# NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1243

## TWO-DIMENSIONAL POTENTIAL FLOWS

By Manfred Schäfer and W. Tollmien

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Outline:



## I. CHARACTERISTIC DIFFERENTIAL EQUATIONS - INITIAL AND BOUNDARY CONDITIONS.

For setting up the characteristic differential equations one starts from the differential equation for the velocity potential since the velocity components can be expressed more simply by the velocity potential than by the stream function. This differential equation reads, according to I(15):

$$\left(1 - \frac{u^2}{a^2}\right)\varphi_{xx} - 2\,\frac{uv}{a^2}\,\varphi_{xy} + \left(1 - \frac{v^2}{a^2}\right)\varphi_{yy} = 0 \tag{1}$$

with

$$u = \varphi_x,\ v = \varphi_y \tag{2}$$

*"Ebene Potentialströmungen." Technische Hochschule Dresden, Archiv Nr. 44/3, Kapitel III, March 22, 1941.

and the sonic velocity $a$ determined by

$$a^2 = \frac{\kappa - 1}{2}\left(1 - q^2\right) \tag{3}$$

in the selected non-dimensional representation.

The characteristic condition (cf. chapter II(8), NACA TM 1242) is

$$\left(1 - \frac{u^2}{a^2}\right)\dot{y}^2 + 2\frac{uv}{a^2}\dot{x}\dot{y} + \left(1 - \frac{v^2}{a^2}\right)\dot{x}^2 = 0 \tag{4}$$

If one puts

$$u = q \cos \vartheta \quad v = q \sin \vartheta \tag{5}$$

the characteristic condition is written

$$\left(a^2 - q^2\cos^2\vartheta\right)\dot{y}^2 + 2q^2\sin\vartheta \; \cos\vartheta \; \dot{x}\dot{y} + \left(a^2 - q^2\sin^2\vartheta\right)\dot{x}^2 = 0 \tag{6}$$

Hence result two roots $\lambda'$ and $\lambda''$ for $\frac{dy}{dx}$, the slope of the characteristic base curves toward the x-axis:

$$\lambda' = \frac{q^2\sin \vartheta \cos \vartheta + a\sqrt{q^2 - a^2}}{q^2\cos^2\vartheta - a^2} \tag{7a}$$

$$\lambda'' = \frac{q^2\sin \vartheta \cos \vartheta - a\sqrt{q^2 - a^2}}{q^2\cos^2\vartheta - a^2} \tag{7b}$$

As differential equation for the first family of the characteristic base curves results

$$dy - \lambda' \, dx = 0 \tag{8a}$$

and for the second

$$dy - \lambda'' \, dx = 0 \tag{8b}$$

For explanation of these relations (somewhat difficult to survey due to the complicated form of $\lambda'$ and $\lambda''$) one uses an artifice which is permissible in two-dimensional flows. Since in the two-dimensional flow, for instance, in contrast to the rotationally-symmetrical flow, no direction is preferred, one may place the x-axis of a Cartesian xy system in the direction of the flow at the location under investigation; thus there becomes

$$\left.\begin{aligned} \vartheta &= 0 \\ \lambda' &= \frac{a}{\sqrt{q^2 - a^2}} \\ \lambda'' &= -\frac{a}{\sqrt{q^2 - a^2}} \end{aligned}\right\} \qquad (9)$$

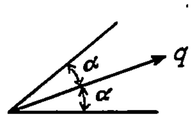
The Mach angle $\alpha$ is defined by

$$\sin \alpha = \frac{a}{q}$$

so that

$$\tan \alpha = \frac{a}{\sqrt{q^2 - a^2}} \qquad (10)$$

This signifies according to (8) that the characteristic base curves form with the stream lines the Mach angle $\alpha$. The first family (8a) of characteristic base curves forms the Mach angle toward the left (looking in the flow direction), the second family (8b) forms the same angle toward the right. The first family of characteristic base curves were thereupon denoted as left-hand, the second as right-hand Mach waves.




The second characteristic equation of the first family of characteristics is, according to II(28b), (NACA TM 1242):

$$du + \frac{q^2 \sin \vartheta \cos \vartheta - a\sqrt{q^2 - a^2}}{q^2 \cos^2 \vartheta - a^2} dv = 0 \qquad (11a)$$

and for the second family of characteristics, according to II(29b) (TM 1242):

$$du + \frac{q^2 \sin \vartheta \cos \vartheta + a\sqrt{q^2 - a^2}}{q^2 \cos^2 \vartheta - a^2} dv = 0 \tag{11b}$$

with

$$\begin{aligned} du &= \cos \vartheta \, dq - q \sin \vartheta \, d\vartheta \\ dv &= \sin \vartheta \, dq + q \cos \vartheta \, d\vartheta \end{aligned} \tag{12}$$

If one applies the same artifice as in selecting the special coordinate system at the considered location, one obtains, since $\vartheta$ there becomes 0,

$$du = dq, \; dv = q \, d\vartheta$$

and one has on the first family of characteristics

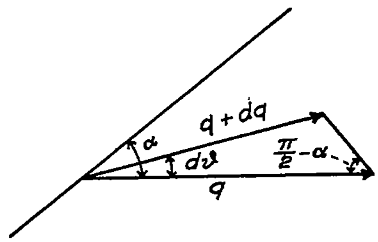
$$dq - \frac{a}{\sqrt{q^2 - a^2}} q \, d\vartheta = 0 \tag{13a}$$

and on the second

$$dq + \frac{a}{\sqrt{q^2 - a^2}} q \, d\vartheta = 0 \tag{13b}$$

The two relations (13) contain only quantities independent of the special selection of a coordinate system. Together with the remark made initially on the admissibility of the last used coordinate system there results, accordingly, the general validity of the relation (13a) for the first, of the relation (13b) for the second family of characteristics.

The relations (13) are often, in an elementary manner, deduced from the fact that the infinitesimal velocity variation is perpendicular to the Mach wave:




$$\frac{q + dq}{q} = \frac{\sin\left(\frac{\pi}{2} - \alpha\right)}{\sin\left(\frac{\pi}{2} + \alpha - d\vartheta\right)}$$

$$= \frac{\cos \alpha}{\cos \alpha + \sin \alpha \, d\vartheta}$$

$$= 1 - \tan \alpha \, d\vartheta$$

$$dq = -q \tan \alpha \, d\vartheta$$

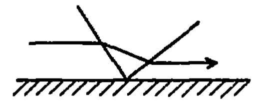
$$= -\frac{a}{\sqrt{q^2 - a^2}} q \, d\vartheta$$

Thus the velocity variation in crossing a left-hand Mach wave is regulated according to this equation (cf. (13b)); the crossing has to take place along a right-hand Mach wave. Regarding this elementary derivation the fundamental remark has to be made that here tacitly the existence of a relation between $u$ and $v$ alone (and between $q$ and $\vartheta$ alone, respectively) is assumed. In the rotationally-symmetrical case where this presupposition no longer holds, one obtains accordingly another characteristic equation although the velocity increment occurs as before perpendicularly to the Mach wave.

A few remarks concerning the secondary conditions which supervene the differential equation of the two-dimensional potential flow are to be inserted at this point.

The secondary conditions may for instance be given by initial conditions, that is, on an initial curve (for example, in a certain cross section of a channel) a few or all flow values are prescribed. The front of a compression shock also may serve as initial curve. The initial distributions for the approximation method discussed here are approximated by distributions constant over small distances. Therein it was often used as approximation principle that the jumps in $\vartheta$ are to be of a certain magnitude, for instance $\pm 1^\circ$ or $\pm 2^\circ$. Correspondingly, the boundary distributions which are given by boundary conditions, the main types of which will now be discussed, are also approximated.

$\vartheta$ is prescribed at a solid wall. If one denotes as compression wave a Mach wave behind which a pressure increase and a velocity decrease takes place, compression waves are reflected at a solid wall as compression waves. This can readily be seen in the figure.



The crossing of the right-hand Mach compression wave occurs along a left-hand Mach wave; thus the presupposed decrease of q according to (13a) is connected with a decrease of $\vartheta$. The crossing of the adjoining left-hand wave along a right-hand wave must cause - due to the boundary condition at the wall which is assumed to be unbroken - an increase of $\vartheta$ which according to (13b) produces a decrease of q, therefore a compression.

At a free jet boundary the pressure p is prescribed as constant; thus the velocity q also is a known constant there. The free jet boundary must - because of the kinematic boundary condition of vanishing normal velocity - coincide with a stream line which will be determined in the course of the solution of the flow problem. A compression wave is reflected at a free jet boundary as rarefaction wave since the drop in velocity which occurred first must be made good again by an increase, in order to satisfy the condition of constant velocity at the free jet boundary.

## 2. INTEGRATION OF THE SECOND CHARACTERISTIC DIFFERENTIAL EQUATIONS.

Of the characteristic equations (8a), (8b), (13a), (13b) the last pair can be very easily integrated; this was already done by Th. Meyer (cf. Th. Meyer, particularly p. 38). Following, a derivation is given which fits into our general theory.

On the left-hand family of characteristics

$$\frac{dq}{d\vartheta} = \frac{a}{\sqrt{q^2 - a^2}} q \qquad (13a)$$

with $a^2 = \frac{\kappa - 1}{2}(1 - q^2)$. Hence $\vartheta$ may be determined as function of q by quadrature. The execution of this elementary integration results in

$$\vartheta = \sqrt{\frac{\kappa + 1}{\kappa - 1}} \text{ arc tan} \left\{ \sqrt{\frac{\kappa - 1}{\kappa + 1}} \sqrt{\frac{2}{(\kappa - 1)(1 - q^2)} - \frac{\kappa + 1}{\kappa - 1}} \right\}$$

$$- \text{ arc tan} \sqrt{\frac{2}{(\kappa - 1)(1 - q^2)} - \frac{\kappa + 1}{\kappa - 1}} + \vartheta_l \tag{14}$$

$$= \frac{1}{a^*} \text{ arc tan} \left\{ a^* \sqrt{\frac{1}{a^2} - \frac{1}{a^{*2}}} \right\} - \text{ arc tan} \sqrt{\frac{1}{a^2} - \frac{1}{a^{*2}}} + \vartheta_l$$

or $\vartheta = \sqrt{\frac{\kappa + 1}{\kappa - 1}} \text{ arc tan} \sqrt{\frac{\kappa - 1}{\kappa + 1}} \sqrt{\frac{q^2}{a^2} - 1} - \text{ arc tan} \sqrt{\frac{q^2}{a^2} - 1} + \vartheta_l$, where $a^*$ is the critical velocity $\sqrt{\frac{(\kappa - 1)}{(\kappa + 1)}}$ and $\vartheta_l$ represents an integration constant. It has to be noted that all velocities have been made nondimensional by $V_m = \sqrt{\frac{2\kappa}{\kappa - 1} \frac{p_o}{\rho_o}}$ ($p_o$ tank pressure, $\rho_o$ tank density). For this relation a table of data particularly convenient for the practical calculation has been given by O. Walchner for air ($\kappa = 1.405$) (cf. pp. 22-23). Aside from the velocities $q$ referred to the sonic velocity $a$ and the critical sonic velocity $a^*$, the pressure $p$, referred to the tank pressure $p_o$ is given, for which the equation

$$\frac{p}{p_o} = (1 - q^2)^{\kappa/(\kappa-1)}$$

is valid. In the last column the Mach angle $\alpha$ is indicated. The integration constant $\vartheta_l$ is selected as 0, $\nu$ is used instead of $\vartheta$ for this special integration constant; the reason for this will be shown in the next paragraph. The application of the Meyer-Walchner table for the approximated construction of two-dimensional potential flows will be discussed in the next paragraph.

For the right-hand characteristics one obtains correspondingly by integration of (13b)

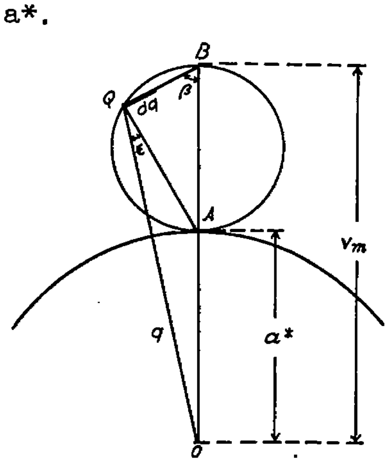
$$-\vartheta = \frac{1}{a^*} \text{ arc tan}\left\{a^* \sqrt{\frac{1}{a^2} - \frac{1}{a^{*2}}}\right\} - \text{arc tan}\sqrt{\frac{1}{a^2} - \frac{1}{a^{*2}}} - \vartheta_r \qquad (15)$$

or

$$-\vartheta = \sqrt{\frac{\kappa + 1}{\kappa - 1}} \text{ arc tan}\sqrt{\frac{\kappa - 1}{\kappa + 1}} \sqrt{\frac{q^2}{a^2} - 1} - \text{arc tan}\sqrt{\frac{q^2}{a^2} - 1} - \vartheta_r$$

with $\vartheta_r$ being an integration constant. On the left-hand characteristics increasing $q$ is, according to (13a), connected with increasing $\vartheta$, on the right-hand characteristics increasing $q$, according to (13b), with decreasing $\vartheta$. Hence the designation left-hand and right-hand, at first introduced through the characteristic base curves of the Mach waves, is immediately comprehensive also for consideration of the "characteristic hodographs" the equation of which is given in polar coordinates ($q$ radius vector, $\vartheta$ angular coordinate) in differential form by (13a) and (13b), in the integrated form by (14) and (15). Cl. Thiessen (1926) first drew attention to an interesting geometrical interpretation of these characteristic hodographs. A geometrical proof which, however, requires longer preparation, has been given by A. Busemann. Following, we give our own proof, relinquishing the non-dimensional representation, in order to conform to customary representation.

According to Thiessen the characteristic hodograph curves are epicycloids originating from the rolling of a circle of diameter $v_m - a^*$ on a circle of radius $a^*$.



Since the rolling circle rolls in the denoted position about the point A, the point Q - which has the distance q from the center of the fixed circle O - traces a small circular arc about A. Thus the increment dq lies on the line QB, since QB is perpendicular to QA.

On the other hand, dq is, according to the end of the previous paragraph, for <u>two-dimensional</u> potential flows <u>unequivocally</u> determined by the fact that dq is perpendicular to the Mach wave which with q (here represented by OQ forms the Mach angle $\alpha = \arc\sin a/q$. Thus the proof will be given if the angle $\measuredangle$ OQA is found to be equal to the Mach angle.

If one puts preliminarily

$$\measuredangle\ OQA = \epsilon$$

and

$$\measuredangle\ QBO = \beta$$

one has

$$\measuredangle\ OQB = \pi/2 + \epsilon$$

$$\measuredangle\ QAO = \pi/2 + \beta$$

According to the sine theorem applied to the triangles OQB and OQA one obtains

$$\frac{\sin\left(\frac{\pi}{2} + \epsilon\right)}{\sin \beta} = \frac{\cos \epsilon}{\sin \beta} = \frac{V_m}{q}$$

$$\frac{\sin \epsilon}{\sin\left(\frac{\pi}{2} + \beta\right)} = \frac{\sin \epsilon}{\cos \beta} = \frac{a^*}{q}$$

From these two equations the angle $\epsilon$ which is of interest to us may be calculated by elimination of $\beta$. This brief calculation may be performed about as follows. One has directly

$$\sin \beta = \frac{q}{V_m} \cos \epsilon$$

$$\cos \beta = \frac{q}{a^*} \sin \epsilon$$

Hence follows by squaring and adding

$$q^2\left(\frac{\cos^2\epsilon}{V_m^{\ 2}} + \frac{\sin^2\epsilon}{a^{*2}}\right) = 1$$

or

$$\frac{q^2}{V_m^{\ 2}}\left(1 - \sin^2\epsilon + \frac{\kappa + 1}{\kappa - 1}\sin^2\epsilon\right) = 1$$

Next, there results

$$\sin^2\epsilon = \frac{\kappa - 1}{2}\,\frac{V_m^{\ 2} - q^2}{q^2} = \frac{a^2}{q^2}$$

This, however, is just the equation for the Mach angle; thus $\epsilon$ equals the Mach angle. Therewith the fundamental statement dealing with the behavior of $dq$ with regard to the characteristics has been obtained again. The directional field produced by rolling of the circle coincides, therefore, with the directional field of the characteristic hodograph curves. The characteristic hodograph curves are epicycloids.

## 3. DIRECT APPLICATION OF MEYER'S CHARACTERISTIC HODOGRAPH TABLE FOR CONSTRUCTION OF TWO-DIMENSIONAL POTENTIAL FLOWS.

Meyer's solution for the characteristic hodographs indicated in the previous paragraph has been used directly for the construction of two-dimensional potential flows by J. Ackeret and, recently, by O. Walchner. We shall explain the method in one of the examples calculated by O. Walchner which for the first time showed the general validity of the method.

First it is to be noted that the one table given actually is sufficient for all characteristic hodographs, since, according to (14), for the left-hand characteristic hodographs

$$\vartheta = \nu + \vartheta_l \qquad (16)$$

according to (15) for the right-hand characteristic hodographs

$$\vartheta = -\nu + \vartheta_r \tag{17}$$

It has to be pointed out that the crossing of a right-hand Mach wave takes place along a left-hand characteristic, so that (16) is valid. The crossing of a left-hand Mach wave occurs along a right-hand characteristic with (17) being valid.

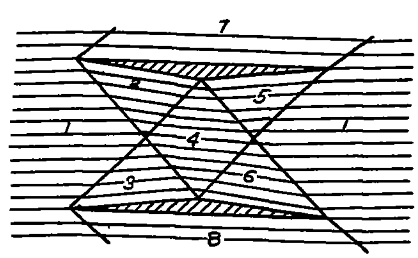
The transition from one field with the index $i$ to the next with the index $i + 1$ is regulated at the crossing of a right-hand Mach wave, thus according to the equations:

$$\vartheta_i = \nu_i + \vartheta_l$$

$$\vartheta_{i+1} = \nu_{i+1} + \vartheta_l$$

whence follows, with $\vartheta_l$ eliminated:

$$\boxed{\nu_{i+1} = \nu_i + \left(\vartheta_{i+1} - \vartheta_i\right)} \tag{18}$$




For crossing of a left-hand Mach wave

$$\vartheta_i = -\nu_i + \vartheta_r$$

$$\vartheta_{i+1} = -\nu_{i+1} + \vartheta_r$$

must be valid or, after elimination of $\vartheta r$:

$$\boxed{\nu_{i+1} = \nu_i - \left(\vartheta_{i+1} - \vartheta_i\right)} \tag{19}$$

According to the two equations (18) and (19) Meyer's table is used for the approximated calculation of two-dimensional potential flows.

The selected example of Walchner (fig. 3 of Walchner's report Lufo Bd. 14 (Aviation Research, vol. 14) p. 55, 1937) deals with the flow about a biplane at an angle of attack, with relatively rough approximation since large jumps in direction from one field to the next are admitted. Furthermore compression shocks are approximated by Mach waves which - in view of the weakness of the occurring shocks - is still justified (cf. chapter V, par. 5).

The field numbers are put as indices to the pertaining flow values. In the initial field $\frac{q_1}{a_1} = 1.64$, $\frac{p_1}{p_o} = 0.221$, $\vartheta_1 = 0$ is prescribed. According to the table the value $\nu_1 = 16^\circ$ pertains to this value of $q/a$ or $p/p_o$. In field 2 $\vartheta_2 = -10^\circ$, due to the geometrical boundary condition. Since in the transition from field 1 to field 2 a right-hand Mach wave is crossed, $\nu_2 = 6^\circ$ according to (18); according to the table, the pertinent values are $\frac{q_1}{a_1} = 1.293$, $\frac{p_1}{p_o} = 0.363$. For field 3 one has, for geometrical reasons, $\vartheta_3 = 4^\circ$; the transition from field 1 to field 2 leads, according to (19), to $\nu_3 = 12^\circ$ with $\frac{q_3}{a_3} = 1.504$, $\frac{p_3}{p_o} = 0.270$. The calculation of the q- and $\vartheta$-values in the next field 4 represents the general case of the method. From field 3 one arrives at field 4 by the crossing of a right-hand Mach wave, thus according to (18): $\nu_4 = \nu_3 + \vartheta_4 - \vartheta_3 = 8^\circ + \vartheta_4$. From field 2 one arrives at field 4 by the crossing of a left-hand Mach wave, thus according to (19): $\nu_4 = \nu_2 - \left(\vartheta_4 - \vartheta_2\right) = -4^\circ - \vartheta_4$. From the two equations for $\vartheta_4$ and $\nu_4$ set up just now follows $\vartheta_4 = -6^\circ$, $\nu_4 = 2^\circ$ with $\frac{q_4}{a_4} = 1.132$, $\frac{p_4}{p_o} = 0.449$. In all remaining fields $\vartheta$ is prescribed by geometrical boundary conditions so that the calculation is easy and takes place very similarly to that for field 2 and field 3. One has

$$\vartheta_5 = 4^\circ, \ \nu_5 = 12^\circ, \ \frac{q_5}{a_5} = 1.504, \ \frac{p_5}{p_o} = 0.270;$$

$$\vartheta_6 = -10^\circ, \ \nu_6 = 6^\circ, \ \frac{q_6}{a_6} = 1.293, \ \frac{p_6}{p_o} = 0.363;$$

$$\vartheta_7 = -3^\circ, \ \nu_7 = 19^\circ, \ \frac{q_7}{a_7} = 1.743, \ \frac{p_7}{p_o} = 0.190;$$

$$\vartheta_8 = -3^\circ, \ \nu_8 = 13^\circ, \ \frac{q_8}{a_8} = 1.538, \ \frac{p_8}{p_o} = 0.257$$

A drawing machine is desirable for plotting the Mach waves which close any newly calculated field.

With a finer subdivision of the angular variations, a greater accuracy seems attainable by means of this method than with the aid of the Prandtl-Busemann method described below.

## 4. PRANDTL-BUSEMANN METHOD:

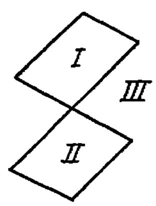
For approximated construction of two-dimensional potential flows mostly the Prandtl-Busemann method is used, the main expedient of which is a diagram with the characteristic hodograph curves (14) and (15), respectively.

The Prandtl-Busemann method is, according to our terminology introduced in chapter II, (NACA TM 1242), a field method, that is, a pair of values $q$, $\vartheta$ is coordinated to each field formed by the characteristic base curves or Mach waves. For the sake of a simple representation of the method we assume this pair of values $q$, $\vartheta$ to be valid precisely for the field center, the definition of which was given in chapter II, paragraph 7, NACA TM 1242.[1] We now visualize the field centers as connected with each other; these connecting curves

[1] The field centers are very useful for explanation of the method; they are, however, in case of two-dimensional potential flows, in contrast to rotationally-symmetrical ones, not required for the construction so that the exact definition of the field centers would here not yet be necessary.

give, as was shown, again characteristic base curves, thus here the Mach waves. To these Mach waves, not perhaps to the Mach waves of the field boundaries, the characteristic hodograph curves were coordinated.

The net of the characteristic hodographs may be drawn once and for all according to the expositions in section 2 for a given $\kappa$. According to former representations the crossing of a left-hand Mach wave in the flow plane is connected with a progressing along a right-hand characteristic hodograph in the velocity plane. Correspondingly, crossing of a right-hand Mach wave is coupled with progressing along a left-hand characteristic hodograph. Busemann and Preiswerk gave a net of characteristic hodographs for $\kappa = 1.405$. This diagram is customarily denoted simply as "characteristics diagram".




In order to calculate from the known pairs of values $q$, $\vartheta$ in field I and II the unknown pair of values $q$, $\vartheta$ in the field III adjoining downstream, one progresses from the point in the characteristics diagram corresponding to field I along a right-hand epicycloid, since one has crossed a left-hand Mach wave in the transition from I to III; correspondingly one progresses from the point of the characteristics diagram corresponding to field II along a left-hand epicycloid. The point of intersection gives the pair of values $q$, $\vartheta$ for the field III. The field that had been open so far is then closed by two Mach waves the direction of which is determined from the values of $q$ and $\vartheta$ found just now. The modifications of the method for fields at the boundary of the region are obvious.

In order to facilitate the reading of the $q$- and $\vartheta$-values from the characteristics diagram, one may take the net of the characteristic hodographs as net of coordinates. According to (14) and (16), respectively, one has for the left-hand epicycloids:

$$\vartheta - \nu(q) = \vartheta_l \tag{20}$$

according to (15) and (17), respectively, for the right-hand epicycloids

$$-\vartheta - \nu(q) = -\vartheta_r \tag{21}$$

Thus $\vartheta$ and - through the tabulated function $\nu(q)$ - $q$ as well may be very easily expressed by the parameters $\vartheta_l$ and $\vartheta_r$ of the epicycloids.

Instead of the parameters $\vartheta_l$ and $\vartheta_r$ which probably first seemed obvious, Busemann selected others with only the starting points of the count shifted from $\vartheta$ and $\nu$. The angles $\vartheta$ and $\nu$ are measured in degrees. The degree sign (°) is omitted below. One may then express Busemann's epicycloid numbering so that as equation of the left-hand epicycloids

$$\vartheta - \nu(q) = 2(\lambda - 400) \tag{22}$$

as equation of the right-hand epicycloids

$$-\vartheta - \nu(q) = 2(\mu - 600) \tag{23}$$

is written with the new parameters $\lambda$ and $\mu$. Hence there results

$$200 - \vartheta = \mu - \lambda \tag{24}$$

$$1000 - \nu(q) = \mu + \lambda \tag{25}$$

Thus the difference of the new parameters $\lambda$ and $\mu$ gives the angle $\vartheta$ except for an insignificant shifting of the initial point and reversal of the sense in which one is counting. The center line of Busemann's characteristics diagram ($\vartheta = 0$) obtains the "direction number" $\mu - \lambda$ equal to 200, whereas the sum of $\lambda$ and $\mu$ yields the function $\nu(q)$ and therewith also $q$ and the pressure $p$. The numbering of the epicycloids according to (22) and (23) is carried out very easily if one considers additionally that $\nu(q)$ just vanishes for $q = a^*$ (critical velocity). Busemann writes the parameters $\lambda$ and $\mu$ as "field numbers" into the fields of the flow plane; a table for the connection of the "pressure number" $\mu + \lambda$ with $q$ and $p$ must be given as supplement.

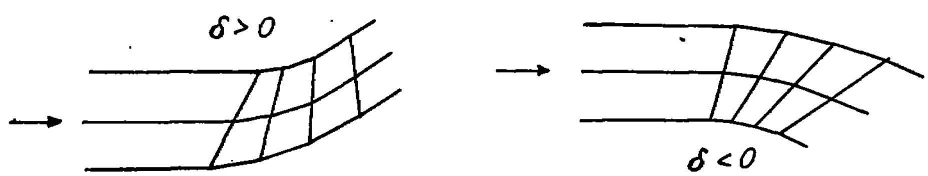
If one approximates the initial and boundary conditions in such a manner that one replaces the prescribed angles by sectionally constant distributions with jumps of $\pm 1°$ or $\pm 2°$, one may assure by a suitably fine-meshed characteristics diagram that one gets by without interpolation. The customary characteristics diagrams are in their main part arranged for angular jumps of 1°.

Additionally developed graphical expedients for facilitated plotting of the Mach waves will not be discussed, since one can dispense with them when a drawing machine is used.

We will be content with these observations regarding the Prandtl-Busemann method and will omit the carrying out of a standard example since the method has been represented in detail by Heybey (HVP - Archiv Nr. 66/31 and 66/32).

## 5. DEVELOPMENT OF THE PRESSURE VARIATION FOR SMALL DEFLECTION ANGLES.

In a flow unilaterally bounded by a wall the flow variations enforced by the boundary conditions are propagated from the wall along one family of Mach waves; in the figure it is the left-hand family.




This property of two-dimensional potential flows follows immediately from the characteristic differential equations (13) which connect velocity and directional variation. This property is by no means transferable to other than two-dimensional potential flows, for instance potential flows with rotational symmetry.

Since for the conditions assumed in the figure the flow variations occur at the crossing of left-hand Mach waves, they may be calculated by progressing along a right-hand characteristic, thus according to (13b) from

$$dq = - \frac{aq}{\sqrt{q^2 - a^2}} d\vartheta \qquad (13b)$$

A development of the pressure difference for deflection of the flow from the angle $\vartheta = 0$ to the angle $\vartheta = \delta$ is to be given; the deflection angle $\delta$ is to be small and the third powers of $\delta$ are still to be included in the development. Busemann has for the first time set up such a development, using a method totally different from ours. In order to develop the pressure, first the velocity $q_2$ which is to pertain to $\vartheta = \delta$ is developed according to equation (13b) with respect to $\delta$. For $\vartheta = 0$, $q$ is to equal $q_1$. We set up:

$$q_2 = q_1 + c_1 \delta + c_2 \delta^2 + c_3 \delta^3 \tag{26}$$

and determine the unknown coefficients $c_1$, $c_2$, $c_3$ from (13b) by comparison of the coefficients. One has only to equate $\frac{dq}{d\vartheta} = c_1 + 2c_2 \delta + 3c_3 \delta^2$ with the expression originating from $-\frac{aq}{\sqrt{q^2 - a^2}}$ when $q$ is replaced by $q_2$ according to (26). Therein is $a^2 = \frac{\kappa - 1}{2}\left(V_m^2 - q^2\right)$. It is most convenient to square the two sides of the relation mentioned before comparing the coefficients. There results

$$\left.\begin{aligned} c_1 &= -\frac{a_1 q_1}{\sqrt{q_1^2 - a_1^2}}, \quad c_2 = -\frac{q_1}{4\left(q_1^2 - a_1^2\right)^2}\left[(\kappa - 1)q_1^4 + 2a_1^4\right] \\ c_3 &= -\frac{a_1 q_1}{12\left(q_1^2 - a_1^2\right)^{7/2}}\left\{(\kappa - 1)(2\kappa - 3)q_1^6\right. \\ &\quad \left. + 9(\kappa - 1)a_1^2 q_1^4 + 6a_1^4 q_1^2 + 2a_1^6\right\} \end{aligned}\right\} \tag{27}$$

In order to obtain from the development for $q_2$ that of the pressure $p_2$, one starts from the isentropic pressure equation

$$\frac{p_2}{p_1} = \left\{\frac{V_m^2 - q_2^2}{V_m^2 - q_1^2}\right\}^{\frac{\kappa}{\kappa-1}} = \left\{1 - \frac{q_2^2 - q_1^2}{V_m^2 - q_1^2}\right\}^{\frac{\kappa}{\kappa-1}} \tag{28}$$

This expression is binomially expanded, making use of (26) and (27). For transformation of some of the terms one applies the formula

$$\frac{2\kappa}{\kappa - 1} \quad \frac{p_1}{V_m^2 - q_1^2} = \rho_1$$

which follows from the energy theorem (I(6)), and

$$a_1^2 = \frac{\kappa - 1}{2}\left(V_m^2 - q_1^2\right)$$

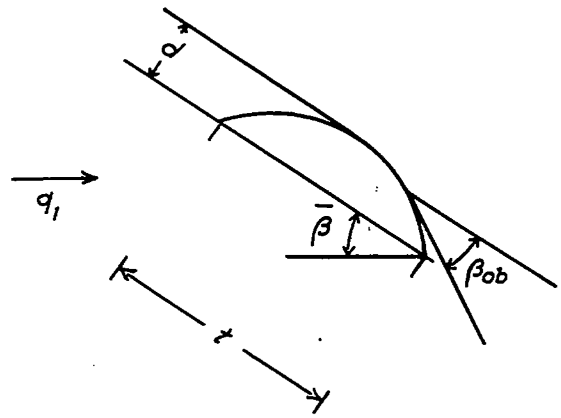
with the Mach number $M = q_1/a_1$ one then finally obtains

$$p_2 - p_1 = \rho_1 q_1^2 \left\{ \frac{\delta}{\sqrt{M^2 - 1}} + \delta^2 \frac{(M^2 - 2)^2 + \kappa M^4}{4(M^2 - 1)^2} + \frac{\delta^3}{12(M^2 - 1)^{7/2}} \left[(\kappa + 1)M^8 + (2\kappa^2 - 7\kappa - 5)M^6 + 10(\kappa - 1)M^4 - 12M^2 + 8\right] \right\} \tag{29}$$

The first term of this development may be found already in Th. Meyer's report; Busemann gives the development up to $\delta^3$ (Volta congress, p. 337). It is true that the coefficient of $\delta^3$ as calculated by us, completely differs from Busemann's. The comparison of the pressure difference occurring for isentropic deflection with the one in case of a compression shock is made in chapter V, paragraph 5.

In transferring formula (29) to the case of the flow variation taking place at the crossing of right-hand Mach waves, one has to establish the connection with the characteristic differential equation (13a). The simple rule results that one has to select the sign of $\delta$ in (29) as it corresponds to a reflection on the freestream direction of the conditions assumed above (cf. the figures on p. 16).

As application of this formula (29) we use the calculation of $c_a$ and $c_w$ for circular segment profiles in second approximation which was already performed by A. Busemann.




According to the figure one has for the upper side (index ob) of the profile $\delta_{ob} = -\bar{\beta} - \beta_{ob}$ and for the lower side (index u) $\delta_u = \beta$. If one replaces cos δ by 1, sin δ by δ, one has

$$c_a = \frac{1}{t\frac{\rho_1 q_1^2}{2}} \int_0^t \left(p_u - p_{ob}\right) dl$$

$$c_w = \frac{1}{t\frac{\rho_1 q_1^2}{2}} \int_0^t \left(p_u\bar{\beta} - p_{ob}\bar{\beta} - p_{ob}\beta_{ob}\right) dl.$$

with the integration to be extended over the entire wing chord. If one uses formula (29) to the second term in δ for the expression of $p_u$ and $p_{ob}$:

$$p - p_1 = \frac{\rho_1 q_1{}^2}{2}\left\{\frac{2\delta}{\sqrt{M^2 - 1}} + \delta^2 \frac{\left(M^2 - 2\right)^2 + \kappa M^4}{4\left(M^2 - 1\right)^2}\right\}$$

for which one writes abbreviatedly:

$$p - p_1 = \frac{\bar{\rho}_1 q_1{}^2}{2}\left(C_1\delta + C_2\delta^2\right)$$

$$C_1 = \frac{2}{\sqrt{M^2 - 1}}, \quad C_2 = \frac{\left(M^2 - 2\right)^2 + \kappa M^4}{4\left(M^2 - 1\right)^2}$$

one obtains

$$c_a = 2C_1\bar{\beta} - \frac{C_1}{t}\int_0^t \beta_{ob}{}^2\, dl$$

Therein it is already taken into account that $\int_0^t \beta_{ob}\, dl$ vanishes (for the expression for $c_w'$ we shall also use the vanishing of $\int_0^t \beta_{ob}{}^3\, dl$ for circular segment profiles). In the foregoing derivation $c_a$ is then determined to include terms which are quadratic in the angles $\bar{\beta}$ and $\beta_{ob}$. For $c_w$ one obtains analogously the expression including terms of the third order in the angles $\bar{\beta}$ and $\beta_{ob}$:

$$c_w = 2C_1\bar{\beta}^2 + \frac{C_1}{t}\int_0^t \beta_{ob}{}^2\, dl - \frac{3C_2}{t}\,\bar{\beta}\int_0^t \beta_{ob}{}^2\, dl$$

Since one has for approximation of the circular arc by a parabolic arc

$$\beta_{ob} = \frac{8ld}{t^2} - \frac{4d}{t}$$

one obtains

$$\int_0^t \beta_{ob}^2 \, dl = \frac{16}{3} \frac{d^2}{t}$$

Therewith becomes

$$c_a = 2C_1\bar{\beta} - \frac{C_2}{3}\left(\frac{4d}{t}\right)^2$$

$$c_w = 2C_1\bar{\beta}^2 + \frac{C_1}{3}\left(\frac{4d}{t}\right)^2 - C_2\bar{\beta}\left(\frac{4d}{t}\right)^2$$

It is noteworthy that for the angle of attack $\bar{\beta} = 0$ a negative lift of the circular segment profiles exists which was also determined experimentally.

TABLE

RELATION BETWEEN DEFLECTION, PRESSURE, VELOCITY, MACH NUMBER AND MACH ANGLE FOR ISENTROPIC CHANGES OF STATE ACCORDING TO PRANDTL-MEYER FOR AIR ($\kappa = 1.405$).

| $\nu$ | $\frac{p}{p_o}$ | $\frac{q}{a^*}$ | $\frac{q}{a}$ | $\alpha = \text{arc sin } \frac{a}{q}$ |
|---|---|---|---|---|
| 0° | 0.527 | 1.000 | 1.000 | 90°00' |
| 1 | .478 | 1.067 | 1.081 | 67 45 |
| 2 | .449 | 1.106 | 1.132 | 62 05 |
| 3 | .424 | 1.140 | 1.177 | 58 10 |
| 4 | .402 | 1.170 | 1.217 | 55 20 |
| 5 | .382 | 1.199 | 1.256 | 52 50 |
| 6 | .363 | 1.225 | 1.293 | 50 40 |
| 7 | .346 | 1.250 | 1.330 | 48 45 |
| 8 | .329 | 1.275 | 1.366 | 47 05 |
| 9 | .314 | 1.299 | 1.401 | 45 35 |
| 10 | .299 | 1.322 | 1.436 | 44 10 |
| 11 | .284 | 1.345 | 1.470 | 42 50 |
| 12 | .270 | 1.367 | 1.504 | 41 40 |
| 13 | .257 | 1.388 | 1.538 | 40 30 |
| 14 | .244 | 1.408 | 1.572 | 39.30 |
| 15 | .232 | 1.428 | 1.606 | 38 30 |
| 16 | .221 | 1.447 | 1.640 | 37 35 |
| 17 | .210 | 1.466 | 1.674 | 36 40 |
| 18 | .200 | 1.485 | 1.708 | 35 50 |
| 19 | .190 | 1.504 | 1.743 | 35 00 |
| 20 | .180 | 1.522 | 1.778 | 34 15 |
| 21 | .170 | 1.540 | 1.812 | 33 30 |
| 22 | .161 | 1.557 | 1.848 | 32 45 |
| 23 | .153 | 1.575 | 1.883 | 32 05 |
| 24 | .145 | 1.592 | 1.918 | 31 25 |
| 25 | .137 | 1.609 | 1.954 | 30 50 |
| 26 | .130 | 1.625 | 1.990 | 30 10 |
| 27 | .123 | 1.642 | 2.027 | 29 35 |
| 28 | .116 | 1.658 | 2.083 | 29 00 |
| 29 | .109 | 1.674 | 2.100 | 28 25 |
| 30 | .103 | 1.689 | 2.138 | 27 55 |

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RELATION BETWEEN DEFLECTION, PRESSURE, VELOCITY, MACH NUMBER AND MACH ANGLE FOR ISENTROPIC CHANGES OF STATE ACCORDING TO PRANDTL-MEYER FOR AIR ($\kappa$ = 1.405). - Continued.

| $\nu$ | $\frac{p}{p_0}$ | $\frac{q}{a^*}$ | $\frac{q}{a}$ | $\alpha = \arc\sin\frac{a}{q}$ |
|---|---|---|---|---|
| 31° | 0.097 | 1.704 | 2.177 | 27°20' |
| 32 | .091 | 1.720 | 2.215 | 26 50 |
| 33 | .086 | 1.735 | 2.255 | 26 20 |
| 34 | .080 | 1.750 | 2.295 | 25 50 |
| 35 | .076 | 1.765 | 2.335 | 25 20 |
| 36 | .071 | 1.780 | 2.376 | 24 55 |
| 37 | .067 | 1.794 | 2.418 | 24 25 |
| 38 | .062 | 1.808 | 2.460 | 24 00 |
| 39 | .058 | 1.822 | 2.502 | 23 35 |
| 40 | .055 | .836 | 2.545 | 23 10 |
| 41 | .051 | 1.850 | 2.590 | 22 45 |
| 42 | .047 | 1.864 | 2.635 | 22 20 |
| 43 | .044 | 1.877 | 2.661 | 21 55 |
| 44 | .041 | 1.890 | 2.728 | 21 30 |
| 45 | .038 | 1.903 | 2.778 | 21 10 |
| 46 | .035 | 1.915 | 2.823 | 20 45 |
| 47 | .033 | 1.928 | 2.872 | 20 20 |
| 48 | .030 | 1.940 | 2.922 | 20 00 |
| 49 | .028 | 1.952 | 2.974 | 19 40 |
| 50 | .026 | 1.984 | 3.027 | 19 20 |
| 51 | .024 | .976 | .081 | 18 55 |
| 52 | .023 | 1.988 | 3.136 | 18.35 |
| 53 | .021 | 1.999 | 3.191 | 18.15 |
| 54 | .019 | 2.011 | 3.247 | 17.53 |
| 55 | .018 | 2.022 | 3.304 | 17 35 |
| 56 | .016 | 2.033 | 3.363 | 17 20 |
| 57 | .015 | 2.044 | 3.424 | 17 00 |
| 58 | .013 | 2.055 | 3.487 | 16 40 |

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