NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1257

VIBRATION OF A WING OF FINITE SPAN IN A SUPERSONIC FLOW

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VIBRATION OF A WING OF FINITE SPAN IN A SUPERSONIC FLOW*

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An investigation was made of the disturbed motion of a gas for the harmonic vibrations of a thin slightly cambered wing of finite span moving forward with supersonic velocity. This problem was considered by E. A. Krasilshchikova (reference 1) who applied the method of Fourier series and obtained a solution of the space problem for the condition that the Mach cones drawn through the leading edge of the wing intersect the wing or are tangent to it.

In this paper, a different method of solution is given, which is free from the previously mentioned condition. In particular, the vibrations of a triangular wing lying within the Mach cone are considered.

1. FUNDAMENTAL EQUATIONS OF PROBLEM

A thin, slightly cambered wing, whose projection on the x,y-plane has the form of an equilateral triangle with vertex angle $2\gamma < 2\alpha$, where α is the Mach angle, is considered. It is assumed that the fundamental motion of the wing consists of a rectilinear translational motion with constant supersonic velocity u parallel to the x-axis. It is also assumed that the coordinate axes move with the same velocity and that the x-axis is taken in the direction of the velocity u. On the fundamental motion of the wing, the additional harmonic of its vibration with frequency ω is superposed, where the possibility of deformation of the wing is not excluded. The equation of the surface of the wing can then be written in the form

$$z(x,y,t) = f_0(x,y) + f_1(x,y) \cos \omega t + f_2(x,y) \sin \omega t \qquad (1.1)$$

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*"Kolebaniia Kryla Konechnogo Razmakha v Sverkhzvukovom Potoke." Prikladnaya Matematika i Mekhanika, Vol. XI, 1947, pp. 371-376.

It is assumed that the ratios of the functions f_k to the linear dimensions of the triangle and the derivatives $\partial f_k/\partial x$ and $\partial f_k/\partial y$ are small.

Considering the nonvortical motion of the gas, for the velocity potential $\Phi(x,y,z,t)$ the following linearized equation may be taken:

$$(1-M^2)\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} + 2\frac{M}{a}\frac{\partial^2\Phi}{x\partial t} - \frac{1}{a^2}\frac{\partial^2\Phi}{\partial t^2} = 0 \qquad (1.2)$$

where $M = u/a$ is the Mach number and a is the velocity of sound.

The flow condition that must be satisfied on the surface of the wing by the velocity potential $\Phi(x,y,z,t)$ shall be considered as satisfied on the projection of the wing in the plane $z = 0$.

The derivation of this equation shall be based on the work of N. E. Kochin (reference 2). A stationary system of coordinates x_1, y_1, and z_1 connected with the moving system of coordinates by the relations $x = x_1 - ut_1$, $y = y_1$, $z = z_1$, and $t = t_1$ is used; when expressed in these coordinates, equation (1.1) becomes

$$z_1 = f_0(x_1-ut_1,y_1) + f_1(x_1-ut_1,y_1)\cos\omega t + f_2(x_1-ut_1,y_1)\sin\omega t$$

For the component of the velocity of the gas particles normal to the wing, there is obtained

$$\frac{dz_1}{dt_1} = -u\frac{\partial f_0}{\partial x} - u\frac{\partial f_1}{\partial x}\cos\omega t - \omega f_1\sin\omega t - u\frac{\partial f_2}{\partial x}\sin\omega t + \omega f_1\cos\omega t$$

By introducing the notation

$$-u\frac{\partial f_0}{\partial x} = Z(x,y) \qquad \omega f_2 - u\frac{\partial f_1}{\partial x} = Z_1(x,y) \qquad -\omega f_1 - u\frac{\partial f_2}{\partial x} = Z_2(x,y)$$

the boundary condition, which must be satisfied by the velocity potential $\Phi(x,y,z,t)$, is represented in the form

$$\left(\frac{\partial\Phi}{\partial z}\right)_{z=0} = Z_0(x,y) + Z_1(x,y)\cos\omega t + Z_2(x,y)\sin\omega t \qquad (1.3)$$

On the surface of the Mach cone, $\Phi(x,y,z,t) = 0$.

1264 Thus the problem is arrived at: To find within the Mach cone the function $\Phi(x,y,z,t)$ satisfying equation (1.2) and condition (1.3) and becoming zero on the Mach cone.

When the steady vibrational character of the motion of the gas is taken into account, equation (1.2) is transformed to a simpler form, setting (reference 3)

$$\Phi(x,y,z,t) = \varphi_0(x,y,z) + \varphi_1(x,y,z)\cos(\omega t + \lambda x) + \varphi_2(x,y,z)\sin(\omega t + \lambda x)$$

$$\left(\lambda = \frac{\omega M}{a(M^2-1)}\right)$$

By introducing the notation

$$Z_1(x,y) + iZ_3(x,y) = Z(x,y)$$

$$\varphi_1(x,y,z) + i\varphi_3(x,y,z) = \varphi(x,y,z)$$

there is obtained for determining the functions $\varphi_0(x,y,z)$ and $\varphi(x,y,z)$ after simple computations from equations (1.2) and (1.3) the following equations:

$$\left.\begin{aligned} &(1-M^2)\frac{\partial^2\varphi_0}{\partial x^2} + \frac{\partial^2\varphi_0}{\partial y^2} + \frac{\partial^2\varphi_0}{\partial z^2} = 0 \\ &\left(\frac{\partial\varphi_0}{\partial z}\right)_{z=0} = Z_0(x,y) \\ &(1-M^2)\frac{\partial^2\varphi_0}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} + k^2\varphi = 0 \\ &\left(\frac{\partial\varphi_0}{\partial z}\right)_{z=0} = iZ(x,y)e^{-i\lambda x} \end{aligned}\right\} \qquad (1.4)$$

$$\left(k = \frac{\omega}{u\sqrt{M^2-1}}\right) \qquad (1.5)$$

The function $\varphi_0(x,y,z)$ determined by equations (1.4) is the velocity potential of the motion of the gas for steady forward motion of the wing and $\mathrm{Re}\left[\varphi(x,y,z)e^{-i(\omega t + \lambda x)}\right]$ is the velocity potential corresponding to the vibrations of the wing.

It is noted that for $k = 0$, the system (1.5) goes over into the system (1.4), so that it is sufficient to find the solution of system (1.5).

2. METHOD OF SOLUTION

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In equations (1.5)

$$x = x^* \sqrt{M^2-1}$$

$$y = y^*$$

$$z = z^*$$

and the curvilinear system of coordinates are introduced

$$\left.\begin{aligned} \xi &= \frac{\sqrt{x^{*2}-y^{*2}-z^{*2}}}{x^*+y^*} \\ \eta &= \frac{z^*}{x^*+y^*} \\ \zeta &= \sqrt{x^{*2}-y^{*2}-z^{*2}} \end{aligned}\right\} \tag{2.1}$$

It is easily seen that within the Mach cone $x^{*2}-y^{*2}-z^{*2} = 0$; the coordinates ξ, η, and ζ uniquely define the position of a point. On the Mach cone $\xi = 0$, and $\zeta = 0$. In the triangular region of the projection of the wing, $z = 0$, $|y| \leqslant \operatorname{tg} \gamma\, x$,

$$\left.\begin{aligned} &\eta = 0 \\ &a \leqslant \xi \leqslant b \\ &\left(a = \frac{\sqrt{\cos 2\gamma}}{\cos\gamma + \sin\gamma} \quad b = \frac{\sqrt{\cos 2\gamma}}{\cos\gamma - \sin\gamma}\right) \end{aligned}\right\} \tag{2.2}$$

In the new coordinates, equation (1.5) assumes the form

$$\zeta^2 \frac{\partial^2\varphi}{\partial\xi^2} + 2\zeta \frac{d\varphi}{d\zeta} - \xi^2 \left(\frac{\partial^2\varphi}{\partial\xi^2} + \frac{\partial^2\varphi}{\partial\eta^2}\right) + k^2\varphi = 0 \tag{2.3}$$

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$$\left(\frac{\partial\varphi}{\partial\eta}\right)_{\eta=0} = f(\xi,\zeta) \qquad (2.4)$$

where

$$f(\xi,\zeta) = \frac{\zeta}{\xi}\, iZ \left(\frac{\zeta(1+\xi^2)}{2\xi}\sqrt{M^2-1} \quad \frac{\zeta(1-\xi^2)}{2\xi}\right) \exp\left(-i\, \frac{\zeta(1+\xi^2)\lambda}{2\xi}\sqrt{M^2-1}\right)$$

By setting in equation (2.3) $\varphi = \chi(\xi,\eta)\psi(\zeta)$ after separation of the variables there results

$$\zeta^2 \frac{d^2\psi}{d\zeta^2} + 2\zeta \frac{d\psi}{d\zeta} + \left[k^2\zeta^2 - n(n+1)\right]\psi = 0 \qquad (2.5)$$

$$\xi^2\left(\frac{\partial^2\chi}{\partial\xi^2} + \frac{\partial^2\chi}{\partial\eta^2}\right) - n(n+1)\chi = 0 \qquad (2.6)$$

The solutions of equation (2.5) are expressed in Bessel functions. When account is taken of the finiteness of the function φ on the Mach cone $(\zeta = 0)$, the following solution of equation (2.5) results:

$$\psi_n(k\zeta) = \sqrt{\frac{\pi}{2k\zeta}}\, J_{n+\frac{1}{2}}(k\zeta) \qquad (n \geqslant 0) \qquad (2.7)$$

In order to choose in equations (2.5) and (2.6) the constant n, the arbitrary function $F(\zeta)$ is expanded, analytic in the region containing the origin of coordinates, into a series of functions $\psi_n(\zeta)$:

$$F(\zeta) = \sum_{n=0}^{\infty} a_n\psi_n(\zeta)$$

In order to establish the possibility of this expansion, use is made of the formula of Gegenbauer (reference 4, p. 283):

$$\frac{\zeta^\nu}{\tau-\zeta} = \sum_{n=0}^{\infty} A_{n\nu}(\tau)J_{\nu+n}(\zeta) \qquad (|\zeta| < |\tau|) \qquad (2.8)$$

where $A_{n\nu}(\tau)$ are the Gegenbauer polynomials determined by the equation

$$A_{n\nu}(\tau) = \frac{2^{\nu+n}(\nu+n)}{\tau^{n+1}} \sum_{m=0}^{\leq \frac{1}{2}n} \frac{\Gamma(\nu+n-m)}{m!} \left(\frac{\tau}{2}\right)^{2m} \qquad (2.9)$$

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By setting $\nu = \frac{1}{2}$ in equations (2.8) and (2.9), and by taking account of equation (2.7), the following series is obtained:

$$\frac{1}{\tau - \zeta} = \sum_{n=0}^{\infty} A_n(\tau)\psi_n(\zeta) \qquad (|\zeta| < |\tau|) \qquad (2.10)$$

where $A_n(\tau)$ according to equation (2.9) will have the form

$$A_n(\tau) = \frac{2^n(2n+1)}{\sqrt{\pi}\tau^{n+1}} \sum_{m=0}^{\leq \frac{1}{2}n} \frac{\Gamma\left(\frac{1}{2}+n-m\right)}{m!} \left(\frac{\tau}{2}\right)^{2m} \qquad (2.11)$$

When series (2.10) and the integral formula of Cauchy are used,

$$\left.\begin{aligned} F(\zeta) &= \sum_{n=0}^{\infty} a_n\psi_n(\zeta) \\ a_n &= \frac{1}{2\pi i} \int_{|\tau|<r} F(\tau)A_n(\tau)d\tau \end{aligned}\right\} \qquad (2.12)$$

The coefficients a_n can easily be expressed in terms of the derivatives of the function $F(\zeta)$ for $\zeta = 0$. Thus, by taking into account series (2.10) and the Cauchy formula for the derivatives,

$$a_n = \frac{2n+1}{2^n} \sum_{m=0}^{\leq \frac{1}{2}n} F^{(n-2m)}(0) \frac{(2n-2m)!}{m!(n-m)!(n-2m)!} \qquad (2.13)$$

For the steady motion in equation (2.5) $k = 0$, then $\psi_n(\zeta) = \zeta^n$ and equations (2.12) become the usual power series

$$F(\zeta) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} \zeta^n$$

It is assumed that in equation (2.4) the function $f(\xi,\zeta)$ is analytic in ξ; then according to equations (2.12), it can be expanded in the functions $\psi_n(k\zeta)$ into the series

$$f(\xi,\zeta) = \sum_{n=0}^{\infty} f_n(\xi)\psi_n(k\zeta) \qquad (2.14)$$

In this connection, the solution of the problem is sought in the form

$$\varphi(\xi,\eta,\zeta) = \sum_{n=0}^{\infty} \psi_n(k\zeta)\chi_n(\xi,\eta) \qquad (2.15)$$

On the basis of equation (2.4) and series (2.14), the boundary condition for the function $\chi_n(\xi,\eta)$, which satisfies equation (2.6), is obtained:

$$\left(\frac{\partial\chi_n}{\partial\eta}\right)_{\eta=0} = f_n(\xi) \qquad (a \leq \xi \leq b) \qquad (2.16)$$

where a and b are determined from equations (2.2). The problem thus reduces to the integration of equation (2.6) for the boundary condition (2.16). By means of the transformation $\chi_n = \xi^{n+1}\chi_n^*$, equation (2.6) reduces to the equation of Darboux for the function $\chi_n^*(\xi,\eta)$ and because in this case n is an integer, the general solution of equation (2.6) can be represented in the form (see reference 5)

$$\chi_n(\xi,\eta) = \xi^{n+1} \Delta^n \frac{U_n(\xi,\eta)}{\xi} \qquad (2.17)$$

where $U_n(\xi,\eta)$ is an arbitrary harmonic function and Δ^n is the Laplace operator of order n.

3. OBTAINING THE HARMONIC FUNCTION U_n

By developing the expression (2.17) and taking into account that $\Delta U_n(\xi,\eta) = 0$, there is obtained

$$X_n(\xi,\eta) = n!\,\xi^{-n}\sum_{m=0}^{n}(-2)^m \frac{(2n-m)!}{m!(n-m)!}\,\xi^m \frac{\partial^m U_n}{\partial \xi^m} \tag{3.1}$$

When condition (2.16) is satisfied, an ordinary differential equation of Euler for $(\partial U_n/\partial \eta)_{\eta=0}$ is obtained:

$$\sum_{m=0}^{n}(-2)^m \frac{(2n-m)!}{m!(n-m)!}\,\xi^m \frac{d^m}{d\xi^m}\left(\frac{\partial U_n}{\partial \eta}\right)_{\eta=0} = \frac{\xi^a f_n(\xi)}{n!} \qquad (a \leq \xi \leq b) \tag{3.2}$$

The general integral of an Euler equation, as is known, has the form

$$\left(\frac{\partial U_n}{\partial \eta}\right)_{\eta=0} = \sum_{m=1}^{n} C_m'\,\xi^{2m-1} + u_n(\xi) \qquad (a \leq \xi \leq b) \tag{3.3}$$

where $u_n(\xi)$ is a particular integral for equation (3.2) and C_m' is an arbitrary constant.

The complex variable $\tau = \xi + i\eta$ is introduced and the function of this variable $W_n(\tau) = U_n + iV_n$ is considered. By making use of the Cauchy-Riemann equations, the boundary conditions that must be satisfied by the function $W_n(\tau)$ are obtained:

$$\left.\begin{aligned} V_n(\xi,0) = v_n(\xi) = \overline{v}_n(\xi) + \sum_{m=0}^{n} C_m \xi^{2m} \\ \left(\overline{v}_n(\xi) = -\int_{\infty}^{\xi} u_n(\xi)\,d\xi\right) \end{aligned}\right\} \tag{3.4}$$

where C_m are new constants of integration and $a \leq \xi \leq b$.

Further, from equations (2.15) and (3.1), it follows that on the Mach cone

$$\varphi(0,\eta,0) = \lim_{\xi\to 0,\zeta\to 0}\sum_{n=0}^{\infty} n!\,\frac{\psi_n(k\xi)}{\xi^n}\sum_{m=0}^{n}(-2)^m\frac{(2n-m)!}{m!(n-m)!}\xi^m\frac{\partial^m U_n}{\partial\xi^m}$$

When the properties of the function $\psi_n(k\zeta)$ and equations (2.1) are taken into account,

$$\varphi(0,\eta,0) = \sum_{n=0}^{\infty} n!(x+y)^n\, U_n(0,\eta)$$

Because on the surface of the Mach cone, the function φ becomes zero, it is necessary that $U_n(0,\eta) = 0$. Moreover, on account of the symmetry

$$U_n(\xi,0) = 0 \qquad (0<\xi<a,\ \xi>b) \tag{3.5}$$

Condition (3.5) permits the analytical continuation of the function $W_n(\tau)$ into the left half-plane.

Finally, the function φ and therefore, according to equation (3.1), the entire function $U_n^{(m)}(\xi,\eta)$ must be finite for $\eta = 0$ and $\xi = a$, and $\xi = b$. This condition will be satisfied if the finiteness of the function $W_n^{(m)}(\tau)$ is required for $m = 0, 1, \ldots, n$. The satisfying of the last condition leads to equations determining the constants C_m in equations (3.4). Thus the following boundary-condition problem of the theory of functions is arrived at: To determine the analytical function $W_n^{(m)}(\tau)$, regular in the upper half-plane, finite at the points $\tau = \pm a$ and $\tau = \pm b$, and satisfying the conditions on the real axis $(\eta = 0)$:

$$\operatorname{Im} W_n^{(m)} = v_n^{(m)}(\xi) \qquad (+a<\xi<+b)$$

$$\operatorname{Re} W_n^{(m)} = 0 \qquad (a>|\xi|,\ |\xi|>b)$$

$$\operatorname{Im} W_n^{(m)} = v_n^{*}(\xi) \qquad (-b<\xi<-a)$$

$$\left(v^{*}(-\xi) = (-1)^m v_n^{(m)}(\xi)\right)$$

By applying the formula of Keldish and Sedov (references 6 and 7),

$$\left.\begin{aligned} W_n^{(m)}(\tau) &= -\frac{2}{\pi i}\sqrt{\frac{\tau^2-a^2}{\tau^2-b^2}}\int_a^b \frac{s v_n^{(m)}(s)}{s^2-\tau^2}\sqrt{\frac{b^2-s^2}{s^2-a^2}}\,ds \quad \text{(m-odd)} \\ W_n^{(m)}(\tau) &= -\frac{2\tau}{\pi i}\sqrt{\frac{\tau^2-a^2}{\tau^2-b^2}}\int_a^b \frac{v_n^{(m)}(s)}{s^2-\tau^2}\sqrt{\frac{b^2-s^2}{s^2-a^2}}\,ds \quad \text{(m-even)} \end{aligned}\right\} \qquad (3.6)$$

Equations (3.6) give for $W_n^{(m)}$ finite values at the points $\tau = \pm a$. The condition of finiteness of $W_n^{(m)}(\tau)$ at $\tau = \pm b$ leads to the system of $n+1$ equations

$$\int_a^b \frac{s^\mu\, v_n^{(m)}(s)\, ds}{\sqrt{(s^2-a^2)(b^2-s^2)}} = 0 \qquad (m=0,1,\ldots,n) \qquad (3.7)$$

where $\mu = 0$ for m even and $\mu = 1$ for m odd.

By computing $v_n^{(m)}(\xi)$ from equations (3.4) and substituting the result in equation (3.7), the system of equation (3.7) can be brought to the form

$$\sum_{l=0}^{n} C_l B_{lm} = D_l \qquad (l=0,1,\ldots,n) \qquad (3.8)$$

where

$$B_{l,2\nu+1} = 2(l-\nu)B_{l,2\nu} = 2l(2l-1)\ldots(2l-2\nu)\,\frac{a^{2(l-\nu)}}{b}\,J_{2(l-\nu)}$$

$$J_{2(l-\nu)} = \int_0^{\frac{1}{k}} \frac{x^{2(l-\nu)}dx}{\sqrt{(x^2-1)(1-\sigma^2x^2)}}$$

$$D_l = -\int_a^b \frac{s^\nu\, \bar{v}_n^{(l)}(s)\, ds}{\sqrt{(s^2-a^2)(b^2-s^2)}} \qquad \left(\sigma = \frac{a}{b},\ \nu = \begin{cases} 0 \text{ for } l \text{ even} \\ 1 \text{ for } l \text{ odd} \end{cases}\right)$$

The integrals J_{2k} can be expressed in terms of complete elliptic integrals of the first and second kind making use of the recurrence relation

$$\left.\begin{aligned} (2l+3)k^2J_{2l+4} &= 2(l+1)(1+k^2)J_{2(l+1)} - (2l+1)J_{2l} \\ J_0 &= K' \\ J_2 &= K' + \left(\frac{k'}{k}\right)^2 E' \\ (k' &= \sqrt{1-k^2}) \end{aligned}\right\} \quad (3.9)$$

where K' and E' are complete elliptic integrals of the first and second kind of the auxiliary modulus k'. All the coefficients B_{lm} of the system (3.8) can thus be computed.

Translated by S. Reiss,
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