NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM

No. 1229

HEAT TRANSMISSION IN THE BOUNDARY LAYER

By L. E. Kalikhman

Translation of "Gazodinamicheckaya Teoriya Teploperedachi"
Prikladnaya Matematika i Mekhanika, Tom X, 1946

Washington
April 1949

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM No. 1229

HEAT TRANSMISSION IN THE BOUNDARY LAYER*

By L. E. Kalikhman

Up to the present time for the heat transfer along a curved wall in a gas flow only such problems have been solved for which the heat transfer between the wall and the incompressible fluid was considered with physical constants that were independent of the temperature (the hydrodynamic theory of heat transfer). On this assumption, valid for gases only, for the case of small Mach numbers (the ratio of the velocity of the gas to that of sound) and small temperature drops between the flow and the wall, the velocity field does not depend on the temperature field.

In 1941 A. A. Dorodnitsyn (reference 1) solved the problem on the effect of the compressibility of the gas on the boundary layer in the absence of heat transfer. In this case the relation between the temperature field and the velocity field is given by the conditions of the problem (constancy of the total energy).

In the present paper which deals with the heat transfer between the gas and the wall for large temperature drops and large velocities use is made of the above-mentioned method of Dorodnitsyn of the introduction of a new independent variable, with this difference, however, that the relation between the temperature field (that is, density) and the velocity field in the general case considered is not assumed given but is determined from the solution of the problem. The effect of the compressibility arising from the heat transfer is thus taken into account (at the same time as the effect of the compressibility at the large velocities). A method is given for determining the coefficients of heat transfer and the friction coefficients required in many technical problems for a curved wall in a gas flow at large Mach numbers and temperature drops. The method proposed is applicable both for Prandtl number $P = 1$ and for $P \neq 1$.

*"Gazodinamichechkaya Teoriya Teploperedachi." Prikladnaya Matematika i Mekhanika, Tom X, 1946, pp. 449-474.

I. FUNDAMENTAL RELATIONS FOR THE LAMINAR AND TURBULENT BOUNDARY LAYER IN A GAS IN THE PRESENCE OF HEAT TRANSFER

1. Statement of the Problem

We consider the flow over an arbitrary contour of the type of a wing profile in a steady two-dimensional gas flow (fig. 1). For supersonic velocities we take into account the existence of an oblique density discontinuity (compression shock) starting at the sharp leading edge or a curvilinear head wave occurring ahead of the profile. For subsonic velocities we assume there are no shock waves (value of the Mach number of the approaching flow is less than the critical).

We denote by u, v the components of the velocity along the axes x, y, where x is the distance along the arc of the profile from the leading edge, y is the distance along the normal, T is the absolute temperature, p the pressure, ρ the density, μ the coefficient of viscosity, λ the coefficient of heat transfer, ϵ the coefficient of turbulence exchange, λ_t the coefficient of turbulent heat conductivity, c_p the specific heat, and J the mechanical equivalent of heat.

$$T^* = T + \frac{u^2}{2Jc_p}$$

is the stagnation temperature

$$a = \sqrt{\frac{\kappa p}{\rho}}$$

is the velocity of sound

$$\kappa = \frac{c_p}{c_v}$$

is the adiabatic coefficient

$$P = \frac{\mu c_p}{\lambda}$$

is the Prandtl number. The remaining notation is explained in the text. The values of the magnitudes in the undisturbed flow are denoted by the subscript ∞, the values of the magnitudes at the wall by the subscript w.

The problem consists in the solution of the system of equations (reference 2)

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left[(\mu+\epsilon)\frac{\partial u}{\partial y}\right] \tag{1.1}$$

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0 \tag{1.2}$$

$$\rho\left(u\frac{\partial T^*}{\partial x}+v\frac{\partial T^*}{\partial y}\right)=\frac{\partial}{\partial y}\left[(\mu+\epsilon)\frac{\partial T^*}{\partial y}\right]+\left(\frac{1}{P}-1\right)\frac{\partial}{\partial y}\left(\mu\frac{\partial T}{\partial y}\right) \tag{1.3}$$

$$\frac{\partial p}{\partial y}=0 \tag{1.4}$$

$$p=\rho\bar{R}T,\quad \mu=CT^n \tag{1.5}$$

where $\bar{R}$ is the gas constant, and C and n are constants.

In the solution of the system equations (1.1) to (1.5) we start out from considerations on the dynamic and thermal boundary layer of a finite (but variable) thickness. The flow outside the dynamic boundary layer approaches the ideal (nonviscous) flow, nonvortical in front of the shock wave and, in general, vortical behind the wave. Equation (1.4) shows that the pressure is transmitted across the boundary layer without change, that is, the pressure $p_0(x)$ and the velocity on the boundary of the layer $U(x)$ may be considered as given functions of x. The flow outside the thermal boundary layer we assume to occur without heat transfer, that is, outside the thermal layer and on its boundary the total energy i_0 is constant:

$$Jc_pT+\frac{u^2}{2}=Jc_pT_\infty+\frac{U_\infty^{\,2}}{2}=Jc_pT_0+\frac{U^2}{2}=i_0 \tag{1.6}$$

From this it follows that the stagnation temperature T^* outside the thermal boundary layer has a constant value

$$T^*=T_{00}=\frac{i_0}{Jc_p} \tag{1.7}$$

Thus the flow outside the thermal boundary layer for small velocities is assumed nearly isothermal while for large subsonic velocities isentropic.

For supersonic velocities the entropy is constant up to the shock wave while after the wave the entropy is constant along each flow line of the external flow about the profile but variable from one flow line to the next.

The boundary conditions of the problem are:

$$u = v = 0, \quad T^* = T_w \quad \text{for} \quad y = 0 \tag{1.8}$$

where $T_w = T_w(x)$ is a given function. In the absence of external heat transfer (across the wall) we have instead of the second condition of equation (1.8) the condition $\left(\frac{\partial T^*}{\partial y}\right)_{y=0} = 0$. Further

$$\begin{aligned} u &= U(x) \quad &\text{for} \quad y = \delta_y(x) \\ T^* &= T_{00} \quad &\text{for} \quad y = \Delta_y(x) \end{aligned} \tag{1.9}$$

where δ_y and Δ_y are the values of y referring, respectively, to the boundary of the dynamic and the boundary of the thermal layer.

2. Fundamental Expressions for the Temperatures

For $P = 1$ equation (1.3) gives the "trivial integral" $T^* =$ Constant. Taking into account the second condition of equation (1.9), we obtain

$$T = T_{00}(1 - \bar{u}^2) \quad \left(\bar{u} = \frac{u}{\sqrt{2i_0}}\right) \tag{2.1}$$

The temperature of the wall is equal to the temperature of the adiabatic stagnation:

$$T_w = T_{00} = T_\infty\left[1 + \frac{1}{2}(\kappa - 1)M_\infty^2\right] \quad \left(M_\infty = \frac{U_\infty}{a_\infty}\right) \tag{2.2}$$

The integral (2.1) corresponding to the case of the absence of external and internal heat transfer in the boundary layer, $\left(\frac{\partial T}{\partial y}\right)_{y=0} = 0$ for $P = 1$, was obtained on the basis of the solution of A. A. Dorodnitsyn (reference 1).

For $P = 1$, U = Constant, and T_w = Constant, equation (1.3) is likewise integrated independent of the solution of the remaining equations of the system and gives the so-called Stodola-Crocco integral

$$T^* = au + b \tag{2.3}$$

Imposing the boundary conditions we obtain

$$T = T_{00}\left[1 - \bar{u}^2 + (\bar{T}_w - 1)\left(1 - \frac{u}{U}\right)\right] \qquad \left(\bar{T}_w = \frac{T_w}{T_{00}}\right) \tag{2.4}$$

From equation (2.3) we also obtain

$$\frac{u}{U} = \frac{t^*}{t_0^*} \qquad \left(t^* = T^* - T_w,\ t_0^* = T_{00} - T_w\right) \tag{2.5}$$

The integral (2.4) corresponds to the case where there is similarity of the velocity field with the field of the stagnation temperature drop (equation (2.5)) and was used in the solution of the problem of the flow about a flat plate (reference 3).

In any more general case $U \neq$ Constant $\left[\text{for } \left(\frac{\partial T}{\partial y}\right)_{y=0} \neq 0\right]$ or $T_w \neq$ Constant or $P \neq 1$ the integral of (1.3) is not known in advance.

The existence of the trivial integral is not, however, the required condition for the solution of the problem and this fact is fundamental for what follows.

Let the function $T^*(x, y)$ or $u(x, y)$ be integrals of the system (1.1) to (1.5) satisfied by the boundary conditions (equations (1.8) and (1.9)). The temperature at an arbitrary point can then be represented in the form

$$\left.\begin{aligned} T &= T_{00}\left(\frac{T^*}{T_{00}} - \bar{u}^2\right) \\ \text{for} \\ T &= T_{00}\left[1 - \bar{u}^2 + (\bar{T}_w - 1)\left(1 - \frac{t^*}{t_0^*}\right)\right] \end{aligned}\right\} \tag{2.6}$$

As is easily seen, the integrals (2.1) and (2.4) are particular cases of the second form of the relation (2.6) for $\overline{T}_w = 1$ and $\dfrac{t^*}{t_0^*} = \dfrac{u}{U}$, respectively. This relation permits expressing together with the temperature also the density and viscosity as a function of the velocity and stagnation temperature drop.

3. Expressions for the Pressure, Density, and Viscosity

The pressure p at any point within the boundary layer is determined by the equation of Bernoulli

$$p = p_0 = p_{02}(1 - \overline{U}^2)^{\frac{\kappa}{\kappa-1}} \qquad \left(\overline{U} = \frac{U}{\sqrt{2i_0}}\right) \tag{3.1}$$

where p_0 is the pressure on the boundary of the layer (thermal or dynamic depending on which of them is thicker), p_{02} is the pressure on adiabatically reducing the velocity to zero in the tube of flow passing through the shock wave. The density ρ at an arbitrary point within the boundary layer is determined by the equation of state (first equation of (1.5)), equations (2.6) and (3.1)

$$\rho = \rho_{02} \frac{1}{1 - \overline{u}^2 + (\overline{T}_w - 1)\left(1 - \dfrac{t^*}{t_0^*}\right)} (1 - \overline{U}^2)^{\frac{\kappa}{\kappa-1}} \tag{3.2}$$

where ρ_{02} is the density on adiabatic reduction of the velocity to zero.

The viscosity μ is determined by the equation

$$\mu = \mu_{00} \left[1 - \overline{u}^2 + (\overline{T}_w - 1)\left(1 - \frac{t^*}{t_0^*}\right)\right]^n \quad (n \approx 0.75) \tag{3.3}$$

where μ_{00} corresponds to the temperature T_{00}, that is, is obtained on the adiabatic reduction of the velocity to zero.

For any point ahead of the shock we have

$$p = p_{01}\left(1 - \bar{U}^2\right)^{\frac{\kappa}{\kappa-1}}$$

$$\rho = \rho_{01}\left(1 - \bar{U}^2\right)^{\frac{1}{\kappa-1}} \tag{3.4}$$

where p_{01} and ρ_{01} are, respectively, the pressure and density on adiabatically reducing the velocity to zero up to the intersection of the streamline with the shock.

As a scale of the velocities it is possible instead of the assumed magnitude $\sqrt{2i_0}$ to take the critical velocity a^* and the local sound velocity a. As is known $a^* = \sqrt{(\kappa - 1)/(\kappa + 1)}\,\sqrt{2i_0}$. Substituting $\frac{U}{a^*} = {}^{**}\lambda_1$ and $\frac{U}{a} = M$ we obtain

$$\bar{U} = \sqrt{\frac{\kappa - 1}{\kappa + 1}}\lambda_1, \quad \bar{U} = \sqrt{\frac{(\kappa - 1)M^2/2}{1 + (\kappa - 1)M^2/2}}$$

$$M = \sqrt{\frac{\lambda_1}{\sqrt{(\kappa + 1)/2 - (\kappa - 1)\lambda_1{}^2/2}}} \tag{2.5}$$

4. Integral Relations of the Momentum and Energy in New Variables

From equations (3.1) and (3.2) we obtain

$$\frac{dp}{dx} = -\rho_0 U \frac{dU}{dx} = -\rho_0 U U' \tag{4.1}$$

**In order to distinguish various uses of the symbol λ herein, subscripts 1 and 2 have been added by the NACA reviewer in the translated version.

where ρ_0 is the density on the boundary of the layer (thermal or dynamic depending on which is the thicker). We represent equations (1.1) and (1.2) in the form

$$\frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho uv) = \rho_0 UU' + \frac{\partial}{\partial y}\left[(\mu + \epsilon)\frac{\partial u}{\partial y}\right]$$

$$\frac{\partial}{\partial x}(\rho uU) + \frac{\partial}{\partial y}(\rho vU) - \rho uU' = 0$$

Subtracting the previous equation from the above we obtain

$$U^2 \frac{\partial}{\partial x}\left[\rho \frac{u}{U}\left(1 - \frac{u}{U}\right)\right] + 2UU'\rho \frac{u}{U}\left(1 - \frac{u}{U}\right) + \frac{\partial}{\partial y}\left[\rho v(U - u)\right]$$

$$+ UU'\rho\left(\frac{\rho_0}{\rho} - 1\right) + UU'\rho\left(1 - \frac{u}{U}\right) = -\frac{\partial}{\partial y}\left[(\mu + \epsilon)\frac{\partial u}{\partial y}\right] \qquad (4.2)$$

From equation (3.2) we find

$$\rho_0 = \rho_{02}(1 - \overline{U}^2)^{\frac{1}{\kappa - 1}}$$

$$\frac{\rho_0}{\rho} - 1 = \frac{\overline{U}^2 - \overline{u}^2 + (\overline{T}_w - 1)(1 - t^*/t_0^*)}{1 - \overline{U}^2} \qquad (4.3)$$

Integrating equation (4.2) term by term from $y = 0$ to $y = \Delta_y$, if $\Delta_y > \delta_y$ and to $y = \delta_y$ if $\delta_y > \Delta_y$ (for definiteness we assume that $\Delta_y > \delta_y$; the same result is obtained if we assume $\delta_y > \Delta_y$), making use of the relation (4.3), and taking into account the fact that starting from the boundary of the dynamic layer the velocity u is constant along x and the friction τ is equal to zero, we obtain

$$U^2 \frac{\partial}{\partial x} \int_0^{\delta_y} \rho \frac{u}{U} \left(1 - \frac{u}{U}\right) dy + UU' \left(2 + \frac{\overline{U}^2}{1 - \overline{U}^2}\right) \int_0^{\delta_y} \rho \frac{u}{U} \left(1 - \frac{u}{U}\right) dy$$

$$+ UU' \left(1 + \frac{\overline{U}^2}{1 - \overline{U}^2}\right) \int_0^{\delta_y} \rho \left(1 - \frac{u}{U}\right) dy + \frac{UU'(\overline{T}_w - 1)}{1 - \overline{U}^2} \int_0^{\Delta_y} \rho \left(1 - \frac{t^*}{t_0^*}\right) dy = \tau_w \tag{4.4}$$

where

$$\tau_y = \left[(\mu + \epsilon) \frac{\partial u}{\partial y}\right]_{y=0} = \mu_w \left(\frac{\partial u}{\partial y}\right)_w \tag{4.5}$$

is the frictional stress at the wall.

We now represent equations (1.2) and (1.3) in the form

$$\frac{\partial}{\partial x} (\rho u t_0^*) + \frac{\partial}{\partial y} (\rho v t_0^*) - \rho u \frac{dt_0^*}{dx} = 0$$

$$\frac{\partial}{\partial x} (\rho u t^*) + \frac{\partial}{\partial y} (\rho v t^*) - \rho u \frac{dt_0^*}{dx} = \frac{\partial}{\partial y} \left[(\mu + \epsilon) \frac{\partial t^*}{\partial y}\right] + \left(\frac{1}{P} - 1\right) \frac{\partial}{\partial y}\left(\mu \frac{\partial T}{\partial y}\right)$$

Subtracting the second equation from the first, integrating the result term by term from $y = 0$ to $y = \Delta_y$ (assuming as above for definiteness that $\Delta_y > \delta_y$), and remembering that for $\Delta_y > \delta_y$ the heat transfer $q = \lambda \frac{\partial T}{\partial y}$ and the friction $\tau = \mu \frac{du}{dy}$ are equal to zero on the boundary of the thermal layer we obtain

$$\frac{d}{dx} U t_0^* \int_0^{\Delta_y} \rho \frac{u}{U} \left(1 - \frac{t^*}{t_0^*}\right) dy = \frac{q_w}{c_p} \tag{4.6}$$

where

$$q_w = c_p\left(\mu\frac{\partial t^*}{\partial y}\right)_{y=0} + \left(\frac{1}{P} - 1\right) c_p\left(\mu\frac{\partial T}{\partial y}\right)_{y=0}$$

$$= \frac{1}{P} c_p\left(\mu\frac{\partial T}{\partial y}\right)_{y=0} = \lambda_w\left(\frac{\partial T}{\partial y}\right)_w \tag{4.7}$$

is the intensity of the heat transfer at the wall.

In solving the problem of the boundary layer without heat transfer between the gas and the wall for large velocities and $P = 1$, A. A. Dorodnitsyn introduced the change in variables

$$\eta = \int_0^y \frac{1}{1 - \bar{u}^2}\left(1 - \bar{U}^2\right)^{\frac{\kappa}{\kappa-1}} dy \tag{4.8}$$

Noting that the function under the integral sign in equation (4.8) agrees with the expression ρ/ρ_{02} for $\bar{T}_w = 1$, we introduce a new independent variable of the analogous equation containing in the function under the integral sign the expression ρ/ρ_{02} for the general case according to equation (3.2)

$$\eta = \int_0^y \frac{1}{1 - \bar{u}^2 + (\bar{T}_w - 1)(1 - t^*/t_0^*)}\left(1 - \bar{U}^2\right)^{\frac{\kappa}{\kappa-1}} dy \tag{4.9}$$

For $\bar{T}_w = 1$ the relation between the coordinates η and y depends on the unknown velocity profile. For $\frac{t^*}{t_0^*} = \frac{u}{U}$ equation (4.9) gives the change in variables applied to the problem of the heat interchange of the plate with the gas flow. In this case the relation between η and y likewise depends only on the velocity profile.

As is seen from equation (4.9) in the general case the relation between the coordinates η and y depends not only on the velocity profile $u(x, y)$, but also on the temperature-drop profile $t^*(x, y)$ which likewise is not initially known but is determined from the solution of the problem.

Replacing the density ρ in equations (4.4) and (4.6) by its expression (3.2) and passing to the variables x, η we obtain

$$U^2 \frac{d}{dx} \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) d\eta + UU'\left(2 + \frac{\overline{U}^2}{1 - \overline{U}^2}\right) \int_0^\delta \frac{u}{U} \left(1 - \frac{u}{U}\right) d\eta$$

$$+ UU'\left(1 + \frac{\overline{U}^2}{1 - \overline{U}^2}\right)\int_0^\delta \left(1 - \frac{u}{U}\right) d\eta + \frac{UU'(\overline{T}_w - 1)}{1 - \overline{U}^2}\int_0^\Delta \left(1 - \frac{t^*}{t_0^*}\right) d\eta = \frac{\tau_w}{\rho_{02}} \quad (4.10)$$

$$\frac{d}{dx}\left[Ut_0^* \int_0^\Delta \frac{u}{U}\left(1 - \frac{t^*}{t_0^*}\right) d\eta\right] = \frac{q}{\rho_{02}c_p} \quad (4.11)$$

where δ and Δ are the values of the variable η referring, respectively, to the boundary of the dynamic and the thermal layers.

We denote the thickness of the loss in momentum and the thickness of the displacement in the plane $x\eta$, respectively, by

$$\vartheta = \int_0^\delta \frac{u}{U} \left(1 - \frac{u}{U}\right) d\eta \qquad \delta^* = \vartheta H = \int_0^\delta \left(1 - \frac{u}{U}\right) d\eta \quad (4.12)$$

we introduce the concept of the thickness of the energy loss (in the plane $x\eta$)

$$\theta = \int_0^\Delta \frac{u}{U}\left(1 - \frac{t^*}{t_0^*}\right) d\eta \quad (4.13)$$

This magnitude has a clear physical meaning; namely, the magnitude θ characterizes the difference between that total energy which the mass of the fluid that flows in unit time through a given section of the thermal boundary layer would have if its stagnation temperature were equal to the stagnation temperature of the external flow and the true total energy of this mass. The magnitude θ thus represents in length units referred to the temperature t_0^* the "loss" in total energy due to the heat transfer. For small velocities the concept of the thickness of energy loss agrees with the previously introduced concept of the thickness of heat-content loss (reference 4). The magnitude

$$\Delta^* = \theta H_T = \int_0^\Delta \left(1 - \frac{t^*}{t_0^*}\right) d\eta \tag{4.14}$$

may be called the thickness of the thermal mixing.

With the aid of the magnitudes defined by equations (4.12), (4.13), and (4.14) we represent the obtained integral relations of the momenta and energy (equations (4.10) and (4.11)) in the final form

$$\frac{d\theta}{dx} + \frac{U'}{U}\left[H + 2\ (H + 1)\ \frac{\overline{U}^2}{1 - U^2}\right]\vartheta + \frac{U'(\overline{T}_w - 1)}{U(1 - \overline{U}^2)} H_T\theta = \frac{\tau_w}{\rho_{02}U^2} \tag{4.15}$$

$$\frac{d\theta}{dx} + \frac{U'}{U}\theta + \frac{t_0^{*\prime}}{t_0^*}\theta = \frac{q_w}{\rho_{02}Uc_pt_0^*} \qquad \left(t_0^{*\prime} = \frac{dt_0^*}{dx} = -\frac{dT_w}{dx}\right) \tag{4.16}$$

We note that the integration with respect to y (or η) may be taken from 0 to ∞ so that the relations (4.15) and (4.16) are general for the theory of the boundary layer of finite thickness and the theory of the asymptotic layer.

For small Mach numbers (the effect of the compressibility due to the temperature drop) the relations (4.15) and (4.16) assume the form

$$\left.\begin{aligned} \frac{d\vartheta}{dx} + \frac{U'}{U}(H + 2)\vartheta + \frac{U'}{U}\left(\frac{T_w}{T_0} - 1\right) H_T\theta &= \frac{\tau_w}{\rho_0U^2} \\ \frac{d\theta}{dx} + \frac{U'}{U}\theta + \frac{t_0'}{t_0}\theta &= \frac{q_w}{\rho_0Uc_pt_0} \end{aligned}\right\} \tag{4.17}$$

where $t_0 = T_0 - T_w$, T_0 and ρ_0 are, respectively, the temperature and density of the isothermal flow outside the thermal boundary layer (it follows from equations (2.6) and (3.2) if we set approximately $U = 0$, that is, according to equation (3.5) $M = 0$). The pressure distribution over the profile is determined by the equation of Bernoulli for an incompressible fluid.[1]

As is seen from equations (4.15) and (4.16) and also from what follows, in the variables x, η the equations of the system (1.1) to (1.3) are simplified and approach in principle the corresponding equations for the incompressible fluids. For this reason the fundamental methods of the theory of the boundary layer in an incompressible fluid may be generalized to the case of a body in a gas flow with heat interchange.

We give below the generalization of the method of Pohlhausen for the case of the laminar layer and the logarithmic method of Prandtl-Kármán for the case of the turbulent layer. The proposed method of the solution of the problems connected with heat interchange permits, of course, generalization of certain other problems in the theory of the boundary layer in an incompressible fluid.

II. LAMINAR BOUNDARY LAYER WITH HEAT INTERCHANGE BETWEEN THE GAS AND THE WALL[2]

5. Transformation of the Differential Equations

Assuming in equations (1.1) to (1.3) $\epsilon = 0$ and $t^* = T^* - T_w$, substituting the values p, ρ, and μ according to equations (3.1) to (3.3), we transform these equations to the new independent variables $x = x$ and η, determined according to equation (4.9). The equations of transformation of the derivatives will be

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}, \qquad \frac{\partial}{\partial y} = \frac{(1 - \bar{U}^2)^{\frac{\kappa}{\kappa-1}}}{1 - \bar{u}^2 + (\bar{T}_w - 1)(1 - t^*/t_0^*)}\frac{\partial}{\partial \eta}$$

---

[1] From equations (3.1) and (3.5) we have

$$p_{00} - p_0 = \left\{\left[1 + \frac{1}{2}(\kappa - 1)M^2\right]^{\frac{\kappa}{\kappa-1}} - 1\right\} p_0 \approx \frac{\rho_0 U^2}{2}\left(1 + \frac{1}{4}M^2 + \frac{2-\kappa}{24}M^4 + \ldots\right)$$

whence setting $M = 0$, we obtain

$$p_0 + \frac{1}{2}(\rho_0 U^2) = \text{Constant}$$

[2] In what follows we restrict ourselves to the case $P = 1$.

We obtain (introducing the notation $\nu_{02} = \dfrac{\mu_{00}}{\rho_{02}}$

$$u\frac{\partial u}{\partial x} + \tilde{v}\frac{\partial u}{\partial \eta} = \frac{1 - \bar{u}^2 + (\bar{T}_w - 1)(1 - t^*/t_0^*)}{1 - \bar{U}^2} UU'$$

$$+ \nu_{02}(1 - \bar{U}^2)^{\frac{\alpha}{\kappa-1}} \frac{\partial}{\partial \eta}\left\{\left[1 - \bar{u}^2 + (\bar{T}_w - 1)\left(1 - \frac{t^*}{t_0^*}\right)\right]^{n-1} \frac{\partial u}{\partial \eta}\right\} \tag{5.1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial \tilde{v}}{\partial \eta} = 0 \quad \left(\tilde{v} = v\,\frac{\rho}{\rho_{02}} + u\,\frac{\partial \eta}{\partial x}\right) \tag{5.2}$$

$$u\frac{\partial t^*}{\partial x} + v\frac{\partial t^*}{\partial \eta} - u\frac{dt_0^*}{dx} = \nu_{02}(1 - \bar{U}^2)^{\frac{\kappa}{\kappa-1}} \frac{\partial}{\partial \eta}\left\{\left[1 - \bar{u}^2 \right.\right.$$

$$\left.\left. + (\bar{T}_w - 1)\left(1 - \frac{t^*}{t_0^*}\right)\right]^{n-1} \frac{\partial t^*}{\partial \eta}\right\} \tag{5.3}$$

If it is assumed approximately that $n = 1$, the obtained system can be still further simplified.

6. Generalization of the Method of Pohlhausen

We represent the velocity profile and the stagnation temperature-drop profile by the polynomials

$$\frac{u}{U} = A_1\left(\frac{\eta}{\delta}\right) + A_2\left(\frac{\eta}{\delta}\right)^2 + A_3\left(\frac{\eta}{\delta}\right)^3 + A_4\left(\frac{\eta}{\delta}\right)^4 \tag{6.1}$$

$$\frac{t^*}{t_0^*} = B_1\left(\frac{\eta}{\Delta}\right) + B_2\left(\frac{\eta}{\Delta}\right)^2 + B_3\left(\frac{\eta}{\Delta}\right)^3 + B_4\left(\frac{\eta}{\Delta}\right)^4 \tag{6.2}$$

To determine the coefficients of the polynomials we set up the conditions

$$\frac{\partial^2}{\partial(\eta/\delta)^2}\frac{u}{U} + \alpha\frac{\delta}{\Delta}\frac{\partial}{\partial(\eta/\Delta)}\frac{t^*}{t_0^*}\frac{\partial}{\partial(\eta/\delta)}\frac{u}{U} + \lambda_2 \quad \text{for} \quad \frac{\eta}{\delta} = 0 \tag{6.3}$$

where

$$\lambda_2 = \frac{\delta^2 U' \overline{T}_w^{\,2-n}}{\nu_{02}(1 - U^2)^{\frac{\kappa}{\kappa-1}+1}}, \qquad \alpha = (1 - n)\frac{T_w - T_{00}}{T_w}$$

Condition (6.3) follows from the equation of motion (5.3) since $u = \tilde{v} = 0$ for $\eta = 0$. Further,

$$\frac{u}{U} = 1, \quad \frac{\partial}{\partial(\eta/\delta)}\frac{u}{U} = 0, \quad \frac{\partial^2}{\partial(\eta/\delta)^2}\frac{u}{U} = 0 \quad \text{for} \quad \frac{\eta}{\delta} = 1 \tag{6.4}$$

From equation (5.3) for $\eta = 0$, $u = \tilde{v} = 0$ we obtain

$$\frac{\partial^2}{\partial(\eta/\Delta)^2}\frac{t^*}{t_0^*} + \alpha\left[\frac{\partial}{\partial(\eta/\Delta)}\frac{t^*}{t_0^*}\right]^2 = 0 \quad \text{for} \quad \frac{\eta}{\Delta} = 0 \tag{6.5}$$

Similarly to conditions (6.4) for the profile u/U we take for the profile t^*/t_0^* the conditions

$$\frac{t^*}{t_0^*} = 1, \quad \frac{\partial}{\partial(\eta/\Delta)}\frac{t^*}{t_0^*} = 0, \quad \frac{\partial^2}{\partial(\eta/\Delta)^2}\frac{t^*}{t_0^*} = 0 \quad \text{for} \quad \frac{\eta}{\Delta} = 1 \tag{6.6}$$

By differentiating equations (5.1) and (5.3) with respect to η with the subsequent equating of η to zero, it is easy to obtain the conditions for the third derivatives of u/U and t^*/t_0^* at the wall. These conditions may be useful for various aspects of the method of Pohlhausen. Using conditions (6.5) and (6.6) we obtain

$$B_1 = \frac{3 - \sqrt{9 - 12\alpha}}{\alpha}, \quad B_2 = 6 - 3B_1, \quad B_3 = -8 + 3B_1, \quad B_4 = 3 - B_1 \qquad (6.7)$$

From conditions (6.3) and (6.4) we now find

$$A_1 = \frac{12 + \lambda_2}{6 - (3 - \sqrt{9 - 12\alpha})\delta/\Delta}, \quad A_2 = 6 - 3A_1, \quad A_3 = -8 + 3A_1, \quad A_4 = 3 - A_1 \qquad (6.8)$$

In relations (4.15) and (4.16) there enter, besides $\vartheta(x)$ and $\theta(x)$, the four unknown functions τ_w, H, q_w, H_T. In constructing the profiles there were also introduced the auxiliary functions δ and Δ. The required six additional equations are obtained by substituting equations (6.1) and (6.2) in equations (4.5), (4.7), and (4.12) to (4.14). We obtain

$$\delta^* = \vartheta H = \delta \frac{8 - A_1}{20}, \quad \vartheta = \delta \frac{-5A_1^2 + 12A_1 + 144}{1260} \qquad (6.9)$$

$$\tau_w = \mu_{00}\overline{T}_w^{\,n-1} \frac{U}{\delta} (1 - \overline{U}^2)^{\frac{\kappa}{\kappa-1}} A_1, \quad q_w = c_p\mu_{00}\overline{T}_w^{\,n-1} \frac{t_0^*}{\Delta} (1 - \overline{U}^2)^{\frac{\kappa}{\kappa-1}} B_1 \qquad (6.10)$$

$$\Delta^* = \theta H_T = \Delta \frac{8 - B_1}{20}, \quad \theta = \Delta(M_1 + N_1A_1) \qquad (6.11)$$

where for $\frac{\Delta}{\delta} < 1$

$$M_1 = 6b_2\left(\frac{\Delta}{\delta}\right)^2 - 8b_3\left(\frac{\Delta}{\delta}\right)^3 + 3b_4\left(\frac{\Delta}{\delta}\right)^4, \quad N_1 = b_1\left(\frac{\Delta}{\delta}\right) - 3b_2\left(\frac{\Delta}{\delta}\right)^2 + 3b_3\left(\frac{\Delta}{\delta}\right)^3 + b_4\left(\frac{\Delta}{\delta}\right)$$

$$b_1 = \frac{6 - B_1}{60}, \quad b_2 = \frac{16 - 3B_1}{420}, \quad b_3 = \frac{5 - B_1}{280}, \quad b_4 = \frac{24 - 5B_1}{2520}$$

for $\frac{\Delta}{\delta} > 1$

$$M_1 = \frac{8 - B_1}{20} + \frac{\delta}{\Delta}\left[-0.4 + 0.2290\left(\frac{\delta}{\Delta}\right)^2 - 0.1430\left(\frac{\delta}{\Delta}\right)^3 + 0.0290\left(\frac{\delta}{\Delta}\right)^4\right]$$

$$+ B_1\left(\frac{\delta}{\Delta}\right)^2\left[0.1 - 0.1143\,\frac{\delta}{\Delta} + 0.0536\left(\frac{\delta}{\Delta}\right)^2 - 0.0096\left(\frac{\delta}{\Delta}\right)^3\right]$$

$$N_1 = \frac{\delta}{\Delta}\left[0.0500 - 0.0429\left(\frac{\delta}{\Delta}\right)^2 + 0.0286\left(\frac{\delta}{\Delta}\right)^3 - 0.006\left(\frac{\delta}{\Delta}\right)^4\right]$$

$$+ B_1\left(\frac{\delta}{\Delta}\right)^2\left[-0.0167 + 0.0214\left(\frac{\delta}{\Delta}\right) - 0.0107\left(\frac{\delta}{\Delta}\right)^2 + 0.002\left(\frac{\delta}{\Delta}\right)^3\right]$$

From equations (6.10) and (6.8) it is seen that the point of separation of the laminar layer for the given boundary conditions is determined by the condition $\lambda_2 = -12$.

7. Determination of the Initial Conditions

For subsonic flow and also supersonic in those cases where there is a head wave in front of the profile a critical point $U(0) = 0$ is formed at the leading edge $(x = 0)$. The latter is a singular point of equations (4.15) and (4.16) in which the derivatives $d\vartheta/dx$, $d\theta/dx$, and so forth, increase to infinity if the initial ϑ, θ are not subjected to certain special conditions.

Substituting the expressions for τ_w, q_w, Δ^* from equations (6.10) and (6.11) into equations (4.15) and (4.16), multiplying the latter by δ and Δ, respectively, and equating to zero the coefficients of $1/U$ these conditions are obtained in the form

$$\lambda_2(H + 2)\frac{\vartheta}{\delta} + \lambda_2(\overline{T}_w - 1)\frac{\Delta}{\delta}\,\frac{8 - B_1}{20} - \overline{T}_w A_1 = 0, \quad \lambda_2\left(\frac{\Delta}{\delta}\right)^2\frac{\theta}{\Delta} - \overline{T}_w B_1 = 0 \qquad (7.1)$$

In the absence of heat transfer $(\bar{T}_w = 1)$ the first relation (7.1) is a cubical equation in λ_2. Of its three roots (7.052, 17.75, and −70) only the root $(\lambda_2)_{x=0} = 7.052$ satisfies the physical conditions. This value of λ_2 is the one generally assumed initial condition in the theory of the laminar layer for the equation of Kármán–Pohlhausen. The equations (7.1) may be conveniently regarded as a system for determining the initial values of $(\lambda_2)_{x=0}$ and $(\Delta/\delta)_{x=0}$. We present the results of the computation of these values for $0 < (\bar{T}_w)_{x=0} < 5$ (for $n = 0.75$).

$\bar{T}_w = 0$	0.05	0.10	0.50	1.00	1.50	2.00	3.00	4.00	5.00
$\lambda_2 = 0$	0.3	0.7	4.05	7.05	7.5	6.0	3.9	2.6	1.9
$\frac{\Delta}{\delta} = 1.12$	1.16	1.18	1.23	1.31	1.50	1.87	2.74	3.70	4.86

The graphs of these results are shown in figure 2.

8. Method of Successive Approximations

Simple computation formulas can be obtained by generalizing the method of H. Lyon (reference 5). At the same time we modify the method of Lyon with the object of improving the convergence.

We multiply (4.15) by $2\bar{\vartheta}$ and have

$$2\bar{\vartheta}\frac{d\bar{\vartheta}}{d\bar{x}} + \frac{\bar{U}'}{\bar{U}}\frac{2(H + 2 - \bar{U}^2)}{1 - \bar{U}^2}\bar{\vartheta}^2 + 2\frac{\bar{U}'}{\bar{U}}\frac{\bar{T}_w - 1}{1 - \bar{U}^2}\bar{\Delta}^*\bar{\vartheta} = \frac{2}{R_{02}\bar{U}}\bar{T}_w^{\,n-1}(1 - \bar{U}^2)^{\frac{\kappa}{\kappa-1}}A_1\frac{\vartheta}{\delta} \tag{8.1}$$

where

$$\bar{\vartheta} = \frac{\vartheta}{L}, \quad \bar{\Delta}^* = \frac{\Delta^*}{L}, \quad \bar{x} = \frac{x}{L}, \quad \bar{U}' = \frac{d\bar{U}}{d\bar{x}}, \quad R_{02} = \frac{\sqrt{2i_0}L\rho_{02}}{\mu_{00}}$$

and L is a characteristic dimension. Setting $k = 2(H + 2)$ we try in the function k to separate a certain principal part constant for a given value of $\bar{T}_w$; that is, we set

$$k = c_1 + (k - c_1), \quad \text{where} \quad c_1 = c_1(\bar{T}_w)$$

After simple transformations we obtain

$$2\bar{\vartheta}\frac{d\bar{\vartheta}}{d\bar{x}} + \frac{\bar{U}'}{\bar{U}}\left[c_1 + 2\left(\bar{T}_w - 1\right)\frac{\Delta^*}{\vartheta}\right]\bar{\vartheta}^2 + \frac{\bar{U}\bar{U}'}{1 - \bar{U}^2}\left[c_1 + 2\left(\bar{T}_w - 1\right)\frac{\Delta^*}{\vartheta} - 2\right]\bar{\vartheta}^2$$

$$= \bar{T}_w^{\,n-2}\frac{1}{\bar{U}R_{02}}\left(1 - \bar{U}^2\right)^{\frac{\kappa}{\kappa-1}}\left[2A_1\frac{\vartheta}{\delta}\bar{T}_w - \lambda_2\left(\frac{\vartheta}{\delta}\right)^2(k - c_1)\right] \tag{8.3}$$

Assuming over a certain part of the boundary layer $\bar{x}_0 < \bar{x} < \bar{x}_1$ that the ratio $\frac{\Delta^*}{\vartheta} = h$ is constant with respect to $\bar{x}$ and equal to the mean value for the given segment, we set[3]

$$c = c_1 + 2(\bar{T}_w - 1)h = c_1 + 2(\bar{T}_w - 1)\frac{\Delta}{\delta}\frac{1260(8 - B_1)}{20(-5A_1^{\,2} + 12A_1 + 144)} \tag{8.4}$$

Multiplying equation (8.3) by $\bar{U}^c$ we obtain

$$\frac{d}{d\bar{x}}\left(\bar{\vartheta}^2\bar{U}^c\right) + (c - 2)\frac{\bar{U}\bar{U}'}{1 - \bar{U}^2}\bar{\vartheta}^2\bar{U}^c$$

$$= \bar{T}_w^{\,n-2}\frac{\bar{U}^{c-1}}{R_{02}}\left(1 - \bar{U}^2\right)^{\frac{\kappa}{\kappa-1}}\left[2A_1\frac{\vartheta}{\delta}\bar{T}_w - \lambda\left(\frac{\vartheta}{\delta}\right)^2(k - c_1)\right]$$

Considering this equation as linear in $\bar{\vartheta}^2\bar{U}^c$ we write its solution in the form

$$\bar{\vartheta}^2R_{02} = \frac{1}{\bar{U}^c\left(1 - \bar{U}^2\right)^{1-\frac{c}{2}}}\left[C + \int_{\bar{x}_0}^{\bar{x}}\bar{T}_w^{\,n-2}\,\bar{U}^{c-1}\left(1 - \bar{U}^2\right)^{1-\frac{c}{2}+\frac{\kappa}{\kappa-1}}\Phi\,d\bar{x}\right] \tag{8.5}$$

---

[3]For $\bar{T}_w \neq$ Constant there is taken in the exponent c a constant mean value of $\bar{T}_w$ for each section.

where

$$\Phi = 2A_1 \frac{\vartheta}{\delta} \overline{T}_w - \lambda_2\left(\frac{\vartheta}{\delta}\right)^2 (k - c_1), \qquad C = \left[\bar{\delta}^2 R_{02} \overline{U}^c \left(1 - \overline{U}^2\right)^{1-\frac{\bar{c}}{2}}\right]_{\bar{x}=\bar{x}_0} \tag{8.6}$$

where $C = 0$ for $\bar{x}_0 = 0$. Substituting under the integral (in the function Φ and exponent c) certain initial values of $(\lambda_2)_0$ and $(\Delta/\delta)_0$ we obtain a first approximate function $\bar{\delta}(\bar{x})$. The arbitrariness in the choice of the magnitude c_1 may be utilized for improving the convergence. For this purpose c_1 must be chosen such that for each value of $\overline{T}_w$ the error due to the assumed (inexact) values of λ_2 and Δ/δ is a minimum.

Since Φ itself depends little on Δ/δ it is sufficient to set up the condition of little variation of Φ with λ_2. Neglecting the relatively small dependence of the functions ϑ/δ and k on λ, that is, setting in the argument A_1 on which they depend, $\lambda_2 = 0$, we obtain the approximate expression

$$\Phi \approx \frac{12 + \lambda_2}{6} (A_1)_{\lambda_2=0} \left(\frac{\vartheta}{\delta}\right)_{\lambda_2=0} \overline{T}_w - \lambda_2\left(\frac{\vartheta}{\delta}\right)^2_{\lambda_2=0} \left[(k)_{\lambda_2=0} - c_1\right]$$

In order that $\Phi = \text{Constant}$ the coefficient of λ_2 must be equal to zero, whence we obtain

$$c_1 = (k)_{\lambda_2=0} - \frac{(A_1)_{\lambda_2=0} \overline{T}_w}{6(\vartheta/\delta)_{\lambda_2=0}}$$

This dependence of c_1 on $\overline{T}_w$ in a wide interval of change of the argument (and practically independent of Δ/δ) is close to a linear one. For $\overline{T}_w = 0.1$ we obtain $c_1 = 9.35$ for $\frac{\Delta}{\delta} = 1$ ($c_1 = 9.12$ for $\frac{\Delta}{\delta} = 2$, $c_1 = 9.7$ for $\frac{\Delta}{\delta} = 0.5$). For $\overline{T}_w = 1$ we obtain $c_1 = 6.26$. Rounding off the last value to $c_1 = 6$ we apply the linear relation

$$c_1 = 9.5 - 3.5\overline{T}_w \tag{8.7}$$

Finally multiplying the equation of energy (equation (4.17)) in nondimensional form term by term by $2\bar{\theta}\overline{U}^2$ $\left(\bar{\theta} = \frac{\theta}{L}\right)$ and integrating as linear we obtain

$$\bar{\theta}^2 R_{02} = \frac{1}{\bar{U}^2(\bar{T}_w - 1)^2}\left[C' + \int_{\bar{x}_0}^{\bar{x}} \bar{T}_w^{\,n-1}\, 2B_1(\bar{T}_w - 1)^2\, \bar{U}(1 - \bar{U}^2)^{\frac{\kappa}{\kappa-1}} \frac{\theta}{\Delta}\, d\bar{x}\right] \quad (8.9)$$

where

$$C' = \left[\bar{\theta}^2 R_{02} \bar{U}^2 (\bar{T}_w - 1)^2\right]_{\bar{x}=\bar{x}_0}$$

and $C' = 0$ for $\bar{x} = 0$.

The computation of the dynamic and thermal layers by equations (8.5) and (8.9) can be carried out in this sequence. We consider, together with the parameter λ_2, the analogous composite parameters

$$\lambda_{2\vartheta} = \frac{\bar{\vartheta}^2 R_{02} \bar{U}' \bar{T}_w^{\,2-n}}{(1 - \bar{U}^2)^{\omega}}, \quad \lambda_{2\Delta} = \frac{\bar{\Delta}^2 R_{02} \bar{U}' \bar{T}_w^{\,2-n}}{(1 - \bar{U}^2)^{\omega}}, \quad \lambda_{2\theta} = \frac{\bar{\theta}^2 R_{02} \bar{U}' \bar{T}_w^{\,2-n}}{(1 - \bar{U}^2)^{\omega}}$$

$$\left(\omega = \frac{\kappa}{\kappa - 1} + 1\right)$$

By equations (6.9) and (6.11) represented in the form

$$\lambda_{2\vartheta} = \lambda_2\left(\frac{-5A_1^{\,2} + 12A_1 + 144}{1260}\right)^2 \qquad \frac{\Delta}{\delta} = \sqrt{\frac{\lambda_{2\Delta}}{\lambda_2}}$$

$$\lambda_{2\theta} = \lambda_{2\Delta}(M_1 + N_1 A_1)^2$$

there are constructed for the given value of $\bar{T}_w$ auxiliary graphs of the dependence of λ_2 on $\lambda_{2\vartheta}$ for various values of $\lambda_{2\Delta}$ and for the dependence of $\lambda_{2\Delta}$ on $\lambda_{2\theta}$ for various values of λ_2. The magnitudes taken as initial in the computation by formula (8.5) are determined from the initial data. If $\bar{U} \neq 0$ for $\bar{x}_0 = 0$, there are taken the values[4] $(\lambda_2)_0 = 0$ and $(\Delta/\delta)_0 = 1$. If $\bar{U} = 0$ for $\bar{x}_0 = 0$, there can be taken as the initial values the values of $(\lambda_2)_0$ and $(\lambda_{2_\Delta})_0 = (\Delta/\delta)_0^{\,2}(\lambda_2)_0$ according to figure 2. By the values of the functions $\bar{\vartheta}(\bar{x})$, that is, $\lambda_{2_0}(\bar{x})$ of the first approximation using the initial value $(\Delta/\delta)_0$ (for the succeeding approximations it is convenient to use the initial values of λ_{2_Δ}), $\lambda_2(x)$ of the first approximation is found with the aid of the graph. Further, by equation (8.9) there is computed $\bar{\theta}(\bar{x})$, that is, $\lambda_{2_\theta}(\bar{x})$ of the first approximation, making use of $\lambda_2(\bar{x})$ of the first approximation and $(\Delta/\delta)_0$ (in the succeeding approximations there is used the function $\lambda_2(\bar{x})$ following and $\lambda_{2_\Delta}(\bar{x})$ preceding). From the values of $\lambda_{2_\theta}(\bar{x})$ and $\lambda_2(\bar{x})$ with the aid of the graph there is obtained $\lambda_{2_\Delta}(\bar{x})$. In those cases where there is a considerable change in the ratio Δ^*/ϑ (or the function $T_w(\bar{x})$) the computation must be conducted over segments. The required data for each succeeding segment are taken equal to the corresponding values obtained at the end of the preceding segment. The local Nusselt number (that is, the coefficient of heat transfer q_w/t_0^* reduced to nondimensional form) and the coefficient of friction are found from the equations

$$N = \frac{q_w L}{\lambda_\infty t_0^*} = \bar{T}_w^{\,n-1}\left[1 + \frac{1}{2}(\kappa - 1)M_\infty^{\,2}\right]^n (1 - \bar{U}^2)^{\frac{\kappa}{\kappa-1}} \frac{B_1}{\bar{\Delta}} \tag{8.10}$$

$$C_f = \frac{2\tau_w}{\rho_\infty U_\infty^{\,2}} = \frac{2}{R_\infty} \bar{T}_w^{\,n-1}\left[1 + \frac{1}{2}(\kappa - 1)M_\infty^{\,2}\right]^n (1 - U^2)^{\frac{\kappa}{\kappa-1}} \frac{U}{U_\infty} \frac{A_1}{\bar{\delta}} \tag{8.11}$$

---

[4]In particular for $\bar{T}_w = 1$ formula (8.5) with $(\lambda_2)_0 = 0$ gives $\left(M_\infty < M_{\infty_{cr}}\right)$

$$\bar{\vartheta}^2 R_{00} = \frac{1}{\bar{U}^c (1 - \bar{U}^2)^{1-\frac{c}{2}}} \, 0.47 \int_0^{\bar{x}} \bar{U}^{c-1} (1 - \bar{U}^2)^{1-\frac{c}{2}+\frac{\kappa}{\kappa-1}} d\bar{x} \qquad (c = 6)$$

which agrees with the equation of L. G. Loitsiansky and A. A. Dorodnitsyn for the computation of the laminar layer without heat transfer (reference 6). In the absence of heat transfer and for small Mach numbers we again obtain from this the quadrature

$$\bar{\vartheta}^2 R = \frac{1}{(U/U_\infty)^6} \, 0.47 \int_0^{\bar{x}} \left(\frac{U}{U_\infty}\right)^5 dx \qquad \left(R = \frac{U_\infty L \rho}{\mu}\right)$$

earlier derived by us on the basis of the method of Kármán-Pohlhausen for the laminar layer in an incompressible fluid. (See Tekhnika Vozdushnogo Flota, No. 5-6, 1942.)

For small Mach numbers equations (8.5) and (8.9) assume the form

$$\bar{\delta}^2 R = \frac{1}{(U/U_\infty)^c} \int_0^{\bar{x}} \left(\frac{T_w}{T_0}\right)^{n-2} \left(\frac{U}{U_\infty}\right)^{c-1} \Phi \, d\bar{x} \qquad \left(R = \frac{U_\infty L \rho_0}{\mu_0}\right) \tag{8.12}$$

$$\bar{\theta}^2 R = \frac{1}{(T_w/T_0 - 1)^2 (U/U_\infty)^2} \int_0^{\bar{x}} \left(\frac{T_w}{T_0}\right)^{n-1} 2B_1 \left(\frac{T_w}{T_0} - 1\right)^2 \frac{U}{U_\infty} \frac{\theta}{\Delta} \, d\bar{x} \tag{8.13}$$

9. Dependence of the Reynolds Number R_{02} on the Parameters of the Flow

Taking account of the fact that according to the equation of state $\frac{\rho_{02}}{\rho_{01}} = \frac{p_{02}}{p_{01}}$ we represent the parameter R_{02} in the form

$$R_{02} = R_{01} \frac{p_{02}}{p_{01}}, \qquad \text{where} \quad R_{01} = \frac{\sqrt{2i_0} L \rho_{01}}{\mu_{00}} \tag{9.1}$$

The parameter R_{01} is expressed directly in terms of the Reynolds number $R_\infty = \frac{U_\infty L \rho_\infty}{\mu_\infty}$ and the Mach number $M_\infty = \frac{U_\infty}{a_\infty}$ of the approaching flow. From equations (3.3) to (3.5) we find

$$R_{01} = R_\infty \frac{1}{\bar{U}_\infty} \left(1 - \bar{U}_\infty^2\right)^{n - \frac{1}{\kappa-1}} = R_\infty \left[1 + \frac{1}{2}(\kappa - 1) M_\infty^2\right]^{\frac{\kappa+1}{2(\kappa-1)} - n} \left[\frac{1}{2}(\kappa - 1) M_\infty^2\right]^{-\frac{1}{2}} \tag{9.2}$$

In the case of subsonic (subcritical) velocities we have $R_{02} = R_{01} = R_{00}$. For supersonic velocities the ratio p_{02}/p_{01} is found from the condition of a line of a flow passing through an oblique shock wave (or a head wave) at the leading edge of the body. Considering each surface of the profile separately we denote by β_0 the angle which the tangent to the surface of the airfoil at any point makes with the direction of the velocity of the undisturbed flow and, by φ the angle which the normal to the surface of discontinuity makes with the same direction. From the equations of the oblique shock wave we obtain

$$\frac{p_{02}}{p_{01}} = \left(\frac{1/2(\kappa + 1)M_\infty^{2}\cos^2\varphi}{1 + 1/2(\kappa - 1)M_\infty^{2}\cos^2\varphi}\right)^{\frac{\kappa}{\kappa-1}} \left(\frac{2\kappa}{\kappa + 1} M_\infty^{2}\cos^2\varphi - \frac{\kappa - 1}{\kappa + 1}\right)^{\frac{1}{1-\kappa}} \tag{9.3}$$

$$\tan\,(\varphi + \beta_0) = \frac{1}{2}\,\frac{1/2(\kappa + 1)M_\infty^{2}\sin 2\varphi}{1 + 1/2(\kappa - 1)M_\infty^{2}\cos^2\varphi} \tag{9.4}$$

In the case of a head wave in front of the body the direction of the velocity after the discontinuity (for the flow line at the profile) may be considered to coincide with the direction of the velocity before the discontinuity; in equation (9.3) there is in this case to be substituted $\varphi = 0$.

10. Boundary Layer in the Flow of a Gas with Axial Symmetry

For any axial flow about a body of revolution the integral relations of the impulse and energy have the following form:

$$\frac{d}{dx}\int_0^{\delta_y} \rho u^2 r\;\,dy - U\,\frac{d}{dx}\int_0^{\delta_y} \rho u r\,dy = -\frac{dp}{dx}\,\delta_y r - \tau_w r \tag{10.1}$$

$$\frac{d}{dx}\int_0^{\Delta_y} c_p \rho u(t^* - t_0{}^*)r\;dy = -q_w r \tag{10.2}$$

In these equations the usual simplifications were made; x is the distance along the arc of the meridional section, d the distance along the normal to the surface, and $r(x)$ the radius of the cross section of the body of rotation (the change of the radius vector within the boundary layer is neglected). The boundary conditions of the problem and also the assumptions with regard to the external flow are taken to be the same as in section 1. Setting up expressions for T, p, ρ, and μ as in sections 2 and 3 and introducing the new independent variable η by formula (4.9) we obtain the integral relations of the momenta and energy in the variables x, η:

$$\frac{d\bar{\vartheta}}{d\bar{x}} + \frac{\bar{U}'}{\bar{U}}\left[H + 2 + (H + 1)\frac{\bar{U}^2}{1 - \bar{U}^2}\right]\bar{\vartheta} + \frac{U'(\bar{T}_w - 1)}{\bar{U}(1 - \bar{U}^2)}\ \bar{\theta}H_T + \frac{\bar{r}'}{\bar{r}}\bar{\vartheta} = \frac{\tau_w}{\rho_{00}U^2} \quad (10.3)$$

$$\frac{d\bar{\theta}}{d\bar{x}} + \frac{\bar{U}'}{\bar{U}}\bar{\theta} + \frac{\bar{r}'}{\bar{r}}\theta + \frac{t_0'^*}{t_0^*}\theta = \frac{q_w}{\rho_{00}Uc_p t_0^*} \qquad \left(\bar{x} = \frac{x}{L},\ \bar{\vartheta} = \frac{\vartheta}{L}, \text{and so forth}\right) \quad (10.4)$$

(L is a characteristic dimension, for example, the length of the body of rotation.) Restricting ourselves to the case $P = 1$ and taking the expressions (6.1) and (6.2) we obtain a closed system of equations (10.3), (10.4), (6.9), and (6.11) the solution of which we write (for the case where there is no shock wave) in the form

$$\bar{\vartheta}^2 R_{00} = \frac{1}{\bar{U}^c(1 - \bar{U}^2)^{1-\frac{c}{2}}\bar{r}^2}\left[C + \int_{\bar{x}_0}^{\bar{x}} \bar{T}_w^{\,n-2}\bar{U}^{c-1}(1 - \bar{U}^2)^{1-\frac{c}{2}+\frac{\kappa}{\kappa-1}}\bar{r}^2\Phi\, d\bar{x}\right] \quad (10.5)$$

$$\bar{\theta}^2 R_{00} = \frac{1}{\bar{U}^2(\bar{T}_w - 1)^2\bar{r}^2}\left[C' + \int_{\bar{x}_0}^{\bar{x}} \bar{T}_w^{\,n-1}2B_1\bar{U}(1 - \bar{U}^2)^{\frac{\kappa}{\kappa-1}}(\bar{T}_w - 1)^2\bar{r}^2\frac{\theta}{\Delta}\, d\bar{x}\right] \quad (10.6)$$

where

$$C = \left[\bar{\vartheta}^2 R_{00}\bar{U}^c(1 - \bar{U}^2)^{1-\frac{c}{2}}\bar{r}^2\right]_{\bar{x}=\bar{x}_0} \qquad C' = \left[\bar{\theta}^2 R_{00}\bar{U}^2(\bar{T}_w - 1)^2\bar{r}^2\right]_{\bar{x}=\bar{x}_0}$$

The method of computation does not differ from the case of the two-dimensional flow. For the coefficients of the heat transfer and friction the equations (8.10) and (8.11) remain valid. In the case of the internal problem (flow in nozzles) the Nusselt number and the friction coefficient are determined by the equations

$$N_{00} = \frac{q_w L}{\lambda_{00}t_0^*} = \bar{T}_w^{\,n-1}(1 - \bar{U}^2)^{\frac{\kappa}{\kappa-1}}\frac{B_1}{\bar{\Delta}}, \qquad c_{f00} = \frac{2\tau_w}{\rho_{00}U^2} = \frac{2}{R_{00}}\bar{T}_w^{\,n-1}(1 - \bar{U}^2)^{\frac{\kappa}{\kappa-1}}\frac{A_1}{\bar{U}\bar{\delta}}$$

where λ_{00} is the coefficient of heat conductivity corresponding to the temperature T_{00}.

III. TURBULENT BOUNDARY LAYER IN THE PRESENCE OF HEAT TRANSFER BETWEEN THE GAS AND WALL

11. Fundamental Assumptions

The functions H, H_T, τ_w, and q_w entering equations (4.15) and (4.16) are determined by equations (4.12), (4.14), (4.5), and (4.7) which express them as functions of ϑ and θ through the medium of the velocity profile and the stagnation temperature-drop profile. The present state of the problem of turbulence does not permit representing the velocity profile (and also the temperature profile) by a single equation which holds true from the wall to the boundary-layer limit. The fundamental dynamic and thermal characteristics of the turbulent layer can nevertheless be computed with an accuracy which is sufficient for practical purposes. A fortunate property of equations (4.15) and (4.16) which can be predicted on the basis of the results with respect to noncompressible fluids is that the functions H and H_T change very little over the length of the turbulent layer and the functions τ_w and q_w connected with ϑ and θ by the equations are little sensitive to the actual conditions which prevail at a given section of the boundary layer. Hence H and H_T (and also magnitudes analogous to them) can be taken as constant over x and the relations between τ_w and ϑ (the resistance law) and between q_w and θ (the heat-transfer law) can be set up starting from the assumption that the conditions at the given section of the boundary layer do not differ from the conditions on the flat plate. On the basis of the derivation of these supplementary equations we assume the simple scheme of Kármán according to which the section of the boundary layer is divided into a purely turbulent "nucleus of the flow" and a "laminar sublayer" in immediate contact with the wall. In the latter the turbulent friction and the temperature drop are small by comparison with the molecular. We assume that in the turbulent "nucleus" the frictional stress is expressed by the formula of Prandtl:

$$\tau = \rho l^2 \left(\frac{du}{dy}\right)^2 \tag{11.1}$$

where l is the length of the mixing path. In other words, in equations (1.1) and (1.3) we set $\epsilon = \rho l^2 \frac{du}{dy}$. It follows directly from this in view of the fact that the turbulence assumption of Prandtl gives $\lambda_t = c_p \epsilon$ that the expression for the heat transfer is

$$q = \lambda_t \frac{dT}{dy} = c_p \rho l^2 \frac{du}{dy} \frac{dT}{dy} \tag{11.2}$$

The thickness δ_{yl} of the laminar sublayer of the dynamic boundary layer, equal for $P = 1$ to the thickness Δ_{yl} of the thermal sublayer, is determined by the critical Reynolds number (the Kármán criterion)

$$\frac{u_l \delta_{yl} \rho_w}{\mu_w} = \alpha^2 \qquad (\alpha \approx 11.5) \tag{11.3}$$

where u_l is the velocity on the boundary of the laminar sublayer, ρ_w and μ_w are the density and viscosity at the wall.

12. Derivation of the Resistance Law

Assuming that as in the case of the noncompressible fluid a linear variation of the velocity in the laminar sublayer is permissible on account of the small thickness, we have

$$\tau_w = \mu_w \frac{u_l}{\delta_{yl}} \tag{12.1}$$

From equations (11.3) and (12.1) we obtain

$$\frac{\sqrt{\tau_w \rho_w}\ \delta_{yl}}{\mu_w} = \alpha \tag{12.2}$$

In equation (12.2) we pass to the variable η. Near the wall on account of the smallness of the terms $\overline{u}^2$ and t^*/t_0^* we have

$$y = \int_0^\eta (1 - \overline{u}^2)^{\frac{\kappa}{1-\kappa}} \left[1 - \overline{u}^2 + (\overline{T}_w - 1) \left(1 - \frac{t^*}{t_0^*}\right)\right] d\eta \approx \overline{T}_w (1 - \overline{U}^2)^{\frac{\kappa}{1-\kappa}} \eta \tag{12.3}$$

hence

$$\delta_{yl} \approx \overline{T}_w (1 - \overline{U}^2)^{\frac{\kappa}{\kappa-1}} \delta_l$$

where $\delta_l = \Delta_l$ is the thickness of the laminar sublayer for the variable η. Further,

$$\rho_w = \rho_{02} \frac{1}{\overline{T}_w} (1 - \overline{U}^2)^{\frac{\kappa}{\kappa-1}}, \quad \mu_w = \mu_{00} \overline{T}_w{}^n$$

Substituting these expressions in equation (12.2) we obtain

$$\frac{\overline{\delta}_l}{\overline{\theta}} = \frac{\alpha\zeta}{R_\vartheta} \overline{T}_w{}^n \quad \left(\overline{\delta}_l = \frac{\delta}{L}\right) \tag{12.4}$$

For the fundamental parameters of the dynamic layer there is here introduced the notation

$$R_\vartheta = \overline{U}\overline{\vartheta}R_{02}, \quad \zeta = \frac{U}{\sqrt{\tau_w/\rho_{02}}} \frac{1}{\overline{T}_w{}^{1/2}} (1 - \overline{U}^2)^{\frac{\kappa}{2(\kappa-1)}} \tag{12.5}$$

Equation (12.1) is with the aid of equations (3.2) and (4.9) transformed into the form

$$\tau = \rho_{02}\tilde{l}^2\left(\frac{du}{d\eta}\right)^2, \quad \tilde{l}^2 = \frac{l^2}{\left[1 - \overline{u}^2 + (T_w - 1)(1 - t^*/t_0^*\right]^3} (1 - \overline{U}^2)^{\frac{3\kappa}{\kappa-1}} \tag{12.6}$$

Since for small η the terms $\overline{u}^2$ and t^*/t_0^* are small and the mixing path $l = ky$ $(k = 0.391)$ where the coordinate y is expressed according to equation (12.3), the "generalized" mixing path $\tilde{l}$ near the wall is a linear function of η:

$$\tilde{l} = k\eta \frac{1}{T_w{}^{1/2}} (1 - \overline{U}^2)^{\frac{\kappa}{2(\kappa-1)}} \tag{12.7}$$

In deriving the resistance law in an incompressible fluid a linear mixing-path distribution and a constant frictional stress are assumed for the entire section of the boundary layer, from the wall to the outer boundary. Actually the mixing path increases at a considerably slower rate than according to the linear law and the friction drops to zero as the outer boundary of the layer is approached. The assumptions made act in opposite directions and lead to a satisfactory relation between the parameters R_ϑ and ζ.

Carrying over this fundamental idea of the logarithmic method into our present theory we set $\tau = \tau_w$ in equation (12.7) and assume the linear law (equation (12.7)) for the entire section of the boundary layer. We thus assume that as in the case of the noncompressible fluid there will be a mutual compensation of the errors committed in the distribution of τ and $\tilde{l}$. Integrating equation (12.6) between η and δ we obtain the approximate velocity profile:

$$\frac{u}{U} = 1 + \frac{1}{k\zeta} \ln \frac{\eta}{\delta} \tag{12.8}$$

From equations (12.1) and (12.4) we obtain the velocity at the boundary of the laminar sublayer

$$\frac{u_l}{U} = \frac{\alpha}{\zeta}$$

The condition of the equality of the velocities of the turbulent and laminar flows on the boundary of the sublayer gives

$$R_\delta = C_1 \overline{T}_w{}^n e^{k\zeta} k\zeta \qquad \left(R_\delta = \overline{U}\delta R_{02},\ C_1 = \frac{\alpha}{k} e^{-k\alpha} = 0.326\right) \tag{12.10}$$

Making use of the velocity profile (12.8) in equations (4.12) we obtain

$$\frac{\vartheta}{\delta} = \frac{1}{k\zeta}\left(1 - \frac{2}{k\zeta}\right), \qquad \frac{\delta^*}{\delta} = \frac{1}{k\zeta} \tag{12.11}$$

Eliminating from equation (12.10) and the first of relations (12.11) the auxiliary variable δ, we obtain the resistance law:

$$R_\vartheta = C_1 \overline{T}_w{}^n e^{k\zeta}\left(1 - \frac{2}{k\zeta}\right) \tag{12.12}$$

We obtain incidentally also the approximate expression for the parameter H:

$$H = \frac{\delta^*}{\vartheta} = \frac{1}{1 - 2/k\zeta} \tag{12.13}$$

13. Derivation of the Heat-Transfer Law

In this section we shall give a generalization to the case of a gas moving with large velocities of the heat-transfer law earlier derived by us (reference 4) for an incompressible fluid.

We construct the function

$$q^* = \lambda \frac{dt^*}{dy} = q + \frac{1}{J}\tau u \tag{13.1}$$

For the turbulent nucleus of the flow we have

$$q^* = \lambda_t \frac{dt^*}{dy} = c_p \epsilon \frac{dt^*}{dy} = c_p \rho l^2 \frac{du}{dy}\frac{dt^*}{dy'} \tag{13.2}$$

Transforming this equation to the variable η we obtain

$$q^* = c_p \rho_{02} \tilde{l}^2 \frac{du}{d\eta}\frac{dt^*}{d\eta}, \quad \tilde{l}^2 = \frac{l^2}{\left[1 - \bar{u}^2 + (\bar{T}_w - 1)(1 - t^*/t_0^*\right]^3}(1 - \bar{U}^2)^{\frac{3\kappa}{\kappa - 1}} \tag{13.3}$$

Near the wall the function q^* behaves like q, that is, differs little from the constant value q_w, and the mixing path $\tilde{l}$ depends linearly on η according to equation (12.7). The common mechanism of the transfer of heat and the transfer of the momentum in the flows along solid walls provides a basis in the derivation of the law of heat transfer for assuming as before a constant value $q^* = q_w$ and the linear law (equation (12.7)) for the entire thickness of the thermal layer. Substituting the expression for $du/d\eta$ obtained from equation (12.8) and integrating equation (13.3) from η to Δ we obtain the approximate profile for the stagnation temperatures

$$\frac{t^*}{t_0^*} = 1 + \frac{q_w U}{c_p t_0^* \tau_w k \zeta} \ln \frac{\eta}{\Delta} \tag{13.4}$$

From equation (13.1), assuming a linear distribution of the stagnation temperatures in the laminar sublayer, we find

$$q_w = c_p \mu_w \frac{t_l^*}{\Delta y l} \tag{13.5}$$

where t_l^* is the stagnation temperature at the boundary of the laminar sublayer.

As the fundamental thermal characteristics of the boundary layer we introduce the following parameters:

$$R_\theta = \overline{U}\theta R_{02}, \quad \zeta_T = \frac{R_{02}}{N_{00}} \frac{\overline{U}}{\zeta} \frac{1}{\overline{T}_w} \left(1 - \overline{U}^2\right)^{\frac{\kappa}{\kappa-1}} \qquad \left(N_{00} = \frac{q_w L}{\lambda_{00} t_0^*}\right) \tag{13.6}$$

From equations (13.4) and (13.5) we obtain the stagnation temperature on the boundary of the laminar sublayer

$$\frac{t^*}{t_0^*} = \frac{\alpha}{\zeta_T} \tag{13.7}$$

Equating the stagnation temperatures on the boundary of the laminar sublayer and making use of condition (12.4) we obtain

$$R_\Delta = C_1 \overline{T}_w^{\,n} \exp(k\zeta_T) k\zeta \qquad \left(R_\Delta = \overline{U}\Delta R_{02}\right) \tag{13.8}$$

Substituting the expressions for u/U and t^*/t_0^* according to equations (12.8) and (13.4) in equations (4.13) and (4.14) we obtain

$$\frac{\theta}{\Delta} = \left(1 + \frac{1}{k\zeta} \ln \frac{\Delta}{\delta}\right) \frac{1}{k\zeta_T} - \frac{2}{k^2 \zeta \zeta_T}, \qquad \frac{\Delta^*}{\Delta} = \frac{1}{k\zeta_T} \tag{13.9}$$

From equations (12.10) and (13.8) it follows that $\frac{\Delta}{\delta} = \exp(k\zeta_T - k\zeta)$ so that we have

$$\frac{\theta}{\Delta} = \frac{1}{k\zeta}\left(1 - \frac{2}{k\zeta_T}\right) \tag{13.10}$$

Eliminating from equations (13.8) and (13.10) the auxiliary parameter Δ we obtain the heat-transfer law

$$R_\theta = C_1 \overline{T}_w{}^n \left(1 - \frac{2}{k\zeta_T}\right) \exp k\zeta_T \tag{13.11}$$

We obtain incidentally also the approximate expression for the parameter H_T:

$$H_T = \frac{1/k\zeta_T}{1/k\zeta(1 - 2/k\zeta_T)} \tag{13.12}$$

14. Solution of the Equation of the Turbulent Dynamic Boundary Layer

We represent equation (4.15) in the form

$$\frac{dR_\vartheta}{d\bar{x}} + \frac{\overline{U}'(H + 1)}{\overline{U}(1 - \overline{U}^2)} R_\vartheta + \frac{\overline{U}'(\overline{T}_w - 1)}{\overline{U}(1 - \overline{U}^2)} \frac{\Delta^*}{\vartheta} R_\vartheta = \frac{1}{\zeta^2} \frac{1}{\overline{T}_w} (1 - \overline{U}^2)^{\frac{\kappa}{\kappa-1}} \overline{U}R_{02} \tag{14.1}$$

We make the change in variables (reference 7):

$$z = e^{k\zeta}(1 - 2/k\zeta)k^2\zeta^2 \tag{14.2}$$

Differentiating this relation with respect to $\bar{x}$ and using equation (12.12) we obtain

$$\frac{dz}{d\bar{x}} = Kz \left(\frac{1}{R_\vartheta} \frac{dR_\vartheta}{d\bar{x}} - \frac{n}{\overline{T}_w} \frac{d\overline{T}_w}{d\bar{x}}\right) \qquad \left(K = \frac{1 - 2/k^2\zeta^2}{1 - 2/k\zeta + 2/k^2\zeta^2}\right) \tag{14.3}$$

From equations (14.1) to (14.3) we obtain

$$\frac{dz}{d\bar{x}} + \frac{\bar{U}'K(H+1)}{\bar{U}(1-\bar{U}^2)}z + \frac{\bar{U}'K(\bar{T}_w - 1)}{\bar{U}(1-\bar{U}^2)}\frac{\Delta^*}{\vartheta}z + nK\frac{\bar{T}_w'}{\bar{T}_w}z = \frac{K\bar{U}R_{02}k^2}{C_1\bar{T}_w^{\,n+1}}(1-\bar{U}^2)^{\frac{\kappa}{\kappa-1}} \quad (14.4)$$

The magnitudes H and K which change little along the boundary layer are assumed constant with respect to $\bar{x}$. If Δ^*/ϑ is considered as a known function of $\bar{x}$ then equation (14.4) is a linear equation with respect to z.

Assuming a constant mean value of the ratio $\frac{\Delta^*}{\vartheta} = h$ over a certain interval $\bar{x}_0 < \bar{x} < \bar{x}_1$ (in the first approximation we may for the entire turbulent layer assume $h = \frac{\Delta^*}{\vartheta} = \frac{\delta^*}{\vartheta} = H$, which holds for the plate), we obtain the solution of equation (14.4) in the form[5]

$$z = \frac{\sqrt{(1-\bar{U}^2)^c}}{\bar{T}_w^{\,nK}\bar{U}^c}\left[C + \frac{KR_{02}k^2}{C_1}\int_{\bar{x}_0}^{\bar{x}} \frac{\bar{T}_w^{\,nK-n-1}\bar{U}^{c+1}}{\sqrt{(1-\bar{U}^2)^c}}(1-\bar{U}^2)^{\frac{\kappa}{\kappa-1}}\,d\bar{x}\right] \quad (14.5)$$

where

$$C = \left[z\bar{T}_w^{\,nK}\bar{U}^c\sqrt{(1-\bar{U}^2)^{-c}}\right]_{\bar{x}=\bar{x}_0}, \qquad c = K(H+1) + K(\bar{T}_w - 1)h$$

The constants must be taken equal to

$$K = 1.20, \quad H = 1.4, \quad h = 1.4, \qquad (C_1 = 0.326;\ k = 0.391)$$

---

[5]See footnote (3) on page 19.

After computation of the dynamic and thermal layer new mean values of the variables K, H, $h = \frac{H_T R_\theta}{R_\vartheta}$ can be found and, if the deviations from the assumed values are considerable, the computation is repeated. If, for greater accuracy, the computation is conducted over segments, the values of the constants for each succeeding segment are determined by the values of R_ϑ, ζ, R_θ, and ζ_T obtained at the end of the preceding segment. In integrating from the point of the transition of the laminar into the turbulent state, the magnitude $(z)_{\bar{x}=\bar{x}_0}$ is determined from the condition of equality of the initial value of R_ϑ to the value of R_ϑ at the end of the laminar segment. In integrating from the leading edge, $C = 0$. By equations (12.2) and (14.2) the auxiliary graphs of the functions $\log\left(R_\vartheta \bar{T}_w^{-n} C_1^{-1}\right)$ and $\log z$ as functions of $k\zeta$ can be constructed once for all.

The local friction coefficient is found from the equation

$$c_f = \frac{2\tau_w}{\rho_\infty U_\infty^2} = \frac{2}{\bar{T}_w}\left[1 + \frac{1}{2}(\kappa - 1) M_\infty^2\right]^{\frac{1}{\kappa-1}} \frac{p_{02}}{p_{01}}\left(\frac{U}{U_\infty}\right)^2 \frac{1}{\zeta^2}\left(1 - \bar{U}^2\right)^{\frac{\kappa}{\kappa-1}} \qquad (14.6)$$

For small Mach numbers equation (14.5) assumes the form

$$z = \frac{1}{(T_w/T_0)^{nK}(U/U_\infty)^c}\left[C + \frac{KRk^2}{C_1}\int_{\bar{x}_0}^{\bar{x}}\left(\frac{T_w}{T_0}\right)^{nK-n-1}\left(\frac{U}{U_\infty}\right)^{c+1} d\bar{x}\right] \qquad (14.7)$$

where

$$R = \frac{U_\infty L\rho_0}{\mu_0}, \qquad R_\vartheta = \frac{U}{U_\infty}\vartheta R, \qquad C = \left[z\left(\frac{T_w}{T_0}\right)^{nK}\left(\frac{U}{U_\infty}\right)^c\right]_{\bar{x}=\bar{x}_0}$$

15. Solution of the Equation of the Turbulent Thermal Layer

We represent equation (4.16) in the form

$$\frac{dR_\theta}{d\bar{x}} + \frac{1}{\bar{t}_0^*}\frac{d\bar{t}_0^*}{d\bar{x}}R_\theta = \frac{1}{\zeta\zeta_T}\frac{1}{\bar{T}_w}\bar{U}R_{02}\left(1 - \bar{U}^2\right)^{\frac{\kappa}{\kappa-1}} \qquad \left(\bar{t}_0^* = \frac{t_0^*}{T_{00}}\right) \qquad (15.1)$$

Carrying out the change in variables (reference 4)

$$z_T = \exp\,(k\zeta_T)\left(1 - \frac{2}{k\zeta_T}\right)k\zeta_T \tag{15.2}$$

and making use of the heat-transfer law (13.1) we find

$$\frac{dz_T}{d\bar{x}} + K_T n \frac{1}{\bar{T}_w}\frac{d\bar{T}_w}{d\bar{x}} z_T + K_T \frac{1}{\bar{t}_0{}^*}\frac{d\bar{t}_0{}^*}{d\bar{x}} z_T = K_T \frac{k^2}{k\zeta C_1 \bar{T}_w{}^{n+1}} \bar{U}R_{02}(1 - \bar{U}^2)^{\frac{\kappa}{\kappa-1}}$$

$$\left(K_T = \frac{1 - 1/k\zeta_T}{1 - 2/k\zeta_T + 2/k^2\zeta_T{}^2}\right) \tag{15.3}$$

The solution has the form[6]

$$z_T = \frac{1}{\bar{T}_w{}^{nK_T}(\bar{T}_w - 1)^{K_T}}\left[C' + \frac{K_T R_{02} k^2}{C_1}\int_{\bar{x}_0}^{x} \bar{T}^{nK_T-n-1}(\bar{T}_w - 1)^{K_T}\frac{\bar{U}}{k\zeta}(1 - \bar{U}^2)^{\frac{\kappa}{\kappa-1}} d\bar{x}\right]$$

$$C' = \left[z_T \bar{T}_w{}^{nK_T}(\bar{T}_w - 1)^{K_T}\right]_{\bar{x}=\bar{x}_0} \tag{15.4}$$

---

[6]The equation of energy (15.1) in connection with the relation between R_θ and ζ_T in the form (13.11) in the case $\bar{T}$ = Constant is an equation with separable variables so that together with the solution in the form (15.4) we can use its accurate solution

$$\exp(k\zeta_T)(k\zeta_T - 3) + 2\mathcal{E}i(k\zeta_T) = C + \frac{k^2 R_{02}}{C_1 \bar{T}_w{}^{n+1}}\int_{\bar{x}_0}^{\bar{x}} \frac{\bar{U}}{k\zeta}(1 - \bar{U}^2)^{\frac{\kappa}{\kappa-1}} d\bar{x}$$

where

$$\mathcal{E}i(k\zeta_T) = \int_{-\infty}^{k\zeta_T} \frac{e^u}{u}\,du, \quad C = \left[\exp(k\zeta_T)(k\zeta_T - 3) + 2\mathcal{E}i(k\zeta_T)\right]_{\bar{x}=\bar{x}_0}$$

$$(\text{for } \bar{x}_0 = 0, \quad C = 0)$$

The constant K_T must be taken equal to 1.15 (with subsequent check by the results of the computation of the thermal layer). The magnitude $(\zeta_T)_{\bar{x}=\bar{x}_0}$ is determined by equation (13.11) from the condition of the equality of the value of R_θ at the start of the turbulent region to its value at the end of the laminar section. In integrating from the leading edge, $C' = 0$. The auxiliary graph of the function $\log z_T$ against $k\zeta_T$ can be constructed once for all.

The local heat transfer is found from the equation

$$N = \frac{q_w L}{\lambda_\infty t_0^*} = \frac{R_{02}\bar{U}}{\zeta\zeta_T}\frac{1}{\bar{T}_w}\left[1 + \frac{1}{2}(\kappa - 1)M_\infty^2\right]^n (1 - \bar{U}^2)^{\frac{\kappa}{\kappa-1}} \tag{15.5}$$

For small values of the Mach number equation (15.4) assumes the form

$$z_T = \frac{1}{(T_w/T_0)^{nK_T}(T_w/T_0 - 1)^{K_T}}\left[C' + \frac{K_T R k^2}{C_1}\int_{\bar{x}_0}^{\bar{x}}\left(\frac{T_w}{T_0}\right)^{nK_T-n-1}\left(\frac{T_w}{T_0} - 1\right)^{K_T}\frac{U}{U_\infty}\frac{1}{k\zeta}\,d\bar{x}\right] \tag{15.6}$$

16. Determination of the Profile Drag for Subsonic Velocities

The drag of a wing of infinite span (over unit length of span) is obtained from the momentum theorem in the form

$$Q = \int_{-\infty}^{\infty}\rho_\infty u_\infty(U_\infty - u_\infty)\,dy = \rho_{00}U_\infty^2\int_{-\infty}^{\infty}\frac{u_\infty}{U_\infty}\left(1 - \frac{u_\infty}{U_\infty}\right)d\eta = \rho_{00}U_\infty^2\sum\vartheta_\infty \tag{16.1}$$

where $\sum\vartheta_\infty$ denotes the sum of the momentum-loss thicknesses referred to the upper and lower surfaces and computed at a great distance from the wing where $\delta \rightarrow \infty$ and $U \rightarrow U_\infty$. For the drag coefficient we have

$$c_{xp} = \frac{2Q}{\rho_\infty U_\infty^2 L} = 2\sum\bar{\vartheta}_\infty\left[1 + \frac{1}{2}(\kappa - 1)M_\infty^2\right]^{\frac{1}{\kappa-1}} \tag{16.2}$$

The problem consists in expressing ϑ_∞ in terms of the dynamic and thermal characteristics of the boundary layer at the trailing edge. Since in the wake behind the body $\tau_w = 0$ and $q_w = 0$, equations (4.16) and (4.17) for the wake assume the form

$$\frac{d\bar{\vartheta}}{d\bar{x}} + \frac{\bar{U}'}{\bar{U}}\left(2 + \frac{H + \bar{U}^2}{1 - \bar{U}^2} + \frac{\bar{T}_w - 1}{1 - \bar{U}^2}\frac{\theta}{\vartheta} H_T\right)\bar{\vartheta} = 0 \tag{16.3}$$

$$\frac{d\bar{\theta}}{d\bar{x}} + \frac{\bar{U}'}{\bar{U}}\bar{\theta} + \frac{1}{\bar{t}_0^*}\frac{d\bar{t}_0^*}{d\bar{x}}\bar{\theta} = 0 \tag{16.4}$$

We introduce the notation

$$G(\bar{x}) = \frac{H + \bar{U}^2}{1 - \bar{U}^2} + \frac{\bar{T}_w - 1}{1 - \bar{U}^2}\frac{\theta}{\vartheta} H_T \tag{16.5}$$

and represent equation (16.3) in the form

$$\frac{1}{\bar{\vartheta}}\frac{d\bar{\vartheta}}{d\bar{x}} = -(2 + G)\frac{d}{dx}\ln\frac{\bar{U}}{\bar{U}_\infty} \tag{16.6}$$

Integrating this equation with respect to $\bar{x}$ from the trailing edge (denoted by the subscript 1) to $\bar{x} = \infty$, we obtain

$$\ln\frac{\bar{\vartheta}_\infty}{\bar{\vartheta}_1} = (2 + G_1)\ln\frac{\bar{U}_1}{\bar{U}_\infty} + \int_{G_1}^{G_\infty}\ln\frac{\bar{U}}{\bar{U}_\infty}\,dG \tag{16.7}$$

For $\bar{U}_\infty \approx 0$, $\bar{T}_w = 1$ the function $G(\bar{x})$ goes over into $H(\bar{x})$. For an incompressible fluid however the hypothesis of Squire and Young (reference 8) on the linear character of the dependence $\ln(U/U_\infty)$ on H holds:

$$\frac{\ln(U/U_\infty)}{H_\infty - H} = \frac{\ln(U_1/U_\infty)}{H_\infty - H_1}$$

Making the analogous assumption

$$\frac{\ln(U/U_\infty)}{G_\infty - G} = \frac{\ln(U_1/U_\infty)}{G_\infty - G_1} \tag{16.8}$$

we obtain[7]

$$\ln \frac{\bar{\vartheta}_\infty}{\vartheta_1} = \left(2 + \frac{G_1 + G_\infty}{2}\right) \ln \frac{\bar{U}_1}{\bar{U}_\infty} \tag{16.9}$$

We write down the expression for G_1:

$$G_1 = \frac{H_1 + \bar{U}_1{}^2}{1 - \bar{U}_1{}^2} + \frac{\bar{T}_{w_1} - 1}{1 - \bar{U}^2} \frac{\theta_1}{\vartheta_1} H_T \tag{16.10}$$

It is easily seen that $H_\infty = 1$ and $H_{T_\infty} = 1$, hence

$$G_\infty = \frac{1 + \bar{U}_\infty{}^2}{1 - \bar{U}_\infty{}^2} + \frac{\bar{T}_{w_\infty} - 1}{1 - \bar{U}_\infty{}^2} \frac{\theta_\infty}{\vartheta_\infty} \tag{16.11}$$

---

[7]The same result can be arrived at from the following elementary considerations. Equation (16.7) may be represented in the form

$$\ln \frac{\bar{\vartheta}_\infty}{\bar{\vartheta}_1} = (2 + G_1) \ln \frac{\bar{U}_1}{\bar{U}_\infty} + (G_\infty - G_1) \ln \frac{\bar{U}_m}{\bar{U}_\infty}$$

where $\bar{U}_m$ is a certain mean value of the velocity $\bar{U}$ that lies between $\bar{U}_1$ and $\bar{U}_\infty$. For the usual profile shapes however the ratio $\bar{U}_1/\bar{U}_\infty$ is, in general, near unity and since the magnitude $2 + G_1$ exceeds the magnitude $G_\infty - G_1$ by several times, therefore for any choice of the mean value of $\bar{U}_m$ the relative error in the determination of ϑ_∞ is not large. Taking the geometric mean $\bar{U}_m = \sqrt{\bar{U}_1\bar{U}_\infty}$ we again arrive at equation (16.9).

Replacing in equation (16.9) G_1 and G_∞ by their values, we obtain finally

$$\bar{\vartheta}_\infty = \bar{\vartheta}_1\left(\frac{\bar{U}_1}{\bar{U}_\infty}\right)^\pi \tag{16.12}$$

$$\pi = \frac{H_1 + 5}{2} + \frac{H_1 + 1}{2}\,\frac{\bar{U}_1^{\,2}}{1 - \bar{U}_1^{\,2}} + \frac{\bar{U}_\infty^{\,2}}{1 - \bar{U}_\infty^{\,2}} + \frac{1}{2}\,\frac{\left(\bar{T}_{w_1} - 1\right)H_{T_1}\bar{\theta}_1/\bar{\vartheta}_1}{1 - \bar{U}_1^{\,2}} + \frac{1}{2}\,\frac{\left(\bar{T}_{w_\infty} - 1\right)\bar{\theta}_\infty/\bar{\vartheta}_\infty}{1 - \bar{U}_\infty^{\,2}}$$

From equation (10.4) it follows that $\bar{\theta}\bar{U}(\bar{T}_w - 1) = \text{Constant}$, hence

$$(\bar{T}_{w_\infty} - 1)\,\bar{\theta}_\infty = (\bar{T}_{w_1} - 1)\bar{\theta}_1\,\frac{\bar{U}_1}{U_\infty} \tag{16.13}$$

For small Mach numbers equations (16.2), (16.12), and (16.13) assume the form

$$c_{xp} = 2\sum\bar{\vartheta}_\infty, \qquad \pi = \frac{H_1 + 5}{2} + \frac{1}{2}\left(\frac{T_w}{T_0} - 1\right)\frac{\bar{\theta}_1 H_{T_1}}{\bar{\vartheta}_1} + \frac{1}{2}\left(\frac{T_{w_\infty}}{T_0} - 1\right)\frac{\bar{\theta}_\infty}{\bar{\vartheta}_\infty}$$

$$\bar{\vartheta}_\infty = \bar{\vartheta}_1\left(\frac{U_1}{U_\infty}\right)^\pi, \quad \left(\frac{T_{w_\infty}}{T_0} - 1\right)\bar{\theta}_\infty = \left(\frac{T_{w1}}{T_0} - 1\right)\bar{\theta}_1\,\frac{U_1}{U_\infty}$$

17. Boundary Layer in a Gas Flow with Axial Symmetry

For the turbulent flow about a body of rotation the equations (10.1) and (10.2) in the variables x, y and the transformed equations (10.3) and (10.4) remain valid. Restricting ourselves to the case of the Prandtl number $P = 1$ and introducing the parameters R_ϑ, ζ, R_θ, and ζ_T we represent the integral relations in the form

$$\frac{dR_\vartheta}{d\bar{x}} + \frac{U'(H+1)}{U(1-U^2)} R_\vartheta + \frac{\bar{U}'(\bar{T}_w - 1)}{\bar{U}(1-\bar{U}^2)} \frac{\Delta^*}{\vartheta} R_\vartheta + \frac{\bar{r}'}{\bar{r}} R_\vartheta = \frac{1}{\zeta^2} \frac{1}{\bar{T}_w} \bar{U} R_{00} (1-\bar{U}^2)^{\frac{\kappa}{\kappa-1}} \tag{17.1}$$

$$\frac{dR_\theta}{d\bar{x}} + \frac{\bar{r}'}{\bar{r}} R_\theta + \frac{1}{\bar{t}_0^*} \frac{d\bar{t}_0^*}{d\bar{x}} R_\theta = \frac{1}{\zeta\zeta_T} \frac{1}{\bar{T}_w} \bar{U} R_{00} (1-\bar{U}^2)^{\frac{\kappa}{\kappa-1}} \tag{17.2}$$

Making use of the drag law equation (12.12), the heat-transfer law equation (13.11), effecting the change in variables

$$z = e^{k\zeta}\left(1 - \frac{2}{k\zeta}\right) k^2\zeta^2, \qquad z_T = \exp\,(k\zeta_T)\left(1 - \frac{2}{k\zeta_T}\right) k\zeta_T$$

assuming the little changing magnitudes H, K, K_T constant with respect to $\bar{x}$, and also a constant mean value for the ratio $\frac{\Delta^*}{\vartheta} = h$, we obtain a system of linear equations the solution of which has the form

$$z = \frac{\sqrt{(1-\bar{U}^2)^c}}{\bar{T}^{nK}\bar{U}^c\bar{r}^K}\left[C + \frac{KR_{00}k^2}{C_1}\int_{\bar{x}_0}^{x} \frac{\bar{T}_w^{\,nK-n-1}\bar{U}^{c+1}\bar{r}^K}{\sqrt{(1-\bar{U}^2)^c}} (1-\bar{U}^2)^{\frac{\kappa}{\kappa-1}}\, d\bar{x}\right] \tag{17.3}$$

$$C = \left[\frac{zU^c\bar{r}^K\bar{T}_w^{\,nK}}{\sqrt{(1-\bar{U}^2)^c}}\right]_{\bar{x}=\bar{x}_0}$$

$$z_T = \frac{1}{T_w^{\,nK_T}\bar{r}^{K_T}(T_w - 1)^{K_T}}\left[C' + \frac{K_T R_{00} k^2}{C_1}\int_{\bar{x}_0}^{\bar{x}} \bar{T}_w^{\,nK_T-n-1}(\bar{T}_w - 1)^{K_T} \frac{\bar{U}}{k\zeta}(1-\bar{U}^2)^{\frac{\kappa}{\kappa-1}}\bar{r}^{K_T}\, d\bar{x}\right]$$

$$C' = \left[z_T\bar{T}_w^{\,nK_T}(\bar{T}_w - 1)^{K_T}\bar{r}^{K_T}\right]_{\bar{x}=\bar{x}_0}$$

For the frictional stress and the Nusselt number, equations (14.6) and (15.5) remain in force. In the case of the internal problem

$$N_{00} = \frac{q_w L}{\lambda_{00} t_0^*} = \frac{1}{\zeta\zeta_T} \frac{1}{\bar{\bar{T}}_w} R_{00}\bar{U}(1 - \bar{U}^2)^{\frac{\kappa}{\kappa-1}}, \quad C_{f00} = \frac{2\tau_w}{\rho_{00}U^2} = \frac{2}{\zeta^2}\frac{1}{\bar{T}_w}(1 - \bar{U}^2)^{\frac{\kappa}{\kappa-1}} \tag{17.5}$$

The theory presented, in particular the integral relations of the momenta and energy established in part I, permits determining the thermal and dynamic characteristics of the boundary layer at a curved wall in the most general cases, that is, in the presence of external and internal heat interchange. The computation of the boundary layer by the equations derived in parts II and III on the assumption of the Prandtl number $P = 1$ permits finding directly for arbitrary Mach numbers (excluding the interval from $M_\infty = M_{\infty_{cr}}$ to $M_\infty = 1$):

(1) The coefficients of the heat transfer from the wall to the gas for a given maintained temperature of the wall through heat supplied outside the body and the coefficients of heat transfer from the gas to the wall, that is, required for maintaining the heat conduction within the body at the given temperature of the wall.

(2) The distribution of the frictional stress along the wall and the profile drag of the wing (in the case $M_\infty < M_{\infty_{cr}}$) for arbitrary ratio of the stagnation temperatures and those of the wall.

For small velocities the obtained equations express the dependence of the heat transfer and the drag on the ratio of the absolute temperatures of the flow and the wall (the effect of the compressibility and the change of the physical constants due to the heat interchange).

In conclusion we give the results of computation of a single example. In figure 3 is given the distribution of the velocities of the external flow for the supersonic flow about a body with two sharp edges. The contour of the body and the position of the discontinuity are also shown. The flow was computed by the method of Donov (reference 9). In figure 4 are given the curves for the Nusselt number N which assure the uniform cooling of the surface up to the temperature $T_w = 0.25T_{00}$ for $R_\infty = 15 \times 10^6$, $M_\infty = 2$, and $M_\infty = 6$ for the laminar (lower curves) and turbulent (upper curves) regimes.

Translated by S. Reiss
National Advisory Committee
for Aeronautics

REFERENCES

1. Dorodnitsyn, A. A.: Boundary Layer in a Compressible Gas. Prikladnaya Matematika i Mekhanika, vol. VI., 1942.

2. Frankl, F. I., Christianovich, S. A., and Alexseyev, R. P.: Fundamentals of Gas Dynamics. CAHI Report No. 364, 1938.

3. Kalikhman, L. E.: Drag and Heat Transfer of a Flat Plate in a Gas Flow at Large Speeds. Prikladnaya Matematika i Mekhanika, vol. IX, 1945.

4. Kalikhman, L. E.: Computation of the Heat Transfer in a Turbulent Flow of an Incompressible Fluid. NII-I Report No. 4, 1945.

5. Lyon, H.: The Drag of Streamline Bodies. Aircraft Engineering, No. 67, 1934.

6. Dorodnitsyn, A. A., and Loitsiansky, L. G.: Boundary Layer of a Wing Profile at Large Velocities. CAHI Report No. 551, 1944.

7. Kalikhman, L. E.: New Method of Computation of the Turbulent Boundary Layer and Determination of the Point of Separation. Doklady Akademii Nauk SSSR, vol. 38, No. 5-6, 1943.

8. Squire, H. B., and Young, A. D.: The Calculation of the Profile Drag of Aerofoils. ARC, R & M No. 1838, 1938.

9. Donov, A.: Plane Wing with Sharp Edges in Supersonic Flight. Izvestia Akademii Nauk SSSR, Ser. Matematichoskaya, 1939, pp. 603-626.

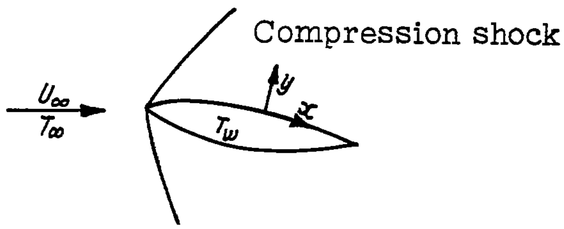

Figure 1.

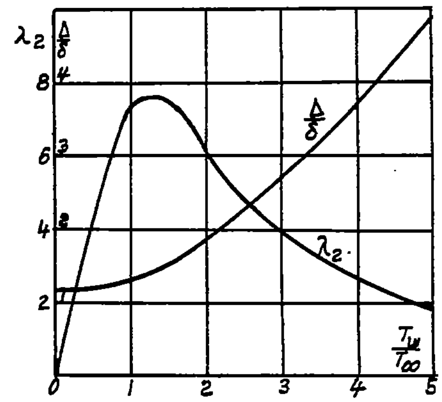

Figure 2.

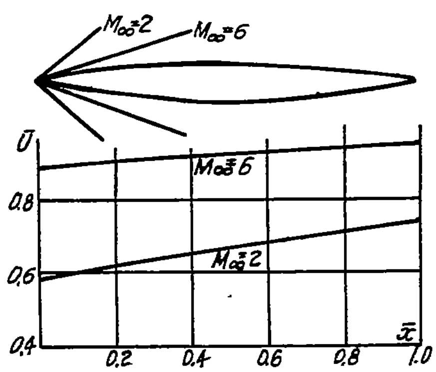

Figure 3.

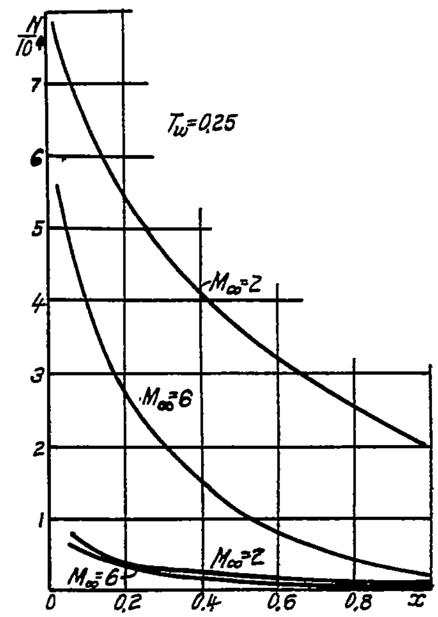

Figure 4.