NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM

No. 1212

ON THE THEORY OF THE LAVAL NOZZLE

By S. V. Falkovich

Translation

"K Teorii Sopla Lavala." Prikladnaya Matematika i Mekhanika.
Vol. 10, no. 4, 1946.

Washington
April 1949

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM NO. 1212

ON THE THEORY OF THE LAVAL NOZZLE*

By S. V. Falkovich

In the present paper, the motion of a gas in a plane-parallel Laval nozzle in the neighborhood of the transition from subsonic to supersonic velocities is studied. This problem was first considered by Meyer (reference 1) who sought to obtain the velocity potential in the form of a power series in the coordinates x,y of the flow plane. The case of the nozzle with plane surface of transition from subsonic to supersonic velocities was further considered in a paper by S. A. Christianovich and his coworkers (reference 2). For computing the supersonic part adjoining the transition line, Christianovich expanded the angle of inclination of the velocity and a specific function of the modulus of the velocity in the power series, using the velocity potential and the stream function as the unknown variables. In a recently published paper, F. I. Frankl (reference 3), applying the hodograph method of Chaplygin, undertook a detailed investigation of the character of the flow near the line of transition from subsonic to supersonic velocities. From the results of Tricomi's investigation on the theory of differential equations of the mixed elliptic-hyperbolic type, Frankl introduced as one of the independent variables in place of the modulus of the velocity, a certain specially chosen function of this modulus. He thereby succeeded in explaining the character of the flow at the point of intersection of the transition line and the axis of symmetry (center of the nozzle) and in studying the behavior of the stream function in the neighborhood of this point by separating out the principal term having, together with its derivatives, the maximum value as compared with the corresponding corrections. This principal term is represented in Frankl's paper in the form of a linear combination of two hypergeometric functions. In order to find this linear combination, it is necessary to solve a number of boundary problems, which results in a complex analysis.

In the investigation of the flow with which this paper is concerned, a second method is applied. This method is based on the transformation of the equations of motion to a form that may be called canonical for the system of differential equations of

*"K Teorii Sopla Lavala." Prikladnaya Matematika i Mekhanika. Vol. 10, no. 4, 1946, pp. 503-512.

the mixed elliptic-hyperbolic type to which the system of equations of the motion of an ideal compressible fluid refers. By studying the behavior of the integrals of this system in the neighborhood of the parabolic line, the principal term of the solution is easily separated out in the form of a polynomial of the third degree. As a result, the computation of the transitional part of the nozzle is considerably simplified.

1. Fundamental equations. - The equations of the two-dimensional, steady, nonvortical motion of an ideal gas in the absence of friction and heat conductivity have the form

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad \frac{\partial (u\rho)}{\partial x} + \frac{\partial (v\rho)}{\partial y} = 0 \tag{1.1}$$

$$\frac{W^2}{2} + \frac{\chi}{\chi-1}\frac{p}{\rho} = \frac{\chi}{\chi-1}\frac{p_0}{\rho_0} \tag{1.2}$$

where u and v are the components of the velocity along the x and y axes, ρ is the density, p is the pressure, $W = \sqrt{u^2 + v^2}$ is the magnitude of the velocity, $\chi = c_p/c_v$, ρ_0 and p_0 are the density and pressure of the gas at rest.

Equations (1.1) represent the condition of the absence of vortices and the equation of continuity. Equation (1.2) is Bernoulli's equation for adiabatic motion for which

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^k \tag{1.3}$$

For the velocity of sound a

$$a^2 = \chi \frac{p}{\rho} \tag{1.4}$$

From equations (1.2), (1.3), and (1.4), the following equation is derived:

$$\rho = \rho_0 \left(1 - \frac{\chi-1}{2}\frac{W^2}{a^2{}_0}\right)^{\frac{1}{\chi-1}} \tag{1.5}$$

($a_0^2 = \chi p_0/\rho_0$ is the velocity of sound in the gas at rest) from which

$$\frac{d}{dW}\left(\frac{\rho_0}{\rho}\right) = \frac{\rho_0}{\rho}\frac{W}{a^2} \tag{1.6}$$

From equation (1.1), it follows that there exist two functions: the velocity potential $\varphi(x,y)$ and the stream function $\psi(x,y)$, which are determined by the equations

$$d\varphi = u dx + v dy \qquad d\psi = \frac{\rho}{\rho_0}(- v dx + u dy) \tag{1.7}$$

In place of the velocity components u and v, the polar coordinates, setting $u = W \cos \theta$ and $v = W \sin \theta$, where θ is the angle between the velocity vector and the x axis, are substituted. Equations (1.7) are solved for dx and dy, thus obtaining

$$dx = \frac{\cos \theta}{W} d\varphi - \frac{\rho_0}{\rho}\frac{\sin \theta}{W} d\psi$$

$$dy = \frac{\sin \theta}{W} d\varphi + \frac{\rho_0}{\rho}\frac{\cos \theta}{W} d\psi \tag{1.8}$$

If x and y as well as W and θ are considered as functions of the variables φ and ψ, then dx and dy must be total differentials, so that the following equations must hold:

$$\frac{\partial}{\partial\psi}\left(\frac{\cos \theta}{W}\right) = - \frac{\partial}{\partial\varphi}\left(\frac{\rho_0}{\rho}\frac{\sin \theta}{W}\right) \qquad \frac{\partial}{\partial\psi}\left(\frac{\sin \theta}{W}\right) = \frac{\partial}{\partial\varphi}\left(\frac{\rho_0}{\rho}\frac{\cos \theta}{W}\right)$$

In carrying out the differentiation, in taking account of the fact that according to equation (1.5) in which ρ_0/ρ depends only on the magnitude of the velocity W, and in making use of equation (1.6), the following equations are obtained:

$$\sin \theta \frac{\partial\theta}{\partial\psi} + \frac{\cos \theta}{W}\frac{\partial W}{\partial\psi} = + \frac{\rho_0}{\rho}\cos \theta \frac{\partial\theta}{\partial\varphi} - \frac{\rho_0}{\rho}\frac{\sin \theta}{W}\left(1 - \frac{W^2}{a^2}\right)\frac{\partial W}{\partial\varphi}$$

$$\cos \theta \frac{\partial\theta}{\partial\psi} - \frac{\sin \theta}{W}\frac{\partial W}{\partial\psi} = - \frac{\rho_0}{\rho}\sin \theta \frac{\partial\theta}{\partial\varphi} - \frac{\rho_0}{\rho}\frac{\sin \theta}{W}\left(1 - \frac{W^2}{a^2}\right)\frac{\partial W}{\partial\varphi}$$

By solving these equations for the derivatives $\partial\theta/\partial\varphi$ and $\partial\theta/\partial\psi$,

$$\frac{\partial\theta}{\partial\varphi} - \frac{\rho}{\rho_0 W}\frac{\partial W}{\partial\psi} = 0 \qquad \frac{\partial\theta}{\partial\psi} + \frac{\rho_0}{\rho}\frac{a^2-W^2}{W^3}\frac{\partial W}{\partial\varphi} = 0 \tag{1.9}$$

This system of differential equations will be of the elliptical type if the magnitude of the velocity W is less than the velocity of sound and will be of the hyperbolical type for supersonic velocities.

The new function η is considered instead of velocity W and is related to W in the following manner (reference 3):

$$\eta = \left(\frac{3}{2}\int_W^{a^*} \frac{\sqrt{a^2-W^2}}{aW}\,dW\right)^{2/3} \tag{1.10}$$

Equations (1.9) then assume the form

$$\frac{\partial\eta}{\partial\psi} + b(\eta)\frac{\partial\theta}{\partial\varphi} = 0 \qquad \eta\frac{\partial\eta}{\partial\varphi} - \frac{1}{b(\eta)}\frac{\partial\theta}{\partial\psi} = 0 \tag{1.11}$$

where

$$b(\eta) = \frac{\rho_0}{\rho}\sqrt{\frac{a^2-W^2}{\eta a^2}} \tag{1.12}$$

as a result of (1.10), is a function of the variable η.

Equations (1.11) are the fundamental equations for the investigation of two-dimensional, nonvortical motion of a gas when the velocity of the flow passes from subsonic to supersonic velocity. In some cases, it is more convenient in these equations to substitute θ and η as the independent variables and take φ and ψ as the required functions. After this transformation, equations (1.11) assume the form

$$\frac{\partial\varphi}{\partial\theta} + b(\eta)\frac{\partial\psi}{\partial\eta} = 0 \qquad \frac{\partial\varphi}{\partial\eta} - \eta b(\eta)\frac{\partial\psi}{\partial\theta} = 0$$

<u>2. Investigation of variable η.</u> - The variable η determined by equation (1.10) is considered in more detail. For

computing the integral entering this equation, the square of the velocity of sound is

1100

$$a^2 = \frac{k+1}{2} a_*^2 - \frac{k-1}{2} W^2 \tag{2.1}$$

In substituting the preceding equation in (1.10)

$$\eta = \left(\frac{3}{2} \int_\lambda^1 \sqrt{\frac{1-\lambda^2}{1-\lambda^2/h^2}} \frac{d\lambda}{\lambda}\right)^{2/3} \qquad \left(\frac{W}{a_*} = \lambda,\ h^2 = \frac{\chi+1}{\chi-1}\right) \tag{2.2}$$

The integration results in

$$\eta = \left\{\frac{3}{4} \ln \left[\left(\frac{1 - \sqrt{\frac{1-\lambda^2}{h^2-\lambda^2}}}{1 + \sqrt{\frac{1-\lambda^2}{h^2-\lambda^2}}}\right)^h \frac{1 + h\sqrt{\frac{1-\lambda^3}{h^2-\lambda^2}}}{1 - h\sqrt{\frac{1-\lambda^2}{h^2-\lambda^2}}}\right]\right\}^{2/3} \tag{2.3}$$

By expanding equation (2.3) in a series

$$\eta = \left(\frac{h(h^2-1)}{2}\right)^{2/3} \frac{1-\lambda^2}{h^2-\lambda^2} \left[1+0(1-\lambda^2)\right]^{2/3} \tag{2.4}$$

From equation (2.3), it follows, that $\eta > 0$ for $\lambda < 1$ and $\eta < 0$ for $\lambda > 1$, that is, in the plane of the variables θ and η, the region lying in the upper half-plane will correspond to the region of subsonic velocities and the region lying in the lower half-plane will correspond to the supersonic velocities. The line of transition from subsonic to supersonic velocity will correspond to the line $\eta = 0$, that is, the axis of abscissas. From equation (1.10), the value of the velocity $W = 0$ in the plane θ,η corresponds to an infinitely distant point. For $\lambda > 1$, equation (2.3) assumes the form

$$\eta = -\left(\frac{3}{2}\right)^{2/3} \left(h \text{ arc tg} \sqrt{\frac{h^2-1}{\lambda^2-\lambda^2}} - \text{arc tg } h\sqrt{\frac{\lambda^2-1}{h^2-\lambda^2}}\right)^{2/3} \tag{2.5}$$

The characteristics in the plane of the hodograph of the velocity for two-dimensional, nonvortical motion of the gas are known as epicycloids (fig. 1), the equations of which are (reference 4).

$$\theta = C \pm \left(h \text{ arc tg} \sqrt{\frac{\lambda^2-1}{h^2-\lambda^2}} - \text{arc tg } h \sqrt{\frac{\lambda^2-1}{h^2-\lambda^2}} \right)$$

Because for a point transformation characteristics go over into characteristics, the following equations of the characteristics in the plane of the variables θ and η are found by using equation (2.5):

$$\theta = \pm \frac{2}{3} \left(-\eta\right)^{3/2} + C \tag{2.6}$$

from which it follows that the characteristics assume the form of semicubical parabolas with the cusps on the axis of abscissas (fig. 2).

3. Differential equations of motion of a gas in neighborhood of transition line. - The flow in a Laval nozzle near the line of transition from the subsonic to the supersonic velocities is considered. This line is hereinafter designated the sound line.

If a straight line perpendicular to the axis of symmetry of the nozzle is directed away from the axis, it will intersect the streamlines with constantly increasing curvatures and will therefore encounter particles of the gas having constantly increasing velocity. The sound line will therefore be a curve that is convex toward the supersonic velocities[1] with vertex on the axis of symmetry (fig. 3). The point of intersection of the sound line with the axis of symmetry is, according to Frankl, denoted as the center of the nozzle.

In the plane of the variables φ and ψ, the region of flow is transformed into a strip the width of which is determined by the amount of gas flowing through the nozzle (fig. 4).

The point of origin of coordinates in the φ,ψ plane corresponds to the center of the nozzle in the flow plane.

The determination of the flow reduces to finding two functions $\eta=\eta(\varphi,\psi)$ and $\theta=\theta(\varphi,\psi)$ that satisfy equations (1.11). Because the flow is to be symmetrical with respect to the streamline $\psi = 0$, it is necessary that the required functions satisfy the conditions

---

[1]When the streamlines have points of zero curvature, the sound line will be a straight line perpendicular to the axis of symmetry; this case was considered by S. A. Christianovich (reference 2).

$$\eta(\varphi,\psi) = \eta(\varphi-\psi) \qquad \theta(\varphi,\psi) = -\,\theta(\varphi-\psi) \qquad \eta(0,0) = 0 \tag{3.1}$$

which are based on equations (2.2).

The solution of equation (1.11), in the form of a power series in the variables φ and ψ, takes into account equations (3.1)

$$\eta = a_1\varphi + a_2\varphi^2 + a_3\psi^2 + a_4\varphi^3 + a_5\varphi\psi^2 + \ldots$$

$$\theta = b_1\varphi\psi + b_2\varphi^2\psi + b_3\psi^3 + b_4\varphi^2\psi + \ldots \tag{3.2}$$

from which it follows that if the flow in the neighborhood of the origin of coordinates is considered, that is, if φ and ψ are assumed to be small magnitudes, the following equations may be obtained from equation (3.2):

$$\eta = O(\varphi) \qquad \theta = O(\varphi\psi) \qquad \frac{\partial\eta}{\partial\psi} = O(\psi)$$

$$\frac{\partial\theta}{\partial\varphi} = O(\psi) \qquad \frac{\partial\eta}{\partial\varphi} = O(1) \qquad \frac{\partial\theta}{\partial\psi} = O(\varphi) \tag{3.3}$$

With the use of equation (2.1) and the notations introduced in equations (2.2), equation (1.12) for the function $b(\eta)$ may be reduced to the form

$$b(\eta) = \frac{\rho_0}{\rho}\sqrt{\frac{h(1-\lambda^2)}{(h^2-\lambda^2)\eta}}$$

In accordance with equation (2.4), the following equation is derived:

$$b(0) = \frac{\rho_0}{\rho}\,(\chi+1)^{1/3} = \left(\frac{k+1}{2}\right)^{\frac{1}{k-1}}\,(k+1)^{1/3}$$

In taking account of the order of smallness of all terms entering equations (1.11), it is concluded that near the origin of coordinates the system of equations (1.11) may be replaced by the following equations:

$$\frac{\partial\eta}{\partial\psi} + b(0)\frac{\partial\theta}{\partial\varphi} = 0 \qquad \eta\frac{\partial\eta}{\partial\varphi} - \frac{1}{b(0)}\frac{\partial\theta}{\partial\psi} = 0$$

By setting $b(0)\psi = \overline{\psi}$, the final result is

$$\frac{\partial\eta}{\partial\psi} + \frac{\partial\theta}{\partial\varphi} = 0 \qquad \eta\frac{\partial\eta}{\partial\varphi} - \frac{\partial\theta}{\partial\psi} = 0 \tag{3.4}$$

where, for simplicity, the bar over ψ has been dropped.

4. Investigation of flow in neighborhood of center of nozzle. - It is evident that the functions

$$\theta = A^2\varphi\psi - \frac{A^3}{3}\psi^3 \qquad \eta = A\varphi - \frac{A^2}{2}\psi^2 \tag{4.1}$$

where A is an arbitrary constant, are integrals of the system of equations (3.4), and satisfy conditions (3.1).

The significance of the constant A will be explained. From the second equation (4.1), $\eta = A\varphi$ along the axis of symmetry of the nozzle $(\psi = 0)$. Differentiation results in

$$A = \frac{d\eta}{d\varphi} = \frac{d\eta}{dW}\frac{dW}{dx}\frac{dx}{d\varphi}$$

Furthermore, by using successively equations (1.10) and (2.1)

$$\frac{d\eta}{dW} = -\frac{1}{aW}\sqrt{\frac{a^2-W^2}{\eta}} = -\frac{ha_*}{\lambda}\sqrt{\frac{1-\lambda^2}{\eta(h^2-\lambda^2)}}$$

Moreover, along the line $\psi = 0$

$$\frac{dx}{d\varphi} = \frac{1}{W} \qquad \frac{dW}{dx} = \frac{\partial u}{\partial x}$$

Hence, for A the following relation is obtained:

$$A = \left(-\frac{h}{W^2}\sqrt{\frac{1-\lambda^2}{\eta(h^2-\lambda^2)}}\frac{\partial u}{\partial x}\right)_{\substack{x=0\\y=0}} = -\frac{(\chi+1)^{1/3}}{a_*^{\,2}}\left(\frac{\partial u}{\partial x}\right)_{\substack{x=0\\y=0}} \tag{4.2}$$

where to obtain the last result, equation (2.4) was used.

The value of A is thus proportional to the value of the derivative of the velocity at the center of the nozzle.

It is assumed that $\partial u/\partial x > 0$ so that A will be a negative quantity.

Along the sound line, $\eta = 0$. Hence, according to the second equation (4.1), the following equation of the sound line is derived:

$$\varphi = \frac{A}{2}\psi^2 \tag{4.3}$$

that is, in the plane φ, ψ, the sound line will be a parabola.

From equation (3.4), the differential equation of the characteristics has the form

$$\left(\frac{d\varphi}{d\psi}\right)^2 = -\eta$$

By substituting the value of η from equation (4.1)

$$\left(\frac{d\varphi}{d\psi}\right)^2 = \frac{A^2\psi^2}{2} - A\varphi$$

In the integration of this equation, set

$$A\varphi = \frac{x^2}{2} - x^2y^2 \qquad A\psi = x$$

The equation then assumes the form

$$\left(1-2y^2-2xy\frac{dy}{dx}\right)^2 = y^2 \qquad \text{or} \qquad 1-2y^2-2xy\frac{dy}{dx} = \pm y$$

After separating the variables and integrating, the following equations of the characteristics are obtained:

$$x(y+1)^{2/3}(2y-1)^{1/3} = C \qquad x(y-1)^{2/3}(2y+1)^{1/3} = C$$

In order to obtain the characteristics passing through the origin of coordinates, set $C = 0$. Thus the variables φ and ψ become

$$\varphi = -\frac{A\psi^2}{2} \qquad \varphi = \frac{A\psi^2}{4} \tag{4.4}$$

Hence, the characteristics passing through the origin of coordinates in the φ,ψ plane are parabolas tangent to each other at this point and tangent to the sound line (fig. 4). The origin of coordinates will therefore be a singular point of the integrals of equations (3.4) determining the flow in the nozzle.

In considering the character of this singularity, it is evident from figure 4 that the characteristics and the sound line divide the neighborhood of the center of the nozzle into six regions. It shall be investigated how the neighborhood of the center of the nozzle is transformed in the plane of the variables θ and η by the integrals of equation (4.1). By eliminating from equation (4.1) the variable φ, the following cubical parabola is used in determining the stream function:

$$A^3\psi^3 + 3A\eta\psi - 3\theta = 0 \tag{4.5}$$

This equation has one real root if its discriminant $\delta = 9\theta^2/4 + \eta^3 > 0$ and three real roots if $\delta < 0$. Because the point $(\varphi = 0,\ \psi = 0)$ corresponds in equation (4.1) to the point $(\theta = 0,\ \eta = 0)$, the equations of the characteristics corresponding to equations (4.1) are in accordance with equation (2.6).

$$\frac{9}{4}\theta^2 + \eta^3 = 0$$

Thus regions I, II, and III of the plane are transformed into the same region of the plane θ,η lying between the characteristics $\theta = \pm\frac{2}{3}(-\eta)^{3/2}$.

Furthermore, the streamlines $\psi = \pm q$ correspond, as seen from equation (4.5), to the straight lines in the θ,η plane.

$$\eta = \pm\frac{\theta}{q} - \frac{A^2q^2}{2}$$

The transformation of the neighborhood of the nozzle in the θ,η plane will thus have the form of the folded surface shown in figure 5. The corresponding regions in figures 4 and 5 are denoted by the same numbers.

In order to compute the streamlines in the flow plane, equations (1.8) are used in which $d\psi$ is set equal to 0, after which they assume the form

$$dx = \frac{\cos\theta}{W}\,d\varphi \qquad dy = \frac{\sin\theta}{W}\,d\varphi$$

By substituting for θ its value from equations (4.1), the magnitude of the velocity W is, according to equation (1.10), a function of the variable η. Thus along the streamlines $\psi = \pm q$

$$dx = \frac{1}{W(\eta)}\cos\left(\frac{A^3q^3}{3} - A^2q\varphi\right)d\varphi \qquad dy = \pm\frac{1}{W(\eta)}\sin\left(\frac{A^3q^3}{3} - A^2q\varphi\right)d\varphi$$

Integration results in

$$x = \int_0^{\varphi}\frac{1}{W(\eta)}\cos\left(\frac{A^3q^3}{3} - A^2q\varphi\right)d\varphi$$

$$y = \pm\left(\int_0^{\varphi}\frac{1}{W(\eta)}\sin\left(\frac{A^3q^3}{3} + A^2q\varphi\right)d\varphi + H\right) \qquad (4.6)$$

where H is the width of the nozzle at the critical section.

In equations (4.6), set according to equations (4.1)

$$\eta = A\varphi - \frac{A^2}{2}q^2$$

The computation of the integrals in equation (4.6) reduces, evidently, to the computation of the two integrals of the type

$$I_1 = \int_0^{\varphi}\frac{\cos A^2q\varphi}{W(\eta)}\,d\varphi \qquad I_2 = \int_0^{\varphi}\frac{\sin A^2q\varphi}{W(\eta)}\,d\varphi$$

with the aid of which x and y are expressed as follows:

$$x = I_1 \cos \frac{A^3q^3}{3} + I_2 \sin \frac{A^3q^3}{3} \qquad y = \pm \left(I_2 \sin \frac{A^3q^3}{3} - I_1 \cos \frac{A^3q^3}{3}\right) \tag{4.7}$$

5. Nozzle with surface of weak discontinuity. - The case in which weak discontinuities are formed along the Mach lines issuing from the center of the nozzle is here considered. For this investigation, it is necessary and sufficient that the derivative $(\partial u/\partial x)_{y=0}$ possess a discontinuity at the center of the nozzle (reference 3). It is assumed that both values $\left[(\partial u/\partial x)_{y=0}\right]_{x=+0}$ and $\left[(\partial u/\partial x)_{y=0}\right]_{x=-0}$ are positive.

From equation (4.1), it is evident that the magnitude A will have the value $A = A_1$ in the regions VI, V, and IV (fig. 4) and the value $A = A_2$ in the region III where, according to equation (4.2), $A_1 < 0$ and $A_2 < 0$.

From equations (4.1), it is concluded that in the regions VI, V, and IV

$$\theta = A_1^2\varphi\psi - \frac{A_1^3}{6}\psi^3 \qquad \eta = A_1\varphi - \frac{A_1^2}{2}\psi^2 \tag{5.1}$$

and for region III

$$\theta = A_2^2\varphi\psi - \frac{A_2^3}{6}\psi^3 \qquad \eta = A_2\varphi - \frac{A_2^2}{2}\psi^2 \tag{5.2}$$

According to equations (4.4), the equations of the characteristics separating the regions IV and V from regions I and II and the equations of the characteristics separating regions I and II from region III have the forms

$$\varphi = \frac{A_1\psi^2}{4} \qquad \varphi = -\frac{A_2\psi^2}{2} \tag{5.3}$$

Substituting the first of these equations in equations (5.1) and the second in equations (5.2)

$$\theta = -\frac{A_1{}^3\psi^3}{12} \qquad \eta = -\frac{A_1{}^2\psi^3}{4} \qquad \varphi = +\frac{A_1\psi^2}{4} \tag{5.4}$$

$$\theta = +\frac{2}{3}A_2{}^3\psi^3 \qquad \eta = -A_2\psi^2 \qquad \varphi = -\frac{A_2\psi^2}{2} \tag{5.5}$$

In order that the flow in the nozzle has no discontinuities, it is necessary to determine θ and η in regions I and II from equations (3.4) in such a manner that the characteristics conditions, equations (5.4) and (5.5), are satisfied. In order to integrate the system, equations (3.4), set

$$\eta = f\left(\frac{\varphi}{\psi^2}\right)\psi^2 \qquad \theta = g\left(\frac{\varphi}{\psi^2}\right)\psi^3 \tag{5.6}$$

where f and g are functions to be determined.

For this substitution, equations (3.4) are transformed into a system of ordinary differential equations with the independent variable $t = \varphi/\psi^2$

$$2f-2tf'-g' = 0 \qquad ff'+3g-2tg' = 0 \tag{5.7}$$

By the elimination of G'

$$g = \frac{1}{3}\left[4tf - (f + 4t^2)\,f'\right] \tag{5.8}$$

By differentiating equation (5.8) and substituting the result in the first equation (5.7), a differential equation of the second order for determining f is obtained

$$(4t^2 + f)\,f'' + f'^2 - 2tf' + 2f = 0 \tag{5.9}$$

From equations (5.6), (5.4), and (5.5), it follows that the boundary conditions for the function f will be

$$f = -\frac{A_1{}^2}{4} \text{ for } t = \frac{A_1}{4} \qquad f = -A_2 \text{ for } t = -\frac{A_2}{2} \tag{5.10}$$

In order to integrate equation (5.9), it is written in the form

$$\left(\frac{f' + 2t}{2tf'-f}\right)' = 0$$

(The solutions $2tf' - f = 0$, that is, $f = c\sqrt{t}$ which do not satisfy equations (5.10) are eliminated.) In carrying out the quadrature

$$\frac{f' + 2t}{2tf'-f} = \frac{1}{2c_1} \quad \text{or} \quad f' - \frac{1}{2(t-c_1)} f = \frac{2c_1 t}{t-c_1}$$

that is, the integration of the linear equation results in

$$f = 4c_1 t - 8c_1^2 + c_2\sqrt{t-c_1} \tag{5.11}$$

The boundary conditions, equations (5.10), which the obtained solution equation (5.11) must satisfy, have the form: $f = f_1$ for $t = t_1$ and $f = f_2$ for $t = t_2$ where it is easily seen from equation (5.10) that the points (t_1, f_1) and (t_2, f_2) lie on the parabola $f = -4t^2$ and that $t_1 < 0$ and $t_2 > 0$. Hence, in order to satisfy the boundary conditions, it is necessary from the family of parabolas equation (5.11) to choose the parabola passing through (t_1, f_1) and (t_2, f_2). Upon satisfying these conditions

$$c_1 = -\frac{t_1^2+t_1t_2+t_2^2}{3(t_1+t_2)} \qquad c_2^2 = \frac{16(t_1-t_2)^2\,(t_1+2t_2)^2\,(2t_1+t_2)^2}{27(t_1+t_2)^3}$$

It is necessary that along a streamline the velocity in the flow direction should increase monotonically, that is, that η should decrease monotonically. Because $\eta = f\psi^2$ according to equation (5.6), $f' < 0$ must be in the range $t_1 < t < t_2$. In order to obtain this result, it is necessary that $c_2 < 0$. This condition is possible only for $2t_1 + t_2 < 0$ and $t_1 + 2t_2 > 0$ when in accordance with equations (5.10), the following condition is obtained

$$A_1 \leq A_2 \leq \frac{A_1}{2}$$

for which a flow without discontinuity is possible.

Translated by S. Reiss
National Advisory Committee
for Aeronautics.

REFERENCES

1. Meyer: Über zweidimensionale Bewegungsvorgänge in einem Gas das mit Überschallgeschwindigkeit strömt. Forschungshefte, Nr. 62, 1908.

2. Levin, Astrov, Pavlov, and Christianovich: On the Computation of Laval Nozzles. Prikladnaya Matematika i Mekhanika, vol. VII, no. 1, 1943.

3. Frankl, F. I.: On the Theory of Laval Nozzles. Izvestia Akademii Nauk SSSR, Ser. Matematika, vol. IX, 1945.

4. Kochin, Kibel, and Rose: Theoretical Hydrodynamics. 1941.

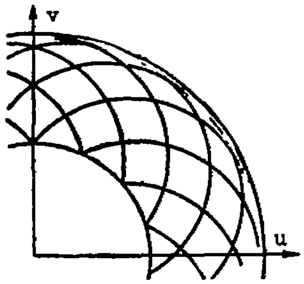

Figure 1.

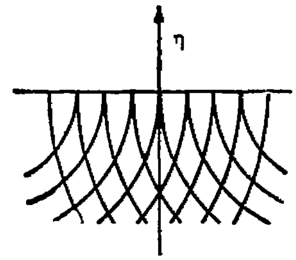

Figure 2.

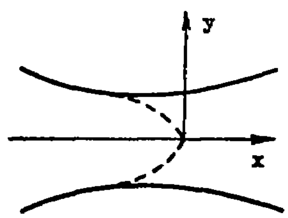

Figure 3.

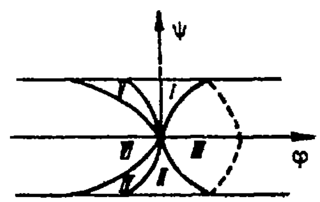

Figure 4.

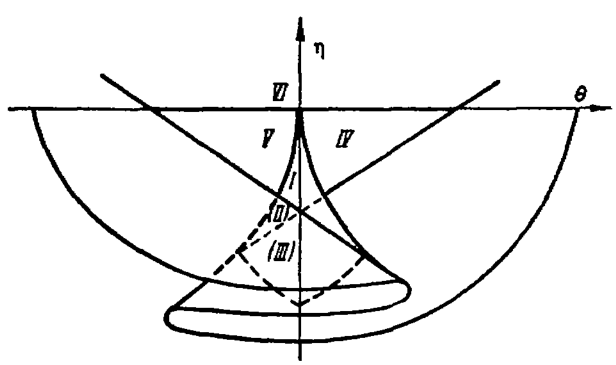

Figure 5.