NATIONAL ADVISORY COMMITTEE
FOR AERONAUTICS

TECHNICAL MEMORANDUM

No. 1151

LIFT AND DRAG OF WINGS WITH SMALL SPAN*

By F. Weinig

Translation

"Auftrieb und Widerstand des Tragflügels kleiner Spannweite"

Deutsche Luftfahrtforschung, Forschungsbericht Nr. 1665

Washington

August 1947

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Abstract: The lift coefficient of a wing of small span at first
shows a linear increase for the increasing angle of
attack, but to a lesser degree than was to be expected
according to the theory of the lifting line; thereafter
the lift coefficient increases more rapidly than linearly,
as contrasted with the theory of the lifting line. The
induced drag coefficient for a given lift coefficient, on
the other hand, is obviously much smaller than it would
be according to this theory. A small change in the
theory of the lifting line will cover these deviations.

Outline: 1. Symbols
          2. The Previous Treatment of the Wing of Small Span
          3. New Treatment Based upon an Enlarged Array of Free
                  Vortices
          4. Summary
          5. Bibliography

1. SYMBOLS

A   lift

F   wing area $\left(F = \frac{\pi}{4} b\bar{t} = \text{Wing area for elliptic base} \right)$

t   wing chord at any point

$\bar{t}$ wing chord at center of the wing

b   span

a   gap of the cascade which is equivalent to the wing profile
            with respect to the deflected mass (compare equation (8))

v   flight velocity

$\dfrac{\rho}{2} v^2$ stagnation pressure

v_{i_∞} induced downwash velocity far behind the wing

v_i induced downwash velocity at the location of wing $\left(\dfrac{v_{i_\infty}}{2}\right)$

q deflection coefficient of cascade equivalent to the wing

       profile (compare equation (13) $\left(\tan \dfrac{\pi t}{2a}\right)$

$\dot{m}$ mass deflected by wing per unit time

ρ air density

α angle of attack

α_i induced angle of attack $\left(\text{arc tan} \dfrac{v_i}{v_\infty}\right)$

Λ aspect ratio, b^2/F $\left(\Lambda = \dfrac{4}{\pi} \dfrac{b}{t} = \text{Aspect ratio of elliptic}\right.$
       wing$\Big)$

c_a lift coefficient according to modified theory, (c_a) lift
       coefficient according to theory of lifting line

c_{w_i} induced drag coefficient according to modified theory,
       $\left(c_{w_i}\right)$ induced drag coefficient according to theory of
       lifting line

Γ circulation at any point

$\bar{\Gamma}$ circulation at the center of wing

ζ_a ratio of lift according to modified theory to lift according
       to theory of lifting line $(c_a/(c_a))$

ζ_w ratio of induced drag coefficient according to modified
       theory to value according to theory of lifting line
       $\left(c_{w_i}/\left(c_{w_i}\right)\right)$

2. THE PREVIOUS TREATMENT OF THE WING OF SMALL SPAN

The comparatively simple theory of the lifting line leads to
satisfactory results for the wing with an aspect ratio b/t of

sufficient magnitude at least in the case of unseparated flow, but
for the wing with small aspect ratio the results with respect to
lift and induced drag are quite unsatisfactory (1), (2). A simple
modification of this theory (3), (4) brought useful results at least
for very small angles of attack. The transition to the theory of
the lifting surface (5), (6) or to the theory of the acceleration
potential which methodically replaces the latter theory (7), (8),
(9), (10) also provided such results for the range of very small
angles of attack. But these theories disappointed for greater angles
of attack, even while the flow was not yet separated.

It is true the simple modification of the theory of the lifting
line which was mentioned above gave a connection between lift and
induced drag coefficient which seemed still useful for the domain
of larger lift coefficient even though the relation between lift
and angle of attack itself was not clarified (11).

For small angles of attack the lift at first increases linearly
with the angle of attack, but the increment grows with increasing
angles of attack when the aspect ratio is small, and the smaller
the aspect ratio the more noticeable is this influence (12). Like-
wise the induced drag at small aspect ratios is obviously considerably
smaller than resulted from the original theory of the lifting line (13).

The conjecture was brought forward that an explanation for these
conditions might perhaps be found in the fact that the vortex area
leaving the wing was somewhat wider than the span (14). Hereby
one could explain the reduction of the induced drag, but not the
behavior of the lift coefficient. Neither can the downwash angle
sufficiently far behind the wing, on the other hand, be larger than
the angle of attack, at the most.

3. NEW TREATMENT BASED UPON AN ENLARGED ARRAY OF FREE VORTICES

There is to be found, however, a natural explanation for
the actual behavior of the wing of small span: One has to start
from the fact that the effective mass which is deflected downward
by the wing is greater than resulted from the original theory of
the lifting line for elliptic lift distribution. according to this
theory, this mass would correspond to the mass of a cylinder of air
density with a diameter equivalent to the span. However, the path
of a wing must no longer be treated as a flat plane as it was in
the case of a large span but rather as a prism with the cross

section $F \sin \alpha$. The mass contained therein, however, must also be deflected downward[1], at least approximately. (See fig. 1.)

With v_{i_∞} as downwash velocity far behind the wing and with

$$\dot{m} = \rho v \left(\frac{\pi}{4} b^2 + F \sin \alpha \right) \qquad (1)$$

as the mass deflected downward per unit time the lift will be, according to the momentum theorem,

$$A = \dot{m} v_{i_\infty} = \rho v \left(\frac{\pi}{4} b^2 + F \sin \alpha \right) \left(v_{i_\infty} \right) \qquad (2)$$

Or, with

$$v_i = \frac{v_{i_\infty}}{2} = v \tan \alpha_i \qquad (3)^2$$

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[1]Mangler (15) shows another way of treatment of this influence. According to his theory, the effect upon the outside flow of the lateral rolling up of the vortex sheet can be replaced by end plates, but this way leads to essentially equivalent results to the one which was used here. Furthermore we should like to point out the treatment by W. Bollay (16); there also the free vortices behind the wing do not lie in a plane. However, the results of the present treatment are much simpler and at least as satisfactory; also they are more readily applicable to any shape of wings.

[2]It is a matter of indifference whether

$$v_i = \frac{v_{i_\infty}}{2} = v \tan \alpha_i \quad \text{or} \quad v_i = \frac{v_{i_\infty}}{2} = v \sin \alpha_i$$

and

$$\tan \alpha_i = \frac{q}{1 + q} \tan \alpha \quad \text{or} \quad \sin \alpha_i = \frac{q}{1 + q} \sin \alpha$$

is introduced for small angles of attack. For larger angles of attack, however, better agreement of the results of the calculation and the measurement is obtained by the use of the first-mentioned form only.

as effective downwash velocity at the location of the wing and the
aspect ratio

$$\frac{b^2}{F} = \Lambda \tag{4}$$

$$A = \frac{\rho}{2} v^2 F \left(\frac{\pi}{4} \Lambda + \sin \alpha \right) 4 \tan \alpha_1 \tag{5}$$

A lift coefficient of

$$c_a = \tan \alpha_1 \left(\pi \Lambda + 4 \sin \alpha \right) = 2\pi \tan \alpha_1 \left(\frac{\Lambda}{2} + \frac{2}{\pi} \sin \alpha \right) \tag{6}$$

would result. In this equation,

$$\tan \alpha_1 = \frac{q}{1 + q} \tan \alpha \tag{7}^{[3]}$$

The deflection coefficient q results from the following consideration
(compare (3) and (4)): To each wing element a flow domain directly
influenced by it is coordinated; this flow domain is bounded in a
first approximation by streamlines as figure 2 shows. This flow
domain is replaced by a strip whose width equals the width of the
wing element and whose length is such as to make the kinetic energy
conveyed by the wing element equivalent to the quantity of energy
which is conveyed to the coordinated flow domain. The length of
this strip will be

$$a = \frac{\Gamma}{v_{1_\infty}} + t \sin \alpha \tag{8}$$

Accordingly, the wing profile influences the coordinated flow like
an identical profile arranged in a cascade (fig. 3) of a stagger
of 90° and a gap a. With

$$\begin{aligned} \Gamma &= \bar{\Gamma} \sin \vartheta \\ t &= \bar{t} \sin \vartheta \end{aligned} \quad \left(\cos \vartheta = \frac{y}{b/2} \right) \tag{9}$$

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[3]Same as footnote 2.

the gap ratio becomes

$$\frac{a}{t} = \frac{\Gamma}{t \, v_{1_\infty}} + \sin \alpha = \frac{\bar{\Gamma}}{t \, V_{1_\infty}} + \sin \alpha \qquad \cdot (10)$$

or, for elliptic lift and circulation distribution, respectively, with

$$v_{1_\infty} = \frac{\Gamma}{t \, v_{1_\infty}} \qquad \Lambda = \frac{4}{\pi} \frac{b}{t} \qquad (11)$$

$$\frac{a}{t} = \frac{b}{t} + \sin \alpha = \frac{\pi}{4} \Lambda + \sin \alpha \qquad (12)$$

The value q becomes, according to the results of the theory of the cascade flow,

$$q = \tan \frac{\pi}{2} \frac{t}{a} = \tan \frac{1}{\frac{\Lambda}{2} + \frac{2}{\pi} \sin \alpha} \qquad \checkmark \qquad (13)$$

and therefore

$$\tan \alpha_i = \frac{\tan \dfrac{1}{\frac{\Lambda}{2} + \frac{2}{\pi} \sin \alpha}}{1 + \tan \dfrac{1}{\frac{\Lambda}{2} + \frac{2}{\pi} \sin \alpha}} \tan \alpha \qquad (14)$$

As a lift coefficient, there results

$$c_a = 2\pi \frac{\tan \dfrac{1}{\frac{\Lambda}{2} + \frac{2}{\pi} \sin \alpha}}{1 + \tan \dfrac{1}{\frac{\Lambda}{2} + \frac{2}{\pi} \sin \alpha}} \left(\frac{\Lambda}{2} + \frac{2}{\pi} \sin \alpha \right) \tan \alpha \qquad (15)$$

The induced drag coefficient becomes

$$c_{w_1} = c_a \tan \alpha_1 = 2\pi \left(\frac{\tan \dfrac{1}{\dfrac{\Lambda}{2} + \dfrac{2}{\pi}\sin\alpha}}{1 + \tan\dfrac{1}{\dfrac{\Lambda}{2} + \dfrac{2}{\pi}\sin\alpha}} \right)^2 \left(\frac{\Lambda}{2} + \frac{2}{\pi}\sin\alpha\right)\tan^2\alpha \quad (16)$$

or

$$c_{w_1} = \frac{c_a^2}{\pi\Lambda + 4\sin\alpha} \quad (16a)$$

There results the ideal normal-force coefficient

$$c_n = c_a \cos\alpha + c_{w_1}\sin\alpha$$

$$c_n = 2\pi \frac{\tan\dfrac{1}{\dfrac{\Lambda}{2} + \dfrac{2}{\pi}\sin\alpha}}{1 + \tan\dfrac{1}{\dfrac{\Lambda}{2} + \dfrac{2}{\pi}\sin\alpha}} \left(\frac{\Lambda}{2} + \frac{2}{\pi}\right)$$

$$\sin\alpha \left(1 + \frac{\tan\dfrac{1}{\dfrac{\Lambda}{2} + \dfrac{2}{\pi}\sin\alpha}}{1 + \tan\dfrac{1}{\dfrac{\Lambda}{2} + \dfrac{2}{\pi}\sin\alpha}} \frac{\sin^2\alpha}{\cos\alpha} \right) \quad (17)$$

In the boundary case of very small span there results for small angles of attack

$$:c_n \big/ \Lambda \to 0 = 2 \sin^2 \alpha \qquad (17a)$$

In the case of large span, the result for small angles of attack is,

$$c_n \big/ \Lambda \to 0 = 2\pi \sin \alpha \cos \alpha \qquad (17b)$$

In figures 4(a), 4(b) and 5(a), 5(b) the results of this calculation are represented for comparison with test results (12): figures 4(a) and 4(b) show the polars $c_a(c_w)$ according to test and $c_a(c_{w_i})$ according to calculation, respectively, (figs. 5(a) and 5(b)) the normal force coefficients $c_n(\alpha)$ as a function of the angle of attack according to test (fig. 5(a)) and to calculation (fig. 5(b)). The agreement is relatively good. The deviations are probably caused by the influence of friction.

From the original theory of the lifting line there results the lift coefficient

$$(c_a) = \frac{2\pi}{1 + \dfrac{2}{\Lambda}} \sin \alpha \qquad (18)$$

and the induced drag coefficient, for a given c_a,

$$\left(c_{w_i} \right) = \frac{c_a{}^2}{\pi \Lambda} \qquad (19)$$

The resulting ratios are, for the lift coefficient,

$$\zeta_a = \frac{c_a}{(c_a)} = \frac{\tan\dfrac{1}{\dfrac{\Lambda}{2} + \dfrac{2}{\pi}\sin\alpha}}{1 + \tan\dfrac{1}{\dfrac{\Lambda}{2} + \dfrac{2}{\pi}\sin\alpha}} \quad \frac{\dfrac{\Lambda}{2} + \dfrac{2}{\pi}\sin\alpha}{\cos\alpha}\left(1 + \frac{2}{\Lambda}\right) \tag{20}$$

and for the induced drag coefficient, for a given lift coefficient,

$$\zeta_w = \frac{c_{w_1}}{\left(c_{w_1}\right)} = \frac{\pi\Lambda}{\pi\Lambda + 4\sin\alpha} \tag{21}$$

Figure 6 shows the value ζ_a for various ratios Λ and angles of attack α.

The value ζ_w as a function of c_a for various ratios is represented in figure 7(b). The relation between the value c_a and the ratio Λ was determined therefrom; figure 8(b) shows this relation for various lift coefficients.

The value ζ_w can also be determined by use of the test results represented in figures 4(a) and 5(a) with the resulting relation between angle of attack α and lift coefficient, and by means of the relations

$$c_{w_1} = c_a\tan\alpha_1 = c_a\tan\alpha\,\frac{\tan\dfrac{1}{\dfrac{\Lambda}{2} + \dfrac{2}{\pi}\sin\alpha}}{1 + \tan\dfrac{1}{\dfrac{\Lambda}{2} + \dfrac{2}{\pi}\sin\alpha}} \approx c_a\tan\alpha\,\frac{\tan\dfrac{1}{\Lambda/2}}{1 + \tan\dfrac{1}{\Lambda/2}}$$

$$c_{w_1} = \frac{c_a^{\,2}}{\pi\Lambda}$$

There results a value

$$\zeta_w = \frac{\pi\Lambda}{c_a} \frac{\tan \dfrac{1}{\dfrac{\Lambda}{2} + \dfrac{2}{\pi}\sin\alpha}}{1 + \tan \dfrac{1}{\dfrac{\Lambda}{2} + \dfrac{2}{\pi}\sin\alpha}} \tan\alpha \approx \frac{\pi\Lambda}{c_a} \frac{\tan \dfrac{1}{\Lambda/2}}{1 + \tan \dfrac{1}{\Lambda/2}} \tan\alpha \qquad (21a)$$

The approximation indicated above was used for the evaluation;
the values neglected in this approximation were very small.
Based upon the connection between c_a and α found by test,
the relations represented in figures 7(a) and 8(a) are determined.

The comparison of figures 7(a) and 8(a) with figures 7(b) and 8(b)
shows certain differences which also may probably be traced mainly
to influences of friction. However, these differences are still so
small as to be negligible.

For large aspect ratios also there results an influence upon
lift coefficient and induced drag coefficient within the range of
the usual angles of attack. The influence on the lift coefficient
will, generally, be negligible; however, such a neglect might not
always be permissible for the induced drag coefficient; for
$\Lambda \gtrsim 5$ c_{wi} may be estimated

$$c_{wi} = \frac{c_a^2}{\pi\Lambda} \left(1 - \frac{2c_a}{\pi^2\Lambda}\right)$$

The results were derived for untwisted wings with elliptic base
planes; however, the proportionality factors ζ_a and ζ_w will
probably furnish results which can also be applied to wings of
different design as shown by a comparison with the test results
for rectangular wings.

4. SUMMARY

The lift coefficient and the induced drag coefficient for a
wing of small span may be determined with sufficient accuracy, as

long as the flow is not separated, in the following way: A three-
dimensional array of free vortices which corresponds to the angle
of attack is taken as a base and a finite flow domain is coordinated
to each wing element. The theory of the lifting line will have to
be changed slightly. Thereby an explanation will, in particular, be
given for the rapid, nonlinear increase of the lift coefficient
which corresponds to an increase of the angle of attack, and for the
relative smallness of the induced drag as compared with the results
of the unmodified theory of the lifting line. These results will
have special importance for the calculation of the lift of the
fuselage and of the lift and drag of vertical tail surfaces.

Translated by Mary L. Mahler
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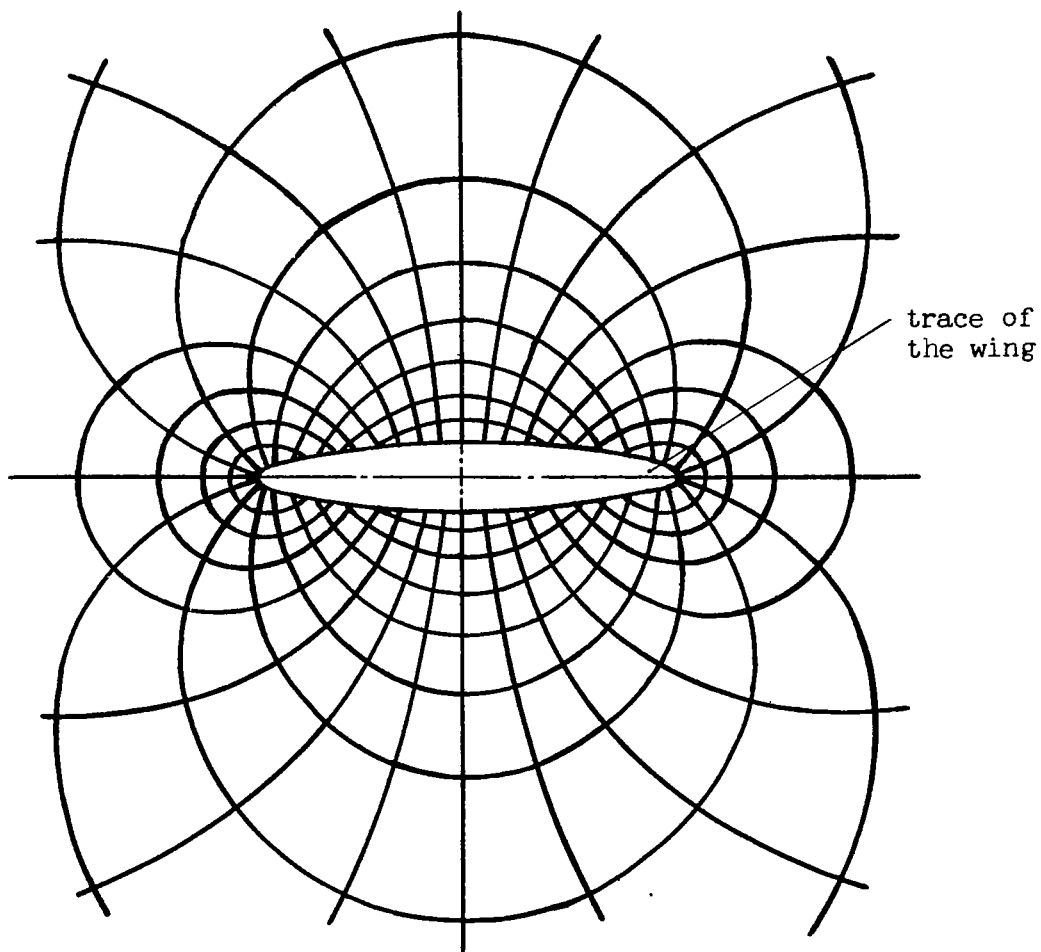   Journal of Aero. Sci   Vol. 4 No. 7   May 1937

trace of
the wing

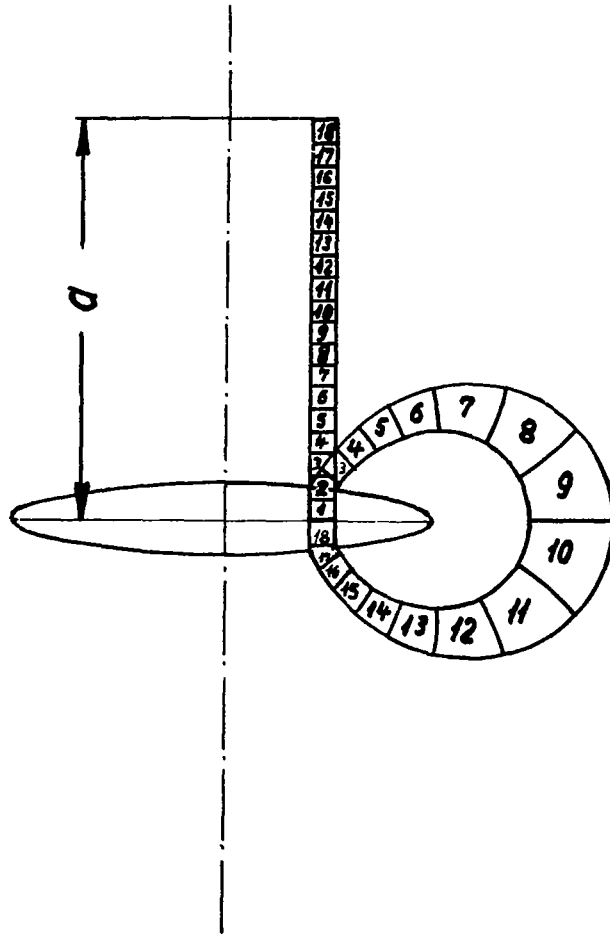Figure 1. Flow about array of vortices behind an elliptic wing of small span.

Fig. 2   NACA TM No. 1151

Figure 2. The flow domain influenced by a wing element.
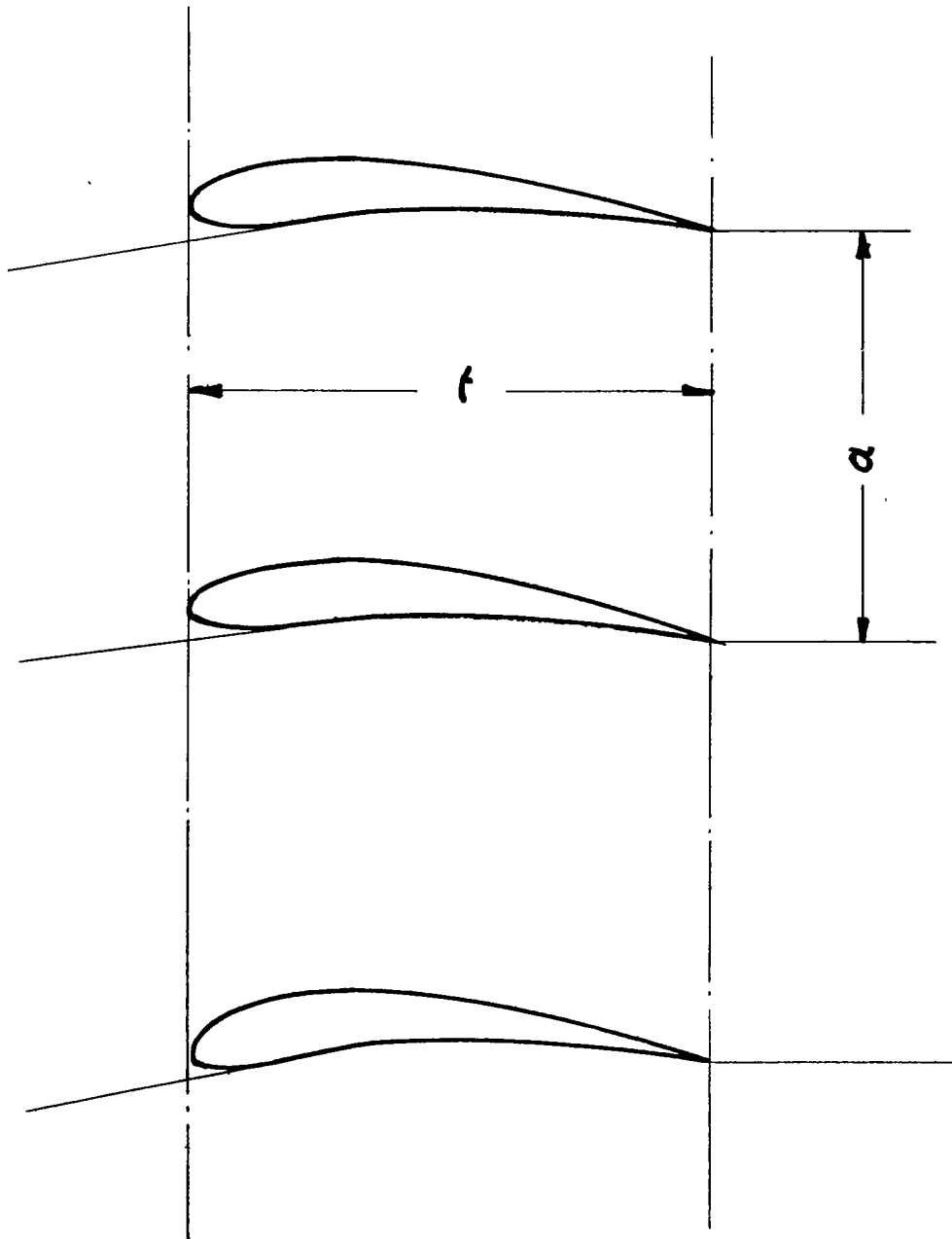
Figure 3. The profile cascade which replaces the wing profile in its effect.
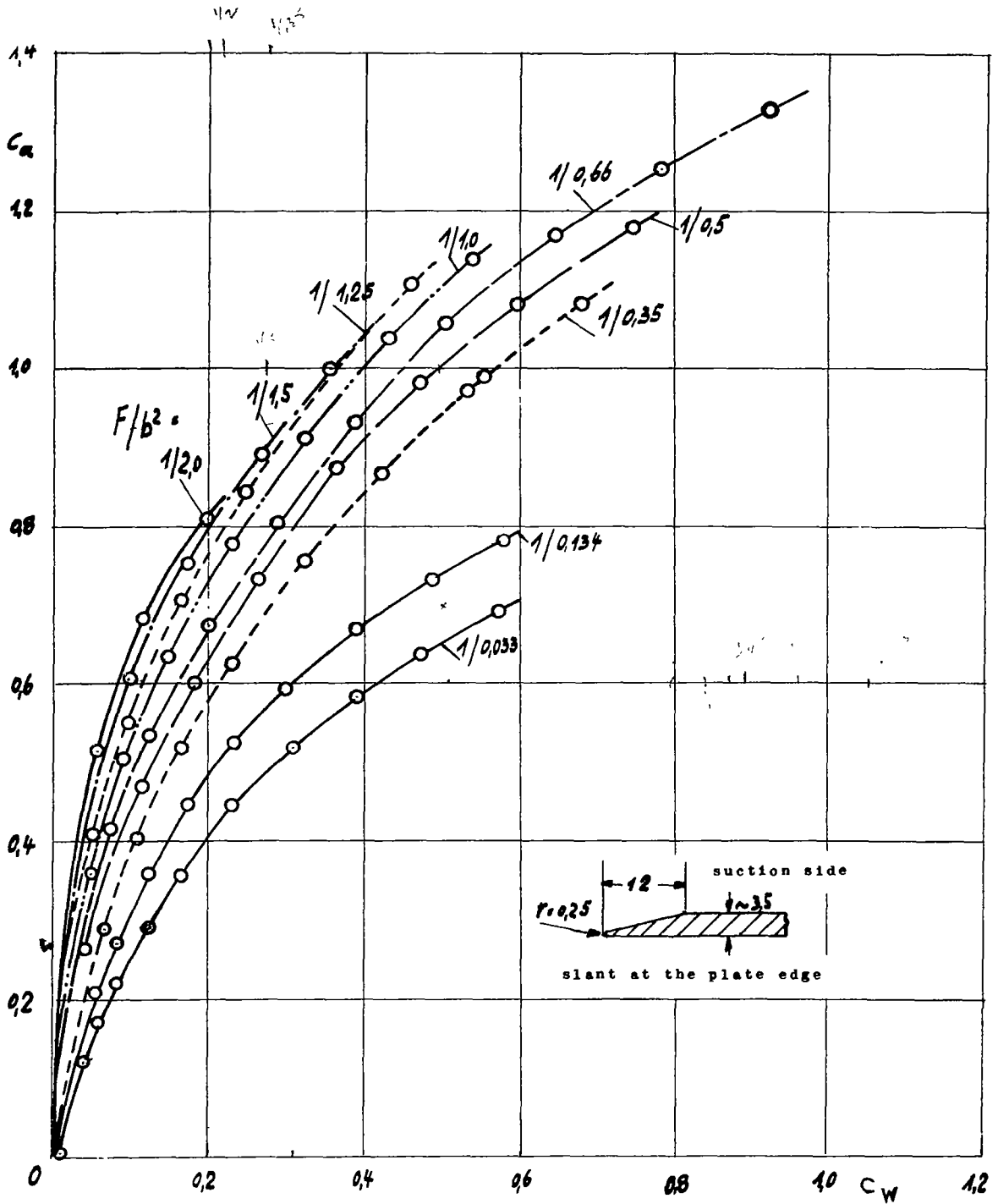
Figure 4a. Polars for wings with small aspect ratio. Drag and lift coefficient
according to measurement.
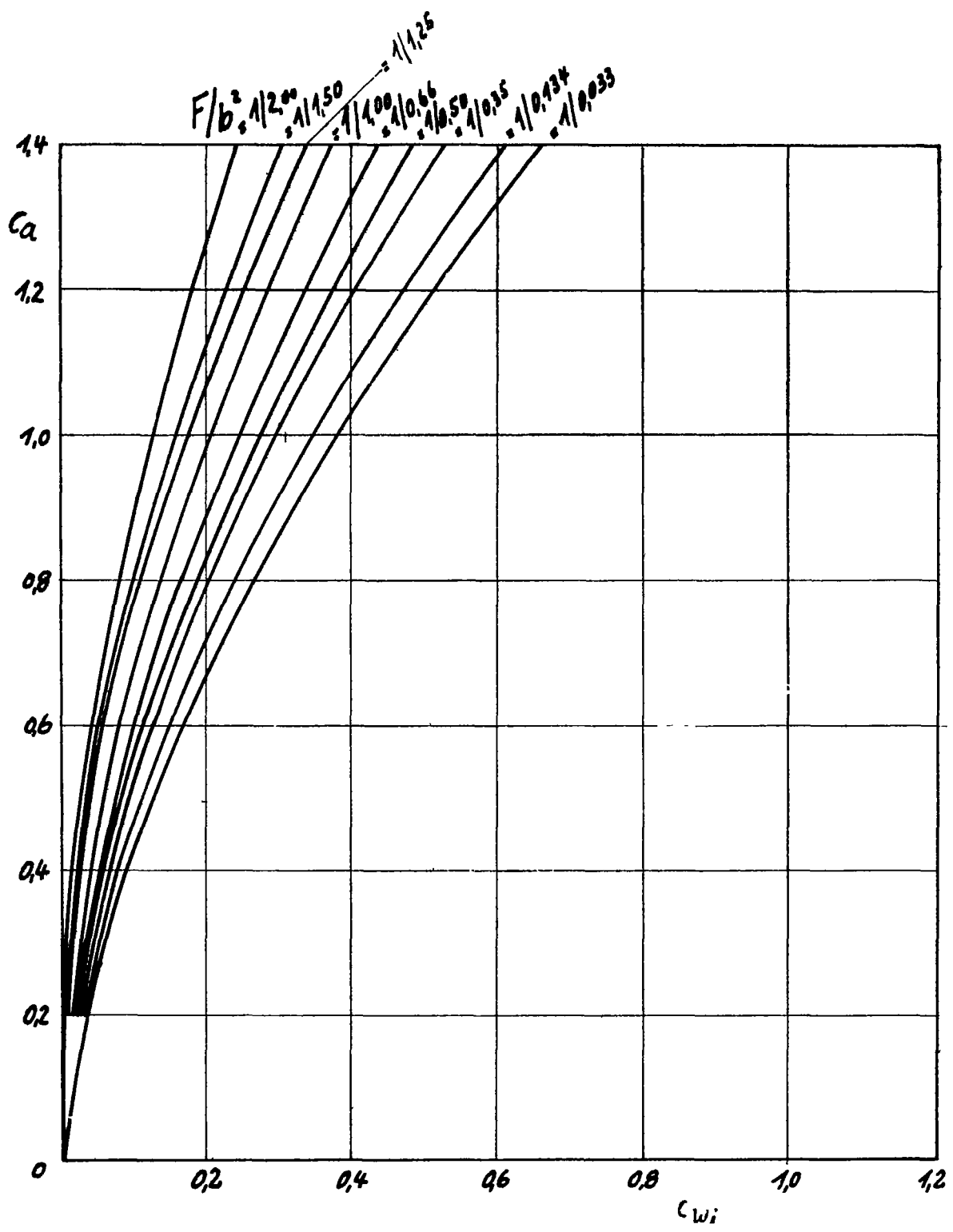
Figure 4b. Polars for wings with small aspect ratio. Induced drag coefficient
and lift coefficient according to calculation.
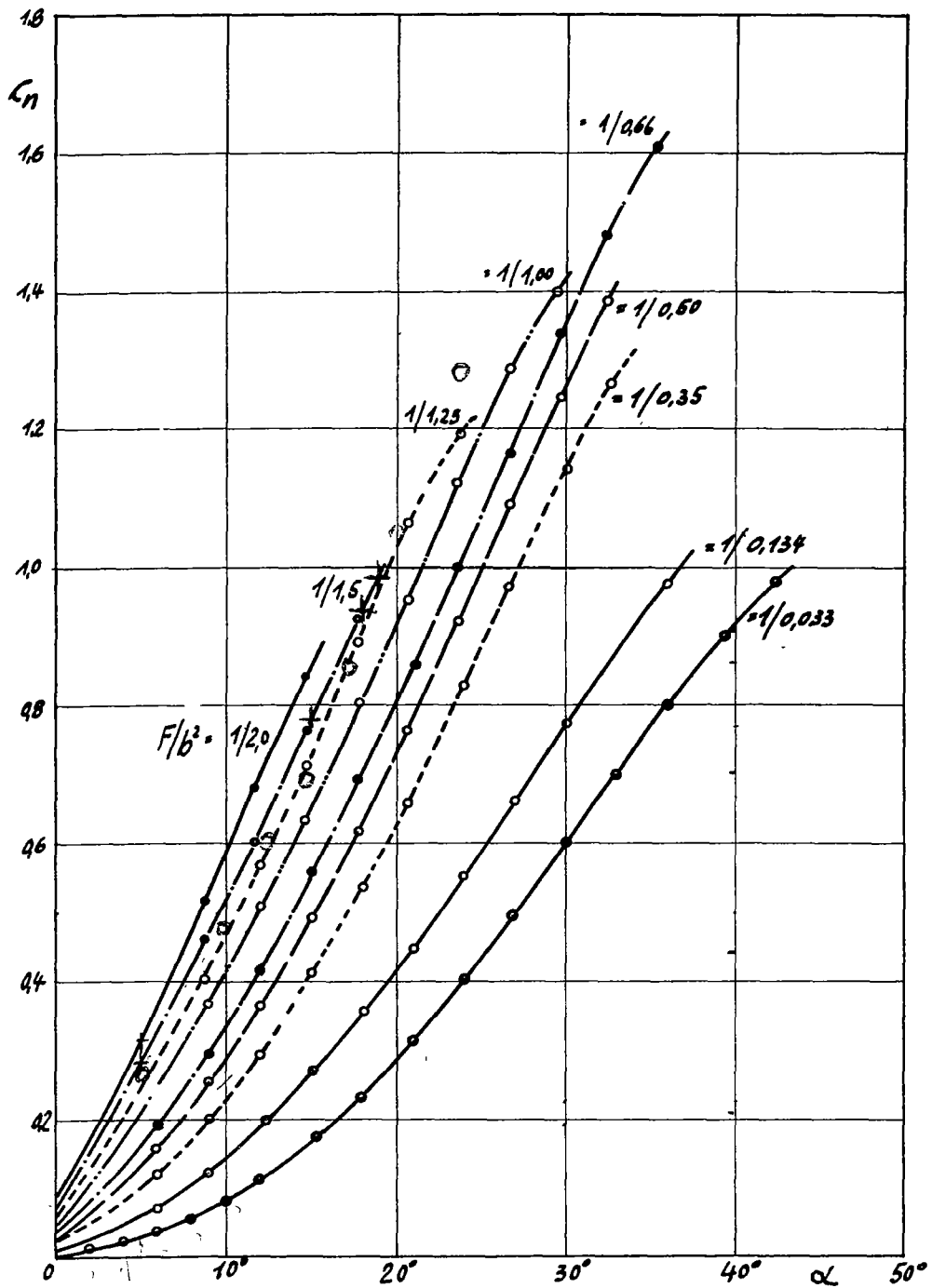
Figure 5a. Normal force coefficient for wing with small aspect ratio as a
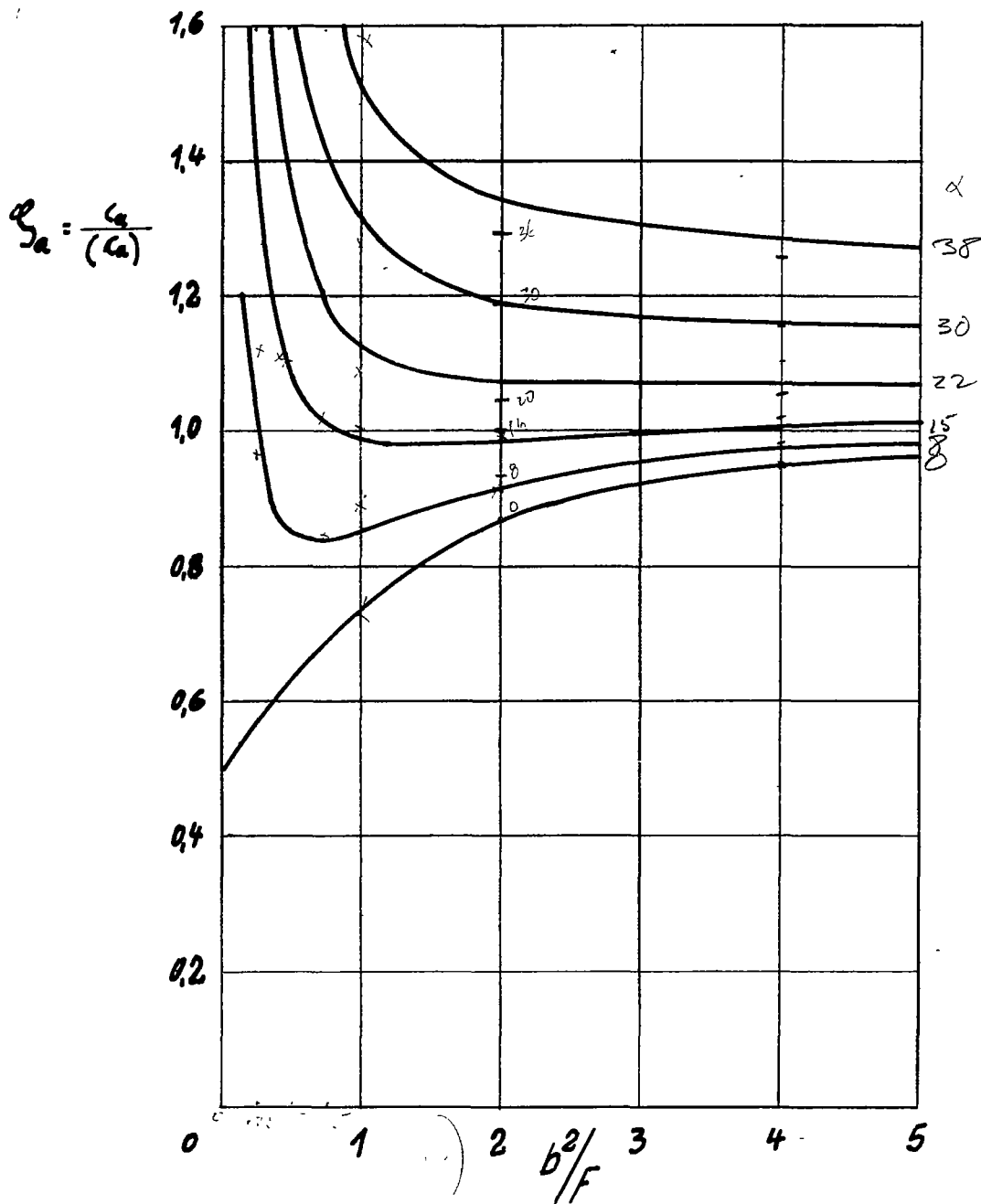function of the angle of attack according to measurement.
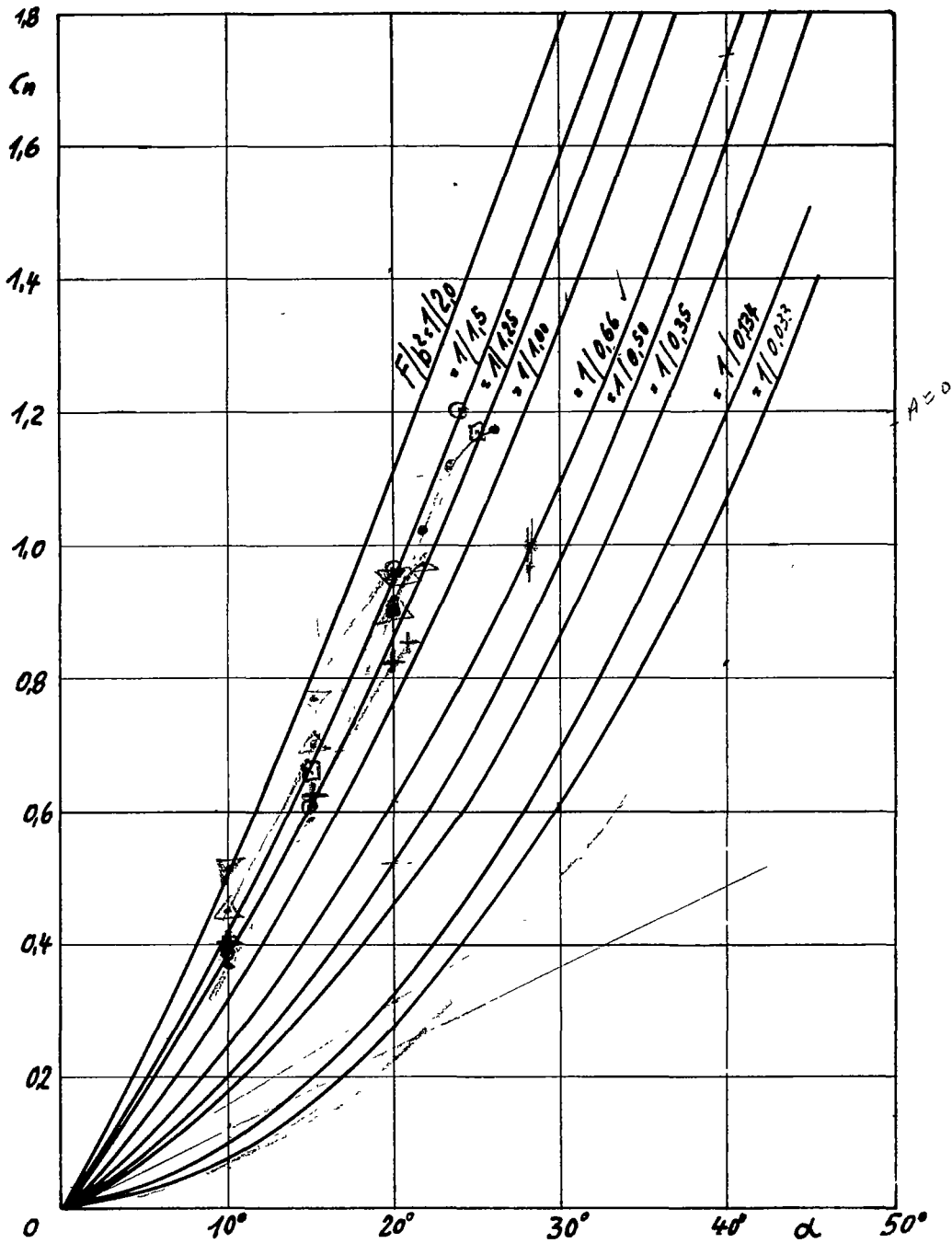
Figure 5b. Normal force coefficient for wing with small aspect ratio as a
function of the angle of attack according to calculation.

Fig. 6   NACA TM No. 1151

Figure 6.- Ratio $\zeta_a = \dfrac{c_a}{(c_a)}$ (ratio of the lift coefficient c_a, according to the modified theory for small span, to the lift coefficient (c_a), according to the theory for large span wings) as a function of the aspect ratio for various angles of attack.
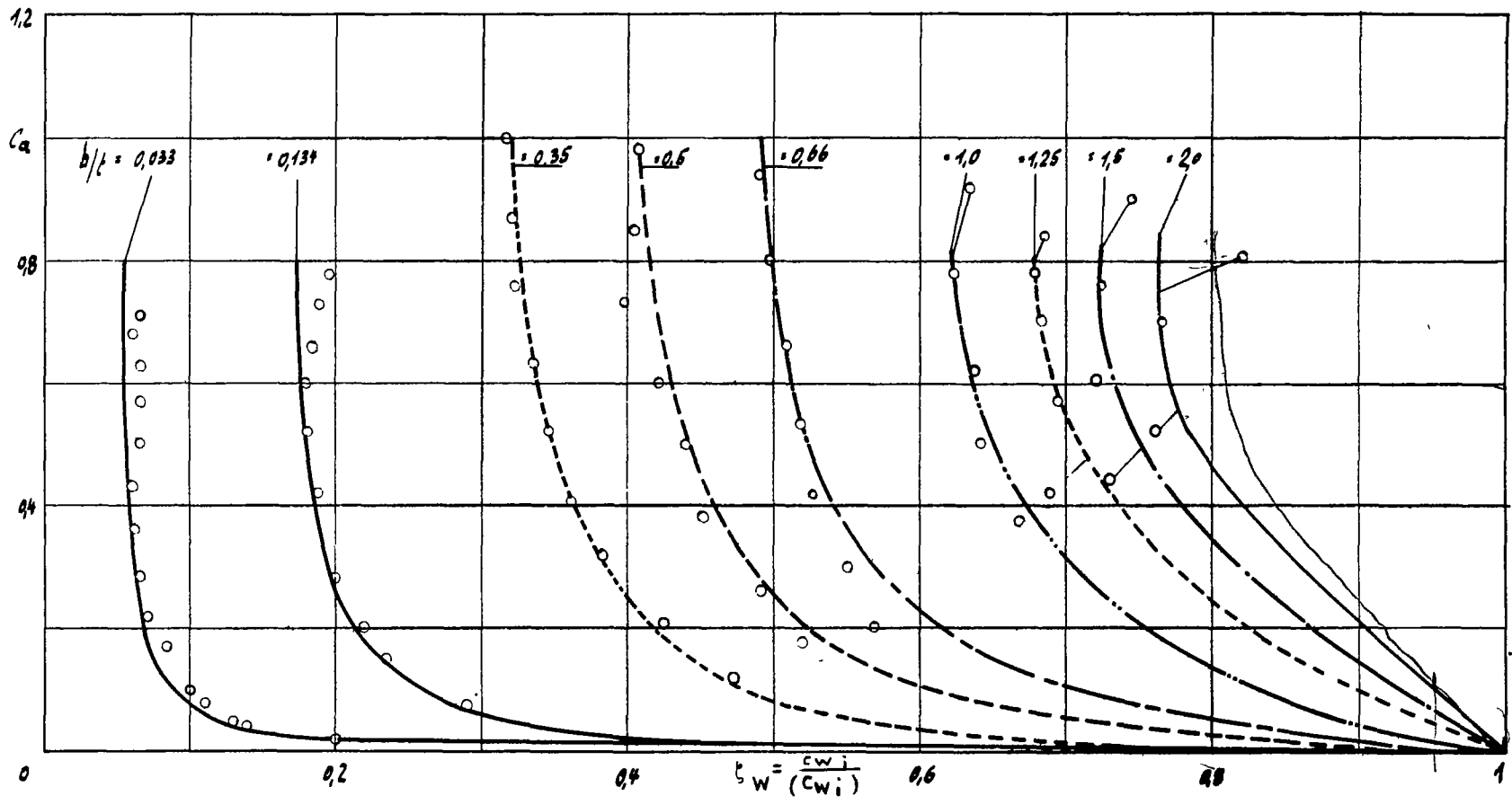
Figure 7a.- Ratio $\zeta_w = \dfrac{c_{w_i}}{(c_{w_i})}$ (ratio of the induced drag coefficient c_{w_i}, according to the theory modified for small span, to the induced drag coefficient (c_{w_i}), according to the theory for large span wings) as a function of the lift coefficient c_a for various aspect ratios, based upon the measured relation between lift coefficient and angle of attack.
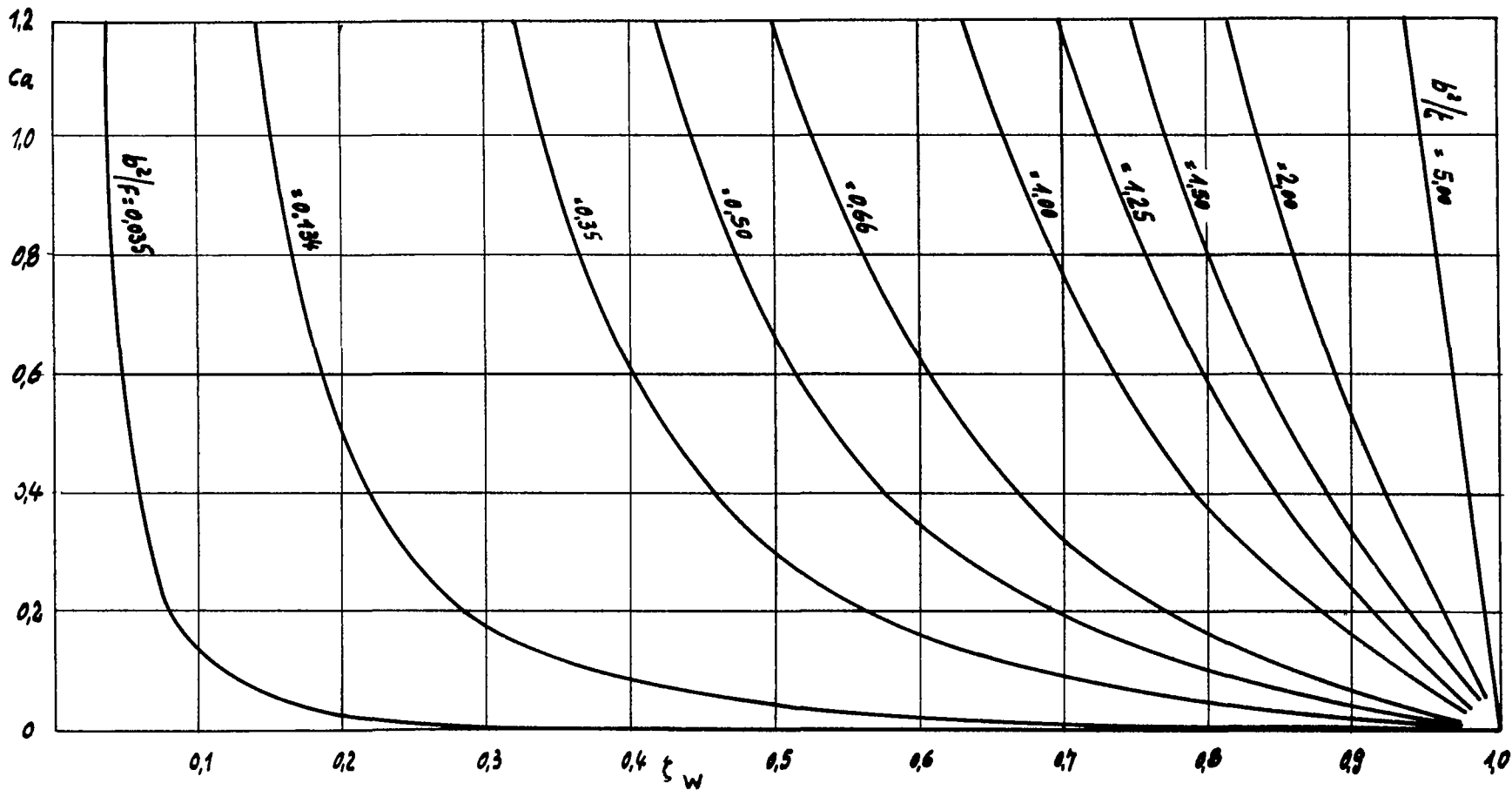
Figure 7b.- The agreement between the ratio $\zeta_w = \dfrac{c_{w_i}}{\left(c_{w_i}\right)}$ (ratio of the induced drag

coefficient c_{w_i}, according to the theory modified for small span, to the induced drag

coefficient $\left(c_{w_i}\right)$, according to the theory for large span wings), shown as a function

of the lift coefficient c_a for various aspect ratios, and the calculated relation between
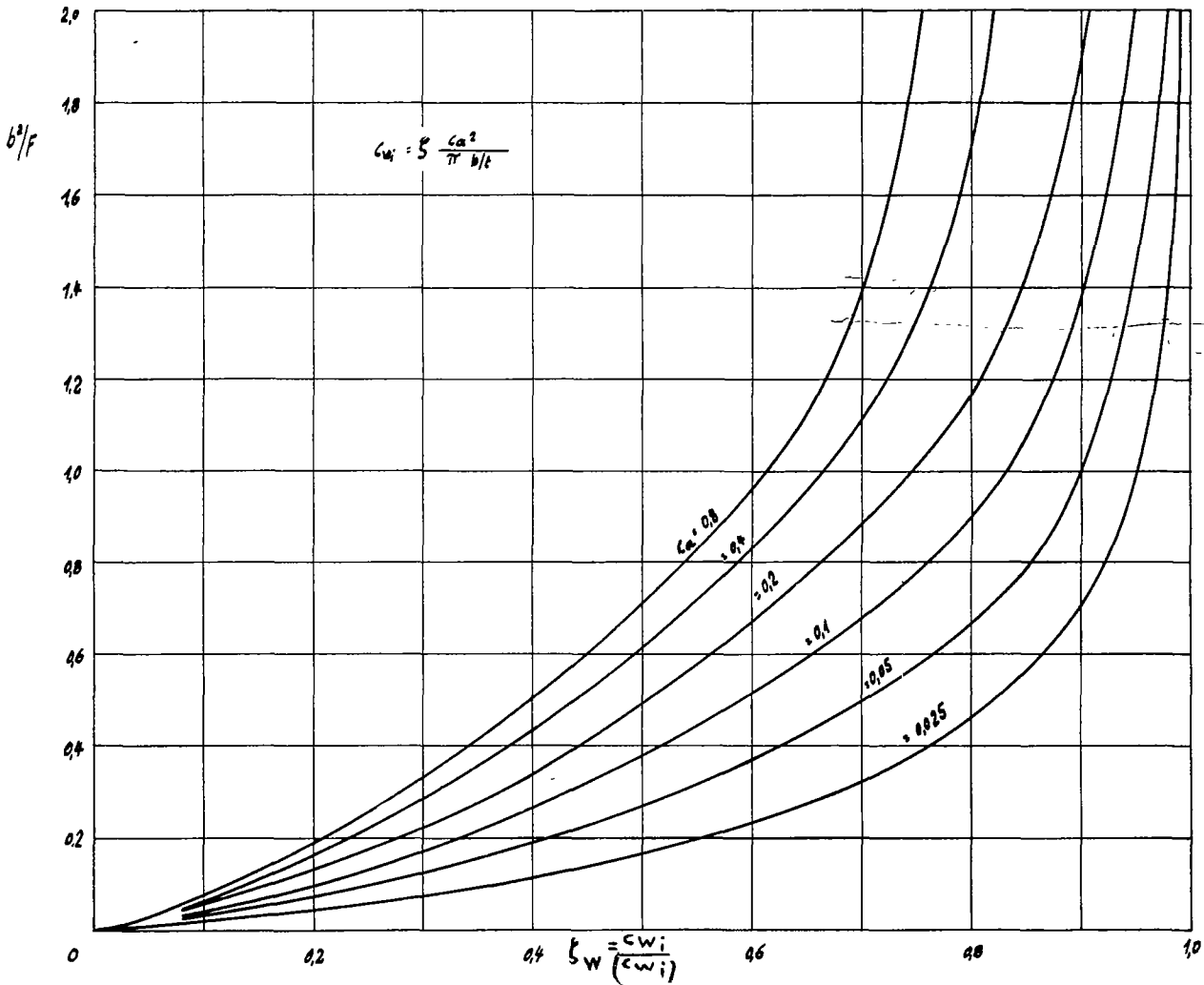lift coefficient and angle of attack.

Figure 8a.- Ratio $\zeta_W = \dfrac{c_{W_i}}{\left(c_{W_i}\right)}$ (ratio of the induced drag coefficient c_{W_i}, according

to the theory modified for small span, to the induced drag coefficient $\left(c_{W_i}\right)$,

according to the theory for large span wings), as a function of the aspect ratio for various lift coefficients based upon the measured relation between lift coefficient and angle of attack.
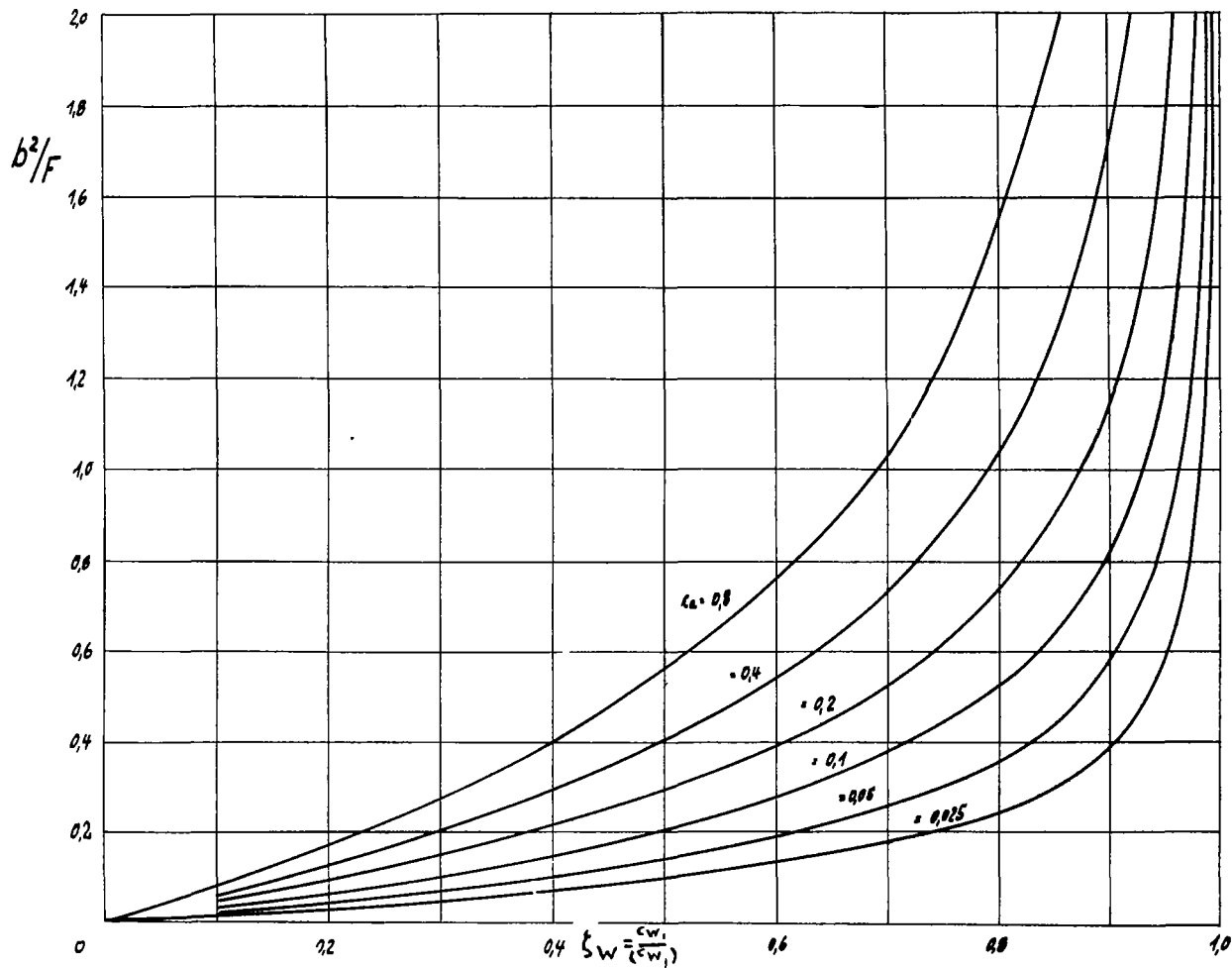
Figure 8b.- Ratio $\zeta_W = \dfrac{c_{w_i}}{(c_{w_i})}$ (ratio of the induced drag coefficient c_{w_i}, according

to the theory modified for small span, to the induced drag coefficient $\left(c_{w_i}\right)$,

according to the theory for large span wings), as a function of the aspect ratio for various lift coefficients in agreement with the calculated relation between lift coefficient and angle of attack.