NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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THE WALL INTERFERENCE OF A WIND TUNNEL OF ELLIPTIC CROSS SECTION*

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The wall interference is obtained for a wind tunnel of elliptic section for the two cases of closed and open working sections. The approximate and exact methods used gave results in practically good agreement. Corresponding to the result given by Glauert for the case of the closed rectangular section, the interference is found to be a minimum for a ratio of minor to major axis of $1:\sqrt{2}$. This, however, is true only for the case where the span of the airfoil is small in comparison with the width of the tunnel. For a longer airfoil the favorable ellipse is flatter. In the case of the open working section the circular shape gives the minimum interference.

INTRODUCTION

The wall interference exerted on a model during wind tunnel tests has been given, for the case of a circular boundary, by Prandtl (reference 1), for the closed rectangular boundary approximately by Glauert (reference 2), and accurately by Terazawa (reference 3). Recently Theodorsen (reference 4) combining the cases of open and closed sections computed the wall-interference coefficient for the rectangular wind tunnel. However, apart from the tests of Knight and Harris (reference 5) on the expected wall interference of the projected large-scale wind tunnel of elliptic section, there have as yet appeared no investigations on the elliptic section. Whether for a large-scale tunnel or one of usual size the determination of the wind-tunnel correction for free-air conditions is an important problem.

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The problem of the interference exerted by the tunnel walls is treated with the aid of the known results of the vortex-filament thoery applied to the boundary. In the present paper the wall interference is treated as a Dirichlet problem. Depending on whether the boundary conditions are completely satisfied or satisfied at only a determined number of points an exact or approximate solution, respectively, is obtained. The reason that both are given is to show that for practical purposes the difference in the results obtained is negligible and to emphasize the usefulness of the latter method when the former is not available.

The closed and open working sections are characterized by the conditions that the normal and tangential components of the velocity at the wall are, respectively, zero. The former is represented by the N.P.L. wind tunnel, the latter by the Göttingen and Eiffel tunnels.

APPROXIMATE METHOD FOR THE CLOSED WORKING SECTION

Figure 1 shows the two symmetrical vortices for the case of the closed section of elliptic boundary. The complex potential $f = \varphi + i\psi$ of the flow within the ellipse is divided into two parts: f_1 due to the effect of the vortex alone, and f_2 due to the existence of the wall. For f_1 it is clear that

$$f_1 = \frac{i\Gamma}{2\pi} \log \frac{z+s}{z-s} \qquad (1)$$

f_2 is as yet unknown, but since it must be an analytic complex function in the field under consideration, put

$$f_2 = \alpha_0 + i\beta_0 + \sum_{n=1}^{\infty} (\alpha_n + i\beta_n) z^n \qquad (2)$$

where α_n, β_n $(n = 0, 1, 2)$ are real constants. Introducing polar coordinates (r, θ) with the aid of the relation $z = re^{i\theta}$ the velocity components in the directions of r and θ from (1) and (2) are obtained, respectively, as

$$
\left.
\begin{aligned}
v_r &= \frac{\partial\phi}{\partial r} = -\frac{\Gamma s}{\pi}\,\frac{(r^2+s^2)\sin\theta}{(r^2-s^2)^2+4r^2s^2\sin^2\theta} \\
&\quad + \sum_{n=1}^{\infty} nr^{n-1}(\alpha_n\cos n\theta - \beta_n\sin n\theta) \\
v_\theta &= \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{\Gamma s}{\pi}\,\frac{(r^2-s^2)\cos\theta}{(r^2-s^2)^2+4r^2s^2\sin^2\theta} \\
&\quad - \sum_{n=1}^{\infty} nr^{n-1}(\alpha_n\sin n\theta + \beta_n\cos n\theta)
\end{aligned}
\right\} \quad (3)
$$

Hence, if λ is the angle made by the normal of the ellipse with the radius vecter, the normal component of the velocity at the boundary is

$$
\begin{aligned}
v_N &= \Big[v_r\Big]_{r=\sigma a}\cos\lambda + \Big[v_\theta\Big]_{r=\sigma a}\sin\lambda \\
&= -\frac{\Gamma s}{\pi a^2}\,\frac{\sigma^2\sin(\theta-\lambda)+k^2\sin(\theta+\lambda)}{(\sigma^2-k^2)^2+4\sigma^2k^2\sin^2\theta} \\
&\quad + \sum_{n=1}^{\infty} na^{n-1}\alpha_n\sigma^{n-1}\cos(n\theta+\lambda) \\
&\quad - \sum_{n=1}^{\infty} na^{n-1}\beta_n\sigma^{n-1}\sin(n\theta+\lambda)
\end{aligned}
\qquad (4)
$$

where

$$
\left.
\begin{aligned}
&k = s/a,\ \sigma = [(1-\epsilon^2)/(1-\epsilon^2\cos^2\theta)]^{1/2} \\
&\tan\lambda = \epsilon(1-\epsilon^2)^{1/2}\sin 2\theta/2(1-\epsilon^2\cos^2\theta) \\
&\epsilon = (1-b^2/a^2)^{1/2} = \text{eccentricity}
\end{aligned}
\right\} \quad (5)
$$

since there is no normal velocity component at the boundary for the closed working section $v_N = 0$, so that from this condition the constants α_n, β_n are determined. From the condition of symmetry there is immediately $\alpha_n = 0$ and, moreover, β_n vanishes for even n. Hence writing

$$na^{n-1}\beta_n = -\Gamma s B_n/\pi a^2 \qquad (6)$$

gives the equation for determining the constants

$$B_1 \sin(\theta+\lambda) + B_3\sigma^2 \sin(3\theta+\lambda) + B_5\sigma^4 \sin(5\theta+\lambda) + \ldots = \frac{\sigma^2 \sin(\theta-\lambda) + k^2 \sin(\theta+\lambda)}{(\sigma^2-k^2)^2 + 4\sigma^2k^2\sin^2\theta} \qquad (7)$$

In this relation θ takes all values from 0 to 2π. Since λ and σ are functions of θ, it is difficult to solve this relation accurately. However, for special values of θ satisfying this condition a relatively easy approximate computation can be made. Since, on account of the symmetry, it is sufficient to consider only one quadrant of figure 1, it is possible to choose, for example, for θ the four values 15°, 30°, 60°, and 90°, keep four terms of series on the left side of (7) and obtain four independent first-degree equations from which to solve for B_1, B_3, B_5, and B_7. Since f_2 is not a very rapidly convergent series the number of equations may be large. It should be checked to see whether the number of values chosen is sufficient.

Now that the constants have been determined the wall interference may be computed. The induced velocity due to the wall in the direction y at right angles to the major axis x is

$$w = \left[\frac{1}{r}\frac{\partial\Phi_2}{\partial\theta}\right]_{\theta=0} = -\beta_1 - 3x^2\beta_3 - 5x^4\beta_5 - \ldots \qquad (8)$$

Take the average between the limits $x = \pm s$

$$\overline{w} = \frac{1}{2s}\int_{-s}^{s} w\,dx = \frac{\Gamma s}{\pi a^2}\left(B_1 + \frac{1}{3}k^2 B_3 + \frac{1}{5}k^4 B_5 + \ldots\right) \qquad (9)$$

The value $\overline{w}/2$ will be used in considering half of an infinite horseshoe vortex filament. Furthermore, use the relation

$$\rho V\ 2s\Gamma = \frac{1}{2}\rho V^2 c_z S$$

where

ρ air density

V air velocity

c_z lift coefficient

S area of the model

and the upward inclination is

$$\left.\begin{aligned}\Delta\alpha = \frac{\overline{w}}{2V} &= \delta c_z \frac{S}{S_0}\\ &= \kappa c_z \frac{1}{\pi\Lambda}\end{aligned}\right\} \qquad (10)$$

where

$S_0 = \pi ab$, tunnel cross-sectional area

$\Lambda = (2s)^2/S$ aspect ratio

$$\left.\begin{aligned}\kappa = \frac{1}{2} k^2 \vartheta, \quad \delta &= \frac{b\vartheta}{8a}\\ \vartheta = B_1 + \frac{1}{3} k^2 B_3 + \frac{1}{5} k^4 B_5 &+ \ldots\end{aligned}\right\} \qquad (11)$$

EXACT SOLUTION FOR CLOSED AND OPEN SECTIONS

Now, consider the more general case. By introduction of elliptic coordinates (ξ, η) with the aid of the relation $z = c \operatorname{ch}(\xi + i\eta)$ where c is a constant, the stream function for the vortex system shown in figure 2 may be written as

$$\psi_1 = \frac{\Gamma}{4\pi} \log \frac{[\operatorname{ch}(\xi+\xi')+\cos(\eta-\eta')][\operatorname{ch}(\xi-\xi')+\cos(\eta+\eta')]}{[\operatorname{ch}(\xi+\xi')-\cos(\eta+\eta')][\operatorname{ch}(\xi-\xi')-\cos(\eta-\eta')]} \qquad (12)$$

which may be expanded into a convergent Fourier series for $\xi > \xi'$, where the indices ' and '' under the summation signs indicate that only odd and even terms, respectively, are taken.

$$\psi_1 = \frac{2\Gamma}{\pi} \left({\sum_{n=1}^{\infty}}' n^{-1} e^{-n\xi} \operatorname{ch} n\xi' \cos n\eta' \cos n\eta + {\sum_{n=2}^{\infty}}'' n^{-1} e^{-n\xi} \operatorname{sh} n\xi' \sin n\eta' \sin n\eta \right) \qquad (13)$$

As in the previous case, the stream function within the ellipse $\psi = \psi_1 + \psi_2$ is divided into two parts, the part ψ_2 due to the existence of the boundary satisfying the condition $\Delta\psi_2 = 0$ continuously within the field and at the boundary satisfying the given boundary conditions. The solution is generally obtained as (reference 6)

$$\psi_2 = \frac{2\Gamma}{\pi} \sum_{n=1}^{\infty} n^{-1}(A_n \operatorname{ch} n\xi \cos n\eta + B_n \operatorname{sh} n\xi \sin n\eta) \qquad (14)$$

The Fourier coefficients A_n, B_n are determined by the boundary conditions.

In the case of the closed section the normal-velocity component at the boundary

$$v_N = \left[h_2 \frac{\partial\psi}{\partial\eta}\right]_{\xi=\xi_0} = \frac{1}{c}(\operatorname{ch}^2\xi_0 - \cos^2\eta)^{-1/2}\frac{2\Gamma}{\pi}$$

$$\left[-\sum_{n=1}^{\infty}{}' e^{-n\xi_0} \operatorname{ch} n\xi' \cos n\eta' \sin n\eta \right.$$

$$+ \sum_{n=2}^{\infty}{}'' e^{-n\xi_0} \operatorname{sh} n\xi' \sin n\eta' \cos n\eta$$

$$\left. + \sum_{n=1}^{\infty}\left(-A_n \operatorname{ch} n\xi_0 \sin n\eta + B_n \operatorname{sh} n\xi_0 \cos n\eta\right)\right] \qquad (15)$$

where $h_2 = \frac{1}{c}(\operatorname{ch}^2\xi - \cos^2\eta)^{-1/2}$ must vanish. Hence,

$$\left.\begin{aligned}
A_n &= -\frac{e^{-n\xi_0}}{\operatorname{ch} n\xi_0} \operatorname{ch} n\xi' \cos n\eta', && (n = 1, 3, 5, \ldots\ldots) \\
&= 0 && (n = 2, 4, 6, \ldots\ldots) \\
B_n &= -\frac{e^{-n\xi_0}}{\operatorname{sh} n\xi_0} \operatorname{sh} n\xi' \sin n\eta', && (n = 2, 4, 6, \ldots\ldots) \\
&= 0, && (n = 1, 3, 5, \ldots\ldots)
\end{aligned}\right\} \qquad (16)$$

In the case of the open section the tangential velocity component at the boundary

$$v_T = -\left[h_1\frac{\partial\psi}{\partial\xi}\right]_{\xi=\xi_0} = -\frac{1}{c}(\mathrm{ch}^2\xi_0 - \cos^2\eta)^{-1/2}\frac{2\Gamma}{\pi}$$

$$\left[-\sum_{n=1}^{\infty}{}' e^{-n\xi_0}\mathrm{ch}\,n\xi'\cos n\eta'\cos n\eta\right.$$

$$-\sum_{n=2}^{\infty}{}'' e^{-n\xi_0}\mathrm{sh}\,n\xi'\sin n\eta'\sin n\eta$$

$$+\sum_{n=1}^{\infty}\Big(A_n\mathrm{sh}\,n\xi_0\cos n\eta$$

$$\left.+ B_n\mathrm{ch}\,n\xi_0\sin n\eta\Big)\right] \qquad (17)$$

where $h_1 = \frac{1}{c}(\mathrm{ch}^2\xi - \cos^2\eta)^{-1/2}$ must vanish. Hence,

$$\left.\begin{aligned} A_n &= \frac{e^{-n\xi_0}}{\mathrm{sh}\,n\xi_0}\mathrm{ch}\,n\xi'\cos n\eta', && (n = 1, 3, 5, \ldots\ldots) \\ &= 0 && (n = 2, 4, 6, \ldots\ldots) \\ B_n &= \frac{e^{-n\xi_0}}{\mathrm{ch}\,n\xi_0}\mathrm{sh}\,n\xi'\sin n\eta', && (n = 2, 4, 6, \ldots\ldots) \\ &= 0 && (n = 1, 3, 5, \ldots\ldots) \end{aligned}\right\} \qquad (18)$$

By determining the coefficients as in the foregoing, the boundary conditions can be completely satisfied. In this manner the wall interference may be computed. However, using elliptic coordinates and obtaining as before the average induced velocity along the wing span is inconvenient. By joining the points with coordinates ξ' and η' $(\pi - \eta')$ the induced velocity in the y direction due to the wall is divided by the airfoil span into two parts

$$\overline{w} = \frac{1}{2s}\left\{[\psi_2]_{\xi=\xi',\,\eta=\pi-\eta'} - [\psi_2]_{\xi=\xi',\,\eta=\eta'}\right\} =$$

$$-\frac{2\Gamma}{\pi s}\left[\sum_{n=1}^{\infty}{}' n^{-1}A_n\mathrm{ch}\,n\xi'\cos n\eta' + \sum_{n=2}^{\infty}{}'' n^{-1}B_n\mathrm{sh}\,n\xi'\sin n\eta'\right] \qquad (19)$$

Taking the value $\overline{w}/2$ for half the infinite horseshoe vortex and with the same notation as the foregoing gives

closed section:

$$\kappa = \sum_{n=1}^{\infty}{}' \frac{e^{-n\xi_o}}{n\,ch\,n\xi_o} ch^2 n\xi' \cos^2 n\eta' + \sum_{n=2}^{\infty}{}'' \frac{e^{-n\xi_o}}{n\,sh\,n\xi_o} sh^2 n\xi' \sin^2 n\eta'$$

open section:

$$\kappa = -\sum_{n=1}^{\infty}{}' \frac{e^{-n\xi_o}}{n\,sh\,n\xi_o} ch^2 n\xi' \cos^2 n\eta' - \sum_{n=2}^{\infty}{}'' \frac{e^{-n\xi_o}}{n\,ch\,n\xi_o} sh^2 n\xi' \sin^2 n\eta' \tag{20}$$

in each case:

$$\delta = \frac{b}{4a} k^2 \kappa$$

Since both of these series converge rapidly, the computation is simple. In the particular case that the airfoil is at the center of the tunnel the second series becomes unnecessary and the computation becomes very simple. Further, in the limiting case when $\xi = \infty$ – that is, for a circular wall – the foregoing relations give the results already known for this case.

DEGREE OF ACCURACY OF THE APPROXIMATE METHOD AND COMPUTATION RESULTS

Table I shows the high degree of accuracy obtained with the approximate method using the values $\theta = 15^\circ, 30^\circ, 60^\circ, 90^\circ$ in equation (7). The values in the table all refer to the case of the closed section with the model mounted at the center of the tunnel. As the ellipse approaches a circle, the difference is small but the law of error distribution is not evident. Nevertheless, an approximation with this degree of accuracy is sufficient for practical purposes.

Figures 3 to 6 all were computed from the exact equations, the wing in all cases being mounted at the center of the tunnel; κ gives the value of the wall interference for fixed aspect ratios; and δ gives the value of the interference for fixed ratio of wing area to tunnel area.

WIND TUNNEL SECTIONS USED IN PRACTICE

An approximate method of solution for a closed elliptic wind tunnel section has been discussed in the foregoing section. The method, however, is of interest in the application to a section shape that approaches an ellipse. The tunnel sections used in practice consist of circular sides joined by their common tangents. The exact solution for this case would be very laborious. The approximate solution may be effected, however, in the following manner. As a particular example, take the ratio of the shorter to the longer axis as $1 : \sqrt{2}$ and, furthermore, consider an open section. This corresponds to the case considered by Knight and Harris (reference 5), who determined the value of δ experimentally.

Since an open section is dealt with here instead of a closed section, the tangential instead of the normal velocity component at the wall must vanish. Hence, the coefficients B_n, instead of by (7), must be determined by

$$B_1 \cos(\theta + \lambda) + B_3\sigma^2 \cos(3\theta + \lambda) + B_5\sigma^4 \cos(5\theta + \lambda) + \ldots$$
$$= \frac{k^2\cos(\theta + \lambda) - \sigma^2\cos(\theta - \lambda)}{(\sigma^2 - k^2)^2 + 4\sigma^2 k^2 \sin^2\theta} \qquad (21)$$

where, however, σ and λ are not computed by the previous relations but for the circular part $(0^\circ \leqq \theta \leqq 67.5^\circ)$ by

$$\sigma = \sin(\theta + \lambda)/\sqrt{2} \sin\theta, \quad \sin\lambda = (\sqrt{2} - 1)\sin\theta$$

and for the straight part by

$$\sigma = 1 \sqrt{2} \sin\theta, \quad \lambda = 90^\circ - \theta$$

Here six terms of the series are taken. By taking the values $\theta = 0^\circ, 15^\circ, 30^\circ, 45^\circ, 60^\circ, 75^\circ$ the coefficients in (21) can be determined. Since, however, there is a discontinuity at $\theta = 65.5^\circ$, the computation is not as simple as in the case of the ellipse. Figure 8 shows the curves obtained by computing the two sides of equation (21) using the value $k=0.6$ and the difference between them may be seen. The difference as compared with the case of the ellipse is graphically indistinguishable. But the normal component at the boundary, as shown in the figure, increases with increasing θ, and at $\theta = 67.5^\circ$ the ratio with the tangential component is about 1:23, and consequently the error in the position of the boundary is below 0.01a. Hence, the error in the value of δ obtained by this computation may be considered to lie within 1 to 2 percent. Further, according to the tests of Knight and Harris the value of δ obtained from the curve of c_z against the angle of attack (denoted by δ_α) differs from the value of δ obtained from the curve of c_z against c_x (denoted by δ_D) while according to the present computation the two values agree. The value of δ obtained from the computation is somewhat smaller than δ_D, the computation being based on the horseshoe vortex hypothesis.

CONCLUSION

A problem of great interest is, for a given ratio S/S_o of the area of the model to the area of the wind tunnel and for a fixed ratio k of the span of the wing to the width of the tunnel, to determine the elliptical section for which the wall interference is a minimum. For the case of the closed working section near the value $k = 0$ the favorable ratio of the minor to the major diameter, as shown in figure 3, is $1:\sqrt{2}$. This agrees with the result obtained by Glauert for a rectangular tunnel. If k is large, however, this is not necessarily the case and a flatter ellipse is more favorable.

In the case of the open-section tunnel the value of δ for the given value of k is smallest for the circular tunnel. But as k becomes larger, the value of δ for the elliptic tunnel gradually decreases. An interesting result is also the fact that for both the closed and the open sections the smallest δ is obtained if the wing tips are at the foci of the ellipse. In the case of the tunnel section consisting of circular arcs joined by their common tangents, the centers of the circles may be considered as the foci.

Translation by S. Reiss,
National Advisory Committee
for Aeronautics.

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TABLE 1

$\epsilon = \frac{\sqrt{3}}{2}$, $\frac{b}{a} = \frac{1}{2}$			$\epsilon = \frac{1}{\sqrt{2}}$, $\frac{b}{a} = \frac{1}{\sqrt{2}}$		
k	κ approximate	κ exact	k	κ approximate	κ exact
0.2	0.038	0.0381	0.2	0.0254	0.0254
.4	.1317	.1308	.4	.0969	.0969
.6	.2542	.2541	.6	.2110	.2110
.8	.4320	.4305	.8	.3932	.3946

TABLE 2

k =	0.45	0.60	0.75
δ_α (test)	−0.249	−0.193	−0.194
δ_D (test)	−.170	−.160	−.164
δ (computation)	−.148	−.145	−.150
δ (ellipse)	−.149	−.147	−.148

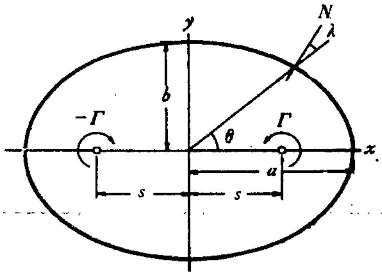

Figure 1.

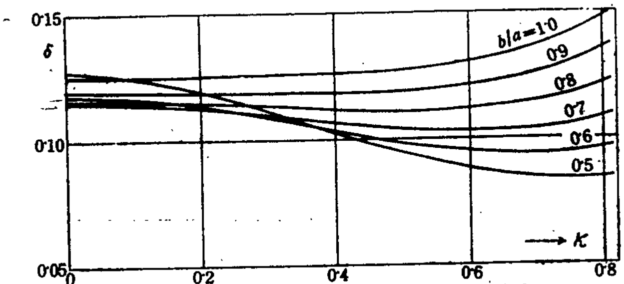

Figure 5.– Closed section.

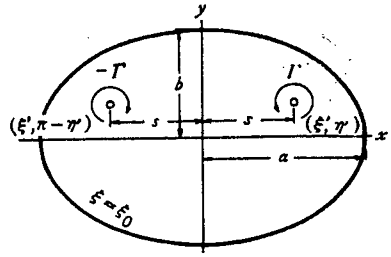

Figure 2.

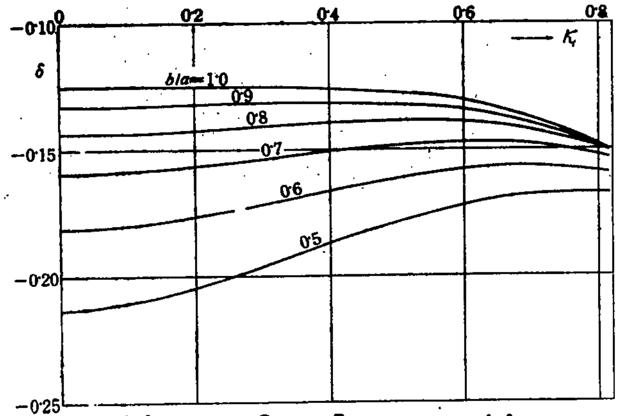

Figure 6.– Open section

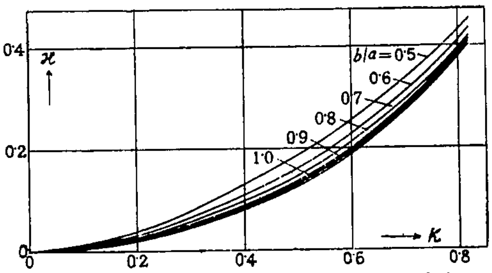

Figure 3.– Closed section.

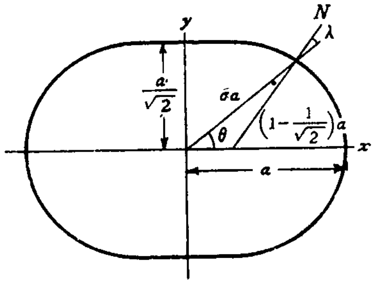

Figure 7.

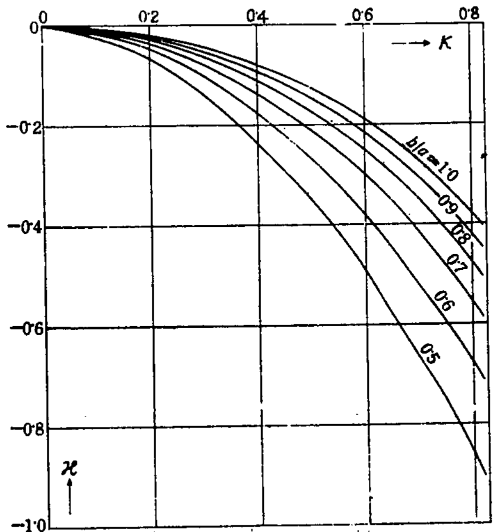

Figure 4.– Open section.

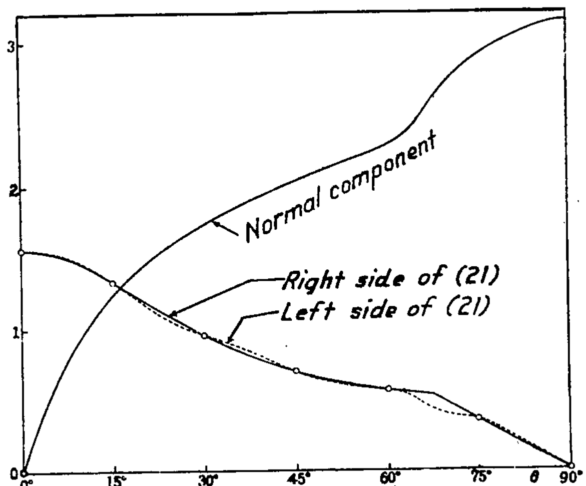

Figure 8.