregoing. As in the case of the "heated" jet the most interesting result obtained was that the effect of the compressibility on the fundamental properties of the high-velocity jet is negligible. This conclusion is valid up to flow velocities attaining the velocity of sound (Ba = 1).

In particular, on passing from small values of the Bairstow number (Ba ~ 0) to Ba = 1, the angle of divergence of the boundary layer decreases by 2 percent; the angle of dissolution of the core of constant mass flow increases by 11 percent; the nondimensional magnitude of the entrained mass decreases by 10 percent; the nondimensional magnitude of the initial mass decreases by 6 percent; and lastly the nondimensional value of the frictional stress at the boundary of the core of constant mass flow decreases by 16.7 percent. The effect of the compressibility on the properties of the jet is so small that at Ba = 0.5 it may be entirely neglected without impairment of the results, hence that dynamic similarity of the stream prevails over a very wide range of variation of the Reynolds number, the temperature above the surroundings and of Ba. The second result in order of importance is that the variation of the fundamental properties with r is linear where r is assumed as the compressibility factor

$$r = 4 \frac{k - 1}{2} Ba_o^2 \left[1 + \frac{k - 1}{2} Ba_o^2\right] \qquad (122)$$

and only the outer boundary of the jet remains unchanged.

V. POSSIBILITY OF DIRECT APPLICATION OF THE PRANDTL LAW OF TURBULENT FRICTION TO THE CASE OF COMPRESSIBLE JET

In section I of the present investigation there was obtained, for the free jet of compressible gas, the friction law

$$\tau = \int 2 l^2 \frac{\partial u}{\partial y} \frac{\partial}{\partial y} \left[\rho \frac{\partial u}{\partial y}\right] dy \qquad (123)$$

which in those cases where the compressibility may be neglected (ρ = constant) reduces to the known Prandtl law obtained for incompressible gas:

$$\tau = \rho\, l^2 \left(\frac{\partial u}{\partial y}\right)^2 \tag{124}$$

In comparing equations (123) and (124) the question naturally arises as to what extent the accuracy of the compressible flow investigation is improved by introducing the new friction law and whether this does not lead to a considerable complication of the problem. Since the solution of the problem of the boundary layer of a high velocity jet was obtained by the new friction law (123) it remains to be explained whether it is possible without introducing large errors to simplify the given problem by direct application of the Prandtl friction law (124). By the latter the differential equation of motion in the case of free turbulence is written as follows:

$$\rho\, u \frac{\partial u}{\partial x} + \rho\, V \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\rho\, l^2 \left(\frac{\partial u}{\partial y}\right)^2\right] \tag{125}$$

Setting, as previously assumed:

$$F'(\varphi) = \frac{\rho u}{\rho_o u_o};\quad \frac{\rho v}{\rho_o v_o} = a\,(\varphi F' - F);\quad l = cx;$$

$$\varphi = \frac{y}{ax};\quad \sqrt[3]{2c^2} = a;\quad \kappa\,(\varphi) = \frac{\rho}{\rho_o}$$

the differential equation is transformed to

$$-F = \frac{1}{2}\,\frac{\partial\left(\frac{F'}{\kappa}\right)}{\partial \varphi}\,\frac{\partial x}{\partial \varphi} + \kappa\,\frac{\partial^2\left(\frac{F'}{\kappa}\right)}{\partial \varphi^2} \tag{126}$$

The differential equation in final form for boundary layer of a two-dimensional free jet then is:

$$-F = F''' + \frac{3}{2}\frac{\kappa'}{\kappa}\left[\frac{F'\kappa'}{\kappa} - F''\right] - \frac{F'\kappa''}{\kappa} \qquad (127)$$

As may be seen, the previous equation obtained by the application of the Prandtl friction law is considerably more complicated than equation (20) which was obtained on the basis of friction law (123) derived specially for free jets of a compressible gas. In the case of large velocities, as shown in section III, a density distribution is obtained for which

$$\frac{\kappa'}{\kappa} = \frac{rF'\,F''}{\sqrt{1 + rF'^2}\,\left[1 + \sqrt{1 + rF'^2}\right]} \qquad (128a)$$

$$\frac{\kappa''}{\kappa} = \frac{r\left[F''^2 + F'\,F''' + rF'^3\,F'''\right]}{\left[1 + rF'^2\right]^{3/2}\,\left[1 + \sqrt{1 + r\,F'^2}\right]} \qquad (128b)$$

Substitution of the foregoing equations in equation (127) leads to the following form of differential equation of the boundary layer of a free high-velocity jet:

$$F''' = -F\sqrt{1 + rF'^2} + rF'^2\,F''^2\,\frac{1 + \frac{3}{2}\sqrt{1 + rF'^2}}{\left[1 + rF'^2\right]\,\left[1 + \sqrt{1 + rF'^2}\right]} \qquad (129)$$

Comparison of the foregoing equation with the corresponding equation (105) gives

$$F''' = -F\sqrt{1 + r\,F'^2} + \frac{rF'F''^2}{1 + rF'^2}$$

obtained with the improved friction law (123) shows conclusively that the direct extension of the Prandtl friction law to the compressible case not only fails to simplify the study of the latter but on the contrary gives a solution considerably more cumbersome than the one obtained by the more accurate friction law. In view of the fact, however, that for a number of aerodynamic problems of a compressible gas it may be more convenient to use the friction law in the Prandtl form, it was considered useful for solving equation (129) by the Adams method and comparing the results with the solution of equation (106).

The following results were obtained:

(a) The qualitative results of the two solutions of the problem are the same (with increasing velocity the nondimensional values of the frictional stresses decrease, the width of the boundary layer decreases, the velocity field is slightly deformed, and the width of the core of constant mass flow increases.

(b) The quantitative results of the two solutions differ negligibly. Particularly by the use of the Prandtl friction law (equation 129) it is found that with increase in the Bairstow number from $Ba_o \sim 0$ to $Ba_o = 1$ the nondimensional frictional stresses decrease by 15 percent; the boundary layer thickness decreases by 1.3 percent; the divergence angle of the core of constant mass flow increases by 16 percent. Correspondingly with the improved friction law (equation (106)) the nondimensional friction force decreased by 17 percent, the boundary layer thickness decreased by 2 percent, and the divergence angle of the constant mass flow core increased by 9 percent.

The application of the friction law of the incompressible fluid to the compressible flow thus leads to results that differ slightly from the results obtained with the use of the corrected friction law. In the case of the free jet, however, it is of advantage to apply the corrected friction law since it leads to less cumbersome and more readily solvable equations.

GENERAL CONCLUSIONS

In the present paper the theory of free turbulence and the two-dimensional free jet was extended to a compressible fluid. In constructing the theory the turbulence hypothesis of Taylor (vorticity interchange) was used in preference to the Prandtl hypothesis (momentum interchange). This was done because the former was confirmed by test results on the velocity and temperature fields whereas the second, while leading to the same velocity fields as the first, strongly deviated from the experimentally obtained temperature fields.

The differential equation of motion (12) for the two-dimensional free compressible jet leads, in the particular case of an incompressible fluid, to the well-known equation of Prandtl-Tollmien-Schlichting.

The boundary layer considered was that of an infinite plane-parallel flow. An infinite flow of this kind assures constancy of the boundary conditions in the flow direction: (1) at the outer limit of the boundary layer the flow velocity is equal to zero ($u = 0$), (2) at the inner limit the undisturbed flow is of constant velocity $u = u_0$. The constancy of the boundary conditions (of the velocities, temperatures, densities) at the edges of the boundary layer justifies the assumption of aerodynamic similarity (similarity of velocity, temperature fields, etc.) at the various cross-sections of the flow, that is, the existence of absolute distribution laws of temperatures, velocities, densities, frictional stresses, and so forth, as was the case with incompressible fluids. The absolute differential equation of motion (20) obtained for the boundary layer of an infinite stream is solved for the two cases:

(1) Jet whose temperature differs from that of the surroundings and whose velocities are small in comparison with the velocity of sound ($Ba \lesssim 0.5$).

(2) Jet of high velocities (up to $Ba = 1$) the temperature of which is equal to that of the surroundings.

The first case involved the effect of compressibility due to the difference in the temperatures inside and outside the jet; the second case the effect of the compressibility due to the high flow velocities.

The fundamental conclusion derived from the results of the present investigation is that the effect of the compressibility of the fluid on the laws of flow in the free jet is small. It is also of interest that a lowering in the temperature of the moving fluid below that of the surrounding medium produces fundamentally the same effect on the properties of the flow independent of whether the lowering was brought about by a cooling of the jet or by the conversion of the heat into kinetic energy associated with the large flow velocities.

In particular:

(a) The nondimensional stresses obtained in the air jet at moderate velocities and at a temperature below that of the surroundings by 60° C were the same as for the jet with large velocities at $Ba_0 = 1.0$ and have a stagnation temperature equal to that of the surroundings. (At Ba = 1.0 the local temperature was lower than the stagnation temperature by about 60° C.)

(b) The nondimensional velocity fields in both cases practically agree (fig. 13).

(c) The angle of divergence of the core of constant mass flow in both jets increases by the same amount of approximately 11 percent as compared with the incompressible jet.

Special compressibility factors are introduced for the nonisothermal jet (S) and for the jet with high velocities (r) and all fundamental properties of the boundary layer were expressed as linear functions of these factors.

The small effect of the compressibility of the fluid permits the conclusion that the free jet maintains its dynamic similarity over a wide variation of Bairstow number and temperature differences as well as for a wide range of values of the Reynolds number.

The author has not undertaken the investigation of jets escaping from openings of finite diameter and of wakes behind bodies in a compressible fluid. In these cases the construction of a theory is effective only at great distances from the nozzle (or body) where the velocity and temperature difference are relatively small and hence the effect of compressibility practically inappreciable though the study of this effect, involves great difficulty.

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TABLE I.- AUXILIARY FUNCTIONS OF NONISOTHERMAL JET.

φ	F_0	F_0'	F_0''	$\frac{3\,\Delta F}{S}$	$\frac{3\,\Delta F'}{S}$	$\frac{3\,\Delta F''}{S}$
0,98	0,9810	1,0000	0	—0,008	—0,037	0,977
0,90	0,8988	0,9977	0,0746	—0,002	—0,118	0,969
0,80	0,8019	0,9858	0,1600	0,019	—0,213	0,933
0,70	0,7021	0,9667	0,2370	0,044	—0,304	0,878
0,60	0,6080	0,9383	0,3010	0,079	—0,391	0,799
0,50	0,5152	0,9064	0,3580	0,122	—0,466	0,711
0,40	0,4271	0,8675	0,4050	0,173	—0,531	0,609
0,30	0,3430	0,8261	0,4430	0,228	—0,584	0,504
0,20	0,2619	0,7789	0,4730	0,287	—0,629	0,389
0,10	0,1868	0,7316	0,4770	0,365	—0,634	0,316
0	0,1161	0,6799	0,5110	0,420	—0,684	0,16
—0,10	0,0508	0,6297	0,5190	0,487	—0,694	0,053
—0,20	—0,0099	0,5760	0,5200	0,558	—0,693	0,001
—0,30	—0,0647	0,5252	0,5120	0,627	—0,682	—0,090
—0,40	—0,1145	0,4728	0,4980	0,693	—0,662	—0,174
—0,50	—0,1590	0,4235	0,4800	0,757	—0,633	—0,246
—0,60	—0,1995	0,3731	0,4590	0,818	—0,595	—0,313
—0,70	—0,2336	0,3280	0,4340	0,924	—0,505	—0,402
—0,80	—0,2653	0,2833	0,4070	0,924	—0,505	—0,402
—0,90	—0,2912	0,2421	0,3770	0,973	—0,450	—0,428
—1,00	—0,3134	0,2035	0,3450	1,014	—0,401	—0,442
—1,10	—0,3322	0,1664	0,3120	1,050	—0,345	—0,448
—1,20	—0,3468	0,1345	0,2770	1,083	—0,291	—0,437
—1,30	—0,3585	0,1062	0,2370	1,108	—0,238	—0,405
—1,40	—0,3681	0,0805	0,2040	1,132	—0,196	—0,376
—1,50	—0,3750	0,0587	0,1670	1,144	—0,140	—0,329
—1,60	—0,3797	0,0412	0,1300	1,155	—0,102	—0,269
—1,70	—0,3833	0,0270	0,0930	1,161	—0,068	—0,199
—1,80	—0,3853	0,0165	0,0557	1,183	—0,040	—0,117
—1,90	—0,3870	0,0104	0,0177	1,171	—0,024	—0,021
—2,00	—0,3874	0,0078	0,0043	1,178	—0,016	0
—2,04	—0,3880	0	0	1,181	—0,003	0

TABLE II.- JET FUNCTIONS FOR $\Delta t_0 = -60^\circ$C.

φ	F	F'	F''	$\varkappa$	$\frac{F'}{\varkappa}$	$\varphi F' - F$	$\frac{\varphi F' - F}{\varkappa}$	$\frac{F'^2}{\varkappa}$	θ
1,005	1,005	1,0	0	1,0	1,0	0	0	1,0	1
0,9	0,8989	0,9951	0,0985	0,9939	1,0012	—0,0040	—0,0040	0,9963	0,9754
0,8	0,8025	0,9807	0,1836	0,9866	1,9941	—0,0179	—0,0181	0,9750	0,9458
0,7	0,7031	0,9595	0,2586	0,9793	0,9798	—0,0315	—0,0322	0,9401	0,9153
0,6	0,6099	0,9294	0,3207	0,9721	0,9560	—0,0523	—0,0538	0,8885	0,8853
0,5	0,5180	0,8945	0,3755	0,9650	0,9270	—0,0708	—0,0734	0,8291	0,8549
0,4	0,4313	0,8549	0,3200	0,9579	0,8925	—0,0893	—0,0932	0,7630	0,8242
0,3	0,3486	0,8166	0,4554	0,9509	0,8535	—0,1051	—0,1105	0,6927	0,7934
0,2	0,2691	0,7635	0,4826	0,9439	0,8089	—0,1164	—0,1233	0,6176	0,7622
0,1	0,1960	0,7164	0,4848	0,9369	0,7646	—0,1244	—0,1328	0,5478	0,7305
0	0,1264	0,6631	0,5149	0,93	0,7130	—0,1264	—0,1359	0,4728	0,6989
—0,1	0,0629	0,6129	0,5203	0,9232	0,6639	—0,1242	—0,1345	0,4068	0,6674
—0,2	0,0039	0,5589	0,5200	0,9164	0,6099	—0,1156	—0,1262	0,3410	0,6351
—0,3	—0,0496	0,5082	0,5098	0,9096	0,5587	—0,1028	—0,1130	0,2840	0,6026
—0,4	—0,0979	0,4567	0,4937	0,9030	0,5058	—0,0848	—0,0039	0,2310	0,5704
—0,5	—0,1403	0,4084	0,4739	0,8963	0,4557	—0,0639	—0,0713	0,1861	0,5371
—0,6	—0,1788	0,3583	0,4513	0,8898	0,4028	—0,0362	—0,0407	0,1443	0,5042
—0,7	—0,2125	0,3143	0,4251	0,8832	0,3559	—0,0075	—0,0085	0,1108	0,4711
—0,8	—0,2422	0,2705	0,3971	0,8767	0,3086	0,0258	0,0294	0,0835	0,4375
—0,9	—0,2670	0,2308	0,3664	0,8702	0,2653	0,0593	0,0681	0,0612	0,4035
—1,0	—0,2880	0,1941	0,3341	0,8639	0,2247	0,0939	0,1087	0,0436	0,3698
—1,1	—0,3061	0,1575	0,3009	0,8575	0,1837	0,1329	0,1550	0,0289	0,3353
—1,2	—0,3203	0,1268	0,2662	0,8512	0,1490	0,1681	0,1975	0,0189	0,3008
—1,3	—0,3317	0,1001	0,2270	0,8449	0,1185	0,2015	0,2385	0,0118	0,2657
—1,4	—0,3401	0,0757	0,1947	0,8387	0,0902	0,2349	0,2801	0,0068	0,2308
—1,5	—0,3468	0,0552	0,1589	0,8326	0,0664	0,2639	0,3170	0,0037	0,1958
—1,6	—0,3515	0,0387	0,1234	0,8264	0,0468	0,2896	0,3504	0,0018	0,1597
—1,7	—0,3544	0,0254	0,0881	0,8203	0,0310	0,3112	0,3794	0,0008	0,1237
—1,8	—0,3558	0,0155	0,0528	0,8143	0,0190	0,3279	0,4027	0,0003	0,0878
—1,9	—0,3581	0,0081	0,0172	0,8083	0,0100	0,3427	0,4249	0,0001	0,0513
—2,0	—0,3581	0,0020	0	0,8024	0,0025	0,3541	0,4414	0,0001	0,0150
—2,04	—0,3581	0	0	0,8000	0	0,3581	0,4475	0	0

TABLE III.- JET FUNCTIONS FOR $\Delta t = 60^{\circ}C$.

φ	F	F'	F''	χ	$\frac{F'}{\chi}$	$\varphi F' - F$	$\frac{\varphi F' - F'}{\chi}$	$\frac{F'^2}{\chi}$	θ
0,96	0,96	1,0	0	1,0	1,0	0	0	1,0	1,0
0,9	0,8990	1,0	0,0537	1,0037	0,9963	0,001	0,0010	0,9964	0,9781
0,8	0,8016	0,9903	0,1398	1,0098	0,9807	—0,0094	—0,0093	0,9712	0,9416
0,7	0,7011	0,9731	0,2180	1,0159	0,9579	- 0,0199	—0,0196	0,9321	0,9060
0,6	0,6064	0,9469	0,2837	1,0221	0,9264	—0,0383	—0,375	0,8772	0,8703
0,5	0,5125	0,9154	0,3426	1,0284	0,8901	—0,0548	—0,0533	0,8149	0,8342
0,4	0,4235	0,8787	0,3918	1,0340	0,8498	—0,0720	—0,0696	0,7467	0,8027
0,3	0,3384	0,8378	0,432	1,0409	0,8049	—0,0871	—0,0837	0,6743	0,7642
0,2	0,2562	0,7917	0,4646	1,0500	0,7540	—0,0979	—0,0932	0,5970	0,7143
0,1	0,1796	0,7448	0,4702	1,0537	0,7068	—0,1051	—0,0997	0,5264	0,6944
0	0,1075	0,6938	0,5075	1,0600	0,6545	—0,1075	—0,1014	0,4542	0,6604
—0,1	0,0411	0,6440	0,5179	1,0660	0,6041	—0,1055	—0,0990	0,3890	0,6285
—0,2	—0,0212	0,5900	0,5200	1,0730	0,5499	—0,0968	—0,0902	0,3244	0,5918
—0,3	—0,0776	0,5387	0,5139	1,0796	0,4990	—0,0840	—0,0778	0,2688	0,5577
—0,4	—0,1290	0,4864	0,5018	1,0861	0,4478	—0,0656	—0,0604	0,2178	0,5244
—0,5	—0,1743	0,4368	0,4853	1,0928	0,3997	—0,0441	—0,0404	0,1746	0,4904
—0,6	—0,2155	0,3850	0,4662	1,0994	0,3502	—0,0155	—0,0141	0,1348	0,4573
—0,7	—0,2516	0,3392	0,4418	1,1061	0,3067	0,0142	+0,0128	0,1041	0,4245
0,8	—0,2836	0,2932	0,4157	1,1130	0,2634	0,0490	0,0440	0,0773	0,3908
—0,9	—0,3106	0,2512	0,3863	1,1197	0,2244	0,0845	0,0755	0,0564	0,3586
—1,0	—0,3334	0,2121	0,3546	1,1265	0,1883	0,1213	0,1077	0,0399	0,3262
—1,1	—0,3532	0,1730	0,3217	1,1334	0,1526	0,1629	0,1437	0,0264	0,2937
—1,2	—0,3688	0,1399	0,2676	1,1403	0,1227	0,2009	0,1762	0,0172	0,2617
—1,3	0,3813	0,1108	0,2283	1,1472	0,0966	0,2373	0,2069	0,0107	0,2302
—1,4	0,3908	0,0844	0,1959	1,1542	0,0732	0,2726	0,2362	0,0062	0,1984
—1,5	0,3981	0,0615	0,1741	1,1613	0,0530	0,3058	0,2633	0,0033	0,1666
—1,6	0,4033	0,0433	0,1358	1,1683	0,0370	0,3341	0,2860	0,0016	0,1356
—1,7	0,4064	0,0285	0,0973	1,1755	0,0242	0,3580	0,3045	0,0007	0,1042
—1,8	0,4089	0,0173	0,0582	1,1826	0,0146	0,3776	0,3193	0,0003	0,0736
—1,9	0,4106	0,0109	0,0182	1,1892	0,0092	0,3899	0,3279	0,0001	0,0449
—2,0	0,4106	0,0081	0	1,1971	0,0068	0,3943	0,3294	0,0001	0,0121
—2,04	0,4106	0	0	1,2000	0	0,4106	0,3420	0	0

TABLE IV.- JET FUNCTIONS FOR $\Delta t_0 = 120^{\circ}C$

φ	F	F'	F''	χ	$\frac{F'}{\chi}$	$\varphi F' - F$	$\frac{\varphi F' - F}{\chi}$	$\frac{F'^2}{\chi}$	θ
0,942	0,942	1,0	0	1,0	1,0	0	0	1,0	1,0
0,9	0,8991	1,0	0,0386	1,0048	0,9952	0,0009	0,0009	0,9952	0,9834
0,8	0,8013	0,9939	0,1253	1,0162	0,9781	—0,0062	—0,0061	0,9721	0,9442
0,7	0,7004	0,9783	0,2044	1,0272	0,9524	—0,0156	—0,0152	0,9318	0,9073
0,6	0,6051	0,9535	0,2713	1,0393	0,9174	—0,0330	—0,0318	0,8748	0,8677
0,5	0,5105	0,9233	0,3316	1,0511	0,8784	—0,0489	—0,0465	0,8111	0,8300
0,4	0,4206	0,8877	0,3824	1,0630	0,8351	—0,0655	—0,0616	0,7413	0,7926
0,3	0,3345	0,8477	0,4243	1,0751	0,7885	—0,0802	—0,0746	0,6684	0,7556
0,2	0,2513	0,8024	0,4586	1,0873	0,7380	—0,0908	—0,0835	0,5921	0,7190
0,1	0,1734	0,7556	0,4652	1,0996	0,6872	—0,0978	—0,0889	0,5192	0,6831
0	0,1004	0,7054	0,5051	1,1121	0,6343	—0,1004	—0,0903	0,4474	0,6473
—0,1	0,0328	0,6558	0,5170	1,1248	0,5830	—0,0984	—0,0375	0,3824	0,6117
—0,2	—0,0306	0,6018	0,5200	1,1375	0,5291	—0,0898	—0,0789	0,3140	0,5770
—0,3	—0,0883	0,5503	0,5153	1,1504	0,4784	—0,0767	—0,0667	0,2632	0,5424
—0,4	—0,1408	0,4976	0,5045	1,1635	0,4277	—0,0582	—0,0500	0,2128	0,5082
—0,5	—0,1871	0,4475	0,4891	1,1767	0,3803	—0,0367	—0,0312	0,1702	0,4744
—0,6	—0,2294	0,3951	0,4707	1,1900	0,3320	—0,0077	—0,0065	0,1312	0,4413
—0,7	—0,2664	0,3486	0,4474	1,2035	0,2897	0,0224	0,0186	0,1010	0,4082
—0,8	—0,2993	0,3018	0,4220	1,2172	0,2480	0,0579	0,0476	0,0748	0,3755
—0,9	—0,3272	0,2589	0,3929	1,2310	0,2103	0,0942	0,0765	0,0544	0,3433
—1,0	—0,3507	0,2189	0,3614	1,2450	0,1758	0,1318	0,1059	0,0385	0,3113
—1,1	—0,3710	0,1788	0,3287	1,2600	0,1419	0,1743	0,1383	0,0254	0,2778
—1,2	—0,3872	0,1448	0,2932	1,2740	0,1137	0,2134	0,1675	0,0165	0,2473
—1,3	—0,4002	0,1148	0,2520	1,2880	0,0891	0,2510	0,1949	0,0102	0,2174
—1,4	—0,4101	0,0878	0,2180	1,3026	0,0674	0,2872	0,2205	0,0059	0,1870
—1,5	—0,4175	0,0639	0,1792	1,3320	0,0480	0,3217	0,2415	0,0031	0,1576
—1,6	—0,4229	0,0450	0,1400	1,3474	0,0334	0,3509	0,2604	0,0015	0,1277
—1,7	—0,4261	0,0296	0,1004	1,3627	0,0217	0,3758	0,2758	0,0007	0,0976
—1,8	—0,4290	0,0180	0,0601	1,3782	0,0131	0,3966	0,2878	0,0002	0,0684
—1,9	—0,4305	0,0113	0,0185	1,3938	0,0081	0,4090	0,2935	0,0001	0,0396
—2,0	—0,4305	0,0084	0	1,4000	0,0060	0,4137	0,2955	0,0001	0,0111
—2,04	—0,4305	0	0	1,4000	0	0,4305	0,3080	0	0

TABLE V.- BASIC FUNCTIONS FOR $Ba_0 = 1.0$.

№ п/п	φ	F	F'	F''	F'''
0	0,923	0,923000	1,000000	0	—1,292200
1	0,873	0,873027	0,998414	0,062860	—1,219354
2	0,823	0,823210	0,993778	0,121941	—1,141705
3	0,773	0,773697	0,986378	0,177036	—1,060580
4	0,723	0,724626	0,976143	0,228006	—0,977252
5	0,673	0,676124	0,963557	0,274762	—0,892915
6	0,623	0,628308	0,948737	0,317299	—0,808650
7	0,573	0,581285	0,931895	0,355643	—0,725433
8	0,523	0,535149	0,913241	0,389875	—0,644099
9	0,473	0,489987	0,892973	0,420096	—0,565365
10	0,423	0,445873	0,871294	0,446463	—0,489797
11	0,373	0,402878	0,848388	0,469158	—0,417885
12	0,323	0,361055	0,824438	0,488317	—0,349931
13	0,273	0,320447	0,799612	0,504203	—0,286158
14	0,223	0,281103	0,774070	0,517004	—0,226710
15	0,173	0,243049	0,747960	0,526947	—0,171613
16	0,123	0,206313	0,721421	0,534237	—0,120842
17	0,073	0,170912	0,694578	0,539099	—0,074289
18	0,023	0,136859	0,667548	0,541734	—0,031810
19	—0,027	0,104160	0,640437	0,542346	+0,006783
20	—0,077	0,072813	0,613346	0,541115	0,041721
21	—0,127	0,042824	0,586354	0,538226	0,073216
22	—0,177	0,014173	0,559549	0,533845	0,101545
23	—0,227	—0,013136	0,532993	0,528120	0,126950
24	—0,277	—0,039131	0,506756	0,521191	0,149703
25	—0,327	—0,063817	0,480891	0,513191	0,170042
26	—0,377	—0,087227	0,455451	0,504225	0,188220
27	—0,427	—0,109373	0,430182	0,494401	0,204462
28	—0,477	—0,130284	0,406020	0,483808	0,218985
29	—0,527	—0,149985	0,382111	0,472528	0,231989
30	—0,577	—0,168506	0,358778	0,460632	0,243657
31	—0,677	—0,202122	0,313971	0,435229	0,263629
32	—0,777	—0,231386	0,271793	0,408024	0,280012
33	—0,877	—0,256572	0,232417	0,379320	0,293660
34	—0,987	—0,277970	0,195973	0,349363	0,305194
35	—1,077	—0,295872	0,162582	0,318342	0,315029
36	—1,177	—0,310593	0,132336	0,286407	0,323442
37	—1,277	—0,322447	0,105324	0,253699	0,330596
38	—1,377	—0,331764	0,081618	0,220329	0,336603
39	—1,477	—0,338884	0,061276	0,186417	0,341531
40	—1,577	—0,344135	0,044348	0,152059	0,345442
41	—1,677	—0,347868	0,030877	0,117360	0,348435
42	—1,777	—0,350428	0,020885	0,082399	0,350637
43	—1,877	—0,352159	0,014402	0,047254	0,352225
44	—1,977	—0,353436	0,011440	0,011970	—
45	—2,04	—0,354000	0	0	—

TABLE V(a).- AUXILIARY FUNCTIONS FOR $Ba_0 = 1.0$.

№ п/п	φ	κ	$\frac{F'}{\kappa}$	$\varphi F' - F$	$\frac{\varphi F' - F}{\kappa}$	$\frac{F'^2}{\kappa}$
0	0,923	1,00	1,00	0	0	1,000000
1	0,823	0,9982	0,9956	—0,005331	—0,005341	0,985817
2	0,723	0,9932	0,9888	—0,018875	—0,019004	0,946376
3	0,623	0,9856	0,9626	—0,037245	—0,037789	0,887141
4	0,523	0,9758	0,9358	—0,057524	—0,058951	0,913826
5	0,423	0,9645	0,9034	—0,077316	—0,080162	0,732153
6	0,323	0,9523	0,8657	—0,094762	—0,099509	0,647276
7	0,223	0,9396	0,8239	—0,108485	—0,115459	0,562993
8	0,123	0,9269	0,7783	—0,117578	—0,126851	0,482403
9	0,023	0,9145	0,7299	—0,121505	—0,132865	0,407519
10	—0,077	0,9028	0,6793	—0,120041	—0,132965	0,339627
11	—0,177	0,8918	0,6274	—0,113213	—0,126949	0,279218
12	—0,277	0,8819	0,5747	—0,101240	—0,114796	0,226474
13	—0,377	0,8729	0,5217	—0,084478	—0,096779	0,181071
14	—0,477	0,8651	0,4693	—0,063388	—0,073272	0,142613
15	—0,577	0,8567	0,4188	—0,038509	—0,044950	0,110276
16	—0,677	0,8526	0,3682	—0,010436	—0,012240	0,084048
17	—0,777	0,8479	0,3206	+0,020203	+0,023827	0,062635
18	—0,877	0,8440	0,2754	0,052742	0,062491	0,045591
19	—0,977	0,8409	0,2331	0,086504	0,102871	0,032295
20	—1,077	0,8386	0,1939	0,120771	0,144015	0,022167
21	—1,177	0,8368	0,1581	0,154834	0,185031	0,014655
22	—1,277	0,8355	0,1260	0,187948	0,224953	0,009268
23	—1,377	0,8347	0,09778	0,219376	0,262820	0,005560
24	—1,477	0,8341	0,07346	0,248379	0,297781	0,003132
25	—1,577	0,8337	0,05320	0,274198	0,328893	0,001640
26	—1,677	0,8334	0,03706	0,296087	0,355276	0,000794
27	—1,777	0,8334	0,02506	0,313315	0,375948	0,000363
28	—1,877	0,8334	0,01728	0,325126	0,390134	0,000172
29	—1,977	0,8340	0,01370	0,326000	0,392000	0,000160
30	—2,04	0,8340	0	0,326009	0,392000	0

TABLE VI.- BASIC FUNCTIONS FOR $Ba_0 = 0.9$.

№ п, п	φ	F	F'	F''	F'''
0	0,934	0,934000	1,00	0	−1,236624
1	0,884	0,884026	0,998482	0,060176	−1,168140
2	0,834	0,834201	0,994042	0,116818	−1,095812
3	0,784	0,784668	0,986862	0,169762	−1,020711
4	0,734	0,735558	0,977130	0,218895	− 0,943871
5	0,684	0,686996	0,965037	0,264152	−0,866263
6	0,634	0,639090	0,950780	0,305526	−0,788765
7	0,584	0,591950	0,934550	0,343044	−0,712178
8	0,534	0,545664	0,916538	0,376771	−0,637192
9	0,484	0,500322	0,896933	0,406801	−0,564401
10	0,434	0,455993	0,875919	0,433255	−0,494290
11	0,384	0,412751	0,853663	0,456280	−0,427252
12	0,334	0,370647	0,830344	0,476038	−0,363573
13	0,284	0,329732	0,806112	0,492695	−0,303458
14	0,234	0,290048	0,781122	0,506443	−0,247031
15	0,184	0,251632	0,729418	0,525938	−0,145390
16	0,134	0,214504	0,729418	0,525938	−0,145390
17	0,084	0,178695	0,702959	0,532063	−0,100106
18	0,034	0,144212	0,676249	0,536007	−0,058384
19	−0,016	0,110720	0,649393	0,537957	−0,020086
20	−0,066	0,079271	0,622483	0,538069	+0,014958
21	−0,116	0,048823	0,595616	0,536510	0,046922
22	−0,166	0,019708	0,568856	0,533426	0,076026
23	−0,216	−0,008067	0,542296	0,528951	0,102454
24	−0,266	−0,034525	0,515982	0,523218	0,126430
25	−0,316	−0,059671	0,489992	0,516343	0,148149
26	−0,366	−0,083531	0,464367	0,508438	0,167823
27	−0,416	−0,106113	0,439162	0,499591	0,185627
28	−0,466	−0,127454	0,414422	0,489903	0,201764
29	−0,516	−0,147566	0,390185	0,479441	0,216385
30	−0,566	−0,166480	0,366489	0,468286	0,229655
31	−0,666	−0,200830	0,320846	0,444128	0,252695
32	−0,766	−0,230735	0,277733	0,417872	0,271854
33	−0,866	−0,256465	0,237331	0,389862	0,287904
34	−0,966	−0,278298	0,199809	0,360375	0,301420
35	−1,066	−0,296528	0,165298	0,329649	0,312816
36	−1,166	−0,311461	0,133915	0,297876	0,322385
37	−1,266	−0,323418	0,105750	0,265232	0,330332
38	−1,366	−0,332721	0,080892	0,231861	0,336799
39	−1,466	−0,339710	0,059398	0,197918	0,341908
40	−1,566	−0,344717	0,041321	0,163524	0,345769
41	−1,666	−0,348088	0,026705	0,128802	0,348505
42	−1,766	−0,350173	0,015568	0,093859	0,350308
43	−1,866	−0,351319	0,007935	0,058765	0,351348
44	−1,966	−0,351876	0,003220	0,023606	0,351880
45	−2,04	−0,352000	0	0	—

TABLE VI(a).- AUXILIARY FUNCTIONS FOR $Ba_0 = 0.9$.

№ п/п	φ	κ	$\frac{F'}{\kappa}$	$\varphi F' - F$	$\frac{\varphi F' - F}{\kappa}$	$\frac{F'^2}{\kappa}$
0	0,934	1,00	1,00	0	0	1,0000
1	0,834	0,9985	0,9955	−0,005170	−0,005178	0,9896
2	0,734	0,9944	0,9826	−0,018345	−0,018448	0,9602
3	0,634	0,9881	0,9622	−0,036295	−0,036732	0,9149
4	0,534	0,9801	0,9351	−0,056233	−0,057375	0,8571
5	0,434	0,9708	0,9023	−0,075844	−0,078125	0,7903
6	0,334	0,9606	0,8644	−0,093312	−0,097139	0,7178
7	0,234	0,9501	0,8221	−0,107265	−0,112899	0,6422
8	0,134	0,9395	0,7764	−0,116762	−0,124281	0,5663
9	0,034	0,9292	0,7278	−0,121220	−0,130456	0,4922
10	−0,066	0,9193	0,6771	−0,120355	−0,130920	0,4215
11	−0,166	0,9102	0,6250	−0,114138	−0,125399	0,3555
12	−0,266	0,9017	0,5722	−0,102726	−0,113925	0,2953
13	−0,366	0,8942	0,5193	−0,086427	−0,096653	0,2412
14	−0,466	0,8876	0,4669	−0,065667	−0,073983	0,1935
15	−0,566	0,8818	0,4156	−0,040953	−0,046443	0,1523
16	−0,666	0,8770	0,3658	−0,012853	−0,014656	0,1174
17	−0,766	0,8789	0,3182	+0,017992	+0,020621	0,08837
18	−0,866	0,8696	0,2729	0,050936	0,058574	0,06477
19	−0,966	0,8670	0,2305	0,085283	0,098366	0,04605
20	−1,066	0,8650	0,1911	0,120320	0,139099	0,03159
21	−1,166	0,8635	0,1551	0,155316	0,179868	0,02077
22	−1,266	0,8624	0,1226	0,189539	0,219781	0,01297
23	−1,366	0,8616	0,09389	0,222223	0,257919	0,00759
24	−1,466	0,8612	0,06897	0,252633	0,293350	0,00409
25	−1,566	0,8609	0,04800	0,280008	0,325250	0,00198
26	−1,666	0,8607	0,03103	0,303598	0,352734	0,00082
27	−1,766	0,8606	0,01809	0,322680	0,374948	0,00028
28	−1,866	0,8606	0,00922	0,336512	0,391020	0,00007
29	−1,966	0,8606	0,00443	0,346000	0,402000	0
30	-2,040	0,8606	0	0,352000	0,409000	0

TABLE VII.- BASIC FUNCTIONS FOR Ba_0 = 0.5.

№ п/п	φ	F	F'	F''	F'''
0	0,966	0,966	1,0	0	—1,062600
1	0,916	0,916022	0,998695	0,051755	—1,006931
2	0,866	0,866173	0,994871	0,100695	—0,950190
3	0,816	0,816575	0,988672	0,146776	—0,892768
4	0,766	0,767343	0,980241	0,189973	—0,835025
5	0,716	0,718586	0,969722	0,230282	—0,777326
6	0,666	0,670403	0,957260	0,267710	—0,719989
7	0,616	0,622889	0,942995	0,302289	—0,663311
8	0,566	0,576129	0,927077	0,334059	—0,607562
9	0,516	0,530205	0,909634	0,363064	—0,552974
10	0,466	0,485188	0,890816	0,389380	—0,409746
11	0,416	0,441143	0,870741	0,413065	—0,448052
12	0,366	0,398132	0,849551	0,434213	—0,398029
13	0,316	0,356202	0,827360	0,452897	—0,349786
14	0,266	0,315409	0,804300	0,469221	—0,303412
15	0,216	0,275784	0,780477	0,483272	—0,258955
16	0,166	0,237371	0,756007	0,495148	—0,216460
17	0,116	0,200195	0,731000	0,504952	—0,175936
18	0,066	0,164277	0,705543	0,512776	—0,137379
19	0,016	0,129646	0,679751	0,512776	—0,137379
20	—0,034	0,096306	0,653702	0,522886	—0,066092
21	—0,084	0,064280	0,627489	0,525362	—0,033293
22	—0,134	0,033562	0,601192	0,526245	—0,002316
23	—0,184	0,004158	0,574889	0,525623	0,026889
24	—0,234	0,023926	0,548653	0,523585	0,054379
25	—0,284	0,050710	0,522553	0,520210	0,080230
26	—0,334	0,076186	0,496554	0,515585	0,104494
27	—0,384	0,100378	0,471015	0,509786	0,127252
28	—0,434	0,123294	0,445696	0,502882	0,148560
29	—0,484	0,144954	0,420745	0,494952	0,168493
30	—0,534	0,165375	0,396214	0,486054	0,187109
31	—0,634	0,202595	0,348603	0,465627	0,220641
32	—0,734	0,235169	0,303191	0,442078	0,249635
33	—0,834	0,263319	0,260276	0,415834	0,274503
34	—0,934	0,287316	0,220102	0,387301	0,295637
35	—1,034	0,307438	0,182877	0,356818	0,313372
36	—1,134	0,323997	0,148791	0,324729	0,328029
37	—1,234	0,337306	0,117976	0,291307	0,339895
38	—1,334	0,347705	0,090562	0,256831	0,349256
39	—1,434	0,355534	0,066637	0,221531	0,356386
40	—1,534	0,361152	0,046275	0,185618	0,361568
41	—1,734	0,367178	0,016430	0,112640	0,367232
42	—1,834	0,368320	0,007004	0,075854	0,368330
43	—1,934	0,368702	0,001261	0,038998	0,368702
44	—2,04	0,369000	0	0	—

TABLE VII(a).- AUXILIARY FUNCTIONS FOR Ba_0 =0.5.

№ п п	φ	χ	$\frac{F'}{\chi}$	$\varphi F' - F$	$\frac{\varphi F' - F}{\chi}$	$\frac{F'^2}{\chi}$
0	0,966	1,00	1,00	0	0	1,0000
1	0,866	0,9996	0,9954	—0,004573	—0,004575	0,9902
2	0,766	0,9983	0,9819	—0,016543	—0,016571	0,9625
3	0,666	0,9963	0,9609	—0,032803	—0,032925	0,9197
4	0,566	0,9936	0,9331	—0,051429	—0,051760	0,8650
5	0,466	0,9903	0,8993	—0,070088	—0,070775	0,3013
6	0,366	0,9872	0,8607	—0,087132	—0,088262	0,7311
7	0,266	0,9838	0,8176	—0,101509	—0,103181	0,6575
8	0,166	0,9802	0,7713	—0,111871	—0,114131	0,5831
9	0,066	0,9767	0,7223	—0,117677	—0,120484	0,5097
10	—0,034	0,9733	0,6717	—0,118506	—0,121757	0,4390
11	—0,134	0,9702	0,6197	—0,114162	—0,117668	0,3725
12	—0,234	0,9672	0,5672	—0,104474	—0,108017	0,3112
13	—0,334	0,9642	0,5148	—0,089714	—0,093006	0,2558
14	—0,434	0,9622	0,4632	—0,070106	—0,072860	0,2064
15	—0,534	0,9603	0,4127	—0,046225	—0,048136	0,1635
16	—0,634	0,9584	0,3637	—0,013405	—0,019204	0,1268
17	—0,734	0,9570	0,3168	+0,012669	+0,013238	0,0960
18	—0,834	0,9557	0,2724	0,046219	0,048361	0,0708
19	—0,934	0,9549	0,2305	0,081716	0,085575	0,0507
20	—1,034	0,9541	0,1917	0,118338	0,124031	0,0350
21	—1,134	0,9535	0,1552	0,155297	0,162870	0,0232
22	—1,234	0,9532	0,1238	0,191706	0,201118	0,0146
23	—1,334	0,9528	0,0951	0,226805	0,238041	0,0086
24	—1,434	0,9527	0,0699	0,260034	0,272944	0,0046
25	—1,534	0,9525	0,0487	0,290152	0,304621	0,0022
26	—1,634	0,9525	0,03098	0,316710	0,332504	0,0009
27	—1,734	0,9524	0,01720	0,338778	0,355710	0,0002
28	—1,834	0,9524	0,00740	0,355520	0,373289	0,0000
29	—1,934	0,9524	0,00140	0,366202	0,384504	0,0000
30	—2,040	0,9524	0	0,3690	0,38750	0

TABLE VIII.- BASIC FUNCTIONS FOR INCOMPRESSIBLE JET.

№ п, п	φ	F	F′	F″	F‴
0	0,98	0,98	1,00	0	— 0,98
1	0,93	0,930021	0,998796	0,047750	— 0,930021
2	0,88	0,880160	0,995268	0,093004	— 0,889160
3	0,83	0,830531	0,989538	0,135770	— 0,830531
4	0,78	0,781241	0,981732	0,176062	— 0,781241
5	0,73	0,732391	0,971970	0,213899	— 0,732391
6	0,68	0,684077	0,960380	0,249309	— 0,684077
7	0,63	0,636380	0,947081	0,282019	— 0,636380
8	0,58	0,589394	0,932186	0,312957	— 0,589394
9	0,53	0,543187	0,915821	0,341271	— 0.543187
10	0,48	0,497832	0,898095	0,367290	— 0,497832
11	0,43	0,453397	0,879128	0,391069	— 0,453397
12	0,38	0,409939	0,859027	0,412645	— 0,409939
13	0,33	0,367510	0,837897	0,432079	— 0,367510
14	0,28	0,326163	0,815852	0,449414	— 0,326163
15	0,23	0,285938	0,792991	0,464714	— 0,285938
16	0,18	0,246874	0,769413	0,478027	— 0,246874
17	0,13	0,209006	0,745221	0,489420	— 0,209006
18	0,08	0,172361	0,720502	0,498948	— 0,172361
19	0,03	0,136963	0,695357	0,506678	— 0,1366963
20	— 0,02	0,102830	0,669864	0,512666	— 0,102830
21	— 0,07	0,069982	0,644120	0,516980	— 0,069982
22	— 0,12	0,038422	0,618195	0,519686	— 0,038422
23	— 0,17	0,008164	0,592177	0,520844	— 0,008164
24	— 0,22	— 0,020794	0,566135	0,520423	+ 0,020794
25	— 0,27	— 0,048451	0,540147	0,518786	0,048451
26	— 0,32	— 0,074810	0,514280	0,515699	0,074810
27	— 0,37	— 0,099884	0,488599	0,511328	0,099884
28	— 0,42	— 0,123676	0,463169	0,505731	0,123676
29	— 0,47	— 0,146204	0,438045	0,498980	0,146204
30	— 0,52	— 0,167487	0,413286	0,491134	0,167487
31	— 0,62	— 0,206388	0,365075	0,472399	0,206388
32	— 0,72	— 0,240567	0,318925	0,450011	0,240567
33	— 0,82	— 0,270252	0,275179	0,424433	0,270252
34	— 0,92	— 0,295694	0,234130	0,396102	0,295694
35	— 1,02	— 0,317174	0,196035	0,365426	0,317174
36	— 1,12	— 0,335008	0,161109	0,332789	0,335008
37	— 1,22	— 0,335008	0,161109	0,332789	0,335008
38	— 1,32	— 0,361031	0,101443	0,262988	0,361031
39	— 1,42	— 0,369918	0,076966	0,226417	0,369918
40	— 1,52	— 0,376548	0,056186	0,189076	0,376548
41	— 1,62	— 0,581283	0,039169	0,151171	0,381283
42	— 1,72	— 0,384508	0,025966	0,112873	0,384508
43	— 1,82	— 0,386606	0,016603	0,074307	0,386606
44	— 1,92	— 0,387955	0,011108	0,035575	0,387955
45	— 2,02	— 0,388955	0,009494	0	—
46	— 2,04	— 0,389000	0	0	—

TABLE VIII(a).- AUXILIARY FUNCTIONS FOR INCOMPRESSIBLE JET FOR Ba ≅ 0.

№ п/п	φ	φF′ — F	F′²	№ п, п	φ	φF′ — F	F′²
0	0,98	0	1,000	16	— 0,62	— 0,0200	0,133
1	0,88	— 0,0042	0,990	17	— 0,72	— 0,0110	0,102
2	0,78	— 0,0155	0,964	18	— 0,82	0,0446	0,0756
3	0,68	— 0,0310	0,922	19	— 0,92	0,0803	0,0384
4	0,58	— 0,0488	0,869	20	— 1,02	0,1173	0,0384
5	0,48	— 0,0667	0,806	21	— 1,12	0,1546	0,0259
6	0,38	— 0,0835	0,738	22	— 1,22	0,1915	0,0166
7	0,28	— 0,0978	0,666	23	— 1,32	0,2272	0,0102
8	0,18	— 0,108	0,591	24	— 1,42	0,2606	0,0059
9	0,08	— 0,114	0,520	25	— 1,52	0,2911	0,0031
10	— 0,02	— 0,1162	0,449	26	— 1,62	0,3178	0,0015
11	— 0,12	— 0,1126	0,382	27	— 1,72	0,3398	0,0006
12	— 0,22	— 0,1037	0,320	28	— 1,82	0,3364	0,0002
13	— 0,32	— 0,0898	0,264	29	— 1,92	0,3666	0,0001
14	— 0,42	— 0,0708	0,214	30	— 2,02	0,3697	0
15	— 0,52	— 0,0474	0,171	31	— 2,04	0,3890	0

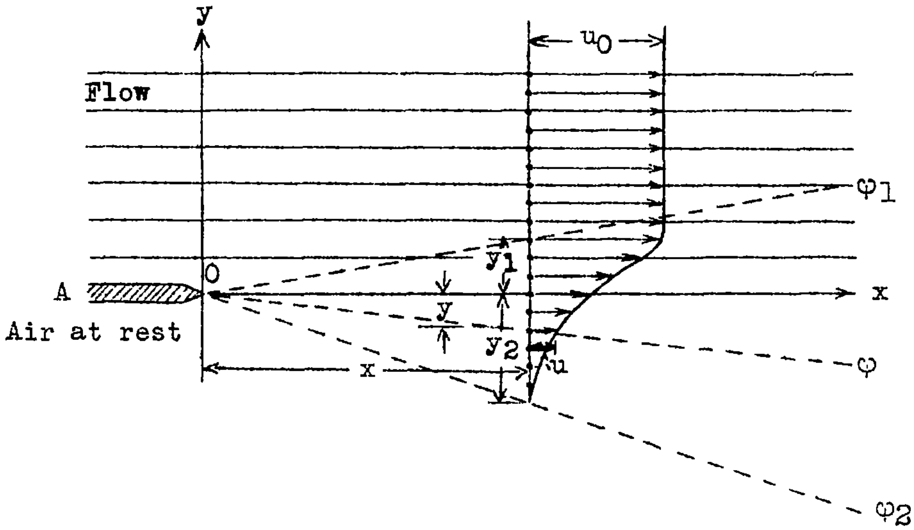

Figure 1.- Boundary layer of free jet.

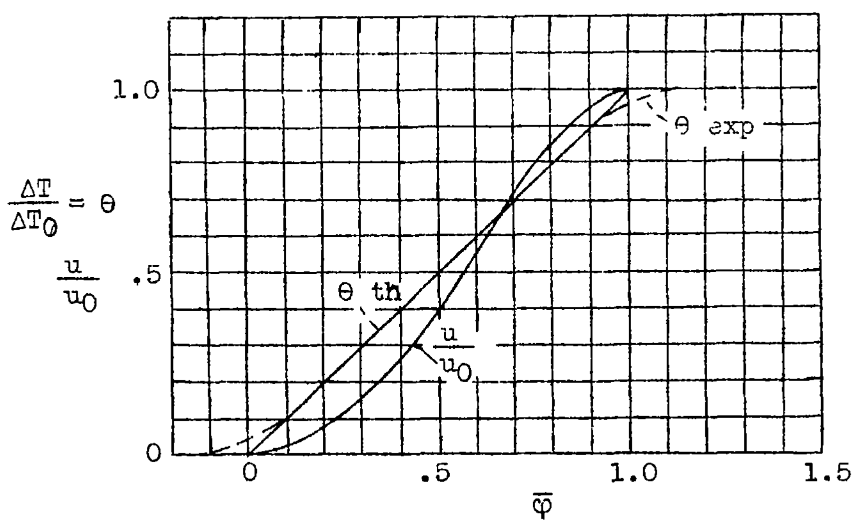

Figure 2.- Temperature and velocity fields in the boundary layer of a jet at the distance x = D from the nozzle according to the tests of Ruden.

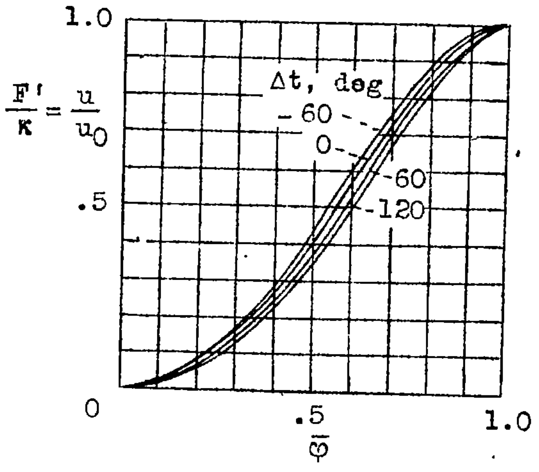

Figure 3.- Velocity fields in the boundary layer of a jet at various temperatures.

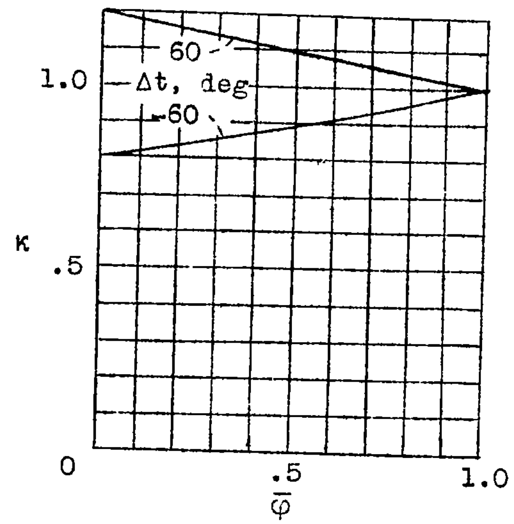

Figure 4.- Density fields in the free jet at Δt = +60°C and Δt = -60°C.

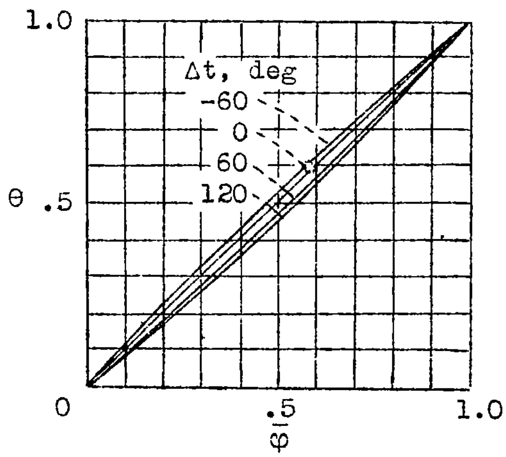

Figure 5.- Temperature fields in the boundary layer of a jet.

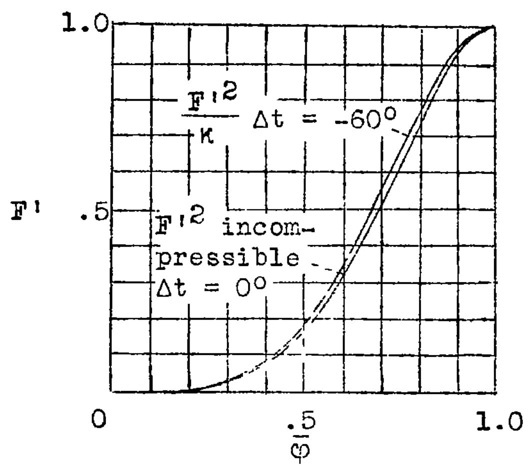

Figure 6.- Field of velocity heads in the free jet at Δt=-60°C.

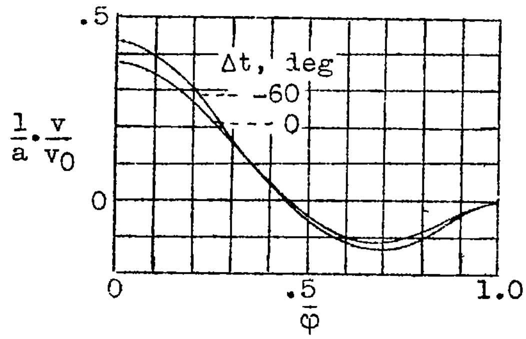

Figure 7.- Fields of transverse velocities in the boundary layer of a jet at Δt = -60°C.

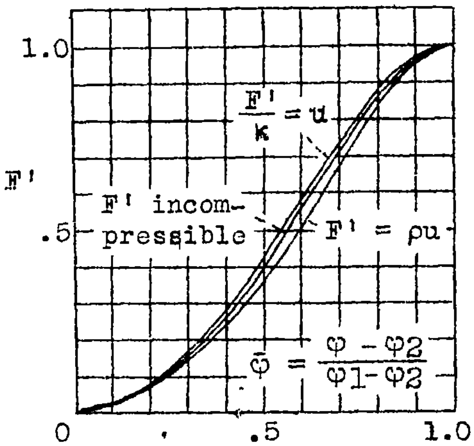

Figure 8.- Velocity fields in the boundary layer of a high velocity jet $(Ba_o = 1)$.

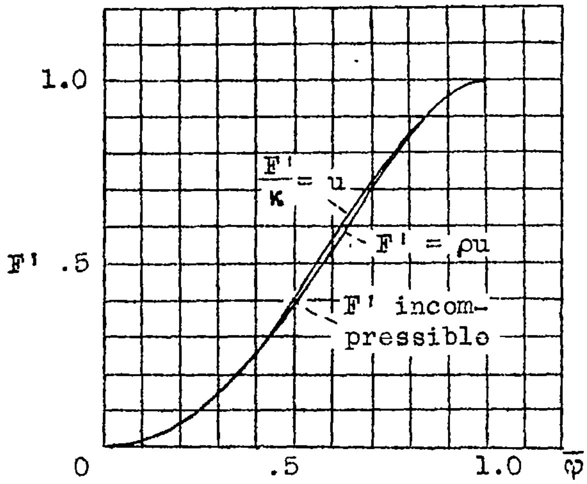

Figure 9.- Velocity fields in the boundary layer of a high velocity jet $(Ba_o = 0.5)$.

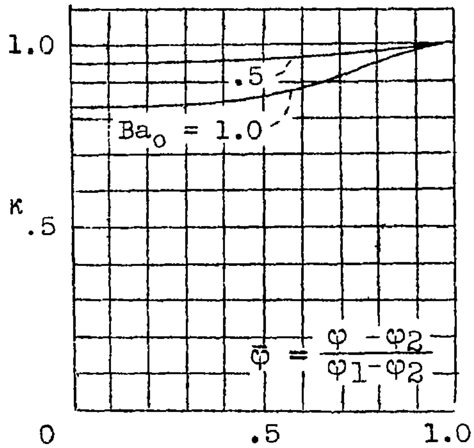

Figure 10.- Density fields in the boundary layer of a high velocity jet.

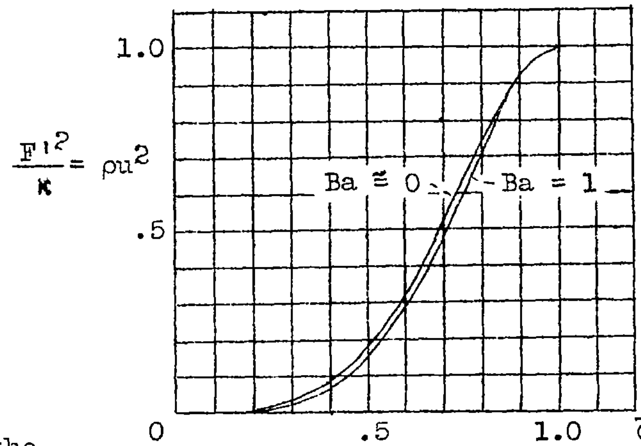

Figure 11.- Fields of velocity heads in a high velocity jet.

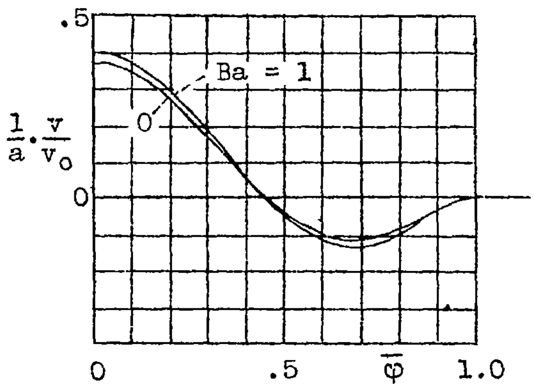

Figure 12.- Transverse velocity field in a high velocity jet.

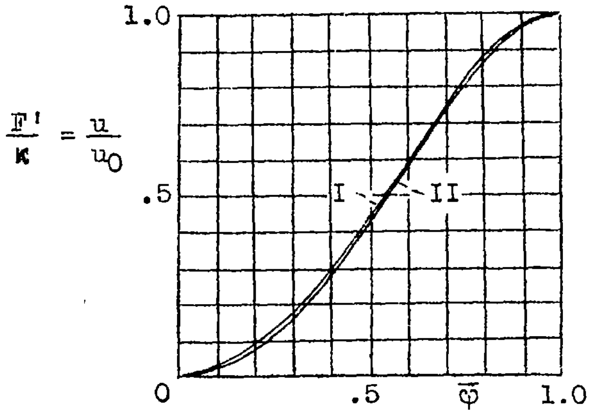

Figure 13.- Comparison of velocity fields of I cooled jet $(\Delta t = 60^o)$ and II jet with high velocities $(Ba_o = 1)$.