

NACA TM 1400

7955

TECH LIBRARY KAFB, NM
014475

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1400

BOUNDARY LAYER

By L. G. Loitsianskii

Translation

"Pogranichnyi sloi." Mechanics in the U. S. S. R. over Thirty Years,
1917-1947.



Washington

May 1956



NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1400

BOUNDARY LAYER*

By L. G. Loitsianskii

1. INTRODUCTION

The fundamental, practically the most important branch of the modern mechanics of a viscous fluid or a gas, is that branch which concerns itself with the study of the boundary layer. The presence of a boundary layer accounts for the origin of the resistance and lift force, the breakdown of the smooth flow about bodies, and other phenomena that are associated with the motion of a body in a real fluid. The concept of boundary layer was clearly formulated by the founder of aerodynamics, N. E. Joukowski, in his well-known work "On the Form of Ships"¹ published as early as 1890. In his book "Theoretical Foundations of Air Navigation," Joukowski gave an account of the most important properties of the boundary layer and pointed out the part played by it in the production of the resistance of bodies to motion. The fundamental differential equations of the motion of a fluid in a laminar boundary layer were given by Prandtl in 1904; the first solutions of these equations date from 1907 to 1910. As regards the turbulent boundary layer, there does not exist even to this day any rigorous formulation of this problem because there is no closed system of equations for the turbulent motion of a fluid.

Soviet scientists have done much toward developing a general theory of the boundary layer, and in that branch of the theory which is of greatest practical importance at the present time, namely the study of the boundary layer at large velocities of the body in a compressed gas, the efforts of the scientists of our country have borne fruit in the creation of a new theory which leaves far behind all that has been done previously in this direction. We shall herein enumerate the most important results by Soviet scientists in the development of the theory of the boundary layer.

*"Pogranichnyi sloi." Mechanics in the U.S.S.R. over Thirty Years, 1917-1947, pp. 300-320.

¹Joukowski, N. E.: O forme sudov. Trudy Otdeleniya fizicheskikh nauk Obshchestva lyubitelei estestvoznania, 1890. (See also N. E. Joukowski, Collected Works. Vol. II. Gostekhizdat, 1949, pp. 627-639.)

2. LAMINAR BOUNDARY LAYER FOR CASE OF PLANE-PARALLEL MOTION
OF INCOMPRESSIBLE FLUID

The solution of the problem of the motion of an incompressible fluid in the stationary laminar boundary layer reduces, as is known, to the obtaining of integrals of a nonlinear system of partial differential equations:

$$\left. \begin{aligned} u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= U \frac{dU}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned} \right\} \quad (2.1)$$

where the unknown functions $u(x,y)$ and $v(x,y)$ are the velocity components along and normal to the surface of the body at the points of the boundary layer, $U(x)$ is the initially given longitudinal velocity component on the outer boundary of the boundary layer, x and y are the coordinates along and normal to the surface of the contour, and $\nu = \mu/\rho$ is the kinematic coefficient of viscosity. The boundary conditions of the problem have the form

$$\left. \begin{aligned} u = 0, v = 0 & \text{ for } y = 0 \\ u \rightarrow U(x) & \text{ for } y \rightarrow \infty \end{aligned} \right\} \quad (2.2)$$

where at times there is the further requirement of satisfying a given distribution of velocities $u = u_0(y)$ at the initial section of the layer $x = 0$.

The conditions of existence and uniqueness of solutions of equations (2.1) have been considered by N. S. Piskunov (ref. 46).

The question of an effective method for solving equations (2.1) for an arbitrary given function $U(x)$ has not yet been answered. The existing exact solutions of the system of equations (2.1) for boundary conditions (eq. (2.2)) refer only to certain special classes of functions $U(x)$ as, for example, a linear function, a monomial to some power, certain very simple exponential combinations, and so forth.

The application of purely numerical devices is not of great use because what is of fundamental importance is the possibility of taking into account the effect of the form of the pressure distribution on the motion in the boundary layer and not the accurate determination of the unknown velocity components in a given special case. This is why from about 1921 extensive use was made of approximate methods for computing the laminar boundary layer that were based on the application of the general integral

theorems of the mechanics of a fluid, especially the momentum theorem. The methods of Kármán and Pohlhausen are primarily methods belonging to this class.

By applying the momentum theorem to an element of the boundary layer, bound by the normal sections of the layer at the points x and $x + dx$ and the outer boundary of the layer $y = \delta(x)$, where the function $\delta(x)$ is conventionally assumed finite even though actually the effect of the viscosity extends asymptotically to infinity, there may be obtained the simple integral condition

$$\frac{d\delta^{**}}{dx} + \frac{U'}{U} (2\delta^{**} + \delta^*) = - \frac{\tau_w}{\rho U^2} \quad (2.3)$$

where the prime denotes differentiation with respect to x . (This equation may also be derived strictly from equations (2.1) by employing the accurate boundary conditions (eq. (2.2)). The two conventional boundary-layer thicknesses $\delta^*(x)$ and $\delta^{**}(x)$ are defined by the integrals

$$\left. \begin{aligned} \delta^* &= \int_0^{\delta, \infty} \left(1 - \frac{u}{U}\right) dy \\ \delta^{**} &= \int_0^{\delta, \infty} \frac{u}{U} \left(1 - \frac{u}{U}\right) dy \end{aligned} \right\} \quad (2.4)$$

denoted, respectively, as the displacement thickness and loss of momentum thickness, while the magnitude τ_w defined by the equation

$$\tau_w = \mu \left(\frac{\partial u}{\partial y} \right)_{y=0}$$

represents the frictional stress on the surface of the body; the symbols δ and ∞ in the upper limit of the integral denote the possibility of employing either the theory of the boundary layer of finite thickness or the asymptotic theory.

Suppose we are given, in a boundary-layer section, the distribution of the velocities expressed in the form of a polynomial of the fourth degree with respect to the nondimensional coordinate $\eta = y/\delta$ with coefficients which are functions of x . Then, by satisfying the conditions

$$\left. \begin{aligned} u &= 0, \quad \frac{\partial^2 u}{\partial y^2} = - \frac{UU'}{\nu} && \text{for } y = 0 \\ u &= U, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial^2 u}{\partial y^2} = 0 && \text{for } y = \delta \end{aligned} \right\} \quad (2.5)$$

the polynomial approximating the velocity distribution may be given the form

$$\frac{u}{U} = \varphi(\eta, \lambda) = 2\eta - 2\eta^3 + \eta^4 + \frac{1}{6} \lambda \eta (1 - \eta)^3 \quad (2.6)$$

where

$$\lambda = \frac{U' \delta^2}{\nu} \quad (2.7)$$

This magnitude, which is a function of x , plays the part of a parameter of the group of curves (eq. (2.6)) determining the form of the velocity profiles in the sections of the boundary layer and is, therefore, often termed the form parameter.

The momentum equation (2.3) may be expressed in the form of an equation for the determination of λ as a function of x :

$$\frac{d\lambda}{dx} = \frac{U'}{U} g(\lambda) + \frac{U''}{U'} k(\lambda) \quad (2.8)$$

where we must put (ref. 35)

$$g(\lambda) = \frac{b_1 - (2H^{**} + H^*)\lambda}{\lambda \frac{dH^{**}}{d\lambda} + \frac{1}{2} H^{**}}$$

$$k(\lambda) = \frac{\frac{1}{2} \lambda H^{**}}{\lambda \frac{dH^{**}}{d\lambda} + \frac{1}{2} H^{**}}$$

with the notation

$$\left. \begin{aligned} H^* &= \frac{\delta^*}{\delta} = \int_0^1 (1 - \varphi) d\eta, & H^{**} &= \frac{\delta^{**}}{\delta} = \int_0^1 \varphi(1 - \varphi) d\eta \\ b_1 &= \left(\frac{\partial \frac{u}{U}}{\partial \frac{y}{\delta}} \right)_{y=0} = \left(\frac{d\varphi}{d\eta} \right)_{\eta=0} \end{aligned} \right\} \quad (2.9)$$

For the given form (eq. (2.6)) of the function $\varphi(\eta, \lambda)$ the magnitudes (eqs. (2.9)) are functions of the parameter λ , and equation (2.8) is a nonlinear ordinary differential equation of the second order for the determination of λ as a function of x . By solving this equation for the initial condition $x = 0$, $\lambda = \lambda_0$, where λ_0 is determined from the

condition that the right-hand side of equation (2.8) is finite at the critical point $x = 0$, we obtain $\lambda(x)$, and hence, by equation (2.7), $\delta(x)$ also. Then there is no difficulty in computing the magnitudes δ^* , δ^{**} , and τ_w whereby the problem is solved.

The abscissa of the point of separation is determined from the condition

$$\tau_w = \mu \left(\frac{\partial u}{\partial y} \right)_{y=0} = 0$$

This briefly is the Pohlhausen method for approximately solving the equations of the laminar boundary layer.

Notwithstanding the roughness and small justification of the assumptions, this method, as numerous computations have shown, has proven itself entirely satisfactory in the region of negative and small positive longitudinal pressure gradients in the boundary layer, but entirely unsuitable in the afterpart of the layer in the presence of a pressure rise sufficiently steep to be accompanied by separation of the boundary layer from the surface of the airfoil. In addition to this deficiency in principle, the method ceased to be of service, also from the point of view of practical application, since for solving the fundamental non-linear differential equation (2.8), it required the use of complicated graphical or analytical computing devices.

A whole series of Soviet investigations may be cited that were concerned with the simplification of the practical application of the above method. Thus, A. P. Melnikov (ref. 41) worked out a method for the numerical integration of the fundamental equation instead of its graphical solution. K. K. Fedyaevskii (ref. 54) showed the possibility of the approximate linearization of this equation and the consequent reduction of the solution for simple quadratures. A. A. Kosmodemyanskii (ref. 19) substituted for the approximating polynomial (eq. (1.6)) the product of a polynomial of the second degree by a trigonometric function and applied the method of successive approximations to solve the differential equation thus obtained.

A. N. Alexandrov (ref. 2) [NACA note: Ref. 2 in turn refers to NACA Rep. 527, "Air Flow in a Separating Laminar Boundary Layer" by G. B. Schubauer, 1935.] worked out a numerical method for integrating equation (2.8), maintaining the velocity profile (eq. (2.6)) in the convergent part of the layer, but for the diffuser part constructing a new polynomial satisfying the boundary conditions obtained from equation (2.5) by adding a new exact condition $d^3u/dy^3 = 0$ for $y = 0$ and dropping the old condition $d^2u/dy^2 = 0$ for $y = \delta$. This device gave good agreement of the computation with experiment for the case of the flow about an

elliptical cylinder at different angles of attack, whereas the old method led to a result which contradicted experimental findings, namely the absence of separation in the after region of an elliptic cylinder with ratio of axes 2.96 to 1 and zero angle of attack.

The method of Alexandrov does not, however, rest on a sufficiently well-founded theoretical basis and possesses little accuracy, being, moreover, extremely complicated computationally, the method was not able to satisfy the increasing demands for a suitable computation of the boundary layer. In the early part of 1941 there appeared in the U.S.S.R. new, very much more accurate methods based on simple theoretical considerations and, in addition, very suitable for practical application.

L. G. Loitsianskii (ref. 35) introduced the following two functions of the form parameter (λ) :

$$f(\lambda) = \lambda H^{**2}, F(\lambda) = 2H^{**}[b_1 - \lambda(2H^{**} + H^*)] \quad (2.10)$$

The functions $g(\lambda)$ and $k(\lambda)$ entering equation (2.8) are expressed in terms of them as follows:

$$g(\lambda) = \frac{F(\lambda)}{\frac{df}{d\lambda}}, k(\lambda) = \frac{f(\lambda)}{\frac{df}{d\lambda}}$$

Equation (2.8) can then be reduced to the form

$$\frac{df}{dx} = \frac{U'}{U} F(f) + \frac{U''}{U'} f \quad (2.11)$$

that is, to a differential equation determining f as a function of x if the system of equations (2.10) is regarded as a parametric relation between F and f through the parameter λ . The parameter λ is thus excluded from consideration and replaced by a new form parameter f , according to equations (2.10) and (2.9):

$$f = \frac{U' \delta^{**2}}{\nu} \quad (2.12)$$

The form parameter f has the principal advantage as compared with the parameter λ because it does not contain the conventional nonphysical magnitude δ and is equally applicable to the theory of the layer of finite thickness as well as to the more strict asymptotic theory. As will be explained below, this form parameter has in addition a number of other advantages.

The problem is thus reduced to that of determining once and for all the functional relations

$$\left. \begin{aligned} F &= F(f), \\ H &= \frac{H^*}{H^{**}} = \frac{\delta^*}{\delta^{**}} = H(f) \\ \zeta &= \left(\frac{\partial \frac{u}{U}}{\partial \frac{y}{\delta^{**}}} \right)_{y=0} = \frac{\tau_w \delta^{**}}{\mu U} = \zeta(f) \end{aligned} \right\} \quad (2.13)$$

after which, by solving equation (2.11), it is possible to find successively $f(x)$, then by equation (2.12) to find $\delta^{**}(x)$, by the third part of equations (2.13) to find $\tau_w(x)$, and finally, if required, by the second part of equation (2.13) to find $\delta^*(x)$. All these magnitudes are encountered in the study of the flow about bodies and their resistance.

To establish equation (2.13), it is possible, for example, to make use of the following one-parameter approximation of the velocity profiles in the sections of the boundary layer (ref. 35):

$$\frac{u}{U} = 1 + a_1(1 - \eta)^n + a_2(1 - \eta)^{n+1} + a_3(1 - \eta)^{n+2} \quad (2.14)$$

where the coefficients a_1 , a_2 , and a_3 are determined from the boundary conditions on the surface of the cylindrical body:

$$u = 0, \quad \frac{\partial^2 u}{\partial y^2} = -\frac{UU'}{v}, \quad \frac{\partial^3 u}{\partial y^3} = 0 \quad \text{for } y = 0 \quad (2.15)$$

and the exponent n , characterizing the degree of contact of the curve of equation (2.14) with the straight line $u = U$ on the outer boundary of the layer, is considered as a function of the parameter λ ; that is, in contrast to the old methods, it changes in passing from the forward part of the layer to the rear part. To determine the relation between n and λ , use was made of a class of exact solutions of the equations of the boundary layer for the case of a prescribed velocity of the external flow in the form of a monomial $U = cx^m$. With a high degree of approximation, it was possible to use the simple linear relation $n = 4 + 0.15\lambda$, which gives good agreement of the magnitude of the non-dimensional friction coefficient ζ computed by the present method, and, from the above-mentioned class of exact solutions, for different exponents of degree m corresponding to different flows of a fluid in converging and diverging channels.

The same idea was more consistently carried out in the cooperative work of N. E. Kochin and L. G. Loitsianskii (ref. 21). Instead of the family of curves of equation (2.14), they made use of tables of values of the velocity $u(x, y)$ that were computed with great accuracy for the class of problems $U = cx^m$.

A. M. Basin (ref. 3) proposed employing the family of velocity profiles for the same purpose in place of equation (2.14)

$$\frac{u}{U} = \left[1 + \frac{2\lambda}{\pi^2} \left(1 - \sin \frac{\pi}{2} \eta \right) \right] \sin \frac{\pi}{2} \eta$$

satisfying at the point of separation all the conditions of equation (2.15) and at the anterior critical point the boundary conditions of equation (2.5).

An ingenious solution of the same problem was given by E. E. Solodkin who showed that it was possible in equation (2.14) to choose a linear relation between n and λ to satisfy approximately, at the same time, both the equation of momentum and the equation of energy. It is then no longer necessary to use a class of exact solutions. According to Solodkin, the relation $n = 4 + 0.27 \lambda$ holds.

All previous examples give approximate the quantitative results for equation (2.13). Omitting in our present review the tables of these functions and the graphs showing their variation, we remark only that the function $F(f)$ deviates little from the simple linear dependence

$$F(f) = a - bf \quad (2.16)$$

where the constants a and b have definite initially computed values fluctuating within certain limits depending on the device used for determining the approximating velocity profiles in the sections of the boundary layer. It is possible, for example, to assume, on the average, the values $a = 0.45$ and $b = 5.7$ leading to a deviation of $F(f)$ from the straight line of equation (2.16) by only a few percent. Because of the equality equation (2.8), equation (2.3) may be integrated in quadratures and the solution has the form

$$r = \frac{aU'}{U^b} \int_0^x U^{b-1}(\xi) d\xi \quad (2.17)$$

If desired, it is possible to take into account the deviation of the function $F(f)$ from the straight line of equation (2.16) and to introduce a correction in the solution of equation (2.17) but, as computations show, there is practically no need for this. From equation (2.17) there are readily obtained $f(x)$, $\delta^{**}(x)$, and so forth. The condition of

separation will be $f = f_s = \text{constant}$, where this constant likewise has different values depending on the approximation used; there may be assumed, for example, $f_s = -0.085$ or other values of f_s close to it. To compute τ_w and δ^* , it is necessary to have recourse to tables for $\zeta(f)$ and $H(f)$ given in the previously cited references.

The one-parameter method described previously is widely applied at the present time for computing the flow of wing profiles and other cylindrical bodies in a two-dimensional flow. The further increase in accuracy of the method by passing to a large number of parameters and using the equation of energy (L. S. Leibenzon, ref. 27, and a number of other authors) is associated with extreme complication, and evidently is not dictated by necessity, since the use of the single-parameter method already gives sufficiently good accuracy for smooth wing profiles.

3. TURBULENT BOUNDARY LAYER IN PLANE-PARALLEL MOTION OF INCOMPRESSIBLE FLUID

Depending on the shape of the cylindrical body in the flow, the condition of its surface, and also the structure of the approach flow, the laminar boundary layer turns into a turbulent boundary layer over a certain small transitional region generally taken to be a point called the transition point. To compute the resistance of the wing and to determine the character of the flow about it and also, of particular importance, to estimate correctly the maximum lift force of the wing, it is essential to be able to predict the position of the transition point. Much had been done in this direction even before the start of the war by Soviet aerodynamicists. Especially to be noted are the numerous experimental investigations serving as the basis for devising empirical methods for determining the position of the transition point. Thus, E. M. Minskii (ref. 43) investigated the effect of the turbulence of the approach flow and of the longitudinal pressure drop on the transition point on the upper surface of a wing.

From the curves presented in the work of Minskii, it may be seen very clearly how the transition point is displaced upstream of the flow with increased turbulence of the flow and also with increase in the angle of attack of the wing. Similar tests were conducted by Minskii for the circular cylinder. On the basis of his investigations and numerous tests of other authors, Minskii proposed a generalized empirical diagram from which it is possible to determine approximately the position of the transition point being given certain averaged characteristics of the test conditions.

Soviet scientists have worked out new experimental devices for determining the position of the transition point under laboratory and natural conditions. N. N. Fomina and E. K. Buchinskaya (ref. 62) have conducted an extensive experimental investigation of the boundary layer on a plate, a wing, and a body of revolution with the aid of total-pressure microtubes. The velocity profiles obtained by them permit estimating the position of the transition point. Similar measurements on the surface of biangular profiles were conducted by I. L. Povkh (ref. 47). The investigations carried out in the last 10 years by P. P. Krasilshchikov, K. K. Fedyaevskii, and others have considerably increased our understanding of the part played by transition phenomena in the development of the interaction force between the body and the fluid (refs. 22, 55, and 60).

The effect of the above-mentioned factors on the heat transfer of bodies in a fluid flow was investigated by a number of authors in the aerodynamics and thermal physics laboratories of the Leningrad Polytechnical Institute (L. G. Loitsianskii, P. I. Tretyakov, V. A. Shvab; refs. 40, 68, and 70). On the basis of these investigations concerned mainly with the intensification of the processes of heat exchange in steam boilers, an original method was proposed for determining the turbulence of a fluid based on the measurement of the heat given off by a calibrated body, a sphere, depending on the displacement of the line of transition (thermal scale of turbulence).

In 1944, an extremely simple semi-empirical theory of the transition of a laminar layer into the turbulent layer was proposed by A. A. Dorodnitsyn and L. G. Loitsianskii (ref. 10). On the basis of the consideration that the principal reason for the transition of the laminar layer to the turbulent layer is the occurrence of premature instantaneous local separations of the laminar boundary layer in the region located farther upstream than the point of stationary separation arising in the absence of external disturbances, the authors proposed the following simple formula for determining the abscissa of the transition point:

$$\left(\frac{vU'}{U^2} + \gamma\right) \left(\frac{U\delta^{**}}{v}\right)^2 = f_s \quad (3.1)$$

where γ is a certain constant, characteristic of the given flow, and is determined by the equation

$$\gamma = f_s \left(\frac{v}{U\delta^{**}}\right)_{U'=0}^2 \quad (3.2)$$

where f_s is the separation value of the form parameter that is given by $f_s = -0.085$. The expression in parentheses on the right side of equation (3.2) is computed, once and for all, for a given aerodynamic

3935

wind tunnel from tests on a plate or other body for which the point of transition coincides with the point of minimum pressure. This very approximate semi-empirical theory was sufficiently well confirmed by numerous Soviet and foreign tests.

The more accurate theory, presented at the end of the paper cited above, shows that, in fact, the constant γ is a function of the non-dimensional velocity at the transition point. There is also given an explicit relation between the magnitude γ and the intensity and scale of the turbulence. It is important to note that the previously mentioned semi-empirical theory can be easily generalized to the case of motion of large velocities where it is no longer permissible to neglect the effect of the compressibility of the air.

Let us now turn to the question of the turbulent boundary layer on a wing profile. The absence of a rational theory of the turbulent boundary layer has not up to the present permitted devising a theoretically justified method for its computation. The first solutions of this problem for the case of the wing profile were based on the utilization in the boundary-layer sections of the velocity distributions corresponding to a known power law, for example, the $1/7$ power law, derived for the steady motion in a pipe. As is known, power laws have the fundamental defect that laws of such type are applicable only within a certain range of Reynolds numbers.

The first investigator to overcome this deficiency was G. A. Gurzhienko (ref. 6) who applied a logarithmic velocity distribution not depending on the Reynolds number to the computation of the turbulent boundary layer. By making use of a logarithmic formula for the velocities in the sections of the boundary layer, Gurzhienko reduced the problem to a certain relatively complicated differential equation and gave a method of integrating it by successive approximations. From its very nature, this method cannot take into account the effect of a longitudinal pressure gradient on the shape of the velocity profile and it is therefore not applicable to those cases where such a gradient is of importance.

The first attempt to take into account the effect of the longitudinal pressure gradient on the velocity distribution in a turbulent boundary layer is that of K. K. Fedyaevskii (ref. 57) who presented a new theory of the turbulent boundary layer based on the application of the idea of "mixing length".

The proposed law of variation of the "mixing length" with the distance from the wall is the same for the boundary layer as for the pipe. By approximating the distribution of the friction stress in a cross section of the layer by a method analogous to the previously mentioned device in laminar motion, Fedyaevskii established the form of the one-parameter family of velocity profiles in the sections of the layer,

choosing for the form parameter a magnitude equal to the ratio of the longitudinal pressure drop over a length equivalent to the thickness of the boundary layer to the friction stress at a given point on the surface of the wing. By generalizing the idea of a laminar sublayer for the case of the presence of a longitudinal pressure drop and applying the formula for the velocity to the boundary of the sublayer, Fedyaevskii obtained a formula for the resistance after which the equations of the problem formed a closed system and the solution was carried to the end.

The method of Fedyaevskii was subsequently developed in the direction of greater convenience of computation by L. E. Kalikhman (ref. 12), who also carried out a large number of computations of the boundary layer for different wing profiles and showed the effect of the shape of the profile, the lift coefficient, and other factors on the flow about the wing.

A somewhat different method was followed by A. P. Melnikov (refs. 41 and 42). Employing the semi-empirical theory of the turbulent motion between two parallel walls in which the "mixing length" is expressed through the derivatives of the longitudinal velocities along the direction normal to the surface, Melnikov applied this theory to the boundary layer and obtained comparatively simple formulas for the one-parameter family of velocity profiles with the same form parameter which figures in the method of Fedyaevskii. Later Melnikov simplified the method, at the same time, made it more accurate, and confirmed its practical applicability by a number of computations.

In the theory of turbulent boundary layer, there is still a third line of attack considerably more simple from the point of view of its applications, which, in contrast to the above-mentioned semi-empirical methods, might be denoted as empirical. This approach has recently received the greatest development.

The underlying basis of all work using the empirical approach is the employment of the momentum equation, which in the case of the turbulent boundary layer maintains the same form (eq. (2.3)) as in the case of the laminar layer. The equation contains essentially three unknown magnitudes δ^{**} , δ^* , and τ_w . In the semi-empirical theories, having chosen a certain one-parameter family of velocity profiles in the sections of the layer, the two unknowns δ^* and δ^{**} are expressed in terms of one unknown, the thickness of the boundary layer δ (see eq. (2.4)); after this there remains only to establish a formula for the resistance connecting τ_w and δ . For this purpose there is employed the concept of laminar sublayer, introduced, strictly speaking, only for the case of the absence of a longitudinal pressure gradient.

In the investigation using the empirical approach, the family of velocity profiles in the sections of the boundary layer remains undetermined, while the unknown magnitudes δ^* , δ^{**} , and τ_w , or their combinations, are connected by approximate relations obtained from tests or from certain assumptions of an intuitive character. Thus, for example, two experimental curves are employed connecting the nondimensional coefficient of resistance

$$\zeta = \frac{w\tau}{\rho U^2} \left(\frac{U\delta}{\nu} \right)^{\frac{1}{4}}$$

and the thickness ratio $\delta^*/\delta^{**} = H$ with the form parameter

$$\Gamma = - \frac{U'\delta^{**}}{U} \left(\frac{U\delta^{**}}{\nu} \right)^{\frac{1}{4}}$$

Instead of using experimental curves connecting the resistance coefficient and the magnitude H with a certain form parameter, curves which incidentally are drawn through a very small number of test points and refer to the region of small Reynolds numbers, it is possible, on the basis of certain general assumptions, to construct a method suitable for computations; the accuracy of this method is found to be entirely sufficient in a number of cases. Thus L. G. Loitsianskii (ref. 36) introduced a form parameter Γ and a reduced resistance coefficient ζ according to the formulas

$$\Gamma = \frac{U'\delta^{**}}{U} G(R^{**}), \quad \zeta = \frac{\tau_w}{\rho U^2} G(R^{**})$$

where $G(R^{**})$ is a certain function of the number $R^{**} = U\delta^{**}/\nu$ determined from tests on plates. In this case, equation (2.3) may be transformed to the form

$$\frac{d\Gamma}{dx} = \frac{U'}{U} F(\Gamma; R^{**}) + \frac{U''}{U'} \Gamma \quad (3.3)$$

which is entirely analogous to equation (2.11) for the determination of the form parameter of the theory of laminar boundary layer. The function $F(\Gamma; R^{**})$ entering above and given by

$$F(\Gamma; R^{**}) = (1+m)\zeta - [3+m+(1+m)H]\Gamma' \quad (3.4)$$

is a weak function of R^{**} because the number R^{**} enters into it chiefly through the magnitude m , which is equal to

$$m = \frac{d \ln G(R^{**})}{d \ln R^{**}} = \frac{R^{**}G'(R^{**})}{G(R^{**})}$$

Making the simple assumption of similarity of the changes of ζ and H as a function of Γ in the turbulent and laminar ($m = 1$) boundary layers easily makes the problem completely determinate, and the functions $F(\Gamma)$, $\zeta(\Gamma)$, and $H(\Gamma)$, which are the same for different cases of flow, can be tabulated. The function $G(R^{**})$ may, however, evidently be well approximated by the empirical formula

$$G(R^{**}) = 153.2R^{**\frac{1}{6}}$$

whence it follows that $m = 1/6$. The function $F(\Gamma)$ is as readily linearized as in the case of the laminar boundary layer. From equation (3.3), which becomes linear, the magnitude Γ is determined by simple quadrature. Computations show satisfactory agreement with test results. The method may be applied also for determining the abscissa of the point of separation, that is, the value $x = x_g$ for which $\zeta(x_g) = 0$.

If the turbulent boundary layer is considered for the case of smooth flow without separation about a wing (small relative thicknesses and small lift coefficient), it is sufficient in equation (3.4) to put simply

$$m = \frac{1}{6}, \quad \zeta = 1, \quad H = 1.4$$

after which equation (3.2) [NACA note: Eq. (3.3).] is easily integrated. For this very simple and also important case from the point of view of practical application a somewhat different, but likewise simple, equation, convenient for solution, was given by L. E. Kalikhman (ref. 15).

To the empirical methods based, as in the method above, on the momentum equation there may be added the method of computing the boundary layer worked out by L. E. Kalikhman (ref. 14).

In the U.S.S.R., as is seen from the previous review, a whole series of original methods of computing the turbulent boundary layer has been developed. The further development of this important field of hydrodynamics requires experimental work on turbulent motion in general and the turbulent boundary layer in particular.

4. CERTAIN SPECIAL PROBLEMS OF THEORY OF BOUNDARY

LAYER IN INCOMPRESSIBLE FLUID

Parallel to the laminar and turbulent internal friction in the boundary layer, the processes of heat transfer occur which are associated with a similar mechanism and which depend on the distribution of the temperatures and velocities in the layer. Investigations along these lines have been conducted principally in U.S.S.R.

G. N. Kruzhilin (ref. 23), making use of the concept introduced by him of a thermal boundary layer of finite thickness, established a simple integral relation for the heat transfer in a laminar layer. Applying a method analogous to that earlier described for the computation of the laminar layer but in a more simplified form, Kruzhilin reduced the problem to quadratures and obtained for $N = \alpha l / \lambda$, $R = V_0 l / \nu$, and $P = \nu / a$ the following general formula which interconnects them:

$$N = \frac{2}{F(\bar{x})} P^{\frac{1}{3}} R^{\frac{1}{2}} \quad (4.1)$$

where $F(\bar{x})$, a function of the nondimensional coordinate $\bar{x} = x/l$, l being an arbitrary scale dimension of the body, is a quadrature depending on the shape of the body; the magnitudes α , λ , a , and ν are respectively, equal to the coefficients of heat transfer, the heat conductivity, the thermal diffusivity, and the kinematic viscosity of the fluid. In the case of the flow along a plate, equation (4.1) assumes the form

$$N = 0.670 P^{\frac{1}{3}} R^{\frac{1}{2}} \quad (4.2)$$

The coefficient entering it differs little from that of the accurate solution. Equations (4.1) and (4.2) are derived on the assumption that the thermal boundary layer is thinner than the velocity boundary layer, that is, P is greater than 1. The equations retain their form, however, also for P less than 1 but greater than $1/2$. In his further studies, Kruzhilin applied equation (4.1) to the forward part of a circular cylinder (ref. 25) and made a comparison with test data obtained by himself and V. A. Shvab (ref. 26). The results of the comparison were found to be entirely satisfactory. In one of his subsequent papers (ref. 24), Kruzhilin studied the effect of a longitudinal pressure gradient on the form of the velocity profile in the boundary layer and also the generation of heat arising from the dissipation of energy due to the internal friction in the rapidly moving fluid in the boundary layer. It should be remarked that at the time of the appearance of Kruzhilin's papers there existed in world literature individual theoretical investigations of the heat transfer of bodies in a forced flow but only for the particular cases of given distribution of the velocities in the outside flow and of the temperatures over the surface of the body and without account taken of the generation of heat due to the dissipation of mechanical energy.

In the U.S.S.R., the first investigations were carried out in the field of heat transfer in a turbulent boundary layer. V. A. Shvab, in a theoretical paper (ref. 69) dating from 1936, first gave a solution of the problem of the heat transfer under the conditions of the external problem in the presence of a turbulent boundary layer in an incompressible fluid. In this paper Shvab makes use of a well-known analogy

between the turbulent transport of momentum and heat and, assuming monomials with various powers for the velocity and temperature distribution, he gave formulas for the heat transfer both for a plate and for a cylindrical body and body of revolution. For P equal to 1, Shvab obtained an equation connecting the numbers N and R in the form

$$N = c \cdot R^{\frac{1+n}{1+3n}}$$

where n is the exponent in the assumed distribution of the velocities in the sections of the boundary layer. With the usual power law $n = 1/7$ there is obtained $N \sim R^{0.8}$ in contrast to the previously mentioned law $N \sim R^{0.5}$ for the laminar boundary layer.

In a second generalizing paper appearing in 1937 (ref. 68), Shvab developed the ideas of the preceding paper, showing how the effect of the point of transition is to be taken into account and comparing the results of the computations with experimental data obtained by him, together with other coworkers, in the aerodynamics laboratory of the Leningrad Polytechnical Institute.

K. K. Fedyaevskii (ref. 56) generalized his method of computing the turbulent boundary layer to the case of a thermal boundary layer. Making use of a polynomial representation of the distribution of the heat transport in a section of the layer, he obtained the distribution of the temperatures over the cross section and then a new integral formula of the dependence of the local value of N on P and R (the latter enters in nonexplicit form through the coefficient of resistance). Comparison with the results of the tests of A. S. Chashchikhin showed good agreement of theory with experiment.

Other studies by Soviet investigators in the field of forced heat transfer of bodies in the boundary layer will be discussed in the following section devoted to the problems of motion of a gas at large velocities, a case which is inseparably connected with heat transfer.

There should be mentioned the investigations of Soviet scientists in the field of free convective heat exchange and also on turbulent jet theory in which so much progress has been made principally by the work of G. N. Abramovich (ref. 1). In these investigations, practical methods are given for the computation of turbulent jets both with and without heat transfer.

Together with turbulent jets, there belongs to the number of problems of the so-called "theory of free turbulence" also the problem of the turbulent motion of a fluid in the aerodynamic wake behind a body, that is, in the region of flow formed by the boundary layer coming from the body. We may mention the interesting experimental investigations

of G. I. Petrov and R. I. Shteinberg (ref. 45) who were concerned with the question of the effect of the shape of the body on the frequency of the pulsations, of pressure or velocity in the wake behind the body, and the work of B. Y. Trubchikov (ref. 49) on the measurement of the temperatures in the wake behind a heated body. These investigations led Trubchikov to establishing a method of measuring the turbulence in wind tunnels.

In considering the flow about the fuselage of an airplane, the interference of the fuselage with the wing, the flow near the tips of a wing of finite span, and also in studying the phenomena of slip and the flow about a back-swept wing, it is of great importance at the present time to study the three-dimensional flows of a liquid or gas in the boundary layer. The problem of the three-dimensional boundary layer in general presents great theoretical difficulties; the simplest case to solve is that of the flow with axial symmetry.

In this field, practical application has been made in the U.S.S.R. of the method for computing the frictional resistance of bodies of revolution worked out by K. K. Fedyaevskii (ref. 52), based on the application of power laws of velocity and resistance with variable exponents. The first application of the logarithmic velocity profile to the computation of the boundary layer and the resistance of bodies of revolution for the case of axially symmetric flow about them was made by G. A. Gurzhienko (ref. 6).

All new methods of computation of plane laminar flow or of the turbulent boundary layer henceforth automatically were carried over to the case of axially symmetric flow about bodies of revolution. The presentation of these methods may be found in the previously cited references. An approximate method of computing the laminar boundary layer analogous to that described in section 2 is given in a separate paper by L. G. Loitsianskii (ref. 31).

Turning to a consideration of the more difficult problem of the computation of a three-dimensional boundary layer, we may note first that L. E. Kalikhman (ref. 16) gave the derivation of the fundamental integral relations which can serve for the development of approximate methods of solution of the problem analogous to those applied in the two-dimensional case.

In the period from 1936 to 1938, Loitsianskii published a number of papers in which, by employing various approximate devices, he was able to solve the following three-dimensional problems:

- (1) The laminar and turbulent motion of a fluid in a boundary layer near the line of intersection of two mutually perpendicular planes (there was applied the method of the finite layer (ref. 29) and the method of the asymptotic layer (ref. 33))

- (2) The analogous problem for planes inclined to each other by a certain angle (ref. 38)
- (3) The laminar boundary layer along the line of intersection of two surfaces (ref. 30)
- (4) The laminar boundary layer near the lateral edge of a plate in an axial flow (ref. 37)

In these papers new phenomena were revealed by mathematical computation, the most interesting of which are: the thickening of the boundary layers and the decreasing of the friction in the region of juncture of the planes or surfaces and, conversely, the thinning of the boundary layer and increase in the friction as the lateral edge of the plate was approached. Consequently, there appears the phenomenon of the premature, as compared with the two-dimensional layer, separation of the boundary layer near the line of intersection of the surfaces. The latter phenomenon, usually aggravated further by the harmful interference of the external potential flows, which are as yet not subject to computation, are actually observed in the region where the wing and fuselage are joined and in other flows where there is an intersection of surfaces in the diffuser region of the layer.

Very recently V. V. Struminskii (ref. 48) gave a theory of the three-dimensional boundary layer on a cylindrical wing of infinite span moving with constant angle of slip. For this purpose he applied the theory of the boundary layer with finite thickness.

We now proceed to consider the investigations on the effect of the roughness of the surface on the boundary layer. The effect of surface roughness on the resistance of a body is determined principally by the ratio of the mean height of the roughness protuberance to the thickness of the laminar sublayer. The semi-empirical theory of the turbulent boundary layer near a rough surface was worked out by the combined efforts of several Soviet specialists. Particularly to be mentioned are a number of systematic studies conducted by K. K. Fedyaevskii and his coworker, N. N. Fomina. Fedyaevskii (ref. 53) in his early work, dating from 1936, provided the answer to two fundamental questions of interest to the designing constructor: what is the "permissible" roughness which does not appreciably increase the resistance of a wing, and what is the effect of a given over-all roughness on the resistance of a wing. Later on, carrying out tests on the resistance of an individual schematized protuberance, Fedyaevskii and Fomina (ref. 61) sharpened the question of the possibility of applying the hypothesis of plane flow to the roughness protuberances. By introducing the notion of the equivalent height of a roughness protuberance, the authors gave a table of conventional heights equivalent to various wing and fuselage surface roughnesses that are encountered in practice. A similar investigation on the roughness of a ship's hull was conducted by I. G. Khanovich (ref. 67). He is also to be credited with a method for computing the boundary layer on a rough surface in the presence of a longitudinal pressure drop.

An analysis of the parameters determining the resistance of a rough surface and also the basis for the derivation of the fundamental formulas of the velocity distribution were given in a note by L. G. Loitsianskii (ref. 34).

The results of the investigations of our aerodynamicists on the problem of the effect of roughness are widely applied in airplane construction practice and in work on the analysis of the effect of roughness on the resistance of ships, on the efficiency of hydraulic turbines (39), and so forth.

The attention of Soviet investigators was likewise drawn to special problems on the decrease of the friction due to changes in the physical constants of the liquid or gas by having the boundary layer consist of a liquid or gas differing in its properties from those of the approach flow and, also, by heating the surface of the body in the flow. An interesting experimental investigation of the surface of a body in a flow was made by K. K. Fedyevskii and E. L. Blokh (ref. 59) who showed that the coefficient of resistance of a body in an air flow with the surface of the body heated decreases as the square root of the squares of the absolute temperatures of the approach flow and the surface of the body.

The effect of a boundary layer consisting of a fluid with other constants was investigated in the theoretical note of L. G. Loitsianskii (ref. 32) where it was shown that of fundamental importance for reducing the resistance is a decrease of the ratio of the density of the fluid in the boundary layer to that in the approach flow since this ratio enters as a power close to unity, in contrast to the very small influence of the ratio of the kinematic coefficients of viscosity.

Fedyaevskii conducted interesting experiments on the effect of the aeration of the boundary layer on the resistance of a body moving in water and showed the practical possibility of decreasing the resistance. Several general considerations on this subject may be found in the theoretical paper (ref. 58) of this author.

In conclusion, we note the investigations of N. A. Zaks (ref. 11) on the control of the boundary layer by suction or blow-off of air on the wing. The theoretical basis of the possibility of obtaining a gain in the lift force from the application of various methods of control of the boundary layer and by adding flaps to the wing was first given by V. V. Golubev. In his investigations on the theory of the slotted wing (ref. 4), Golubev showed that the presence of a forward flap retards the separation of the boundary layer toward the region of larger angles of attack than for the wing without flap and, in connection with this fact, he advanced several general considerations on the structural parameters of the wing with flap. Later Golubev (ref. 5) occupied himself with the theoretical investigation of other forms of mechanization, in particular, with the suction and blow-off of the boundary layer.

5. BOUNDARY LAYER AND RESISTANCE IN COMPRESSIBLE

GAS AT LARGE VELOCITIES

The investigation of the effect of compressibility of a gas on the motion in the boundary layer, the resistance, and the heat transfer is the newest branch of the theory of the boundary layer.

The first theoretical study in which a method was given for the complete computation of the distribution of the velocities and temperatures in a laminar boundary layer in a compressible gas was the work of F. I. Frankl (ref. 63). In this paper, Frankl generalized the usual method of the boundary layer of finite thickness to the case of a compressible gas. In his later papers (refs. 64 to 66) dating from 1935 to 1937, Frankl solved the problem of the heat transfer and friction in the turbulent boundary layer on a plate. The latter problem, as well as the analogous problem of the laminar boundary layer, presented serious computational difficulties but the author carried his investigation far enough to give quantitative conclusions.

An extremely simple approximate theory of the turbulent friction on a plate in a compressible fluid flow was given by K. K. Fedyevskii and N. N. Fomina (ref. 61) who showed that if the usual quadratic distribution formula for the turbulent friction is assumed for the cross sections of a compressible-flow boundary layer, the effect of compressibility on the resistance of the plate may at first approximation be taken into account through a change in the physical constants in the boundary layer and reduced to the previously mentioned law of the square root of the square of the ratio of the temperatures at the wall to those of the approach flow.

A fundamental step forward in the solution of the problem of the boundary layer in a compressed gas was the investigation of A. A. Dorodnitsyn (ref. 7) conducted by him even before the war but published only at the beginning of 1942. In this work, Dorodnitsyn showed that at P equal to unity, and in the absence of heat transfer, the system of differential equations of motion of a gas in a laminar boundary layer can be reduced to a form differing slightly from the equations of the boundary layer in an incompressible gas if we pass from the coordinates x and y to the new coordinates ξ and η , connected with the old coordinates by the integral relations

$$\xi = \int_0^x \frac{p}{p_{00}} dx, \quad \eta = \int_0^y \frac{\rho}{\rho_{00}} dy \quad (5.1)$$

where p_{00} and ρ_{00} are the pressure and density in the gas adiabatically brought to rest.

In the particular case of the plate in an axial flow, Dorodnitsyn obtained the following equation for the resistance coefficient:

$$c_f = \frac{2\phi_0''(0)}{\sqrt{R_\infty}} \left(1 + \frac{k-1}{2} M_\infty^2 \right)^{\frac{n}{2} - \frac{1}{2}}$$

where the magnitude $\phi_0''(0)$ represents a certain function, computed by the author, of M_∞ equal to the ratio of the velocity at infinity to the velocity of sound at infinity, $R_\infty = \rho_\infty V_\infty L / \mu_\infty$, $k = c_p / c_v$, and n is the exponent in the assumed law of dependence of the coefficient of viscosity μ on the temperature.

The previous transformation (eq. (5.1)) can be successfully applied also to the turbulent boundary layer if we make use of the usual averaged equations or the momentum equation derived from them and carry over the fundamental equations of the semi-empirical theory of turbulence to a compressible gas. By following this method, Dorodnitsyn (ref. 8) obtained the equation for the local coefficient of resistance of a plate in the presence of a turbulent boundary layer over its entire surface

$$\frac{0.242}{\sqrt{c_f}} = \sqrt{1 + \frac{k-1}{2} M_\infty^2} \left[\ln(R_x c_f) + \eta \ln\left(1 + \frac{k-1}{2} M_\infty^2\right) + 0.15 \right] \quad (5.2)$$

where

$$R_x = \frac{\rho_\infty V_\infty x}{\mu_\infty}$$

I. A. Kibel (ref. 18) solved the problem of the laminar boundary layer on a plate for P equal to unity and in the absence of heat transport across the wall, but with the presence of radiation. At large values of M_∞ of the approaching flow the plate temperature established in the presence of radiation was found to be much less than in the absence of radiation.

By employing, in a somewhat generalized form, the method of simplifying the fundamental equations given by Dorodnitsyn, L. E. Kalikhman (ref. 17) solved the problem of the laminar and turbulent flow of a compressible gas on a plate in the presence of heat transfer.

The investigations of the boundary layer in a compressible gas on the wing and on a body of revolution were carried out principally in the U.S.S.R. In the work referred to previously (ref. 8), Dorodnitsyn considers not only these cases but, employing the transformation of equation (5.1), also solves much more complicated problems. In the

general case of a laminar boundary layer, he applies primarily a method analogous to that described in section 2 of this review, while, the turbulent boundary layer, he has recourse to the general devices of the semi-empirical theory.

To determine the coefficient of profile resistance, a simple formula is established serving as a generalization of the well-known resistance formula of a body in an incompressible fluid. Dorodnitsyn (ref. 9) carried out wide computations of the resistance coefficients of wing profiles at large velocities and brought to light specific effects of the compressibility of the air on the resistance coefficients of wing profiles of various geometric shapes for various conditions of the flow about them.

The comparative complexity of the method on the one hand, and the impossibility of its application to the solution of the problem of the separation of the boundary layer, on the other, made it necessary to generalize the method of computing the laminar boundary layer, described in section 2 of this review, to the case of the motion of a compressible gas with large velocities. A. A. Dorodnitsyn and L. G. Loitsianskii (ref. 10) showed that equation (2.3), for P equal to unity and in the absence of heat transfer, may be brought to the form

$$\frac{df}{dx} = F(f) \ln \frac{V}{\sqrt{1 - \alpha_0^2}} + f \frac{d}{dx} \left(\ln \frac{V'}{\left(1 - \alpha_0^2\right)^{2 + \frac{k}{k-1}}} \right) \quad (5.3)$$

where $\alpha_0 = V/\sqrt{2i_0}$ represents the nondimensional velocity at the outer boundary of the layer, $i_0 = Jc_p T_{00}$ is the total energy, and the form parameter f has the form

$$f = \frac{V' \delta^{**2}}{v_{00} \left(1 - \alpha_0^2\right)^{1 + \frac{k}{k-1}}}$$

In the above equation and equation (5.3), V' denotes the derivative with respect to x , while the momentum thickness loss δ^{**} is determined by the formula

$$\delta^{**} = \int_0^{\infty} \frac{u}{V} \left(1 - \frac{u}{V}\right) d\eta$$

It is of interest to remark that the structure of the expression for the function $F(f)$ in terms of $\zeta(f)$ and $H(f)$ (see section 2) in no way differs from the corresponding expression in the case of the

incompressible fluid. If we make the assumption that, at least for not too large values of M_∞ , the functions $\zeta(f)$ and $H(f)$ will be the same, as in the case of the incompressible fluid, we may employ the tables of functions for $\zeta(f)$, $H(f)$, and $F(f)$ computed for the incompressible fluid. For a first approximation we obtain the following generalization of equation (2.17):

$$f = \frac{\alpha V^1}{v^b (1 - \alpha_0^2)^m} \int_0^x v^{b-1} (1 - \alpha_0^2)^{m-1} dx \quad (5.4)$$

where a and b are the same constants as in section 2 and the magnitude m is determined by the equation

$$m = 2 + \frac{k}{k-1} - \frac{b}{2}$$

The solution of the problem has thus been reduced, as before, to a simple quadrature.

L. E. Kalikhman (ref. 13) investigated the laminar and turbulent boundary layers on a wing in two-dimensional flow and on a body of revolution with axially symmetric flow for the case of the presence of heat transfer from the surface of the body. Introducing a transformation of coordinates representing a generalization of the transformation of A. A. Dorodnitsyn (eq. (5.1)), Kalikhman constructed the integral relations of the moments and energies; then assuming a polynomial distribution of velocities and temperatures in the cross sections of the boundary layer, he converted these relations into differential equations relative to several complexes containing the thicknesses of loss of momentum and energy. The equations are integrated by the method of successive approximations. In the first approximation, the solution is represented as a simple quadrature. To solve the analogous problem for the turbulent boundary layer, Kalikhman applies a semi-empirical theory of turbulence in which he assumes a linear dependence of the mixing length on the coordinates. The solution of the fundamental differential equations in this case likewise lead to quadratures. At the conclusion of the work an equation is established for the coefficient of profile resistance serving as a generalization of the formula of Dorodnitsyn for the case of a body in a compressible gas flow with the presence of heat transfer.

The theory of the boundary layer occupies an important place in the Soviet manuals on hydrodynamics (ref. 20) and constitutes a subject of special monographs (ref. 28).

REFERENCES

1. Abramovich, G. N.: Turbulent Free Streams of Liquids and Gases. Trudy CAHI, no. 512, 1940.
2. Aleksandrov, A. N.: Theoretical and Experimental Investigation of the Dependence of the Point of Separation of a Laminar Boundary Layer on an Elliptical Cylinder on the Angle of Attack. JTF, vol. VIII, no. 22, 1938.
3. Basin, A. M.: An Approximate Method of Computing the Laminar Boundary Layer. DAN, vol. XL, no. 1, 1943.
4. Golubev, V. V.: Investigations on the Theory of a Slotted Wing. Pt. 1 - Theory of a Flap in Plane-Parallel Flow. Trudy CAHI, no. 147, 1933.
5. Golubev, V. V.: Theoretical Foundations of the Methods of Increasing the Lift of a Wing. Trudy VVA im. N. E. Zhukovskogo, no. 46, 1939.
6. Gurzhienko, G. A.: Application of the Universal Logarithmic Law of Velocity Distribution to the Flow About Bodies at Large Reynolds Numbers. Trudy CAHI, no. 257, 1936.
7. Dorodnitsyn, A. A.: Laminar Boundary Layer in a Compressible Gas. DAN, vol. XXXIV, no. 8, 1942.
8. Dorodnitsyn, A. A.: Boundary Layer in a Compressible Gas. Prikl. matem. i mekh., vol. VI, no. 6, 1942.
9. Dorodnitsyn, A. A.: Computation of the Coefficient of Resistance of Wing Airfoils with Account Taken of the Compressibility of the Air. Trudy CAHI, no. 549, 1944.
10. Dorodnitsyn, A. A., and Loitsianskii, L. G.: Transition of the Laminar into the Turbulent Boundary Layer and Laminar Profiles. Trudy CAHI, no. 563, 1945. (See also Prikl. matem. i mekh., vol. IX, no. 4, 1945.)
11. Zaks, N. A.: Aerodynamics of a Wing with Suction and Blowoff of the Boundary Layer. Trudy VVA im. N. E. Zhukovskogo, no. 54, 1940.
12. Kalikhman, L. E.: Effect of Shape of Profile on the Frictional Resistance. Investigation of the Turbulent Boundary Layer in the Region of Separation. Trudy CAHI, no. 333, 1937.
13. Kalikhman, L. E.: Gasdynamic Theory of Heat Transfer. Prikl. matem. i mekh., vol. X, no. 4, 1946.

- 3935
14. Kalikhman, L. E.: New Method of Computing the Turbulent Boundary Layer and Determining the Point of Separation. DAN, vol. XXXVIII, nos. 5-6, 1943.
 15. Kalikhman, L. E.: A simple Method for Computing the Turbulent Boundary Layer of a Wing and Body of Revolution. Tekhn. Vozd. Flota, no. 1, 1942.
 16. Kalikhman, L. E.: Three-Dimensional Boundary Layer. Trudy CAHI, no. 371, 1939.
 17. Kalikhman, L. E.: Resistance and Heat Transfer of a Flat Plate in the Flow of a Gas at Large Velocities. Prikl. matem. i mekh., vol. IX, no. 3, 1945.
 18. Kibel, I. A.: Boundary Layer in a Compressible Fluid with Account Taken of Radiation. DAN, vol. XXV, no. 4, 1939.
 19. Kosmodemyanskii, A. A.: Theory of Frontal Resistance. Pt. 1 - Approximate Integration of the Differential Equation of the Laminar Boundary Layer. Trudy CAHI, no. 215, 1935.
 20. Kochin, N. E., Kibel, I. A., and Roze, N. V.: Theoretical Hydro-mechanics, pt. II. Gostekhizdat, 1948, p. 427-520.
 21. Kochin, N. E., and Loitsianskii, L. G.: An Approximate Method of Computing the Laminar Boundary Layer. DAN, vol. XXXVI, no. 9, 1942.
 22. Krasilshchikov, P. P.: Effect of the Reynolds Number and the Turbulence of the Flow on the Maximum Lift of a Wing. Trudy CAHI, no. 268, 1936; no. 339, 1937.
 23. Kruzhilin, G. N.: Investigation of the Thermal Boundary Layer. JTF, vol. VI, no. 3, 1936.
 24. Kruzhilin, G. N.: Determination of the Temperature of the Surface of a Non-heat Conducting Body in a Very Rapidly Moving Stream of an Incompressible Fluid. JTF, vol. VI, no. 9, 1936.
 25. Kruzhilin, G. N.: Theory of the Heat Transfer of a Circular Cylinder. JTF, vol. VI, no. 5, 1936.
 26. Kruzhilin, G. N., and Shvab, V. A.: Investigation of the α -Field on the Surface of a Circular Cylinder in a Transverse Air Flow. JTF, vol. V, no. 4, 1935.
 27. Leibenzon, L. S.: Energy Form of the Integral Condition in the Theory of the Boundary Layer. Trudy CAHI, no. 240, 1935.

28. Loitsianskii, L. G.: Aerodynamics of the Boundary Layer. Gostekhizdat, 1941.
29. Loitsianskii, L. G.: Interaction of Boundary Layers. Trudy CAHI, no. 249, 1936.
30. Loitsianskii, L. G.: Motion of a Fluid in Boundary Layer Along the Line of Intersection of Two Surfaces. Trudy LII, vol. 6, no. 3, 1937.
31. Loitsianskii, L. G.: Laminar Boundary Layer on a Body of Revolution DAN, vol. XXXVI, no. 6, 1942.
32. Loitsianskii, L. G.: Change in the Resistance of Bodies by Having the Boundary Layer Consist of a Fluid with Other Physical Constants. Prikl. matem. i mekh., vol. VI, no. 1, 1942.
33. Loitsianskii, L. G.: On a Problem of the Three-Dimensional Boundary Layer. Trudy LII, razdel fiz.-matem. nauk, no. 1, 1937.
34. Loitsianskii, L. G.: Universal Formulas in the Theory of the Resistance of Rough Pipes. Trudy CAHI, no. 250, 1936.
35. Loitsianskii, L. G.: Approximate Method of Computing the Laminar Boundary Layer on a Wing. DAN, vol. XXXV, no. 8, 1942.
36. Loitsianskii, L. G.: Approximate Method of Computing the Turbulent Boundary Layer on a Wing Profile. Prikl. matem. i mekh., vol. IX, no. 6, 1945.
37. Loitsianskii, L. G.: Three-Dimensional Boundary Layer and Friction Near the Lateral Edge of a Plate in a Longitudinal Viscous Flow. Prikl. matem. i mekh., vol. II, no. 2, 1938.
38. Loitsianskii, L. G., and Bolshakov, V. P.: The Motion of a Fluid in the Boundary Layer Near the Line of Intersection of Two Surfaces. Trudy CAHI, no. 279, 1936.
39. Loitsianskii, L. G., and Simonov, I. A.: The Scale Effect in Hydro-turbines. Inzhenernyi sb., vol. I, no. 1, 1941.
40. Loitsianskii, L. G., and Shvab, V. A.: Thermal Scale of Turbulence. Trudy CAHI, no. 239, 1935.
41. Melnikov, A. P.: The Theory of the Boundary Layer of a Wing. Trudy Leningrad. inst. inzhenerov grazhd. vozd. flota, no. 12, 1937.

42. Melnikov, A. P.: Turbulent Friction on a Wing and Its Computation with Account Taken of the Effect of a Pressure Gradient. Trudy Leningrad. inst. inzhenerov grazhd. vozd. flota, no. 19, 1939.
43. Minskii, E. M.: The Effect of the Turbulence of the Approach Flow on the Transition from the Laminar to the Turbulent State of the Boundary Layer and on the Separation of the Layer. Trudy CAHI, no. 415, 1939.
44. Minskii, E. M.: The Effect of the Turbulence of the Approach Flow on the Boundary Layer. Trudy CAHI, no. 290, 1936.
45. Petrov, G. I., and Shteinberg, R. I.: Investigation of the Nonsmooth Flow About Bodies. Trudy CAHI, no. 482, 1940.
46. Piskunov, N. S.: Integration of the Equations of the Theory of the Boundary Layer. Izv. AN SSSR, ser. matem., vol. 7, no. 1, 1943, pp. 35-48.
47. Povkh, I. L.: Experimental Investigation of the Boundary Layer on a Profile of Biangular Form. JTF, vol. XIV, nos. 10-11, 1944.
48. Struminskii, V. V.: Slip of a Wing in a Viscous Fluid. DAN. nov. ser., vol. LIV, no. 7, 1946.
49. Trubchinkov, B. Y.: Thermal Method of Measuring the Turbulence in Wind Tunnels. Trudy CAHI, no. 372, 1938.
50. Fedyaevskii, K. K.: Critical Review of the Work on Retarded and Accelerated Turbulent Boundary Layers. Tekhn. zametki CAHI, no. 158, 1937.
51. Fedyaevskii, K. K.: Surface Friction in the Turbulent Boundary Layer of a Compressible Gas. Trudy CAHI, no. 516, 1940.
52. Fedyaevskii, K. K.: Boundary Layer and Frontal Resistance of Bodies of Revolution at Large Reynolds Numbers. Trudy CAHI, no. 179, 1934.
53. Fedyaevskii, K. K.: Approximate Computation of the Intensity of the Friction and "Permissible" Heights of Roughnesses for a Wing. Computation of the Friction of Surfaces with Local and General Roughness. Trudy CAHI, no. 250, 1936.
54. Fedyaevskii, K. K.: Frictional Resistance of Wing Profiles for Various Positions of the Place of Transition from Laminar to Turbulent Boundary Layer. Tekhn. vozd. flota, no. 7, 1939.

55. Fedyaevskii, K. K.: Frictional Resistance of Wings for Various Positions of the Place of Transition from Laminar to Turbulent Boundary Layer. Lecture at Conference on Physical Aerodynamics, Moscow, CAHI, 1939. (See also Tekhn. vozd. flota, nos. 7-8, 1939.)
56. Fedyaevskii, K. K.: Thermal Boundary Layer of a Wing. Trudy CAHI, no. 361, 1938.
57. Fedyaevskii, K. K.: Turbulent Boundary Layer of a Wing. Pt. I - The Profile of Frictional Stress and Velocity. Trudy CAHI, no. 282, 1936; Pt. II - The Law of Resistance. Trudy CAHI, no. 316, 1937.
58. Fedyaevskii, K. K.: Decrease in the Frictional Resistance by Changing the Physical Constants of the Fluid at the Wall. Izv. OTN AN SSSR., vol. 7, nos. 9-10, 1943.
59. Fedyaevskii, K. K., and Blokh, E. L.: Experimental Investigation of the Boundary Layer and Frictional Resistance of a Heated Body. Trudy CAHI, no. 516, 1940.
60. Fedyaevskii, K. K., and Fomina, N. N.: Effect of the Degree of Turbulence of the Flow on the Resistance of Bodies. Tekhn. zametki CAHI, no. 126, 1936.
61. Fedyaevskii, K. K., and Fomina, N. N.: Investigation of the Effect of Roughness on the Resistance and State of the Boundary Layer. Trudy CAHI, no. 441, 1939.
62. Fomina, N. N., and Buchinskaya, E. K.: Experimental Investigation of Two-Dimensional Boundary Layer. Trudy CAHI, no. 374, 1938.
63. Frankl, F. I.: Theory of the Laminar Boundary Layer in a Compressible Gas. Trudy CAHI, no. 176, 1934.
64. Frankl, F. I.: Heat Transfer in a Turbulent Boundary Layer at Large Velocities in a Compressible Gas. Trudy CAHI, no. 240, 1935.
65. Frankl, F. I.: Friction in a Turbulent Boundary Layer at Large Velocities in a Compressible Gas. Trudy CAHI, no. 240, 1935.
66. Frankl, F. I., and Veishel, V. V.: Friction in a Turbulent Boundary Layer About a Plate in Plane-Parallel Flow of a Compressible Gas at Large Velocities. Trudy CAHI, no. 321, 1937.
67. Khanovich, I. G.: Resistance of Continuously Rough Bodies. Izv. VMA VMF, no. 1, 1939.

68. Shvab, V. A.: Theory of Heat Transfer in Turbulent Flow. Trudy Leningr. industr. inst., razdel fiz. matem. nauk, no. 1, 1937.
69. Shvab, V. A.: Heat Transfer Under the Conditions of the External Problem in the Presence of a Turbulent Boundary Layer. JTF, vol. VI, no. 7, 1936.
70. Shvab, V. A., and Tretyankov, P. I.: Choice of Optimal Form of Thermal Turbulence Meter. Trudy Leningr. industr. inst., razdel fiz. matem. nauk, no. 1, 1937.

Translated by S. Reiss
National Advisory Committee
for Aeronautics