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## A CLASS OF de LAVAL NOZZLES

By S. V. Falkovich

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A study is made herein of the irrotational adiabatic motion of a gas in the transition from subsonic to supersonic velocities. A shape of the de Laval nozzle is given, which transforms a homogeneous plane-parallel flow at large subsonic velocity into a supersonic flow without any shock waves beyond the transition line from the subsonic to the supersonic regions of flow. The method of solution is based on integration near the transition line of the gas equations of motion in the form investigated by S. A. Christianovich (reference 1).

1. Fundamental equations. - A plane, steady, irrotational, adiabatic flow of a gas is considered. In this case, the equations of motion, as is known, have the form

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \qquad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{1.1}$$

$$\frac{W^2}{2} + \frac{\kappa}{\kappa-1}\frac{p}{\rho} = \frac{\kappa}{\kappa-1}\frac{p_0}{\rho_0} \tag{1.2}$$

where

$\rho$ density of gas

$u,v$ components of velocity along x- and y-axes

$p$ pressure

$W$ absolute value of velocity

$\kappa$ adiabatic exponent

Subscript 0 denotes condition of gas at rest.

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From equations (1.1) there exist two functions, the velocity potential $\varphi(x,y)$ and the stream function $\psi(x,y)$, determined by the equations

$$d\varphi = u\,dx + v\,dy \qquad d\psi = \frac{\rho}{\rho_0}(-v\,dx + u\,dy) \tag{1.3}$$

If $u = W\cos\theta$ and $v = W\sin\theta$, where $\theta$ is the angle between the velocity vector and the x-axis, are substituted in equations (1.3) and are solved for dx and dy,

$$dx = \frac{\cos\theta}{W}d\varphi - \frac{\rho_0}{\rho}\frac{\sin\theta}{W}d\psi \qquad dy = \frac{\sin\theta}{W}d\varphi + \frac{\rho_0}{\rho}\frac{\cos\theta}{W}d\psi \tag{1.4}$$

The concepts x and y and $\varphi$ and $\psi$ are functions of the variables W and $\theta$; then

$$d\varphi = \frac{\partial\varphi}{\partial W}dW + \frac{\partial\varphi}{\partial\theta}d\theta$$

$$d\psi = \frac{\partial\psi}{\partial W}dW + \frac{\partial\psi}{\partial\theta}d\theta$$

If these expressions are substituted for $d\varphi$ and $d\psi$ in equations (1.4),

$$\left.\begin{aligned} dx &= \left(\frac{\cos\theta}{W}\frac{\partial\varphi}{\partial W} - \frac{\rho_0}{\rho}\frac{\sin\theta}{W}\frac{\partial\psi}{\partial W}\right)dW + \left(\frac{\cos\theta}{W}\frac{\partial\varphi}{\partial\theta} - \frac{\rho_0}{\rho}\frac{\sin\theta}{W}\frac{\partial\psi}{\partial\theta}\right)d\theta \\ dy &= \left(\frac{\sin\theta}{W}\frac{\partial\varphi}{\partial W} + \frac{\rho}{\rho_0}\frac{\cos\theta}{W}\frac{\partial\psi}{\partial W}\right)dW + \left(\frac{\sin\theta}{W}\frac{\partial\varphi}{\partial\theta} + \frac{\rho_0}{\rho}\frac{\cos\theta}{W}\frac{\partial\psi}{\partial\theta}\right)d\theta \end{aligned}\right\} \tag{1.5}$$

In order for dx and dy, determined from equations (1.5), to be total differentials, it is necessary and sufficient that the following equalities hold:

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$$\left.\begin{aligned}\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{W}\frac{\partial\varphi}{\partial W}-\frac{\rho_0}{\rho}\frac{\sin\theta}{W}\frac{\partial\psi}{\partial W}\right)=\frac{\partial}{\partial W}\left(\frac{\cos\theta}{W}\frac{\partial\varphi}{\partial\theta}-\frac{\rho_0}{\rho}\frac{\sin\theta}{W}\frac{\partial\psi}{\partial\theta}\right)\\ \frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{W}\frac{\partial\varphi}{\partial W}+\frac{\rho_0}{\rho}\frac{\cos\theta}{W}\frac{\partial\psi}{\partial W}\right)=\frac{\partial}{\partial W}\left(\frac{\sin\theta}{W}\frac{\partial\varphi}{\partial\theta}+\frac{\rho_0}{\rho}\frac{\cos\theta}{W}\frac{\partial\psi}{\partial\theta}\right)\end{aligned}\right\} \quad (1.6)$$

From equation (1.2) and the condition of adiabatic flow,

$$\frac{d}{dW}\left(\frac{\rho_0}{\rho}\right)=\frac{\rho_0}{\rho}\frac{W}{a^2}$$

where

a velocity of sound

If the differentiation in equations (1.6) is carried out,

$$\frac{\partial\varphi}{\partial\theta}=\frac{\rho_0}{\rho}W\frac{\partial\varphi}{\partial W} \qquad \frac{\partial\varphi}{\partial W}=-\frac{\rho_0}{\rho}(1-M^2)\frac{1}{W}\frac{\partial\psi}{\partial\theta} \quad (1.7)$$

where $M = W/a$ is the Mach number.

In equations (1.7), as the independent variable, in place of the velocity $W$ a new variable $s$ (reference 1) is introduced, which is connected with $W$ by the relation

$$s=\int_W^a\frac{\sqrt{a^2-W^2}}{aW}\,dW \quad (1.8)$$

Equations (1.7) then assume the form

$$\frac{\partial\varphi}{\partial\theta}=-\sqrt{K}\frac{\partial\psi}{\partial s} \qquad \frac{\partial\varphi}{\partial s}=\sqrt{K}\frac{\partial\psi}{\partial\theta} \quad (1.9)$$

where

$$K(s)=\left(\frac{\rho_0}{\rho}\right)^2(1-M^2) \quad (1.10)$$

inasmuch as equation (1.8) is a function of $s$.

The system of equations (1.9) in the regions where the flow velocity is subsonic, $W < a$, is of the elliptic type. Hence, any solution $\varphi = \varphi(\theta,s)$ and $\psi = \psi(\theta,s)$ of equations (1.9) represents two functions analytic in the variables $s$ and $\theta$ up to the line of transition to the region of supersonic velocities.

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If the Jacobian of the transformation of the region in the plane $\theta,s$ on the plane $\varphi,\psi$

$$\frac{D(\varphi,\psi)}{D(\theta,s)} = \sqrt{K}\left[\left(\frac{\partial\psi}{\partial\theta}\right)^2 + \left(\frac{\partial\psi}{\partial s}\right)^2\right]$$

does not become zero over a certain segment of the transition line and therefore also in a certain region on the supersonic side, the functions $\varphi(\theta,s)$ and $\psi(\theta,s)$ may be analytically continued across this segment into the supersonic region of the flow. Hence, the flow in the subsonic region determines the flow in the supersonic region near the transition line. Equations (1.9) retain their meaning also for supersonic velocities. For, if $W > a$, $s$ will be purely imaginary and $K$ negative. Setting $s = i\overline{s}$ and $K = -\overline{K}$ in equations (1.9) gives

$$\frac{\partial\varphi}{\partial\theta} = -\sqrt{\overline{K}}\,\frac{\partial\psi}{\partial\overline{s}} \qquad \frac{\partial\varphi}{\partial\overline{s}} = -\sqrt{\overline{K}}\,\frac{\partial\psi}{\partial\theta} \tag{1.11}$$

Thus, if in the solution $\varphi(\theta,s)$ and $\psi(\theta,s)$ of equations (1.9) determining the flow in the subsonic region, $s$ is set equal to $i\overline{s}$ and the real parts of the expressions thus obtained are taken, the solution of equations (1.11) determining the flow of the gas in the supersonic region near the transition line is obtained.

2. Investigation of equations (1.9) near the transonic line $W = a_*$. From equation (1.8), it follows that in the plane of the variables $\theta,s$ the upper half-plane $s > 0$ will correspond to the region of subsonic velocity and the axis of the abscissa $s = 0$ to the line of sound velocity. It is therefore necessary to consider the behavior of the function $K(s)$ for small values of the variable $s$.

Equation (1.2) is represented in the form

$$W^2 = \frac{(\kappa+1)a_*^2 M^2}{2+(\kappa-1) M^2} \tag{2.1}$$

Substituting this value of W in equation (1.8) gives

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$$s = \int_0^t \frac{(h^2-1)t^2}{(1-t^2)(h^2-t^2)}\,dt \qquad \left(t = \sqrt{1-M^2},\ h^2 = \frac{\kappa+1}{\kappa-1}\right)$$

Computing the integral gives

$$s = \frac{1}{2}\log\left[\left(\frac{h-t}{h+t}\right)^h \frac{1+t}{1-t}\right] \tag{2.2}$$

If the equation is expanded in a power series in $t$,

$$s = \frac{h^2-1}{3h^2}t^3 + \frac{h^4-1}{5h^4}t^5 + \frac{h^6-1}{7h^6}t^7 + \ldots$$

Then,

$$t = a_1 s^{1/3} + a_3 s^{3/3} + a_5 s^{5/3} + \ldots \tag{2.3}$$

where

$$a_1 = \sqrt[3]{\frac{3h^2}{h^2-1}}$$

Further, from equation (1.2) and the adiabatic condition after simple transformations,

$$\frac{\rho_0}{\rho} = \frac{\rho_0}{\rho_*}\left(1-\frac{t^2}{h^2}\right)^{\frac{1}{\kappa-1}} \tag{2.4}$$

Substituting this value in expression (1.10) gives for $K(s)$

$$\sqrt{K(s)} = \frac{\rho_0}{\rho_*}\,t\left(1-\frac{t^2}{h^2}\right)^{\frac{1}{\kappa-1}} \tag{2.5}$$

Substituting in equation (2.5) the series for $t$ (equation 2.3) gives

$$\sqrt{K(s)} = b_1 s^{1/3} + b_3 s^{3/3} + b_5 s^{5/3} + \ldots \quad \left(b_1 = \frac{\rho_0}{\rho_*}\sqrt{\frac{3h^2}{h^2-1}}\right) \tag{2.6}$$

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It follows from equation (2.6) that for a velocity near that of sound, that is, for small $s$, consideration may be restricted in series (2.6) to the first term by setting $\sqrt{K} \approx b_1 s^{1/3}$. The fundamental equations (1.9) then assume the form

$$\frac{\partial\varphi}{\partial\theta} = -b_1 s^{1/3}\frac{\partial\psi}{\partial s} \qquad \frac{\partial\varphi}{\partial s} = b_1 s^{1/3}\frac{\partial\psi}{\partial\theta} \tag{2.7}$$

whence for the functions $\psi(\theta,s)$ and $\varphi(\theta,s)$ there is obtained

$$\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial s^2} + \frac{1}{3s}\frac{\partial\psi}{\partial s} = 0 \qquad \frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial^2\varphi}{\partial s^2} - \frac{1}{3s}\frac{\partial\varphi}{\partial s} = 0 \tag{2.8}$$

Having determined from equations (2.8) the velocity potential $\varphi(\theta,s)$ and the stream function $\psi(\theta,s)$, by equations (1.5) the coordinates $x$ and $y$ are found in the flow plane. It is easily seen that equations (1.5) can be represented in the form

$$\left.\begin{aligned} dx &= \frac{1}{W}\left[\left(\cos\theta\frac{\partial\varphi}{\partial s} - \frac{\rho_0}{\rho}\sin\theta\frac{\partial\psi}{\partial s}\right)ds + \left(\cos\theta\frac{\partial\varphi}{\partial\theta} - \frac{\rho_0}{\rho}\sin\theta\frac{\partial\psi}{\partial\theta}\right)d\theta\right] \\ dy &= \frac{1}{W}\left[\left(\sin\theta\frac{\partial\varphi}{\partial s} + \frac{\rho_0}{\rho}\cos\theta\frac{\partial\psi}{\partial s}\right)ds + \left(\sin\theta\frac{\partial\varphi}{\partial\theta} + \frac{\rho_0}{\rho}\cos\theta\frac{\partial\psi}{\partial\theta}\right)d\theta\right] \end{aligned}\right\} \tag{2.9}$$

If, however, we pass from the exact equations (1.9) to the approximate equations (2.7), the expressions (2.9) cease to be exact differentials. Hence, simultaneously with the passing from equations (1.9) to equations (2.7) it is necessary to introduce instead of the relation (1.8) between s and W a new relation such that the expressions (2.9) remain exact differentials. In order to obtain this relation in equations (2.9) are substituted the values of the derivatives of the function $\varphi$ from equations (2.7) after which the condition that dx from equations (2.9) is a total differential assumes the form

$$\frac{\partial}{\partial\theta}\left[\left(b_1 s^{1/3}\cos\theta\,\frac{\partial\psi}{\partial\theta} - \frac{\rho_0}{\rho}\sin\theta\,\frac{\partial\psi}{\partial s}\right)\frac{1}{W}\right]$$

$$= -\frac{\partial}{\partial s}\left[\left(b_1 s^{1/3}\cos\theta\,\frac{\partial\psi}{\partial s} + \frac{\rho_0}{\rho}\sin\theta\,\frac{\partial\psi}{\partial\theta}\right)\frac{1}{W}\right]$$

Carrying out the differentiation and making use of the first of equations (2.8) satisfies this condition if the following equations are satisfied (the expression for dy likewise then becomes a total differential):

$$\frac{\rho_0}{\rho W} = b_1 s^{1/3}\,\frac{d}{ds}\left(\frac{1}{W}\right) \qquad \frac{b_1 s^{1/3}}{W} = \frac{d}{ds}\left(\frac{\rho_0}{\rho W}\right)$$

Setting $s = \frac{2}{3}\eta^{3/2}$ gives

$$\frac{\rho_0}{\rho W} = \left(\frac{2}{3}\right)^{1/3} b_1\,\frac{d}{d\eta}\left(\frac{1}{W}\right) \qquad \left(\frac{2}{3}\right)^{1/3}\frac{b_1\eta}{W} = \frac{d}{d\eta}\,\frac{\rho_0}{\rho W} \tag{2.10}$$

Thus, the equation determining $1/W$ is obtained, namely,

$$\frac{d^2}{d\eta^2}\left(\frac{1}{W}\right) - \frac{\eta}{W} = 0 \tag{2.11}$$

The functions satisfying this equation are called Airy functions. Tables of these functions have been computed by V. A. Fock (reference 5). Thus

$$\frac{1}{W} = C_1\,u(\eta) + C_2\,v(\eta) \tag{2.12}$$

where $u(\eta)$ and $v(\eta)$ are two linearly independent tabulated integrals of the equation (2.11). The constants of integration are determined from the conditions

$$\left(\frac{1}{W}\right)_{\eta=0} = \frac{1}{a_*} \qquad \frac{d}{d\eta}\left(\frac{1}{W}\right)_{\eta=0} = \left(\frac{3}{2}\right)^{1/3} \frac{\rho_0}{b_1 \rho_* a_*} = \frac{\gamma}{a_*} \qquad \left(\gamma = \sqrt[3]{\frac{h^2-1}{2h^2}}\right)$$

where the latter relation is obtained from equations (2.10) and (2.6). After computation,

$$C_1 = \frac{1}{a_*}\left[\gamma v(0) - v'(0)\right] \qquad C_2 = \frac{1}{a_*}\left[u'(0) - \gamma u(0)\right] \quad (2.13)$$

where from reference 5,

$$u(0) + iv(0) = \frac{2\sqrt{\pi}}{3^{4/3}\Gamma(2/3)} \exp \frac{i\pi}{6}$$

$$u'(0) + iv'(0) = \frac{2\sqrt{\pi}}{3^{4/3}\Gamma(4/3)} \exp \frac{-i\pi}{6}$$

From the first of equations (2.10)' and (2.12),

$$\frac{\rho_0}{\rho} = \left(\frac{2}{3}\right)^{1/3} b_1 \frac{C_1 u'(\eta) + C_2 v'(\eta)}{C_1 u(\eta) + C_2 v(\eta)} \qquad (2.14)$$

The expressions (2.12) and (2.14) thus found for $1/W$ and $\rho_0/\rho$ must be substituted in equations (2.9).

3. Determination of flow in feed part of de Laval nozzle. - The shape of the de Laval nozzle was determined such that the distribution of the velocity over the section tended, with increasing distance from the critical section upstream of the flow, to a uniform flow with a certain subsonic velocity $W_0$ near the velocity

of sound. It was necessary that the walls of the nozzle for $x \to -\infty$ have a horizontal asymptote $y = \pm H$ (fig. 1), the magnitude $H$ being determined from the amount of gas flow and the velocity $W_0$.

In the plane of the variables $\theta, s$, there corresponds to the infinitely distant section in which the velocity is constant and equal to $W_0$, the point A with coordinates $\theta = 0$, $s = s_0$. To the lines of flow there correspond in the $\theta, s$ plane a bundle of curves issuing from the point A. In order to obtain a solution of equations (2.8) having the stated properties, in the upper half-plane of $\theta, s$ bipolar coordinates are introduced (fig. 2).

$$\alpha = \log\sqrt{\frac{\theta^2+(s+s_0)^2}{\theta^2+(s-s_0)^2}} \qquad \beta = \text{arc tg}\,\frac{2s_0\theta}{\theta^2+s^2-s_0^2} \tag{3.1}$$

Thus the lines $\alpha = \text{constant}$ constitute a family of circles with centers on the s-axis and the lines $\beta = \text{constant}$ constitute a family of circles with centers on the $\theta$-axis passing through the point A (fig. 2). The equations of the families of these circles are, respectively,

$$\theta^2+(s+s_0\,\text{cth}\,\alpha)^2 = \frac{s_0^2}{\text{sh}^2\alpha} \qquad (\theta-s_0\,\text{ctg}\,\beta)^2 + s^2 = \frac{s_0^2}{\sin^2\beta}$$

The first of equations (2.8) transformed into bipolar coordinates has the form

$$\frac{\partial^2\psi}{\partial\alpha^2} + \frac{\partial^2\psi}{\partial\beta^2} + \frac{1}{3\,\text{sh}\,\alpha(\text{ch}\,\alpha+\cos\beta)}\left[(1+\text{ch}\,\alpha\cos\beta)\frac{\partial\psi}{\partial\alpha} + \text{sh}\,\alpha\sin\beta\,\frac{\partial\psi}{\partial\beta}\right] = 0 \tag{3.2}$$

$$\psi = (\text{ch}\,\alpha + \cos\beta)^{1/6}\,\chi(\alpha,\beta) \tag{3.3}$$

Equation (3.2) is reduced to the form

$$\frac{\partial^2\chi}{\partial\alpha^2} + \frac{\partial^2\chi}{\partial^2\beta} + \frac{\text{cth}\,\alpha}{3}\frac{\partial\chi}{\partial\alpha} + \frac{1}{36}\chi = 0 \tag{3.4}$$

in which the variables are separable.

In seeking a solution of equation (3.4) of the form $\chi = X(\alpha)\,Y(\beta)$, the ordinary equations for $X(\alpha)$ and $Y(\beta)$ are obtained:

$$\frac{d^2Y}{d\beta^2} + n^2Y = 0$$

$$\frac{d^2X}{d\alpha^2} + \frac{\operatorname{cth}\alpha}{3}\frac{dX}{d\alpha} + \left(\frac{1}{36} - n^2\right)X = 0$$

where $n$ is an arbitrary constant. If $n$ is set equal to 0,

$$\frac{d^2Y}{d\beta^2} = 0 \qquad \frac{d^2X}{d\alpha^2} + \frac{\operatorname{cth}\alpha}{3}\frac{dX}{d\alpha} + \frac{1}{36}X = 0 \tag{3.5}$$

The first of equations (3.5) gives $Y = C_1 + C_2\beta$ and the second, by the substitution $t = \operatorname{ch}^2\alpha$, reduces to the hypergeometric equation

$$t(1-t)\frac{d^2X}{dt^2} + \left(\frac{1}{2} - \frac{7}{6}t\right)\frac{dX}{dt} - \frac{1}{144}X = 0 \tag{3.6}$$

the general integral of which can be represented in the form

$$X = C_3 t^{-1/12} F\left(\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{t}\right) + C_4 F\left(\frac{1}{12}, \frac{1}{12}, \frac{1}{2}, t\right)$$

If $C_1 = C_4 = 0$ and considering equation (3.3), the required solution of equation (3.2) is in the form

$$\psi_0 = \left(\frac{\operatorname{ch}\alpha + \cos\beta}{\operatorname{ch}\alpha}\right)^{1/6} F\left(\frac{1}{12}, \frac{1}{12}, 1, \frac{1}{\operatorname{ch}^2\alpha}\right)\beta \tag{3.7}$$

Returning to the initial variables $\theta$ and $s$ according to equations (3.1), after simple transformations

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$$\psi_0(\theta,s)=\left(\frac{2s_0}{\theta^2+s^2+s_0^2}\right)^{1/6} F\left(\frac{1}{12},\frac{7}{12},1,\frac{\left[\theta^2+(s-s_0)\right]\left[\theta^2+(s+s_0)^2\right]}{(\theta^2+s^2+s_0^2)^2}\right) \tan^{-1}\frac{2s_0\theta}{\theta^2+s^2-s_0^2} \tag{3.8}$$

This solution corresponds to the flow of gas in a nozzle having the shape shown in figure 1. Such flow cannot, however, be continued into the region of supersonic velocities. In order that a certain subsonic flow having a straight streamline may be continued into the supersonic region, it is necessary and sufficient, as has been shown by F. I. Frankl (reference 2) (see also S. A. Christianovich, reference 3, ch. V.), that the stream function $\Psi(\theta,s)$ have on the transition line the form

$$\Psi(\theta,0) = A_1\theta^{1/3} + A_3\theta^{3/3} + A_5\theta^{5/3} + \ldots \tag{3.9}$$

The solution (3.8) does not, however, satisfy this condition inasmuch as on the transition line $(s = 0)$ it has the form

$$\psi_0(\theta,0) = \left(\frac{2s_0}{\theta^2+s_0^2}\right)^{1/6} F\left(\frac{1}{12}, \frac{7}{12}, 1, 1\right) \tan^{-1}\frac{2s_0\theta}{\theta^2-s_0^2}$$

In order to continue the flow with the stream function of equation (3.8) into the supersonic region, it is necessary to add to equation (3.8) the solution of the first of equations (2.8) satisfying the condition $\psi_1(\theta,0) = \theta^{1/3}$. Such a solution, as is shown in reference 4, has the form

$$A^3\psi_1^{\,3} + 3A\eta\psi_1 - 3\theta = 0 \qquad \eta = \left(\frac{3}{2}s\right)^{2/3}$$

whence setting $A = 3^{1/3}$, we obtain

$$\psi_1 = \left(\frac{1}{2}\right)^{1/3}\left(\sqrt[3]{\theta + \sqrt{\theta^2 + s^2}} + \sqrt[3]{\theta - \sqrt{\theta^2 + s^2}}\right) \tag{3.10}$$

Hence, in order to construct the flow in the de Laval nozzle having the shape shown in figure 1, it is sufficient for the stream function $\Psi(\theta,s)$ to assume the form

$$\Psi(\theta,s) = A_0\,\Psi_0 + A_1\Psi_1 + A_3\Psi_3 \qquad (\Psi_3 = \theta) \qquad (3.11)$$

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where $A_0$, $A_1$, and $A_3$ are arbitrary constants and the functions $\Psi_0$ and $\Psi_1$ are given by equations (3.8) and (3.10).

Equation (3.11) for $\Psi(\theta,s)$ is obtained if in the expansion (3.9) the first two terms are retained. With the values of the constants $A_0$, $A_1$, and $A_3$, a nozzle may be constructed sufficiently near the given nozzle of the shape under consideration.

The equations determining the functions $\Psi_0$ and $\Psi_1$ in the supersonic region will be determined. In the expression (3.8) for $s = 0$, the argument of the hypergeometric function attains the value unity and for the supersonic velocities, that is, for imaginary $s$, although remaining real, becomes greater than unity.

The formula giving the analytic continuation of the hypergeometric series (reference 6) is used

$$F\,(a,b,c,t) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}\,F\,(a,b,a+b-c+1,\ 1-t) +$$

$$\frac{\Gamma(c)\,\Gamma(a+b-c)}{\Gamma(a)\,\Gamma(b)}\,(1-t)^{c-a-b}\,F\,(c-b,\ c-a,\ c-a-b+1,\ 1-t)$$

which in the case considered has the form

$$F\left(\frac{1}{12},\ \frac{7}{12},\ 1,\ t\right) = \frac{\Gamma(1/3)}{\Gamma(11/12)\,\Gamma(5/12)}\,F\left(\frac{1}{12},\ \frac{7}{12},\ \frac{2}{3},\ 1-t\right) +$$

$$\frac{\Gamma(-1/3)}{\Gamma(1/12)\,\Gamma(7/12)}\,(1-t)^{1/3}\,F\left(\frac{5}{12},\ \frac{11}{12},\ \frac{4}{3},\ 1-t\right)$$

and the characteristic coordinates $\lambda = \theta - is$ and $\mu = \theta + is$ are determined. After computations, the following equations are obtained:

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$$\psi_0(\lambda,\mu)=\left(\frac{2s_0}{\lambda\mu+s_0^2}\right)^{1/6}\left[\frac{\Gamma(1/3)}{\Gamma(11/12)\,\Gamma(5/12)}\,F\left(\frac{1}{12},\frac{7}{12},\frac{2}{3},\ -\frac{s_0^{2}(\mu-\lambda)^2}{(\lambda\mu+s_0^2)^2}\right)-\right.$$

$$\left.\frac{\Gamma(-1/3)}{\Gamma(1/12)\,\Gamma(7/12)}\left(\frac{s_0(\mu-\lambda)}{\lambda\mu+s_0^2}\right)^{2/3}F\left(\frac{5}{12},\frac{11}{12},\frac{4}{3},\ -\frac{s_0^2(\mu-\lambda)^2}{(\lambda\mu+s_0^2)^2}\right)\right]\text{arc tg}\,\frac{2s_0(\lambda+\mu)}{\lambda\mu-s_0^{2}} \tag{3.12}$$

$$\psi_1(\lambda,\mu)=\left(\frac{1}{4}\right)^{1/3}\left(\sqrt[3]{\sqrt{\lambda}+\sqrt{\mu}}+\sqrt[3]{\sqrt{\lambda}-\sqrt{\mu}}\right) \tag{3.13}$$

These expressions determine the flow in the regions 1 and 2 between the transition line and the characteristic passing through the center of the nozzle and directed upstream of the flow (fig. 1). The further computation of the supersonic part of the nozzle can be carried out by the method of characteristics.

Translated by Samuel Reiss
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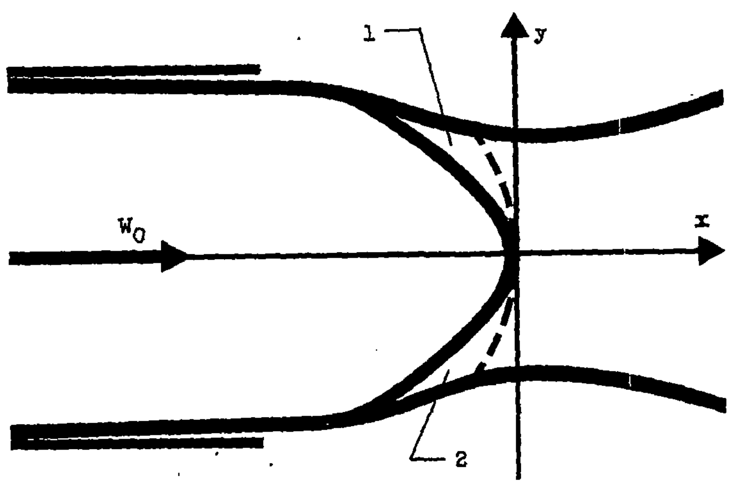


Figure 1.

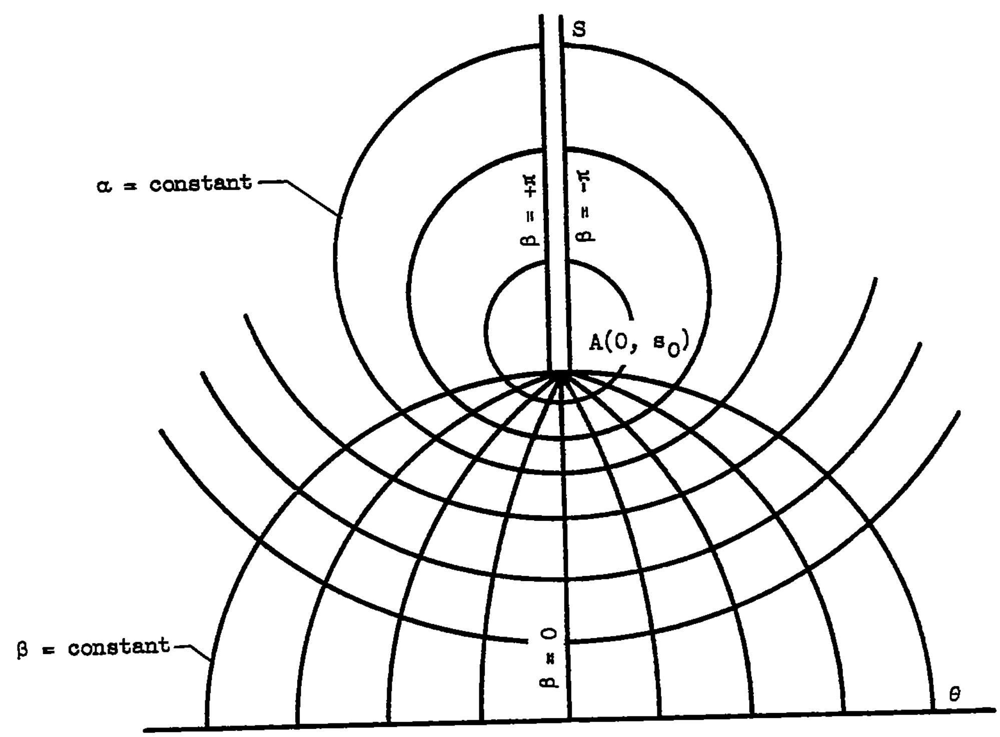


Figure 2.