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Central Aero-Hydrodynamical Institute

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ON THE THEORY OF THE UNSTEADY MOTION OF AN AIRFOIL*

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The paper presents a systematical analysis of the problem of the determination of the unsteady motion about an airfoil moving in an infinite fluid that contains a system of vortices and the determination of the hydrodynamical forces acting on the airfoil. The hydrodynamical problem is reduced to the determination of the function f (ζ) which transforms conformally the external region of the airfoil into the interior of a circle. The proposed methods of determining the irrotational motion of a fluid that is produced by any motion of the airfoil are especially simple and effective if the function f (ζ) is rational. As an example the flow is determined for the case of an arbitrary motion of an airfoil of the Joukowsky type. The formulas obtained for the determination of the hydrodynamical forces by means of contour integration are similar to those given by S. Chaplygin. These formulas are used to determine the force acting on the airfoil in the cases where the unsteady motion is potential throughout and the circulation about the airfoil is constant and also when the fluid contains a system of vortices. A full discussion is given of the concept of virtual masses together with practical formulas for computing the virtual mass coefficients. A table is added giving the virtual mass coefficients for different types of Joukowsky airfoils. For the case of motion with constant circulation the following theorems were obtained. Every airfoil possesses a fixed point such that the force, depending on the circulation, for any motion of the airfoil is determined by Joukowsky's law in terms of the velocity of this point, the moment of the hydrodynamical forces about this point being independent of the circulation. Formulas are also given for the determination of the forces and moments acting on a thin, slightly curved airfoil of Joukowsky profile for any motion with constant circulation. For the case where the circulation about the airfoil is zero the condition that the fluid velocity at the sharp edge of the airfoil should be finite for a continuously potential flow is reduced to the requirement that the traveling polhode should be a fixed straight line, perpendicular to the first axis for every airfoil. We have the similar geometrical condition for the case where the angular velocity of the airfoil and the circulation are independent of the time. We have also given formulas for the computation of the hydrodynamical forces when the flow contains any system of vortices. These forces depend on the

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position and on the velocities of the system of vortices. The developed general formulas give, for the case of steady motion of the fluid, the results obtained by Lagally. For an unsteady motion of a system of free vortices which are very distant from the airfoil the results of H. Wagner are given in the generalized form, namely, for the case of an arbitrary motion of the airfoil. The analysis is illustrated by the computation of the forces acting on a plate when the fluid contains only a single vortex filament.

1. INTRODUCTION

In recent years the unsteady motion of a fluid has become with increasing frequency the subject of theoretical and experimental hydrodynamical investigations. For the practical study of such motions it is particularly important in view of the fact that they include a large number of phenomena which are encountered in the solution of definite technical problems. It is sufficient to mention for example the problem of the determination of the aerodynamic forces for arbitrary motion of a rigid and elastic wing, the problem of the dynamic stability of airplanes moving within or on the boundary of fluids, the problem of the falling of a body on water, etc.

The unsteady motion of a viscous fluid about wing-shaped sections is generally accompanied by separation of the boundary layer which extends in the form of a thin film from the point of separation into the fluid. The boundary layer in separating off, is generally speaking, greatly deformed and serves as a source of vortex motion within the fluid. These vortices gradually diffuse and are dissipated in the general mass of the fluid. These are actually observed phenomena. In the case of small viscosity it is sometimes succeeded in obtaining for them a sufficiently accurate description by studying the motion of an ideal fluid under the corresponding conditions.

As is known, the motion of an ideal fluid under the action of conservative inertia forces starting from a state of rest is always potential. The physical requirement of positive pressures is generally impossible to satisfy if the study is restricted to continuously potential flows and it is necessary to introduce into consideration fields of velocity potential having surfaces of discontinuity. The surfaces of discontinuity coincide with the boundaries of the body and they may be considered as a schematic representation of a very thin vortex layer arising in the viscous fluid for the case of very small viscosity.

Starting from flows with surfaces of discontinuity within in fluid L. Prandtl developed theoretically on an experimental basis the approximate theory of a wing of finite span for steady motion (reference 1). He also gave examples of the accurate solution of certain two-dimensional problems on the unsteady motion of an ideal fluid with a line of discontinuity in the form of a spiral (reference 2).

Birnbaum set up an approximate theory for unsteady motion, the body being replaced by a system of bound vortices (references 3, 4).

In studying the unsteady motion with surfaces of velocity discontinuity a fundamental difficulty lies in the fact that the mechanical characteristics of the body-fluid system depends not only on the condition of motion of the body at a given instant of time but also on the preceding motion of the body. The motion of the system at a given instant of time is thus a function of a combination of all preceding states of motion of the body. In particular, the exact solution of the two-dimensional problem of unsteady motion, with lines of velocity discontinuity starting out from the trailing edge, encounters very great mathematical difficulties and as yet no solution has been obtained. This problem was characterized by Prandtl as "hopeless" of solution.

The forces acting in a two-dimensional flow on a plate from the trailing edge of which a line of velocity discontinuity is given off were investigated by Wagner (reference 5), Glauert (reference 6) and Keldish and Lavrentev (reference 7). In these investigations it is assumed that the line of discontinuity coincides with the straight line which is a prolongation of the plate. This assumption is associated with another, namely, that the angle of attack of the plate remains infinitesimally small.*

In the general case it is convenient to isolate from the hydrodynamic forces those which depend only on the velocities and accelerations of the body. These forces agree with the total force for the motion of the body in an infinite fluid on the assumption that the motion of the fluid is continuously potential. The determination

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*In the given case by angle of attack is meant the angle between the velocity of the center of the plate and the direction of the plate.

of the motion of the fluid and the forces in this case is a classical problem (reference 6). The additions to these forces thus isolated will depend essentially on the position and motion of the singularities of the flow outside the body.*

By the above approach the study of the continuously potential flows enters the study of discontinuously potential motion of an ideal fluid as a component part.

The general theorem of the forces independent of the singularities of the flow was worked out by Thomson and Tait (reference 9) and Kirchhoff (reference 10) as far back as the middle of the last century. The theory of Kirchhoff was generalized by S. A. Chaplygin (reference 11) in the case of the two-dimensional problem to the motion of a wing with constant circulation. In this work methods were given by Chaplygin for the determination of the potential flow of a fluid and the computation of the forces based on the theory of functions of a complex variable. The present paper gives a systematic presentation of the theory of unsteady motions for the case of Chaplygin and the case where a number of isolated vortices exist within the fluid. There are a large number of unsteady motions, the character of which will be clear from what follows, for which these theories may have a direct practical value. Moreover, as has already been pointed out, the results of these theories may enter as component parts of investigations dealing with the most general case of the motion of a fluid.

2. FUNDAMENTAL RELATIONS

We consider an arbitrary plane-parallel motion of a cylindrical wing within an incompressible fluid such that the plane of the motion is at right angles to the generators of the wing. All mechanical characteristics will be computed with the aid of a Cartesian system of coordinates xOy fixed invariably to the wing. For convenience the vectors will be considered as complex numbers. For example the position of a point will be determined by the vector

$$z = x + iy \qquad i = \sqrt{-1}$$

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*These singularities may be vortices, boundaries of the fluid, surface of discontinuity, etc.

Denoting the velocity of the origin of coordinates by q_0 we have

$$q_0 = U_0 + iV_0$$

where U_0 and V_0 are the projections of the origin on the axes of coordinates. For the velocity of a point of the wing with coordinates x, y we shall have

$$q = U + iV = U_0 - \omega y + i\ (V_0 + \omega x)$$

that is

$$q = q_0 + i\omega z$$

where ω denotes the angular velocity of the wing.

The velocities of the particles of the fluid we denote by $\vec{v} = u + iv$. Letters with a horizontal stroke above will denote the conjugate complex of the same letters without the strokes.

$$\bar{z} = x - iy;\ \bar{\vec{v}} = u - iv, \text{ etc.}$$

The stream function is denoted by $\psi(x, y, t)$. Since the flow is continuous we have on the contour C:

$$d\psi = \vec{v}_n ds = udy - vdx = Udy - Vdx \tag{1}$$

Remembering that $U = U_0 - \omega y;\ V = V_0 + \omega x$ and integrating relation (1) along C we obtain

$$\psi = U_0 y - V_0 x - \frac{\omega}{2}\left(x^2 + y^2\right) + \text{const.} \tag{2}$$

The value of the stream function ψ on the wing contour is always determined by equation (2) which is true for any motion of the incompressible fluid. Equation (2) is the mathematical expression of the condition imposed by the moving liquid due to the presence of a solid body within it.

In the region of potential motion the potential $\varphi(x, y, t)$ and the stream function $\psi(x, y, t)$ satisfy the equation of Laplace

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0; \quad \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

while the functions themselves are connected by the Cauchy-Riemann conditions

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y}; \qquad \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x}$$

so that

$$\varphi + i\psi = w(z)$$

whence

$$\vec{\bar{v}} = u - iv = \frac{dw}{dz}$$

$w(z)$ is known as the complex potential function.

For points of the region of potential motion the pressure $p(x, y, t)$ may be computed with the aid of the Lagrange integral. In the absence of external mass forces we shall have

$$p = p_0(t) - \rho\frac{\partial\varphi}{\partial t} - \frac{\rho}{2}\left(u^2 + v^2\right)$$

where ρ is the density of the fluid, $p_0(t)$ is a function depending only on the time. This function is determined by the given pressure at any one point of the fluid. In the above formula of Lagrange the partial derivative $-\partial\varphi/\partial t$ is taken on the assumption that the potential φ is expressed as a function of the time and the coordinates ξ, η defined in the stationary system of axes. Between the coordinates ξ, η and the coordinates x, y in the moving system we have the relation:

$$\xi = \xi(x, y, t); \qquad \eta = \eta(x, y, t)$$

Evidently $\partial\xi/\partial t$ and $\partial\eta/\partial t$ are the projections on the stationary axes of the velocity of the transporting motion.

We thus have

$$\frac{\partial\varphi(x, y, t)}{\partial t} = \frac{\partial\varphi(\xi, \eta, t)}{\partial t} + \frac{\partial\varphi}{\partial\xi}\frac{\partial\xi}{\partial t} + \frac{\partial\varphi}{\partial y}\frac{\partial\eta}{\partial t}$$

or

$$\frac{\partial\varphi(\xi, \eta, t)}{\partial t} = \frac{\partial\varphi(x, y, t)}{\partial t} - (\text{grad } \varphi \cdot \vec{q}) \tag{3}$$

where $\vec{q}$ is the velocity of the transporting motion.

Making use of relation (3) we rewrite the integral of Lagrange for the case where the potential is determined as a function of the time and coordinates of the moving system:

$$p = p_0(t) - \rho\frac{\partial\varphi}{\partial t} - \frac{\rho}{2}(u^2 + v^2) + \rho(uU + vV) \tag{4}$$

Since in what follows we shall throughout make use of the moving system of coordinates we shall always compute the pressures by the above formula.

In differentiating vectors with respect to time we shall distinguish derivatives with respect to the moving and stationary systems of coordinates. The first we shall denote by the symbol $\partial/\partial t$ and the second by $\delta/\delta t$. The former are used because the vectors under consideration depend also on the points of space. These derivates are connected, as is known, by the relation

$$\frac{\delta\vec{a}}{\delta t} = \frac{\partial\vec{a}}{\partial t} + [\vec{\omega}, \vec{a}]$$

which on replacing the vector $\vec{a}$ by the complex number a becomes

$$\frac{\delta a}{\delta t} = \frac{\partial a}{\partial t} + i\omega a \tag{5}$$

3. FORMULAS FOR THE FORCE AND MOMENT EXERTED BY THE FLUID ON A WING FOR UNSTEADY MOTION

In this section we shall consider the general formulas expressing the total hydrodynamic force and the moment of hydrodynamic forces with respect to the origin of coordinates by means of integrals, taken around the contour of the wing, of functions of a complex variable. With respect to the velocity field of the fluid we assume only that in the vicinity of the contour of the wing C the field has the potential $\varphi(P,t)$ where P is a point within the fluid and t the time. At the trailing sharp edge of the wing in the plane of motion of the fluid a line of velocity discontinuity may start out. In passing through the line of discontinuity at the sharp edge M the potential φ in the general case changes by a discontinuous amount the value of which may depend on the time.

We denote by n the unit vector directed along the normal to the contour of the wing C into the fluid, by dz the element of the contour C in the positive direction of passing round the contour (counter-clockwise). To obtain the direction dz, n must be rotated by a right angle from right to left, hence the equation:

$$n\,|dz| = -\,i dz$$

Let $dX + idY$ be the force acting on an element of the wing dz of unit width. Making use of formula (4) section 2 for the pressure we have

$$dX + idY = -\,pn\,|dz| = ipdz =$$

$$= ip_0\,(t)\,dz - i\rho\frac{\partial\varphi}{\partial t}\,dz - \frac{i\rho}{2}\,(u + iv)\,(u - iv)\,dz + i\rho\,(uU + vV)\,dz \tag{1}$$

Since $(u - iv)\,dz = dw$, $udx + vdy = d\varphi$ and along the wing $Udy - Vdx = d\psi$ it can be readily found that on the wing contour the following identities are valid:

$$-\,\frac{i}{2}\Big[(u+iv)(u-iv)dz - 2\,(uU+vV)dz\Big] = -\,\frac{i}{2}\Big[(u+iv)(d\varphi - id\psi) +$$

$$+ 2i\,(u+iv)\,d\psi - 2\,(uU+vV)dz\Big] = -\,\frac{i}{2}\left[\overline{\frac{dw}{dz}}\,dw - 2(U+iV)d\varphi\right] =$$

$$= -\,\frac{i}{2}\overline{\left(\frac{dw}{dz}\right)^2}\,dz + iq_0 d\varphi - \omega z d\varphi \tag{2}$$

Moreover, on the basis of relation (5), section 2 we have

$$-\,i\,\frac{\partial\varphi}{\partial t}\,dz - \omega z d\varphi = -\,id\left(\frac{\partial\varphi z}{\partial t}\right) + i\,\frac{\delta}{\delta t}\,zd\varphi =$$

$$= -\,id\;\;z\left(\frac{\partial\varphi}{\partial t}\right) + i\,\frac{\delta}{\delta t}\,zdw + \frac{\delta}{\delta t}\,zd\psi \tag{3}$$

With the aid of (2) and (3) we reduce relation (1) to the form:

$$\frac{1}{\rho}(dX+idY) = \frac{p_0}{\rho}idz + iq_0 d\varphi - id\left(z\frac{\partial\varphi}{\partial t}\right) - \frac{i}{2}\overline{\left(\frac{dw}{dz}\right)^2}dz + \\ + \frac{\delta}{\delta t}zd\psi + \frac{\delta}{\delta t}zdw \qquad (4)$$

Integrating this equation we obtain the formula for the total force on a unit width of wing:

$$\frac{1}{\rho}(X+iY) = iq_0\Gamma - iz_M\frac{d\Gamma}{dt} - \frac{i}{2}\oint_C\overline{\left(\frac{dw}{dz}\right)^2}dz + \frac{\delta}{\delta t}\oint_C zd\psi + \\ + i\frac{\delta}{\delta t}\oint_C z\frac{dw}{dz}dz \qquad (5)$$

In the above equation Γ denotes the circulation taken on the contour of the wing counterclockwise and z_M is the coordinate of the starting point of the line of velocity discontinuity.

Remembering that along $C\psi = U_0 y - V_0 x - \frac{\omega}{2}(x^2+y^2) + \text{const.}$ we compute the integral $\oint_C zd\psi$. Integrating by parts we obtain

$$\oint_C zd\psi = -\oint\left[U_0 y - V_0 x - \frac{\omega}{2}(x^2+y^2)\right](dx+idy)$$

But

$$\oint xdx = \oint ydy = \oint x^2dx = \oint y^2dy = 0$$

Thus we may write

$$\oint_C zd\psi = -U_0\oint ydx + iV_0\oint xdy + \frac{\omega}{2}\oint y^2dx + ix^2dy$$

The formula of Green, applied to the present case, gives

$$-\oint ydx = \oint xdy = \iint dxdy = S\oint y^2dx = -2\iint ydxdy = \\ -2Sy^*\oint x^2dy = 2\iint xdxdy = 2Sx^*$$

where S is the area bounded by the wing contour, x* and y* are the coordinates of the center of gravity of the wing area. Hence

$$\oint_C z\,d\psi = \left[U_0 + iV_0 + \omega\,(-\,y^* + ix^*)\right] S = Sq^* \tag{6}$$

The magnitude here denoted by q^* is evidently no other than the velocity of the center of gravity of the wing area. Substituting the values of the computed integral in formula (5) we obtain the general formula for the force exerted by an unsteady flow on a wing

$$X + iY = i\rho q_0 \Gamma - i\rho z_M \frac{d\Gamma}{dt} + \overline{\frac{i\rho}{2} \oint\left(\frac{dw}{dz}\right)^2} dz + \frac{\delta}{\delta t}\left(\rho S q^* + i\rho \oint \frac{dw}{dz}\,dz\right) \tag{I}$$

The above equation is analogous to the Blasius-Chaplygin equation for steady flow. The integral in the equation may be taken about any contour L enclosing the wing if between L and C there are no singular points of the complex potential function. This fact makes equation (I) suitable for the computation of the acting forces.

If: 1) the wing moves forward with constant velocity $\delta q^*/\delta t = \omega = 0$

2) the circulation about the wing is constant Γ = const, $d\Gamma/dt = 0$

3) the fluid is infinite, over a finite distance there are no vortices and the fluid is at rest at infinity, then the last term in equation (I) is equal to zero since in this case the integral does not depend on the time. Since in this case dw/dz outside the wing is throughout holomorphic and at infinity is of the order $1/z$, $(dw/dz)^2$ is at infinity of the order $1/z^2$ and therefore the first integral likewise vanishes. Thus the acting force is reduced to the Jourkowsky force

$$X + iY = i\rho q_0 \Gamma$$

If there are external mass forces an Archimedes force is to be added on the right of formula (I).

We shall now compute the moment. For the moment of the force acting on an element dz taken with respect to the origin of coordinates we may write

$$d\vec{M} = xdY - ydX = \text{Real} - i\bar{z}\ (dX + idY)$$

or, on the basis of (1) and (2)

$$\frac{1}{\rho}\, d\vec{M} = \text{Real}\ \bar{z}\left[\frac{p_0}{\rho}\, dz - \frac{\partial\varphi}{\partial t}\, dz - \frac{1}{2}\overline{\left(\frac{dw}{dz}\right)^2 dz} + q_0 d\varphi + i\omega z d\varphi\right]$$

Setting $z\bar{z} = x^2 + y^2 = r^2$ we shall have: Real $\bar{z}dz = dr^2/2$ and considering too the fact that $i\omega z\bar{z}d\varphi$ is purely imaginary we obtain

$$\frac{1}{\rho}\, d\vec{M} = \frac{1}{2}\,\frac{p_0}{\rho}\, dr^2 - \underline{\frac{\partial}{\partial t}\,\frac{1}{2}}\, d\,(r^2\varphi) + \text{Real}\left[iq_0\bar{z}d\psi + q_0\overline{zdw} - \right.$$
$$\left. - \frac{1}{2}\, z\left(\frac{dw}{dz}\right)^2 dz + \frac{\partial}{\partial t}\ \frac{1}{2}\, z\ \bar{z}dw\right]$$

Integrating over the contour of the wing we obtain

$$\frac{1}{\rho}\,\vec{M} = -\frac{1}{2}\, r_M{}^2\,\frac{d\Gamma}{dt} +$$

$$+ \text{Real}\left[-i\overline{q_0}\oint z d\psi + \overline{q_0}\oint z\frac{dw}{dz}\, dz - \frac{\rho}{2}\oint z\left(\frac{dw}{dz}\right)^2 dz + \frac{\partial}{\partial t}\,\frac{\rho}{2}\oint z\bar{z}\,\frac{dw}{dz}\, dz\right]$$

where r_M is the distance of the point of separation to the origin of coordinates. (In the expression contained in the square brackets in the first three terms i has been replaced by $-i$. This is permissible as we are interested only in the real part). Taking equation (6) into account we finally obtain

$$\vec{M} = -\frac{\rho r^2{}_M}{2}\,\frac{d\Gamma}{dt} +$$

$$+ \text{Real}\left[-i\overline{q_0}\left(\rho Sq^* + i\rho\oint z\frac{dw}{dz}\, dz\right) - \frac{\rho}{2}\oint z\left(\frac{dw}{dz}\right)^2 dz + \frac{\partial}{\partial t}\,\frac{\rho}{2}\oint z\bar{z}\,\frac{dw}{dz}\, dz\right] \quad \text{(II)}$$

Evidently in both formulas (I) and (II) the two first integrals may be taken over any contour L enclosing the wing if between L and C there are no singular points of the complex potential function.

If the fluid is infinite and there are only isolated singularities within the flow (sources, vortices, dipoles, etc.) both integrals in equation (I) and the two first integrals in equation (II) may be determined as the residues about the singularities of the functions under the integral signs. The last integral in equation (II) in this case may also be computed with the aid of residues if it is possible to construct an analytical function of the complex variable z having isolated singularities outside C and assuming on the contour C itself the values $\bar{z}$.

From what follows it will be seen that the presence of isolated singularities in the flow, if the transformation of the outside region of the wing into the interior of a circle is effected by a rational function, the last integral in equation (II) may be obtained as a sum of residues.

4. DETERMINATION OF POTENTIAL FLOWS

A solution of the problem of determining the potential flow of an infinite fluid for any motion of the wing with the aid of a function conformally mapping the external region of the wing on the upper half-plane has previously been given by S. Chaplygin. In this section we shall give methods of solving this problem with the aid of the function $z = f(\zeta)$ that transforms the outer region of the wing in the z-plane into the interior of a unit circle K in the ζ-plane.

For definiteness we shall assume that the center of the circle K coincides with the origin of coordinates and that for $z = \infty$, $\zeta = 0$. The function $f(\zeta)$ near the origin of coordinates has the following form

$$z = f(\zeta) = \frac{k}{\zeta} + k_0 + k_1\zeta + k_2\zeta^2 + \ldots \tag{1}$$

where $P(\zeta) = k_0 + k_1\zeta + k_2\zeta^2 + \ldots$ holomorphic everywhere within K.

We denote the complex potential function of the flow under consideration by $w_0\ (z) = \varphi_0 + i\psi_0$ where $\varphi_0\ (x, y, t)$ and $\psi_0\ (x, y, t)$ are single-valued and harmonic functions everywhere outside the wing. To determine $w_0\ (z)$ we have on the wing contour the condition

$$\psi_0 = U_0 y - V_0 x - \frac{\omega}{2} (x^2 + y^2) \tag{2}$$

We represent $w_0\ (z)$ in the form

$$w_0\ (z) = U_0 w_1\ (z) + V_0 w_2\ (z) + \omega w_3 (z)$$

so that $w_1\ (z)$ $w_2\ (z)$ $w_3\ (z)$ are holomorphic everywhere outside the wing and their imaginary parts ψ_1, ψ_2, ψ_3 on the contour of the wing C satisfy the following conditions:

$$\psi_1 = y;\ \psi_2 = -x;\ \psi_3 = -\frac{1}{2} (x^2 + y^2) \tag{3}$$

It is readily seen that $w_1\ (z)$ is a potential function of the potential flow for forward motion of the wing along the x-axis with unit velocity, $w_2\ (z)$ corresponds to the forward motion of the wing with unit velocity along the y-axis and $w_3 (z)$ gives the potential flow of the fluid in rotating the wing about the origin of coordinates with an angular velocity equal to unity. The functions $w_1\ (z)$, $w_2\ (z)$, $w_3\ (z)$ are determined by the geometric properties of the wing contour.

Replacing in $w_0\ (z)$, z by ζ we obtain a function $w_0\ (\zeta)$ holomorphic within K. In a similar manner we obtain the functions $w_1\ (z)$, $w_2\ (z)$, $w_3\ (z)$ holomorphic within K. To determine these functions we write boundary condition (3) in the ζ-plane on the contour of the circle K in the following form:

$$\text{Imag } w_1 = \text{Imag } f(\zeta)$$

$$\text{Imag } w_2 = \text{Imag } - if(\zeta) \tag{4}$$

$$\text{Imag } w_3 = -\frac{1}{2} (x^2 + y^2) = -\frac{1}{2} f(\zeta)\ \overline{f\ (\zeta)}$$

From the above equations we can, knowing $f(\zeta)$, determine $w_1(\zeta)$, $w_2(\zeta)$, $w_3(\zeta)$. The first two of these functions can readily be expressed through $f(\zeta)$ in finite form as may be shown as follows: The functions $w_1(\zeta)$ and $f(\zeta)$ have the same imaginary parts on the circle K, $f(\zeta)$ having a pole at $\zeta = 0$ with principal part k/ζ. Evidently k/ζ and $-\bar{k}\zeta$ have on K the same imaginary parts. Hence

$$w_1(\zeta) = f(\zeta) - \frac{k}{\zeta} - \bar{k}\zeta \tag{5}$$

or

$$w_1(\zeta) = k_0 + (k_1 - \bar{k})\zeta + k_2\zeta^2 + \ldots \tag{5a}$$

From boundary condition (4) we conclude that the functions $w_2(\zeta)$ and $-if(\zeta)$ have the same imaginary parts on K. Furthermore the function $-if(\zeta)$ has a pole at $\zeta = 0$ with principal part $-ik/\zeta$, the negative part of which on K coincides with the negative part $-i\bar{k}\zeta$. Therefore

$$w_2(\zeta) = -if(\zeta) + \frac{ik}{\zeta} - i\bar{k}\zeta \tag{6}$$

or

$$w_2(\zeta) = -ik_0 - i(k_1 + \bar{k})\zeta - ik_2\zeta^2 - \ldots \tag{6a}$$

In what follows we shall use the notation

$$\bar{f}\left(\frac{1}{\zeta}\right) = \bar{k}\zeta + k_0 + \frac{k_1}{\zeta} + \frac{\bar{k_2}}{\zeta} + \ldots$$

On the unit circle $x^2 + y^2 = f(\zeta)\,\bar{f}(1/\zeta)$ hence the last of conditions (4) may also be written thus:

$$\text{Imag } w_3(\zeta) = -\frac{i}{2} f(\zeta)\,\bar{f}\left(\frac{1}{\zeta}\right) \tag{7}$$

Restricting ourselves for simplicity to the case where the contour of the wing is an analytical curve and therefore $f(\zeta)$ is holomorphic at $|\zeta| = 1$ we shall show that the problem of determining $w_3(\zeta)$ is equivalent to the problem of splitting the function $-\frac{i}{2} f(\zeta)\,\bar{f}\left(\frac{1}{\zeta}\right)$ into a sum of two functions

$f_1(\zeta)$ and $f_2(\zeta)$ in such a manner that $f_1(\zeta)$ is holomorphic everywhere within and on K, and $f_2(\zeta)$ everywhere outside and on K and that

$$w_3(\zeta) = 2f_1(\zeta) \tag{8}$$

Thus for $|\zeta| = 1$, $-\frac{1}{2} f(\zeta) \overline{f}\left(\frac{1}{\zeta}\right) = f_1(\zeta) + f_2(\zeta)$ is purely imaginary, hence on the circle K

$$\text{Real } f_1(\zeta) = - \text{Real } f_2(\zeta) = - \text{Real } \overline{f_2}\left(\frac{1}{\zeta}\right) \tag{9}$$

It is evident also that on K

$$\text{Imag } f_2(\zeta) = - \text{Imag } \overline{f}_2\left(\frac{1}{\zeta}\right) \tag{10}$$

The functions $f_1(\zeta)$ and $-\overline{f_2}(1/\zeta)$ are holomorphic within and on K, and according to (9), on the circle their real parts are equal. Hence throughout the ζ-plane the equation holds:

$$f_1(\zeta) = - \overline{f_2}\left(\frac{1}{\zeta}\right) + q \tag{11}$$

where q is a purely imaginary constant.

By virtue of relations (10) and (11) it is evident that for $|\zeta| = 1$

$$\text{Imag } f_1(\zeta) = \text{Imag } f_2(\zeta) + q$$

Hence the function $2f_1(\zeta) - q$ is holomorphic within K and for $|\zeta| = 1$ the imaginary part is equal to $-i(x^2 + y^2)/2$ and therefore with an accuracy up to an unessential additive constant relation (8) holds. The above splitting of the function is particularly easy to carry out in the case where $f(\zeta)$ is a rational function.

Evidently $w_3(\zeta)$ may be determined also with the aid of the Schwarz integral (reference 12). Since the real part of the function $iw_3(\zeta)$ on the circle K is equal to $1/2\, f(\zeta)\, \overline{f}(1/\zeta)$

$$iw_3(\zeta) = C + \frac{1}{4\pi} \int_0^{2\pi} f(e^{i\theta})\, \overline{f}(e^{-i\theta}) \frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta}\, d\theta$$

where C is a purely imaginary constant which may be neglected.

Transforming, we obtain

$$w_3(\zeta) = -\frac{1}{4\pi} \oint_K f(u)\, \bar{f}\left(\frac{1}{u}\right) \frac{u+\zeta}{u-\zeta} \frac{du}{u} \qquad (8_1)$$

If $f(\zeta)$ is single valued and has only an isolated singularity, the integral in equation (8_1) may be computed with the aid of residues. Setting

$$w_3(\zeta) = c_1\zeta + c_2\zeta^2 + c_3\zeta^3 + \ldots \qquad (8_{11})$$

With the aid of equation (8_1) for the coefficient c_1 we obtain the formula

$$c_1 = \left(\frac{dw_3}{d\zeta}\right)_{\zeta=0} = -\frac{1}{2\pi} \oint_K f(u)\, \bar{f}\left(\frac{1}{u}\right) \frac{du}{u^2} \qquad (12)$$

The series (8_{11}) for $w_3(\zeta)$ may readily be written out if we first expand in a Fourier series the function

$\psi(\theta) = -\frac{1}{2} f(e^{i\theta})\, \bar{f}(e^{-i\theta})$. For, let

$$\psi(\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\,\theta)$$

The trigonometric series for $\varphi(\theta)$ conjugate to $\psi(\theta)$ is (reference 12)

$$\varphi(\theta) = \sum_{n=1}^{\infty} (-\,a_n \sin n\,\theta + b_n \cos\, n\,\theta)$$

Setting $b_n + ia_n = c_n$, we have

$$w_3(\zeta) = \sum_{n=1}^{\infty} c_n\, \zeta^n$$

The complex potential functions $w_1 (z)$, $w_2 (z)$, $w_3 (z)$ the methods for obtaining which we have considered are expressed in terms of the variable $z = x + iy$, referred to a moving system of coordinates, and give the absolute flow of the fluid. In order to obtain the relative flow of the fluid it is necessary from the vector of the absolute velocity to subtract the vector of the transporting velocity. Thus the relative velocity field is determined by the complex function

$$\frac{dw_0}{dz} - \bar{q}_0 + i\omega\bar{z}$$

If the motion of the wing is translational $\omega = 0$, the relative flow is potential and has the complex potential function

$$w_0 (z) - \bar{q}_0 z$$

This is the so-called characteristic stream function of the wing for translational flow at infinity.

It should be remarked that the potential flow considered is not always physically possible since it is determined only by kinematic conditions without taking account of the dynamic condition of the positiveness of the hydrodynamic pressure. It can readily be shown that with the presence of projecting angles on the wing contour the velocity of the potential flow of the fluid near the angles tends to infinity and this, according to the integral of Lagrange leads to infinite negative pressures physically impossible within the fluid. In this case if as before an ideal fluid is assumed it is necessary to consider motions with discontinuous velocity distribution along lines starting out from the walls of the body.

5. ELLIPTIC WING AND JOUKOWSKY PROFILE

We shall consider examples of the application of the methods presented above to the determination of potential flows. We shall first take a wing bounded by an ellipse with semi-axes a and b (fig. 3). The external region of the ellipse is conformally transformed into the interior of a unit circle K in the ζ-plane with the aid of the function

$$z = f(\zeta) = -\frac{1}{2}\left[(a-b)\zeta + (a+b)\frac{1}{\zeta}\right]$$

so that

$$k = -\frac{1}{2}(a+b); \quad k_1 = -\frac{1}{2}(a-b)$$

By formulas (5a) and (6a), section 4, we obtain immediately

$$w_1(\zeta) = b\zeta$$
$$w_2(\zeta) = ia\zeta$$

Further, we have

$$-\frac{i}{2} f(\zeta)\,\overline{f}\left(\frac{1}{\zeta}\right) = -\frac{i}{8}\left[(a^2-b^2)\zeta^2 + 2(a^2+b^2) + \frac{a^2-b^2}{\zeta^2}\right]$$

By (8) section 4, we obtain

$$w_3(\zeta) = -\frac{i}{4}(a^2-b^2)\zeta^2$$

If $b = 0$ the ellipse degenerates into a plate of width $2a$ and then

$$w_1 = 0$$

$$w_2 = ia\zeta$$

$$w_3 = -\frac{ia^2}{4}\zeta^2$$

As a second example we shall consider the problem of determining the potential flow of an infinite fluid for any motion within it of a Joukowsky wing. Incidentally, we shall compare certain geometric characteristics of Joukowsky profiles which we require for determining the hydrodynamic forces.

As is known (reference 13) the outer region of the circle K, (fig. 4) in the plane $z_1 = x_1 + iy_1$, is conformally mapped on the external region of the Joukowsky wing in the plan $z' = x' + iy'$ with the aid of the function

$$z' - a = \frac{1}{2}\left(z_1 + \frac{a^2}{z_1}\right) \qquad (1)$$

The Joukowsky wing is given completely by the parameters a, α, R, the geometric meaning of which on the z_1-plane is clear from figure 4. The angle α may be taken as a number characterizing the concavity of the profile; $\epsilon = R \cos \alpha - 1$ characterizes the thickness of the profile; if $\epsilon = 0$ we have an arc of a circle. For $R = \text{const.}$, $\alpha = \text{const.}$, and the variable a in the z_1 and z' planes vary similarly, hence the number a determines only the linear scale of the Joukowsky profile. The point M_1 transforms into the sharp edge of the wing M. At this point the correspondence will be quasi-conformal. All angles in the z'-plane will be twice as large as the corresponding angles in the z_1-plane. Evidently at the point M_1 the direction M_1P_1 will correspond to the direction M P and the direction of the radius OM_1N_1 goes over into the direction of the principal tangent at the inflection point M of the Joukowsky profile. Hence $\angle PMN = 2\angle P_1M_1N_1 = 2\alpha$.

Transforming the irrotational flow about the cylinder K_1 with the aid of relation (1), we obtain the flow about the Joukowsky profile, the angle between the velocity at infinity and the real axis in both planes being the same since $(dz'/dz_1)_\infty = 1/2$. The point M_1 will be a critical branch point of the flow if the angle of inclination of the velocity at infinity to the x_1-axis is equal to α or $\pi+\alpha$. Only in this case will the velocity of the transformed flow at point M be finite and therefore only in this case is an irrotational motion of a potential flow about a Joukowsky profile possible. Evidently the direction of a possible irrotational flow makes the angle α with the direction of the principal tangent at the sharp edge. This direction is known as the first axis of the profile. The angle α may thus be implicitly defined with respect to the profile as the angle between the principal tangent and the first axis indicating by definition the direction of a possible potential irrotational flow about a Joukowsky profile. In what follows we shall assume the direction of the first axis as the real axis of a Cartesian system of coordinates xy, the origin of which is at the sharp edge and we shall set $x+iy = z$. Evidently

$$z = z'e^{-i\alpha} \qquad (2)$$

The external region of the circle K_1 is mapped on the interior of the unit circle K in the ζ-plane with the aid of the transformation

$$z_1 = a\left[Re^{+i\alpha}\ \frac{\zeta-1}{\zeta} - 1\right] \tag{3}$$

This transformation is so determined that to the point M_1, $z_1 = -a$ corresponds the point M', $\zeta = 1$.

We make use of equations (1), (2), (3) in order to express z in terms of ζ. We obtain

$$z = f(\zeta) = -\frac{aR}{2}\left[\frac{1}{\zeta} + \mu - 2 + (1-\mu)^2\ \frac{\zeta}{1-\mu\zeta}\right] \tag{4}$$

where, for briefness, we set $1 - \frac{e^{-i\alpha}}{R} = \mu$; evidently $|\mu| < 1$.

We shall now compute the area of the Joukowsky profile and the coordinates of the center of gravity x^* and y^*, $x^*+iy^* = z^*$.

There are immediately evident the following formulas, valid for a profile of any contour:

$$S = \iint dxdy = -\oint_C ydx = \oint_C xdy = -\frac{i}{2}\oint_C \bar{z}dz = -\frac{i}{2}\oint_K \overline{z(\zeta)}\ \frac{dz}{d\zeta}\ d\zeta$$

$$Sx^* = \iint xdxdy = \frac{1}{2}\oint_C x^2dy = \frac{1}{2}\oint_C (x^2 + y^2)\ dy$$

$$Sy^* = \iint ydxdy = -\frac{1}{2}\oint_C y^2dx = -\frac{1}{2}\oint_C (x^2 + y^2)\ dx$$

whence

$$Sz^* = -\frac{i}{2}\oint_C z\bar{z}dz = -\frac{i}{2}\oint_K z(\zeta)\ \overline{z(\zeta)}\ \frac{dz}{d\zeta}\ d\zeta$$

To simplify the computations we introduce the variable z_0

$$z_0 = z + \frac{aR}{2}(\mu - 2) = -\frac{aR}{2}\left[\frac{1}{\zeta} + (1 - \mu)^2 \frac{\zeta}{1 - \mu\zeta}\right]$$

Further for $|\zeta| = 1$, we shall have

$$\overline{z_0(\zeta)} = -\frac{aR}{2}\left[\zeta + (1 - \bar{\mu})^2 \frac{1}{\zeta - \bar{\mu}}\right]$$

$$\frac{dz_0}{d\zeta} = \frac{aR}{2}\left[\frac{1}{\zeta^2} - \frac{(1 - \mu)^2}{(1 - \mu\zeta)^2}\right]$$

Thus, for determining S we arrive at the computation of the integral

$$S = \frac{ia^2R^2}{8} \oint_K \left[\zeta + (1 - \bar{\mu})^2 \frac{1}{\zeta - \bar{\mu}}\right]\left[\frac{1}{\zeta^2} - \frac{(1 - \mu)^2}{(1 - \mu\zeta)^2}\right] d\zeta$$

Within K the function under the integral has poles at the points $\zeta = 0$ and $\zeta = \mu$. Taking the residues about these poles we obtain

$$S = \frac{\pi a^2R^2}{4}\left[1 - \frac{(1 - \mu)^2 (1 - \bar{\mu})^2}{(1 - \mu\bar{\mu})^2}\right] =$$

$$= \frac{\pi a^2R^2}{4} \frac{(2 - \mu - \bar{\mu})(\mu + \bar{\mu} - 2\mu\bar{\mu})}{(1 - \mu\mu)^2} \qquad (5)$$

or replacing μ by $1 - \frac{e^{-i\alpha}}{R}$:

$$S = \pi a^2R^2 \cos\alpha \frac{R\cos\alpha - 1}{(2R\cos\alpha - 1)^2}$$

For computing the coordinate $z_0{*}$ we have

$$Sz_0{*} = -\frac{ia^3R^3}{16} \oint_K \left[\frac{1}{\zeta} + (1-\mu)^2 \frac{\zeta}{1-\mu\zeta}\right]\left[\zeta + (1-\bar{\mu})^2 \frac{1}{\zeta-\bar{\mu}}\right]\left[\frac{1}{\zeta^2} - \frac{(1 - \mu)^2}{(1 - \mu\zeta)^2}\right] d\zeta$$

Taking the residues about the poles of the function under the integral and transforming the obtained expression we find

$$Sz_0^* = \frac{\pi a^3 R^3}{8} \frac{(1-\mu)^2 (1-\bar{\mu})^2}{(1-\mu\bar{\mu})^3} (\mu+\bar{\mu}-2\mu\bar{\mu})$$

Returning to the variable z we may write

$$Sz^* = Sz_0^* - S \frac{aR}{2} (\mu-2)$$

or

$$Sz^* = \frac{\pi a^3 R^3}{8} \frac{\mu+\bar{\mu}-2\mu\bar{\mu}}{(1-\mu\bar{\mu})^2} \left[\frac{(1-\mu)^2 (1-\bar{\mu})^2}{1-\mu\bar{\mu}} + (\mu-2)(\mu+\bar{\mu}-2)\right] \tag{6}$$

Equations (5) and (6) may be simplified somewhat by taking into consideration the radius of the circle K_2 into which the circle K_1 is transformed, by the inversion $z_{11} = a^2/z_1$. This circle as is known (reference 14) figures in the graphical construction of the Joukowsky profile by the method of Trefftz. Denoting the radius of the circle K_2 by r, we obtain

$$r = \frac{R}{2R \cos \alpha - 1} \tag{7}$$

With the aid of the radius r we may write

$$S = \frac{\pi a^2}{4} (R^2-r^2) \tag{5a}$$

$$z^* = \frac{a}{2} \left\{\frac{R^3-r^3}{R^2-r^2} + e^{-i\alpha}\right\} \tag{6a}$$

When $\alpha = 0$ the Joukowsky profile is symmetrical and has the form shown in figure 5. We denote by l the maximum diameter of the symmetrical profile. Setting $\zeta = -1$ in formula (4) and taking account of (7) we obtain for l

$$l = 2Rr = R+r$$

For the value of the area and the position of the center of gravity of the symmetrical profile, we obtain from equations (5a) and (6a)

$$S = \frac{\pi a^2}{4} l^2 \sqrt{1 - \frac{2}{l}} \tag{5b}$$

$$x^* = \frac{a}{2}\left[l + \frac{1}{2}\right] \; ; \quad y^* = 0 \tag{6b}$$

Having $f(\zeta)$ in explicit form (equation (4)) and making use of equations (5a) and (6a), section 4, we find $w_1(\zeta)$ and $w_2(\zeta)$. Neglecting the additive constants we obtain

$$w_1(\zeta) = \frac{aR}{2}\zeta \left[1 - (1-\mu)^2 \frac{1}{1-\mu\zeta}\right] \tag{8}$$

$$w_2(\zeta) = \frac{iaR}{2}\zeta \left[1 + (1-\mu)^2 \frac{1}{1-\mu\zeta}\right] \tag{9}$$

Further, we have

$$-\frac{i}{2} f(\zeta)\,\overline{f}\left(\frac{1}{\zeta}\right) =$$

$$= -\frac{ia^2R^2}{8}\left[\frac{1}{\zeta} + (\mu-2) + (1-\mu)^2 \frac{\zeta}{1-\mu\zeta}\right]\left[\zeta + (\overline{\mu}-2) + (1-\overline{\mu})^2 \frac{1}{\zeta-\overline{\mu}}\right]$$

Neglecting the principal parts corresponding to the poles within K, the additive constants, and multiplying by two we obtain

$$w_3(\zeta) = -\frac{ia^2R^2}{4}\frac{\zeta}{1-\mu\zeta}\left[\zeta + (\mu-2) + \frac{(1-\mu)^2\,(\mu+\overline{\mu}-2)}{1-\mu\overline{\mu}}\right] \tag{10}$$

or after transforming:

$$w_3(\zeta) = \frac{ia^2R^2}{4}\frac{1}{\mu}\left[\zeta - \frac{(1-\mu)^4}{1-\mu\overline{\mu}}\frac{\zeta}{1-\mu\zeta}\right] \tag{10'}$$

Thus the potential function of the potential flow for arbitrary motion of the Joukowsky wing is expressed by the equation

$$w_0(\zeta) = \frac{aR}{2}\left\{q_0\zeta - \frac{(1-\mu)^2\zeta}{1-\mu\zeta}\overline{q}_0 - \frac{iaR}{2}\frac{\omega\zeta}{1-\mu\zeta}\left[\zeta + (\mu-2) + \frac{(1-\mu)^2(\mu+\overline{\mu}-2)}{1-\mu\overline{\mu}}\right]\right\}$$

and the complex velocity in the ζ-plane by

$$\frac{dw_0}{d\zeta} = \frac{aR}{2}\left\{q_0 - \frac{(1-\mu)^2}{(1-\mu\zeta)^2}\bar{q}_0 - \right.$$

$$\left. - \frac{iaR}{2}\frac{\omega}{(1-\mu\zeta)^2}\left[(2-\mu-\mu\zeta)(\zeta-1) + \frac{(1-\mu)^2(\mu+\bar{\mu}-2)}{1-\mu\bar{\mu}}\right]\right\} \tag{11}$$

Further, we have

$$\frac{dw_0}{dz} = \frac{dw_0}{d\zeta}\frac{d\zeta}{dz}$$

where according to equation (4)

$$\frac{dz}{d\zeta} = \frac{aR}{2}(1-\zeta)\left[\frac{1+\zeta-2\mu\zeta}{\zeta^2(1-\mu\zeta)^2}\right] \tag{12}$$

Evidently at the point $\zeta = 1$ corresponding to the sharp edge $d\zeta/dz$ has a pole and therefore dw_0/dz remains finite only if

$$\left(\frac{dw_0}{d\zeta}\right)_{\zeta=1} = 0 \tag{13}$$

The condition (13) assures the finiteness of the velocities and is therefore necessary in order that the unsteady continuous potential flow be physically possible.

Making use of the explicit expression (`1) for $dw_0/d\zeta$, we reduce condition (13) to the form

$$2V_0 - \frac{aR}{2}\frac{(\mu+\bar{\mu}-2)}{1-\mu\bar{\mu}}\omega = 0 \tag{14}$$

or

$$V_0 = -\frac{a(R+r)}{4}\omega \tag{14a}$$

The velocity component U_0 does not enter into condition (14) and therefore the forward motion along the x-axis may be with any velocity, the velocity at the trailing edge being finite. Condition (14a) denotes geometrically that the instantaneous center of rotation lies on the straight line PP' (fig. 6) perpendicular to the x-axis and at a distance $a(R+r)/4$ from the origin. The addition of a velocity parallel to the x-axis displaces the instantaneous center of rotation along the straight line PP'. We thus arrive at the conclusion that the velocities at the trailing edge are finite if the moving polhode coincides with the straight line PP'. This result, established for a Joukowsky wing, is evidently valid in a similar form for any wing with sharp edge.

For $\alpha = 0$, $R = 1$, $\mu = 0$ the Joukowsky profile degenerates into a plate. In this case

$$\frac{dw_0}{d\zeta} = iaV_0 - \frac{ia^2}{2} (\zeta-2)\, \omega \tag{11'}$$

and instead of (14) we have

$$V_0 = - \frac{a}{2}\, \omega \tag{14'}$$

The straight line PP' for the plate is at a distance of 1/4 the width of the plate from the trailing edge (fig. 7).

6. FORCES ACTING ON THE WING FOR A POTENTIAL FLOW OF THE FLUID

We shall study the hydrodynamic properties of the forces and establish methods of computing them for the case where the motion of the infinite fluid is throughout potential and the circulation about the wing is equal to zero:

$$\Gamma = 0$$

We shall start with equations (I) and (II), section 3. The complex potential function $w_0(z)$ is holomorphic and single-valued outside the wing and therefore the expansion of dw_0/dz at infinity starts with the terms of the order $1/z^2$; $(dw_0/dz)^2$ is therefore of the order $1/z^4$ and hence the integrals

$$P_0 = \frac{1}{2} \oint \left(\frac{dw_0}{dz}\right)^2 dz; \quad R_0 = -\frac{1}{2} \oint z \left(\frac{dw_0}{dz}\right)^2 dz$$

taken over the contour of the wing are equal to zero:

$$P_0 = R_0 = 0$$

Remembering that

$$Q_0 = i \oint_C z \frac{dw_0}{dz} dz = i \oint_K f(\zeta) \frac{dw_0}{d\zeta} d\zeta = 2\pi k \left(\frac{dw_0}{d\zeta}\right)_{\zeta=0}$$

since the function to be integrated is holomorphic within the circle K and only at the point $\zeta = 0$ does it have a pole of the first order. Applying equations (I) and (II), section 3, we obtain

$$X_0 + iY_0 = \frac{\delta I_0}{\delta t} = \frac{\partial I_0}{\partial t} + i\omega I_0 \tag{III}$$

$$\vec{M}_0 = \text{Real} \left[- i\bar{q}_0 I_0 + \frac{\partial N_0}{\delta t} \right] \tag{IV}$$

where

$$I_0 = \rho \left[Sq^* + 2\pi k \left(\frac{dw_0}{d\zeta}\right)_{\zeta=0} \right] \tag{1}$$

and

$$N_0 = \frac{\rho}{2} \oint_C z\bar{z} \frac{dw}{dz} dz \tag{2}$$

First of all we shall make certain observations that follow directly from equations (III) and (IV). If the wing moves in translation with constant velocity it is evident that I_0 and N_0 do not depend on the time, and it then follows from equation (III) that $X_0 + iY_0 = 0$. This is the well known paradox of D'Alembert. In this case $\vec{M}_0 = \text{Real}\left[-\bar{q}_0 I_0\right]$, hence the moment of the hydrodynamic forces, generally speaking, is different from zero also for uniform rectilinear motion. The moment $\vec{M} = 0$ if q_0 and I_0 have the same direction because then $\bar{q}_0 I_0$ is real. For translational motion the components I_0, I_x, and I_y depend linearly on the velocity components q_0, U_0, and V_0. Hence for a wing there exist in general two directions having the property that for translational motion in any of them the moment of the hydrodynamic forces is equal to zero.

If the motion of the wing is a rotation with constant angular velocity about a stationary center, we find by choosing the origin of coordinates at the center of rotation that the moment $\vec{M}_0$ with respect to the center of rotation remains at all times equal to zero. Hence the hydrodynamic force in this case must be considered as applied at the center of rotation.

The potential motion of the fluid may be considered as starting from a state of rest as a result impact or suddenly applied system of impulsive pressures $p_t = -\rho\varphi$ (reference 8) along the contour of the wing C. It is not difficult to show that $B = -I_0$ and $M = -\text{Real } N_0$ are the sum of the impulsive pressures applied to the fluid and the sum of the moments of all these impulsive forces with respect to the origin of coordinates. For we have

$$B = -\rho\left[Sq^* + i\oint z\,dw\right] = -i\rho\oint z\,d\varphi_0$$

where Sq^* is replaced according to equation (6), section 3, by $\oint z\,d\psi_0$. Further integrating by parts we obtain on account of the single-valuedness of φ_0,

$$B = i\rho\oint \varphi_0\,dz = -i\oint p_t\,dz \tag{3}$$

The integral at the right denotes the sum of all the impulsive forces.

In a similar manner we have

$$M = -\text{Real}\,\frac{\rho}{2}\oint z\bar{z}\,dw_0 = \rho\oint \varphi_0\,\frac{dz\bar{z}}{2} \tag{4}$$

or

$$M = \text{Real}\,\rho\oint z\varphi_0 dz = -\text{Real}\oint \bar{z}p_t dz \tag{5}$$

since

$$\frac{dz\bar{z}}{2} = \text{Real}\,\bar{z}dz$$

The right-hand side of (5) represents the sum of the moments of the impulsive forces with respect to the origin of coordinates. It is thus clear that B and M are respectively the momentum and moment of momentum of the fluid.

Equations (III) and (IV) express the theorems on the momentum and moment of momentum for unsteady potential flow of the fluid. The term $-i\bar{q}_0 I_0$ in formula (IV) is added on the right on account of the displacement of the moment center.

If the motion is such that for the two instants of time t_0 and t_1 the orientation of the wing in stationary space is the same and the translational and angular velocities are the same the impulse of the hydrodynamic forces for the interval of time $t_1 - t_0$ is equal to zero since the change in momentum of the fluid for this interval is equal to zero. In particular for periodic motions of the wing the mean value of the hydrodynamic forces over the period T equals zero, that is,

$$\frac{1}{T}\int_t^{t+T} (X_0' + iY_0')\,dt = 0$$

where X_0' and Y_0' denote the projections of the forces on the stationary axes of coordinates. The mean value over a period of the moment of the hydrodynamic forces about any point of the wing for periodic motion is in the general case different from zero. Remembering that the change in the moment of momentum relative to the origin of coordinates over a period is equal to zero we may write

$$\vec{M}_m = \frac{1}{T} \int_t^{t+T} \vec{M}_0 dt = \text{Real} - \frac{i\rho}{T} \int_t^{t+T} \bar{q}_0 I_0 \, dt$$

We shall now consider B and N more in detail. According to equations (3) and (4), and the boundary condition (3), section 4, we may write

$$B = i\rho \oint \varphi_0 dz = -\rho \oint \varphi_0 dy + i\rho \oint \varphi_0 dx = -\rho \oint \varphi_0 d\psi_1 - i \oint \varphi_0 d\psi_2$$

$$M = \rho \oint \varphi_0 \frac{dz\bar{z}}{2} = -\rho \oint \varphi_0 d\psi_3$$

Setting

$$B = B_x + iB_y$$

and

$$-\rho \oint_C \varphi_i d\psi_k = \lambda_{ik} \qquad (i,\ k = 1,\ 2,\ 3)$$

we shall have

$$\left.\begin{aligned} B_x &= \lambda_{11}U_0 + \lambda_{21} V_0 + \lambda_{31}\omega \\ B_y &= \lambda_{12}U_0 + \lambda_{22} V_0 + \lambda_{32}\omega \\ M &= \lambda_{13}U_0 + \lambda_{23} V_0 + \lambda_{33}\omega \end{aligned}\right\} \qquad (6)$$

It is readily shown that the matrix $\|\lambda_{ik}\|$ is symmetrical, $\lambda_{ik} = \lambda_{ki}$ since φ_k and ψ_k are harmonic conjugate functions. For, on the contour C we have

$$\frac{\partial \psi_k}{\partial s} = \frac{\partial \varphi_k}{\partial n}$$

Therefore,

$$\oint \varphi_i \frac{\partial\psi_k}{\partial s} ds = \oint \varphi_i \frac{\partial\varphi_k}{\partial n} ds$$

Applying the Green formula we obtain

$$\oint \left(\varphi_i \frac{\partial\varphi_k}{\partial n} - \varphi_k \frac{\partial\varphi_i}{\partial n}\right) ds = 0$$

whence follows the symmetry

$$\oint \varphi_i d\psi_k = \oint \varphi_k d\psi_i$$

To explain the physical significance of λ_{ik} we shall also compute the kinetic energy of the fluid T. We have

$$T = \frac{\rho}{2} \iint \left|\text{grad}\varphi\right|^2 dxdy = -\frac{\rho}{2} \oint \varphi_0 \frac{\partial\varphi_0}{\partial n} ds = -\frac{\rho}{2} \oint \varphi_0 d\psi$$

whence

$$2T = \lambda_{11}U_0^2 + \lambda_{22}V_0^2 + \lambda_{33}\omega^2 + 2\lambda_{21}U_0V_0 + 2\lambda_{31}U_0\omega + 2\lambda_{32}V_0\omega$$

or

$$2T = B_xU_0 + B_yV_0 + M\omega$$

It is thus clear that the coefficients λ_{ik} play in the given case the same part as the masses and moments of inertia in the dynamics of a solid body. These coefficients are known as the virtual masses.

In what follows we shall use the following notation:

$$\left.\begin{aligned} \lambda_{11} = \lambda_x;\ \lambda_{22} = \lambda_y;\ \lambda_{33} = \omega \\ \lambda_{21} = \lambda_{12} = \lambda_{xy};\ \lambda_{31} = \lambda_{13} = \lambda_{x\omega};\ \lambda_{32} = \lambda_{23} = \lambda_{y\omega} \end{aligned}\right\} \quad (7)$$

It is not difficult to derive formulas for the transformation of the coefficients of virtual masses in passing to a new system of coordinates. Let us together with the system of coordinates xoy assume the system x'o'y' where the coordinates of the new origin o' in the system xoy are ξ, η and the x'-axis makes with the x-axis the angle β. Denoting by $\lambda_{x'}$ $\lambda_{y'}$ $\lambda_{\omega'}$ $\lambda_{x'y'}$ $\lambda_{x'\omega'}$ $\lambda_{y'\omega'}$ the coefficients of the virtual masses referred to the new axes of coordinates we shall have the following transformation formulas:

$$\lambda_{x'} = \lambda_x \cos^2 \beta + \lambda_y \sin^2 \beta + \lambda_{xy} \sin 2\beta$$

$$\lambda_{y'} = \lambda_x \sin^2 \beta + \lambda_y \cos^2 \beta - \lambda_{xy} \sin 2\beta$$

$$\lambda_{x'y'} = \frac{1}{2} (\lambda_y - \lambda_x) \sin 2\beta + \lambda_{xy} \cos 2\beta$$

$$\lambda_{x'\omega'} = (\lambda_x\eta - \lambda_{xy}\xi + \lambda_{x\omega}) \cos \beta + (\lambda_{xy}\eta - \lambda_y\xi + \lambda_{y\omega}) \sin \beta$$

$$\lambda_{y'\omega'} = -(\lambda_x\eta - \lambda_{xy}\xi + \lambda_{x\omega}) \sin \beta + (\lambda_{xy}\eta - \lambda_y\xi + \lambda_{y\omega}) \cos \beta$$

$$\lambda_{\omega'} = \lambda_x\eta^2 + \lambda_y\xi^2 - 2\lambda_{xy}\xi\eta - 2(\lambda_{x\omega}\eta - \lambda_{y\omega}\xi) + \lambda_\omega$$

The numerical values of the coefficients λ_x, λ_y, and λ_{xy} depend only on the direction of the assumed system of coordinates, while the values of $\lambda_{x\omega}$ and $\lambda_{y\omega}$ depend also on the position of the origin; λ_ω depends only on the position of the origin. According to Mises the combination λ_{ik} considered as a whole constitutes a motor dyad (reference 15). The latter is analagous to the motor dyad of inertia of a solid body and is called the motor dyad of the virtual masses.

It is readily seen that in the general case the determinant $\begin{vmatrix} \lambda_x & \lambda_{xy} \\ \lambda_{xy} & \lambda_y \end{vmatrix} = \lambda_x\lambda_y - \lambda_{xy}^2 > 0$ because the quadratic form

$$2T = \lambda_x U_0^2 + \lambda_y V_0^2 + 2\lambda_{xy} U_0 V_0$$

is positive definite since it gives the kinetic energy in translational motion. Hence it follows that the equations

$$\lambda_x\eta^* - \lambda_{xy}\xi^* + \lambda_{x\omega} = 0; \; \lambda_{xy}\eta^* - \lambda_y\xi^* \lambda_{y\omega} = 0$$

have the only solution

$$\xi^* = \frac{\lambda_x \lambda_{y\omega} - \lambda_{xy}\lambda_{x\omega}}{\lambda_x \lambda_y - \lambda_{xy}^2} \; ; \; \eta^* = \frac{\lambda_{xy}\lambda_{y\omega} - \lambda_y \lambda_{x\omega}}{\lambda_x \lambda_y - \lambda_{xy}^2}$$

The point with coordinates ξ^* and η^* is called the central point (reference 16). If the origin of coordinates is chosen at the central point then

$$\lambda_{x\omega} = \lambda_{y\omega} = 0$$

The momentum vector is expressed through the velocity of the central point. The moment of momentum with respect to the central point does not depend on its velocity. Since $\lambda_x \lambda_y - \lambda_{xy}^2 > 0$ the equation

$$\begin{vmatrix} \lambda_x - \lambda & \lambda_{xy} \\ \lambda_{xy} & \lambda_y - \lambda \end{vmatrix} = 0$$

has two real roots λ_1 and λ_2. Therefore there always exist two mutually perpendicular directions such that for translational motion of the body

$$I_{01} = -\lambda_1 q_1; \; I_{02} = -\lambda_2 q_2$$

where $-I_{01}$, $-I_{02}$, and q_1, q_2 are the components of the momentum of the fluid and the velocity of the body in these directions.

In the case of translational motion in one of these two directions, the moment of the hydrodynamic forces about the central point is equal to zero, that is, in this case the hydrodynamic force is applied at the central point. This evidently follows directly from equation (IV) if it is remembered that for the central point $\lambda_{x\omega} = \lambda_{y\omega} = 0$. If in addition the velocity of the motion is constant the hydrodynamic force and moment are equal to zero.

We shall now give formulas for computing the coefficients of the virtual masses. We have

$$B = -\rho \left[Sq^* + 2\pi k \left(\frac{dx_0}{d\zeta}\right)_{\zeta=0} \right]$$

As before let

$$z = f(\zeta) = \frac{k}{\zeta} + k_0 + k_1\zeta + k_2\zeta^2 + \dots$$

be a function that transforms conformally the outer region of the wing into the interior of a unit circle. According to equations (5a) and (6a), section 4,

$$\left(\frac{dw_0}{d\zeta}\right)_{\zeta=0} = (k_1-\bar{k})\, U_0 - i(k_1+\bar{k})\, V_0 + c_1\omega \tag{8}$$

where $c_1 = (dw_3/d\zeta)_{\zeta=0}$; c_1 may be computed for example by equation (12) section 4. With the aid of (8) we may write

$$B = -\rho \left\{ [S+2\pi k(k_1-\bar{k})]U_0 + i\,[S-2\pi k(k_1+\bar{k})]\; V_0 + [iSz^*+2\pi kc_1]\,\omega \right\} \tag{9}$$

Separating in (9) the real and imaginary parts we obtain on the basis of equations (6) and notations (7)

$$\lambda_x = -\rho\,[S-2\pi k\bar{k}+\pi(kk_1+\overline{kk_1})] \tag{10}$$

$$\lambda_y = -\rho\,[S-2\pi k\bar{k}-\pi(kk_1+\overline{kk_1})] \tag{11}$$

$$\lambda_{xy} = i\rho\pi\,(kk_1-\overline{kk_1}) \tag{12}$$

$$\lambda_{x\omega} = \rho\,[Sy^* - \pi(kc_1+\overline{kc_1})] \tag{13}$$

$$\lambda_{y\omega} = \rho\,[-Sx^* + \pi i(kc_1-\overline{kc_1})] \tag{14}$$

For λ_ω we have

$$\lambda_\omega = -\rho \oint_C \varphi_3 d\psi_3 = -\rho \oint_C \overline{w_3}d\psi_3 = -\frac{1}{2i}\oint_C \overline{w_3}d(w_3-\overline{w_3}) = -\frac{\rho}{2i}\oint_C \overline{w_3}dw_3$$

but for $|\zeta| = 1$ $\overline{w(\zeta)} = \bar{w}\,(1/\zeta)$*, hence

$$\lambda_\omega = \frac{i\rho}{2} \oint_K \overline{w_3}\left(\frac{1}{\zeta}\right) \frac{dw_3}{d\zeta}\, d\zeta \qquad (15)$$

Formula (15) is suitable for computing λ_ω if $w_3(\zeta)$ is determined. It is evident finally that on the basis of (2) there also holds for λ_ω the formula

$$\lambda_\omega = \frac{\rho}{2} \oint_K f(\zeta)\, \bar{f}\left(\frac{1}{\zeta}\right) \frac{dw_3}{d\zeta}\, d\zeta \qquad (15a)$$

If $f(\zeta)$ is rational $w_3(\zeta)$ is also rational and the integrals in formulas (15) and (15a) are computed with the aid of residues.

We shall now write in explicit form the expressions for the hydrodynamic forces in terms of the introduced coefficients of virtual masses. From equation (III) we obtain

$$X_0 = -\left[\lambda_x \frac{dU_0}{dt} + \lambda_{xy} \frac{dV_0}{dt} + \lambda_{x\omega} \frac{d\omega}{dt} - \omega\left(\lambda_{xy} U_0 + \lambda_y V_0 + \lambda_{y\omega}\omega\right)\right] \qquad (16)$$

$$Y_0 = -\left[\lambda_{xy} \frac{dU_0}{dt} + \lambda_y \frac{dV_0}{dt} + \lambda_{y\omega} \frac{d\omega}{dt} + \omega\left(\lambda_x U_0 + \lambda_{xy} V_0 + \lambda_{x\omega}\omega\right)\right] \qquad (17)$$

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*If

$$w(\zeta) = c_1\zeta + c_2\zeta^2 + \ldots$$

then

$$\bar{w}\left(\frac{1}{\zeta}\right) = \bar{c}_1 \frac{1}{\zeta} + \bar{c}_2 \frac{1}{\zeta^2} + \ldots$$

From equation (IV)

$$\vec{M}_0 = -\frac{dM}{dt} + \text{Real}\ [i(U_0 - iV_0)\ B]$$

whence

$$\vec{M}_0 = -\left[\lambda_x \frac{dU_0}{dt} + \lambda_{y\omega} \frac{dV_0}{dt} + \lambda_\omega \frac{d\omega}{dt} + \lambda_{xy}(U_0{}^2 - V_0{}^2) + \right.$$

$$\left. + (\lambda_y - \lambda_x)\ U_0V_0 + (\lambda_{y\omega}U_0 - \lambda_{x\omega}V_0)\ \omega\right] \qquad (18)$$

We recall that in formulas (16), (17), (18), U_0 and V_0 are the projections of the velocity of the origin of coordinates on the moving axes.

With the aid of the formulas established in this section and the potential functions obtained in section 5 for the potential flows in the motion of an elliptic cylinder and Joukowsky wing in an infinite fluid, it is not difficult to write out the values of the coefficients of the virtual masses for the above profiles.

Choosing the axes of coordinates along the principal axes of the elliptic cylinder we have

$$z = f(\zeta) = -\frac{1}{2}\ (a-b)\ \ \zeta - \frac{1}{2}\ (a+b)\ \frac{1}{\zeta}$$

$$w_3(\zeta) = -\frac{i}{4}\ (a^2-b^2)\ \zeta^2$$

whence

$$k = -\frac{1}{2}\ (a+b); \quad k_1 = -\frac{1}{2}\ (a-b); \quad c_1 = 0$$

Moreover we have

$$S = \pi ab \text{ and } x^* = y^* = 0.$$

Making use of these relations we obtain from equations (10) to (14)

$$\lambda_x = \rho\pi b^2; \ \lambda_y = \rho\pi a^2; \ \lambda_{xy} = \lambda_{x\omega} = \lambda_{y\omega} = 0$$

From equation (15)

$$\lambda_\omega = -\frac{\rho}{8} \oint_K (a^2-b^2) \frac{1}{\zeta^2} \left[-\frac{1}{2}(a^2-b^2) \right] \zeta \, d\zeta = \frac{\rho\pi}{8}(a^2-b^2)$$

The center of the ellipse is the central point.

Applying formulas (16), (17), and (18) for the projections of the force and moment we obtain the following values

$$\left.\begin{aligned} X_0 &= -\rho\pi b^2 \frac{dU_0}{dt} + \rho\pi a^2 \omega V_0 \\ Y_0 &= -\rho\pi a^2 \frac{dV_0}{dt} - \rho\pi b^2 \omega U_0 \\ \vec{M}_0 &= -\frac{\rho\pi(a^2-b^2)^2}{8} \frac{d\omega}{dt} - \rho\pi(a^2-b^2) U_0 V_0 \end{aligned}\right\} \quad (19)$$

For $b = 0$ the ellipse degenerates into a plate for which

$$\lambda_x = 0; \ \lambda_y = \rho\pi a^2; \ \lambda_{x\omega} = \lambda_{xy} = \lambda_{y\omega} = 0; \ \lambda_\omega = \frac{\rho\pi a^4}{8}$$

$$X_0 = \rho\pi a^2 \omega V_0; \ Y_0 = -\rho\pi a^2 \frac{dV_0}{dt} \quad (20)$$

$$\vec{M}_0 = -\frac{\rho\pi a^4}{8} \frac{d\omega}{dt} - \rho\pi a^2 U_0 V_0$$

With the aid of similar but more laborious computations, on the basis of the results obtained in section 5 and the formulas of the present section, we computed the values of the coefficients of virtual masses for the Joukowsky profiles. The results are given in table I. Examining this table we find that the concavity characterized by the angle α plays a more important part than the thickness which is characterized by the parameter $\epsilon = R \cos \alpha - 1$. A very thin, slightly curved profile is in its hydrodynamic properties similar to an arc of a circle.

The coordinates of the central point of a symmetrical Joukowsky profile with respect to the origin coinciding with the sharp edge, if the x-axis is directed along the axis of symmetry, is determined by the formulas

$$\xi^* = \frac{al}{4}\left(1 + \frac{l^2}{l^2 - l + 2}\right); \quad \eta^* = 0$$

For an arc of a circle the central point is the center of the circle of which the arc is a part.

7. MOTION WITH CONSTANT CIRCULATION

We shall study the special case where the motion of the fluid outside the wing is potential, the velocity is equal to zero at infinity and the circulation about the wing is different from zero, $\Gamma \neq 0$. Assuming that the motion of the fluid is continuous throughout we may consider the circulation Γ as constant in time. For, the pressure from its physical meaning is a single-valued function of the points of the fluid, but only for $\Gamma =$ const. is the term $\partial\varphi/\partial t$, in the Lagrange formula for the pressure, a single-valued function.

The problem of determining the hydrodynamic forces for constant circulation was proposed by S. Chaplygin and the fundamental results were obtained by him (reference 11). In the case under consideration the characteristic stream function of the fluid near an infinitely distant point is of the form

$$w(z) = \frac{\Gamma}{2\pi i} \ln z + c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \ldots \tag{1}$$

whence

$$\frac{dw}{dz} = \frac{\Gamma}{2\pi i}\frac{1}{z} - \frac{c_1}{z^2} - \frac{2c_2}{z^3} - \ldots \tag{2}$$

and

$$\left(\frac{dw}{dz}\right)^2 = -\frac{\Gamma^2}{4\pi^2}\frac{1}{z^2} - \frac{\Gamma c_1}{\pi i z^3} + \ldots \tag{3}$$

On substituting ζ for z where ζ as before is a variable considered within the unit circle K, we shall have

$$w(\zeta) = w_0(\zeta) - \frac{\Gamma}{2\pi i}\ln\zeta \tag{4}$$

where $w_0(\zeta)$ is the characteristic function of the irrotational flow. The term $-\frac{\Gamma}{2\pi i}\ln\zeta$ is added because of the presence of circulation about the wing. Evidently on the boundary for $|\zeta| = 1$ $\text{Imag}\,\frac{\Gamma}{2\pi i}\ln\zeta = 0$. Thus the function $w(\zeta)$ satisfies the required boundary conditions and gives the flow with circulation Γ about the wing.

We shall now investigate the forces. Everywhere outside the wing dw/dz is holomorphic and near an infinitely distant point the expansions (2) and (3) are valid, hence evidently

$$\frac{i}{2}\oint\left(\frac{dw}{dz}\right)^2 dz = 0; \quad \text{Real}\oint z\left(\frac{dw}{dz}\right)^2 dz = 0$$

Further, we have

$$Q = i\oint_C z\frac{dw}{dz}dz = Q_0 - i\oint_K\left[\frac{k}{\zeta}+k_0+k_1\zeta+\ldots\right]\frac{\Gamma}{2\pi i}\frac{1}{\zeta}d\zeta = Q_0 + i\Gamma k_0$$

and

$$N = \frac{\rho}{2}\oint z\bar{z}\frac{dw}{dz}dz = N_0 - \frac{\rho\Gamma}{4\pi i}\oint_K f(\zeta)\,\bar{f}\left(\frac{1}{\zeta}\right)\frac{d\zeta}{\zeta} \tag{5}$$

whereas before

$$Q_0 = i\oint z\frac{dw_0}{dz}dz; \quad N_0 = \frac{\rho}{2}\oint z\bar{z}\frac{dw_0}{dz}dz$$

Remembering that $\frac{\delta i\Gamma k_0}{\delta t} = -\omega\Gamma k_0$ and that $\partial N/\partial t = \partial N_0/\partial t$ since the second term on the right in (5) does not depend on the time we obtain by equations (I) and (II), section 3, and (III) and (IV), section 6

$$X + iY = X_0 = iY_0 + i\rho\Gamma\ (q_0 + i\omega k_0) \qquad (V)$$

$$\vec{M} = \vec{M}_0 + \rho\Gamma\ \text{Real}\ \overline{q}_0 k_0 \qquad (VI)$$

where $X_0 + iY_0$ and $\vec{M}_0$ denote respectively the force and moment which would act on the wing if for the same motion the circulation were equal to zero. Evidently $q_0 + i\omega k_0$ is the velocity vector of the point $z = k_0$ as it moves with the wing. This point, as is readily seen, is invariably fixed to the wing and therefore its position on the wing does not depend on the choice of the system of axes. Setting $q_0 + i\omega k_0 = q_k$ we may write formula (V) in the following form:

$$X + iY = X_0 + iY_0 + i\rho\Gamma q_k \qquad (Va)$$

The above equation is a generalization of the Joukowsky theorem for the arbitrary motion of a wing in a fluid with rotational potential motion. Formula (VI) is equivalent to the following:

$$\vec{M} = \vec{M}_0 + \text{Real}\ ik_0\ (-\ i\rho\Gamma\overline{q}_k) \qquad (VIa)$$

The second term in the above equation represents the moment of the Joukowsky force $i\rho\Gamma q_k$ applied at the point $z = k_0$.

The point $z = k_0$ possesses very remarkable dynamic properties which are evident from the following conclusions drawn from formulas (Va) and (VIa):

I. If the wing moves in an infinite fluid with constant circulation the total hydrodynamic force consists of the force which would exist in the absence of circulation and of the Joukowsky force $i\rho\Gamma q_k$ where q_k is the velocity of the point $z = k_0$.

II. The total moment of the hydrodynamic forces about the point $z = k_0$ for motion of a wing in an infinite fluid with constant circulation does not depend on the value of the circulation.

III. The impulse of the Joukowsky force is equal to $i\rho\Gamma s$, where s is the displacement vector of the point $z = k_0$ in the stationary space.

From the above we obtain directly the following results:

1. The mean value of the Joukowsky force for any interval of time $t - t_0$ is equal to $i\rho\Gamma q_k{}^*$ where $q^* = s/(t - t_0)$ is the mean velocity vector of the point $z = k_0$.

2. If after a certain interval of time the point $z = k_0$ returns to its initial position the mean value of the Joukowsky force for this interval of time is equal to zero.

In the preceding section we have found that in the periodic motion of the wing the mean value of the forces over a period in the absence of circulation is equal to zero. Taking this into account we obtain on the basis of result (1) the theorem of Chaplygin: If the motion of a wing is a periodic oscillation associated with translational motion with constant velocity q_0 then for constant circulation the mean value over a period of the hydrodynamic forces is equal to the Joukowsky force $i\rho\Gamma q_0$ corresponding to the translational motion. This theorem may evidently be given a more general formulation, namely: If at the instant t_0 and t_1 the translational and angular velocity and the orientation of the wing are the same, the mean value of the hydrodynamic forces for constant circulation in the time $t - t_0$ is equal to $i\rho\Gamma q_k{}^*$ where $q_k{}^*$ is the mean velocity of the point $z = k_0$.

As an example we shall consider the motion of a Joukowsky wing with constant circulation. According to the rule of Joukowsky, the circulation about a wing will be determined from the condition of finiteness of the velocity at the trailing edge.

In Section 5 we have seen that the requirement of finite velocity is expressed by the condition

$$\left(\frac{dw}{d\zeta}\right)_{\zeta=1} = 0$$

(where $\zeta = 1$ is the point corresponding to the sharp edge on the wing) which according to (4) may also be written thus:

$$\left(\frac{dw_0}{d\zeta}\right)_{\zeta=1} - \frac{\Gamma}{2\pi i} = 0 \tag{6}$$

Setting in the expression for dw_0/dz (formula 11, section 5) $\zeta = 1$ we obtain

$$iaRV_0 - \frac{ia^2R^2}{4}\frac{(\mu+\bar{\mu}-2)}{(1-\mu\bar{\mu})}\omega + \frac{i\Gamma}{2\pi} = 0$$

whence

$$\Gamma = -2\pi aR\left(V_0 + \frac{a}{4}(R+r)\omega\right) \tag{7}$$

If $\Gamma = 0$, relation (7) becomes condition (14a) section 5. We note that the motion along the moving x-axis is in no way constrained by the condition of finite velocity at the trailing edge.

We shall now explain the kinematic character of the possible motions of the wing for which condition (7) for Γ = const is satisfied. Condition (7) expresses the requirement of the constancy of the projection of the velocity of the points of the wing on the moving straight line PP' perpendicular to the first axis and at a distance $a(R+r)/4$ from the sharp edge.

If the angular velocity of the wing ω is constant (7) is satisfied if the instantaneous center of rotation lies on the straight line QQ' (fig. 8) parallel to the straight line PP' at the distance $\frac{|\Gamma|}{2\pi aR\omega}$. Thus for Γ = const; ω = const only such motion is physically possible for which the moving polhode is a straight line perpendicular to the first axis. If the equation of this straight line is $x = h$ we may give h any previously assigned value, choosing the corresponding circulation by the formula

$$\Gamma = \left[h - \frac{a}{4}(R+r)\right] 2\pi aR\omega \tag{8}$$

In particular a rotation of the wing is possible with constant angular velocity about any stationary center in the z-plane.

We shall now consider condition (7) assuming that the point A (fig. 9) belonging to the wing and having in the moving system the coordinates ξ and η moves in stationary space with velocity $\vec{V}$ constant in direction. Denoting by β the angle between the velocity vector $\vec{V}$ and the x-axis we shall have

$$V_0 = - V \sin \beta - \omega \xi ; \quad \frac{d\beta}{dt} = \omega$$

Condition (7) now assumes the following form:

$$\Gamma = 2\pi a R \left\{ V \sin \beta + \left[\xi - \frac{a}{4} (R + r) \right] \omega \right\}$$

whence if $\xi \neq a (R + r)/4$ we obtain for the angular velocity

$$\omega = \frac{1}{\xi - \frac{a}{4} (R + r)} \left[\frac{\Gamma}{2\pi a R} - V \sin \beta \right] \tag{9}$$

Setting

$$\frac{\Gamma}{2\pi a R \left[\xi - \frac{a}{4} (R + r) \right]} = \omega_0 \quad \text{and} \quad \frac{V \sin \beta}{\xi - \frac{a}{4} (R + r)} = \omega'$$

we have

$$\omega = \omega_0 - \omega'$$

For a thin, slightly curved Joukowsky profile relations (7), (8), and (9) assume the form

$$\Gamma = - 2\pi a(1+\epsilon) \left[V_0 + \frac{a\omega}{2} \right] \tag{7a}$$

$$\Gamma = \pi a(1+\epsilon) [2h-a] \omega \tag{8a}$$

$$\omega = \frac{2}{2\xi - a} \left[\frac{\Gamma}{2\pi a(1+\epsilon)} - V \sin \beta \right] \tag{9a}$$

For a plate $\epsilon = 0$.

The position of the point $z = k_0$ for the Joukowsky profile, on the basis of the transformations (4) section 5, is determined by the formula

$$k_0 = \frac{aR}{2} (2 - \mu) = \frac{a}{2} (R + e^{-i\alpha}) \tag{10}$$

If α and $\epsilon = R \cos \alpha - 1$ are small then

$$k_0 = \frac{a}{2} (2 + \epsilon - i\alpha) \tag{10_1}$$

For the arc of a circle the point $z = k_0$ lies at the center of AB (fig. 10); for the plate the point $z = k_0$ coincides with the center.

The force and moment in the motion of a thin and slightly curved Joukowsky wing moving with constant circulation are expressed by the equations

$$X = - \rho\pi a^2 \left[\left(\frac{dV_0}{dt} + \frac{5}{4} a \frac{d\omega}{dt}\right) \alpha - \omega (\alpha U_0 + V_0 + a\omega)\right] - \rho\Gamma V_k$$

$$Y = - \rho\pi a^2 \left[\alpha \frac{dU_0}{dt} + \frac{dV_0}{dt} + a \frac{d\omega}{dt} + \omega\alpha \left(V_0 + \frac{5}{4} a\omega\right)\right] + \rho\Gamma U_k$$

where U_k, V_k are the projections of the velocity of the point $z = k_0$ on the moving axes

$$U_k = U_0 + \frac{a\alpha}{2} \omega; \quad V_k = V_0 + \frac{a}{2} (2 + \epsilon)\omega$$

$$\vec{M} = - \rho\pi a^2 \left[\frac{5}{4} a\alpha \frac{dU_0}{dt} + a \frac{dV_0}{dt} + \frac{9}{8} a^2 \frac{d\omega}{dt} + \alpha (U_0{}^2 - V_0{}^2) + \right.$$

$$\left. + \left(aU_0 - \frac{5}{4} a\alpha V_0\right) \omega + U_0V_0\right] + \rho a\Gamma \left[U_0 + \frac{1}{2} \epsilon U_0 - \frac{1}{2} \alpha V_0\right]$$

In particular for a plate with the origin of coordinates at the center we obtain

$$X = \rho\pi a^2 V_0\omega - \rho\Gamma V_0$$

$$Y = -\rho\pi a^2 \frac{dV_0}{dt} + \rho\Gamma U_0$$

$$\vec{M} = -\frac{\rho\pi a^4}{8}\frac{d\omega}{dt} - \rho\pi a^2 U_0 V_0$$

8. FORCES ACTING ON THE WING IN THE PRESENCE OF A SYSTEM OF POINT VORTICES WITHIN THE FLUID

We shall consider first the case where there is only a single vortex within the fluid. Let a be the point where the vortex is located at the instant of time under consideration and let $-\Gamma$ be the circulation about it. Evidently the circulation about the wing is equal to Γ. From the fact that the pressure is single-valued it follows that the circulation Γ is constant in time.

If within the fluid there are several isolated vortices the circulation about each vortex is constant in time. The complex potential function is holomorphic everywhere outside the contour C with the exception of the point a which is a logarithmic singularity of $w(z)$. Near the point a the function $w(z)$ is of the form

$$w(z) = -\frac{\Gamma}{2\pi i}\ln(z-a) + c_0 + c_1(z-a) + c_2(z-a)^2 + \ldots \tag{1}$$

$$\frac{dw}{dz} = -\frac{\Gamma}{2\pi i}\frac{1}{z-a} + c_1 + 2c_2(z-a) + \ldots \tag{2}$$

The velocity function dw/dz is single-valued and holomorphic outside the contour C and has a pole of the first order at the point a.

Denoting by v_a the velocity induced at point a we shall have $\bar{\vec{v}}_a = c_1$. If the vortex is free it will travel with velocity $\vec{v}_a$.

From expansion (2) for $(dw/dz)^2$ near the point a we obtain

$$\left(\frac{dw}{dz}\right)^2 = -\frac{\Gamma^2}{4\pi^2}\frac{1}{(z-a)^2} - \frac{\Gamma\vec{v}_a}{\pi i(z-a)} + \vec{v}_a^{\,2} + \ldots \tag{3}$$

Since the velocity at infinity and the circulation over an infinitely removed contour are equal to zero hence near the infinitely removed point, the expansion holds:

$$w(z) = c_0' + \frac{c_1'}{z} + \frac{c_2'}{z^2} \ldots$$

whence

$$\frac{dw}{dz} = -\frac{c_1'}{z^2} - \frac{2c_2'}{z^3} - \ldots$$

and

$$\left(\frac{dw}{dz}\right)^2 = \frac{c_1'^2}{z^4} + \frac{4c_1'\ c_2'}{z^5} + \ldots \tag{4}$$

We shall compute, taking account of the behavior of $w(z)$ outside the contour C, the integral

$$P = \frac{i}{2}\oint_C \left(\frac{dw}{dz}\right)^2 dz$$

In place of the contour C we may take for the integration the contour L and the circle L_1. The direction of integration is indicated in figure 11. The contour L encloses the wing and the vortex and all its points may be taken as far as we please from the origin of coordinates; L_1 is a circle of small radius with center at point a.

From the expansion (4) for $(dw/dz)^2$ about an infinitely removed point it is evident that

$$\frac{i}{2}\oint_L \left(\frac{dw}{dz}\right)^2 dz = 0$$

The integral over the contour L_1 is computed by replacing $(dw/dz)^2$ by its expansion (3). We obtain

$$P = i\vec{v}_a\Gamma \tag{5}$$

To compute the integral denoted in section 7 by Q, we represent the complex potential function $w = \varphi + i\psi$ in the form

$$w(z) = w_0(z) + w_1(z)$$

where

$$w_0 = \varphi_0 + i\psi_0; \quad w_1 = \varphi_1 + i\psi_1$$

$w_0(z)$ is the complex potential function of the motion of an infinite fluid in the absence of any singularity within the fluid. The conditions and methods for determining $w_0(z)$ were explained in section 4. $w_1(z)$ is the complex potential function of the added motion of the fluid outside the wing due to the presence of a point vortex at the point a with circulation $-\Gamma$. $w_1(z)$ is determined from the conditions

1. Along C $\psi_1 = 0$

2. Near the point a

$$w_1(z) = -\frac{\Gamma}{2\pi i}\ln(z-a) + P(z-a)$$

and therefore

$$\frac{dw_1}{dz} = -\frac{\Gamma}{2\pi i}\frac{1}{z-a} + P'(z-a) \tag{6}$$

3. At infinity $w_1(z)$ is holomorphic

The function $P(z - a)$ is holomorphic everywhere outside the contour C. Transforming the outer region of the contour C into the interior of a unit circle in the ζ-plane and substituting ζ for z in $w_1(z)$ we obtain

$$w_1(\zeta) = \frac{\Gamma}{2\pi i}\left[\ln\left(\zeta - \frac{1}{\bar{b}}\right) - \ln(\zeta - b)\right] + \text{const} \tag{7}$$

where b denotes the transformed point a in the ζ-plane. If the transformation is such that $z = \infty$ corresponds to $\zeta = 0$ then near $z = \infty$, $\zeta(z)$ will be of the form

$$\zeta = \frac{k}{z} + \frac{d_1}{z^2} + \frac{d_2}{z^3} + \ldots .$$

Making use of these expansions and (7) we obtain the expansion of $w_1(z)$ near an infinitely removed point. Neglecting the constant terms we obtain

$$w_1(z) = \frac{\Gamma}{2\pi i} k\left(\frac{1}{b} - \bar{b}\right)\frac{1}{z} + \ldots$$

and

$$\frac{dw_1}{dz} = -\frac{\Gamma}{2\pi i} k\left(\frac{1}{b} - \bar{b}\right)\frac{1}{z^2} + \ldots \tag{8}$$

Replacing w by $w_0 + w_1$ we may write

$$Q = i\oint_C z\frac{dw}{dz} = Q_0 + i\int_{L+L_1} z\frac{dw_1}{dz}\,dz$$

Computing the integrals $i\oint_L z\frac{dw}{dz}\,dz$ and $i\oint_{L_1} z\frac{dw_1}{dz}\,dz$ with the aid of expansions (6) and (8) we obtain

$$Q = Q_0 + ik\left(\bar{b} - \frac{1}{b}\right)\Gamma + ia\Gamma \tag{9}$$

For the hydrodynamic force formulas (I) and (III) with (5) and (9) gives

$$X + iY = \frac{\delta I_0}{\delta t} + i\rho q_0\Gamma + i\rho\frac{\delta a}{\delta t}\Gamma - i\rho\vec{v}_a\Gamma + i\rho\frac{\delta}{\delta t} k\left(\bar{b} - \frac{1}{b}\right)\Gamma$$

It is readily seen that $q_0 + \delta a/\delta t$ is the absolute velocity of the vortex. It is evident also that $q_0 + \delta a/\delta t - \vec{v}_a$ is the velocity of the vortex relative to the fluid which we shall denote by $\vec{v}_{rel}$. If the vortices are free $\vec{v}_{rel} = 0$. Setting $i\rho k\,(\bar{b}-1/b) = I_1$ we may write

$$X + iY = X_0 + iY_0 + \frac{\delta I_1}{\delta t} + i\rho\,\vec{v}_{rel}\Gamma \qquad \text{(VII)}$$

We note that $-I_1$ may be considered as the momentum of the fluid induced by the vortex in the presence of the wing. The concentrated force causing the vortex to move with a given velocity for unsteady motion (reference 17) is equal to $i\rho\,\vec{v}_{rel}\Gamma$. If the vortices are free

$$X + iY = \frac{\delta(I_0 + I_1)}{\delta t} \qquad \text{(VIIa)}$$

If the vortices are invariably fixed to the wing and the wing is in translational motion, $b = \text{const}$, $\omega = 0$, and therefore $\delta I_1/\delta t = 0$.

For the acting force we obtain the formula

$$X + iY = X_0 + iY_0 + i\rho\,\vec{v}_{rel}\Gamma \qquad \text{(VIIb)}$$

The forces, if the vortex is fixed to the wing, are computed by the Joukowsky formula if the corresponding values for $\vec{v}_{rel}$ are taken. The reaction force of the fluid on the vortex is equal to $-i\rho\vec{v}_{rel}\Gamma$ and therefore in this case the total reaction of the fluid on the vortex and on the wings is equal to zero.

Thus if the vortex idealizes a small wing invariably fixed to the given wing the total hydrodynamic force acting on the system consisting of the wing and the small wing reduces to $X_0 + iY_0$. Such an infinitely small wing has no effect whatever on the general hydrodynamic force. This result is of course valid only in the case where there are no singularities in the flow except the vortex under consideration. In particular the circulation over the contour enclosing the wing and vortex in the case under consideration is equal to zero.

We shall now suppose that we have a system of n vortices at the points a_k with the circulation $-\Gamma_k$ $(k = 1,2,3. . .n)$. Evidently the circulation about the wing is $\Gamma = \Sigma\Gamma_k$. In this case P is represented as the sum of the residues about the poles of the integrated function which will be the points a_k.

$$P = i\sum_{k=1}^{n} \vec{\bar{v}}_{ak}\, \Gamma_k = i\vec{\bar{v}}^*\Gamma \tag{5a}$$

where $\vec{v}^*$ denotes the velocity of the motion of the center of gravity of the system of free vortices.

Decomposing w into the sum: $w_0 + w_1 + w_2 + . . . + w_n$ where w_k corresponds to the motion of the fluid about a stationary wing in the presence of a vortex at point a_k with the circulation $-\Gamma_k$ and proceeding in the same manner as for the case of a single vortex we obtain

$$Q = Q_0 + i\sum_{s=1}^{n} a_s \Gamma_s + i\sum_{s=1}^{n} k\left(\bar{b}_s - \frac{1}{b_s}\right)\Gamma_s = Q_0 + ia^*\Gamma + \frac{1}{\rho} I$$

where b_s is the image of a_s in the ζ-plane, a^* is the coordinate of the center of gravity of the vortices and I denotes the expression $i\rho\sum_{s=1}^{n} k\left(\bar{b}_s - \frac{1}{\bar{b}_s}\right)\Gamma_s$ and $-I$ is the momentum of the fluid induced by the system of vortices in the presence of a stationary wing. In place of formula (VII) we obtain in the given case the analogous formula

$$X + iY = \frac{\delta(I_0 + I)}{\delta t} + i\rho\, \vec{v}^*_{rel}\Gamma \tag{VII'}$$

where

$$\vec{v}^*_{rel}\Gamma = \sum_{s=1}^{n} \left(q_0 + \frac{\delta a_s}{\delta t} - \vec{v}a_s \right) \Gamma_s$$

If $\Gamma = 0$ the concept of center of gravity becomes meaningless and in formula (VII') and in the previous relations the summation signs must be retained.

If the wing is in translational motion $(\omega = 0)$ with constant velocity and the motion of the fluid with respect to the wing is steady, and the vortices within the fluid are free, then it is evident that $b_k = \text{const}$ and therefore

$$X + iY = 0$$

This is a generalization of the paradox of D'Alembert to the case where the motion is steady and within the fluid there is an arbitrary system of free vortices at a finite distance.

If the motion of the wing is translational as before and the motion of the fluid is steady but the vortices are bound then the force acting on the wing is

$$X + iY = i\rho \vec{v}^*_{rel} \Gamma$$

The force due to these vortices is likewise equal to $i\rho\vec{v}^*_{rel}\Gamma$ and therefore this force is entirely transmitted by the fluid to the wing. The latter case has been considered by Lagally (reference 18).

Formula (VII') permits determining the hydrodynamic forces if the positions and velocities of the vortices are known. If the vortices are free or invariably fixed to the wing it is sufficient for determining the forces to give only the positions of the vortices, because in these two cases the absolute and relative velocities of the vortices are completely determined by the positions of the vortices.

The problem of determining the positions of the free vortices as a function of the time in any particular case, is generally speaking, very difficult. It is sometimes possible to estimate $a_s(t)$ and therefore also $b_s(t)$ approximately from various physical considerations.

Let us consider the limiting case where the system of free vortices is very far removed from the wing $|a_s| \to \infty$.

We have

$$a_s = \frac{k}{b_s} + k_0 + k_1 b_s + k_2 b_s^2 + \ldots; \quad b_s = \frac{k}{a_s} + \frac{d_1}{a_s^2} + \ldots$$

Therefore

$$\sum_{s=1}^{n} k\left(\bar{b}_s - \frac{1}{\bar{b}_s}\right)\Gamma_s = -\sum_{s=1}^{n}(a_s - k_0)\,\Gamma_s +$$

$$+ 0\left(\frac{1}{a_s}\right) = -(a^* - k_0)\,\Gamma + 0\left(\frac{1}{a_s}\right)$$

where a^* is the center of gravity of the system of vortices. The expression $0(1/a_s)$ approaches zero together with $1/a_s$ for $|a_s| \to \infty$. With the aid of formula (VII') we obtain for the force

$$X + iY = X_0 + iY_0 - i\rho\,\frac{\delta(a^* - k_0)}{\delta t}\,\Gamma + 0\left(\frac{1}{a_s}\right)$$

Evidently

$$\frac{\delta(a^* - k_0)}{\delta t} = \vec{v}^* - q_k$$

where the right side represents the velocity of the center of gravity of the system of vortices with respect to the point $z = k_0$. For $|a_s| \to \infty$ keeping only the principal terms we obtain

$$X + iY = X_0 + iY_0 + i\rho(q_k - \vec{v}^*)\,\Gamma \tag{10}$$

Thus the additional force acting on the wing due to the presence of a system of free vortices very far removed from the wing is computed by the Joukowsky formula.

If $\vec{v}^* = 0$ we obtain a stationary vortex at infinity and we shall then have the force found in section 7. The result contained in equation (10) applicable to the translational motion of the wing was obtained by Wagner from other considerations (reference 19).

We shall now turn to the computation of the moment of the hydrodynamic forces acting on the wing in the presence of a vortex at point a. We consider the integral

$$R = \frac{1}{2} \oint_C z \left(\frac{dw}{dz}\right)^2 dz = \frac{1}{2} \oint_L + \frac{1}{2} \oint_{L_1}$$

This integral taken over L is equal to zero since $(dw/dz)^2$ is of the order $1/z^4$ at infinity. Further, it is evident that

$$\frac{1}{2} \oint_{L_1} z\left(\frac{dw}{dz}\right)^2 = \frac{1}{2} \oint_{L_1} (z - a) \left(\frac{dw}{dz}\right)^2 dz + \frac{a}{2} \oint_{L_1} \left(\frac{dw}{dz}\right)^2 dz$$

From the expansion (3) for $(dw/dz)^2$ near the point $z = a$ it follows that

$$\text{Real } R = \text{Real } [-ia\ P] = \text{Real } a\ \vec{\bar{v}}_a \Gamma$$

since the real part of the first integral is equal to zero. Taking into account (7) and (9) we obtain with the aid of formulas (II) and (IV)

$$\vec{M} = \vec{M}_0 + \rho \text{ Real} \left\{ \overline{q_0} \left[k\left(\bar{b} - \frac{1}{b}\right) + a \right] \Gamma - a\vec{\bar{v}}_a \Gamma \right\} + \frac{d\Gamma}{dt} \qquad \text{(VIII)}$$

where

$$L = \frac{\Gamma}{2\pi i} \oint_K f(\zeta)\ \bar{f}\left(\frac{1}{\zeta}\right) \left(\frac{1}{\zeta - \frac{1}{\bar{b}}} - \frac{1}{\zeta - b}\right) d\zeta$$

The sign Real before dL/dt is omitted since L is always real.

For $\frac{\Gamma}{2\pi i}\left(\frac{1}{\zeta - \frac{1}{\bar{\bar{b}}}} - \frac{1}{\zeta - b}\right) d\zeta = dw_1$ but along the circle $K\psi_1 = 0$ hence $dw_1 = d\varphi_1$.

If there is a system of vortices within the fluid then

$$\vec{M} = \vec{M}_0 + \rho \sum_s \left\{ \text{Real}\left[k\overline{q_0}\left(\bar{b}_s - \frac{1}{\bar{b}_s}\right) + a_s(\bar{q}_0 - \bar{\vec{v}}_{as})\right]\Gamma_s + \frac{\partial L_s}{dt}\right\} \qquad \text{(VIIIa)}$$

Finally in the general case where we have a system of vortices and a circulation over an infinitely removed contour $\Gamma_0 \neq 0$ we shall have for the force and moment the equations

$$X + iY = X_0 + iY_0 + i\rho \sum_s \left\{\frac{\delta}{\delta t}\left[k\left(\bar{b}_s - \frac{1}{\bar{b}_s}\right) + \vec{v}_{s\ rel}\right]\Gamma_s + \right.$$

$$\left. + i\rho\, q_k \Gamma_0 \right\} \qquad \text{(VIIb)}$$

$$\vec{M} = \vec{M}_0 + \rho\ \text{Real}\left\{\sum_s\left[k\overline{q_0}\left(\bar{b}_s - \frac{1}{\bar{b}_s}\right) + a_s\ (\bar{q}_0 - \bar{\vec{v}}_{as})\right]\Gamma_s + k_0\bar{q}_0\Gamma_0\right\} +$$

$$+ \frac{d}{dt}\sum_s \frac{dL_s}{dt} \qquad \text{(VIIIb)}$$

As we have seen, in the formulas determining the hydrodynamic forces there enters the vector $\vec{v}_a$ representing the velocity induced at the point where the vortex is situated. We shall now give a rule for computing $\vec{v}_a$ if the complex potential function $w(\zeta)$ is known, and $z = f(\zeta)$ where $f(\zeta)$ is a function holomorphic near the point $\zeta = b$, the image of the point a at which the vortex is situated.

By the expansion (2)

$$\bar{\vec{v}}_a = c_1 = \left(\frac{dw}{dz} + \frac{\Gamma}{2\pi i}\frac{1}{z-a}\right)_{z=a} \qquad (\alpha)$$

Near the point b, $w(\zeta)$ is of the form

$$w(\zeta) = \Phi(\zeta - b) - \frac{\Gamma}{2\pi i} \ln (\zeta - b) \tag{β}$$

where $\Phi(\zeta - b)$ is holomorphic near the point b. Moreover we have

$$(z - a) = z_0' \ (\zeta - b) + \frac{1}{2} z_0'' \ (\zeta - b)^2 + \ . \ . \ . \tag{γ}$$

where

$$z_0' = \left(\frac{dz}{d\zeta}\right)_{\zeta=b}; \quad z_0'' = \left(\frac{d^2 z}{d\zeta^2}\right)_{\zeta=b}$$

Combining (α), (β), (γ) we may write

$$\vec{\bar{v}}_a = \left[\Phi'(0) \frac{1}{z'} - \frac{\Gamma}{2\pi i} \frac{1}{\zeta - b} \frac{1}{z'} + \frac{\Gamma}{2\pi i} \frac{1}{z_0'(\zeta - b) + \frac{1}{2} z_0'' \ (\zeta - b)^2 + \ . \ . \ .}\right]_{\zeta=b}$$

or

$$\vec{\bar{v}}_a = \left(\Phi'(0) - \frac{\Gamma}{4\pi i} \frac{z_0''}{z_0'}\right) \frac{1}{z_0'} - \frac{\Gamma}{2\pi i} \left[\frac{1}{\zeta - b} \left(\frac{1}{z'} - \frac{1}{z_0'}\right)\right]_{\zeta=b}$$

Noting also that

$$\frac{1}{z'} - \frac{1}{z_0''} = - \frac{z_0''}{z_0'} (\zeta - b) + \ . \ . \ .$$

we obtain finally

$$\vec{\bar{v}}_a = \left(\Phi' \ (0) + \frac{\Gamma}{4\pi i} \frac{z_0''}{z_0'}\right) \frac{1}{z_0'} \tag{11}$$

We recall that the circulation about the vortex is $-\Gamma$.

As an example we shall determine the hydrodynamic forces acting on a plate when there is a vortex in the fluid. For simplicity we assume that the point A at which the vortex is situated lies on the x-axis coinciding with the direction of the plate (fig. 12).

With the aid of the function

$$z = f(\zeta) = -\frac{a}{2}\left(\zeta + \frac{1}{\zeta}\right); \quad \zeta = -\frac{1}{a}\left(z + \sqrt{z^2 - a^2}\right)$$

the external region of the plate is transformed conformally into the interior of the unit circle K in the ζ-plane. Making use of the expression for $w_0(\zeta)$ found in section 5, we obtain the complex potential function in the presence of a vortex in the form

$$w(\zeta) = V_0 ia\zeta - \frac{\omega ia^2}{4}\zeta^2 + \frac{\Gamma}{2\pi i}\ln\left(\zeta - \frac{1}{\bar{\zeta}_0}\right) - \frac{\Gamma}{2\pi i}\ln(\zeta - \zeta_0)$$

where ζ_0 is the image of the point z_0 at which the vortex is situated. Remembering that $z_0 = x_0$ is real we obtain by formula (11) for the velocity of the fluid at point A

$$\overline{\vec{v}}_{x_0} = -i\left[V_0 + \frac{V_0 x_0}{\sqrt{x_0^2 - a^2}} + \frac{\omega}{2}\frac{(x_0 + \sqrt{x_0^2 - a^2})^2}{\sqrt{x_0^2 - a^2}}\right] + \frac{\Gamma}{2\pi i}\frac{x_0}{x_0^2 - a^2} \qquad (12)$$

For the force by formula (VII), after passing to the variables in the z-plane, we may write

$$X + iY = X_0 + iY_0 + \frac{i\rho}{2}\left[\left(\frac{\overline{dz_0}}{dt} - \frac{dz_0}{dt}\right) + \frac{x_0}{\sqrt{x_0^2 - a^2}}\left(\frac{\overline{dz_0}}{dt} + \frac{dz_0}{dt}\right)\right]\Gamma -$$

$$-\rho\omega\sqrt{x_0^2 - a^2}\,\Gamma + i\rho\overline{\vec{v}}_{rel}\,\Gamma \qquad (13)$$

where dz_0/dt denotes the velocity of the vortex relative to the plate.

Denoting the absolute velocity of the vortex by $\vec{v}_{ab}$ we shall have

$$\vec{v}_{ab} = \vec{v}_{rel} + \vec{v}_{x_0}; \quad \vec{v}_{ab} = \frac{dz_0}{dt} + q_0 + i\omega x_0$$

In particular:

1. If the vortex is free $\vec{v}_{rel} = 0$; $\vec{v}_{ab} = \vec{v}_{x_0}$; $\frac{dz_0}{dt} = \vec{v}_{x0} - q_0 - i\omega x_0$;

2. If the vortex is bound $dz_0/dt = 0$

$$\vec{v}_{ab} = q_0 + i\omega x_0; \quad \vec{v}_{rel} = q_0 + i\omega x_0 - \vec{v}_{x_0}$$

By formula (12) in the first case:

$$\frac{dz_0}{dt} = - U_0 + \frac{iV_0 x_0}{\sqrt{x_0^2 - a^2}} + \frac{i\omega}{2} \frac{2x_0^2 - a^2}{\sqrt{x_0^2 - a^2}} - \frac{\Gamma}{2\pi i} \frac{x_0}{x_0^2 - a^2}$$

and in the second case:

$$\vec{v}_{rel} = U_0 - \frac{iV_0 x_0}{\sqrt{x_0^2 - a^2}} - \frac{i\omega}{2} \frac{2x_0^2 - a^2}{\sqrt{x_0^2 - a^2}} + \frac{\Gamma}{2\pi i} \frac{x_0}{x_0^2 - a^2}$$

Thus if the vortex is free there is obtained from formula (13)

$$X + iY = X_0 + iY_0 - i\rho \frac{x_0}{\sqrt{x_0^2 - a^2}} (U_0 + iV_0)\, \Gamma + \\ + \frac{\rho\omega a^2 \Gamma}{2\sqrt{x_0^2 - a^2}} + \frac{\rho x_0 \Gamma^2}{2\pi (x_0^2 - a^2)} \qquad (14)$$

For the vortex bound to the plate we obtain

$$X + iY = X_0 + iY_0 + i\rho(U_0+iV_0)\Gamma + \rho \frac{x_0 + \sqrt{x_0^2-a^2}}{\sqrt{x_0^2-a^2}} V_0\Gamma +$$

$$+ \frac{\rho\omega a^2\Gamma}{2\sqrt{x_0^2-a^2}} + \frac{\rho x_0\Gamma^2}{2\pi(x_0^2-a^2)} \qquad (15)$$

From formulas (14) and (15) for $x_0 \rightarrow -\infty$ there is obtained

$$X + iY = X_0 + iY_0 + i\rho(U_0+iV_0)\Gamma$$

Thus in both cases the force depending on the circulation is a Joukowsky force.

If the plate is stationary we obtain from formulas (14) and (15)

$$X = \frac{\rho x_0\Gamma^2}{2\pi(x_0^2-a^2)} = -\frac{\rho}{2\pi}\frac{AO}{AM\cdot AB}\Gamma^2;\ Y = 0$$

that is, the vortex acts on the plate with a force which is the same in both cases. The suction of the tip of the wing on the side of the vortex is greater.

From formulas (14) and (15) it follows that the forces increase without limit when $x_0 \rightarrow -a$. This indicates the impossibility of the shedding of a vortex with finite velocity from the trailing edge. As x_0 approaches $-a$ the forces will be finite if the circulation Γ is small of the order of $\sqrt{|x| - a}$.

In conclusion, we compute the moment of hydrodynamic forces applied at a plate. By formula (VIII) we have

$$\vec{M} = \vec{M}_0 + \rho\,\text{Real}\left\{\bar{q}_0\left(x_0+\sqrt{x_0^2-a^2}\right)\Gamma - x_0\,\vec{\bar{v}}_{x_0}\Gamma\right\} + \frac{dL}{dt}$$

The expression for L in the given case is of the form

$$L = \frac{\Gamma}{4\pi i} \oint_K \frac{a^2}{4} \left(\zeta + \frac{1}{\zeta}\right)^2 \left[\frac{1}{\zeta - \frac{1}{\bar{\zeta}_0}} - \frac{1}{\zeta - \zeta_0}\right] d\zeta = \frac{a^2}{8} \left(\zeta_0^{\,2} + \bar{\zeta}_0^{\,2} + 2\right) \Gamma$$

Passing to variables in the z-plane and differentiating L with respect to t, we find

$$\frac{dL}{dt} = \frac{1}{4} \frac{\left(x_0 + \sqrt{x_0^2 - a^2}\right)^2}{\sqrt{x_0^2 - a^2}} \left(\frac{dz_0}{dt} + \frac{d\bar{z}_0}{dt}\right) \Gamma$$

Thus if the vortex is free

$$\vec{M} = -\frac{\rho\pi a^4}{8} \frac{d\omega}{dt} - \rho\pi a^2 U_0 V_0 - \frac{\rho a^2 U_0}{\sqrt{x_0^2 - a^2}} \Gamma \tag{16}$$

For a vortex fixed invariably to the plate

$$\frac{dL}{dt} = 0$$

and therefore

$$\vec{M} = -\frac{\rho\pi a^4}{8} \frac{d\omega}{dt} - \rho\pi a^2 U_0 V_0 + \rho\left(x_0 + \sqrt{x_0^2 - a^2}\right) U_0 \Gamma \tag{17}$$

From formulas (16) and (17) it is seen that for $x_0 \rightarrow -\infty$ the terms depending on the circulation approach zero. If the plate it stationary then in both cases $\vec{M} = 0$.

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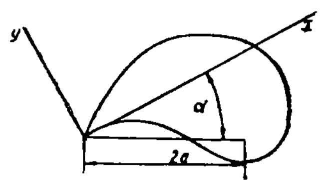
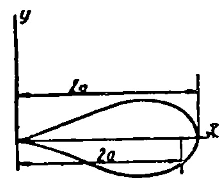
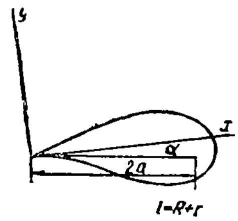
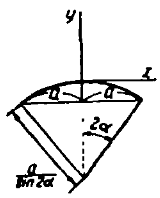
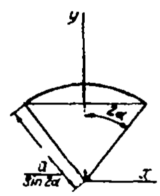
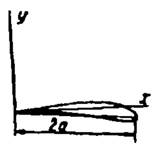
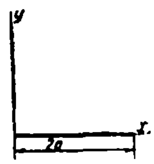
TABLE I - VALUES OF THE VIRTUAL MASS

	Joukowsky profile of arbitrary form	Symmetric profile	Thick, slightly curved profile. Only first order terms in α retained
			
λ_x	$\frac{\rho\pi a^2}{4}(r^2+R^2-2\cos 2\alpha)$	$\frac{\rho\pi a^2}{4}(l-2)(l+1)$	$\frac{\rho\pi a^2}{4}(l-2)(l+1)$
λ_y	$\frac{\rho\pi a^2}{4}(r^2+R^2+2\cos 2\alpha)$	$\frac{\rho\pi a^2}{4}[4+(l-2)(l+1)]$	$\frac{\rho\pi a^2}{4}[4+(l-2)(l+1)]$
λ_{xy}	$\frac{\rho\pi a^2}{2}\sin 2\alpha$	0	$\rho\pi a^2\alpha$
$\lambda_{x\omega}$	$\frac{\rho\pi a^3}{8}\sin\alpha\,[r^2+R^2+4(r+R)\cos\alpha]$		$\frac{\rho\pi a^3}{8}l(l+3)\alpha$
$\lambda_{y\omega}$	$\frac{\rho\pi a^3}{8}[r^3+R^3+(r^2+R^2)\cos\alpha+2(r+R)\cos 2\alpha]$	$\frac{\rho\pi a^3}{16}l(2l^2-l+2)$	$\frac{\rho\pi a^3}{16}l(2l^2-l+2)$
λ_ω	$\frac{\rho\pi a^4}{8}r^2R^2(8r^2R^2\cos^4\alpha-2rR\sin^2 2\alpha+\cos 4\alpha)$	$\frac{\rho\pi a^4}{32}l^2(2l^2+1)$	$\frac{\rho\pi a^4}{32}l^2(2l^2+1)$

COEFFICIENTS FOR JOUKOWSKY PROFILES

Strongly curved thin profile or arc of circle	Various positions of axes of coordinates for arc of circle		Thin, slightly curved profile. Only terms in α retained	Plate
				
$\frac{\rho\pi a^2}{2}\left(\frac{1}{\cos^2\alpha}-\cos 2\alpha\right)$	$\frac{\rho\pi a^2}{2}\,\mathrm{tg}^2\,\alpha$	$\frac{\rho\pi a^2}{2}\,\mathrm{tg}^2\,\alpha$	0	0
$\frac{\rho\pi a^2}{2}\left(\frac{1}{\cos^2\alpha}+\cos 2\alpha\right)$	$\frac{\rho\pi a^2}{2}\left(1+\frac{1}{\cos^2\alpha}\right)$	$\frac{\rho\pi a^2}{2}\left(1+\frac{1}{\cos^2\alpha}\right)$	$\rho\pi a^2$	$\rho\pi a^2$
$\frac{\rho\pi a^2}{2}\sin 2\alpha$		0	$\rho\pi a^2\alpha$	0
$\rho\pi a^3\sin\alpha\left(1+\frac{1}{4\cos^2\alpha}\right)$	$\frac{\rho\pi a^3}{4}\frac{\sin\alpha}{\cos^3\alpha}$	0	$\frac{5}{4}\rho\pi a^3\alpha$	0
$\frac{\rho\pi a^3}{4}\left(4\cos\alpha+\frac{\sin^2\alpha}{\cos^3\alpha}\right)$	0	0	$\rho\pi a^3$	$\rho\pi a^3$
$\rho\pi a^4\left(1+\frac{1}{8\cos^4\alpha}\right)$	$\frac{\rho\pi a^4}{8}\frac{1}{\cos^4\alpha}$	0	$\frac{9}{8}\rho\pi a^4$	$\frac{9}{8}\rho\pi a^4$

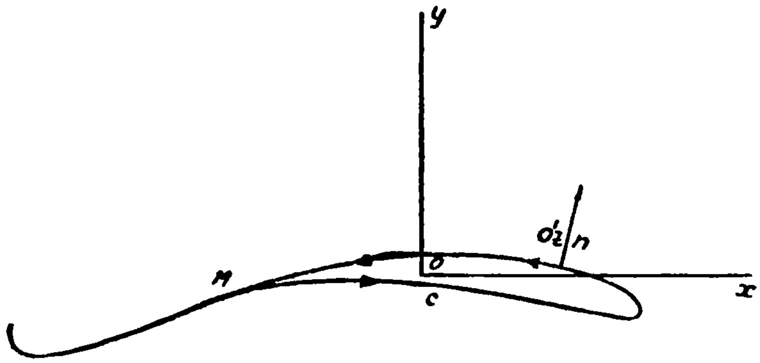


Fig. 1

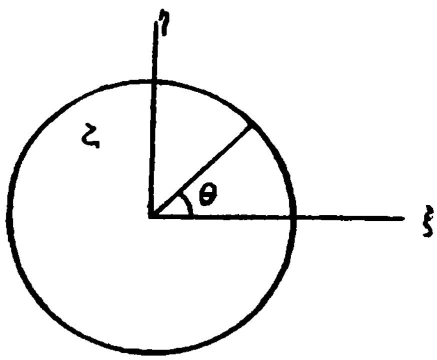


Fig. 2

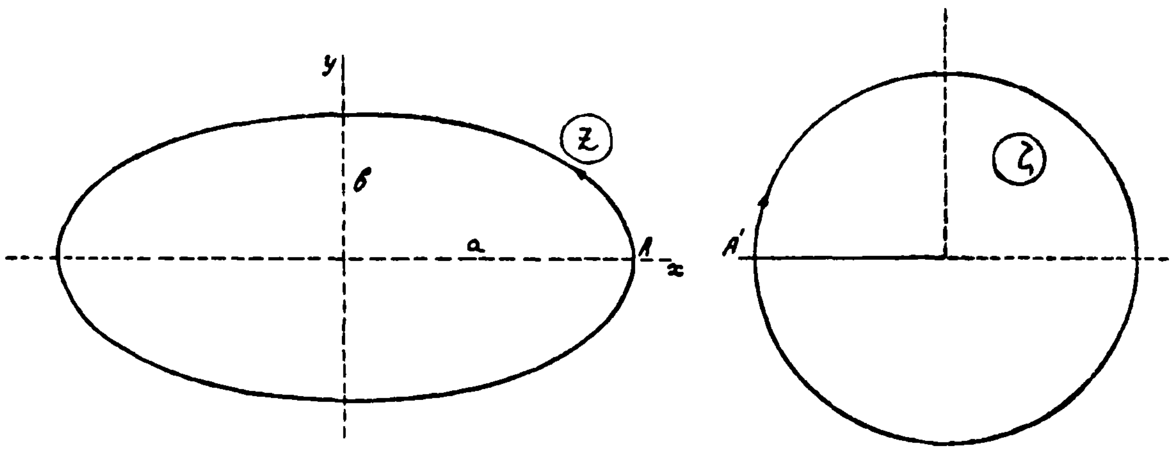


Fig. 3

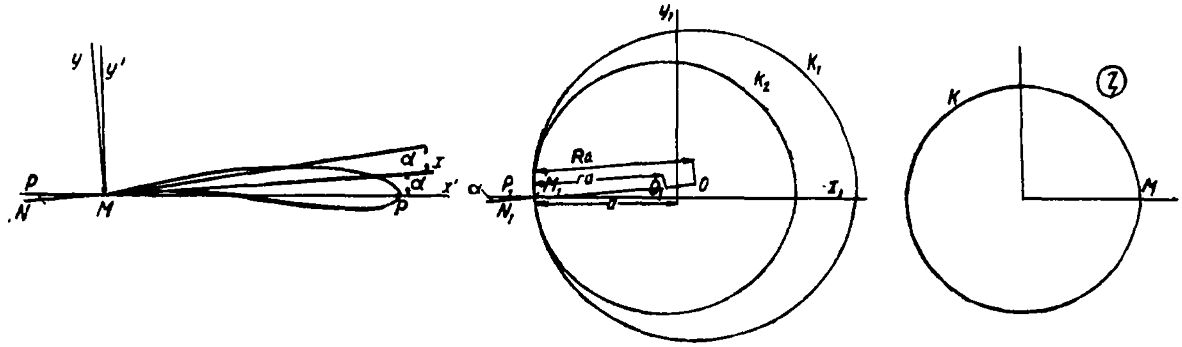


Fig. 4

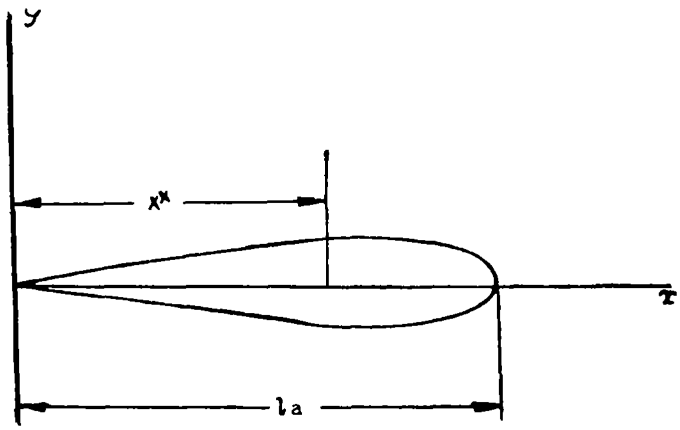


Fig. 5

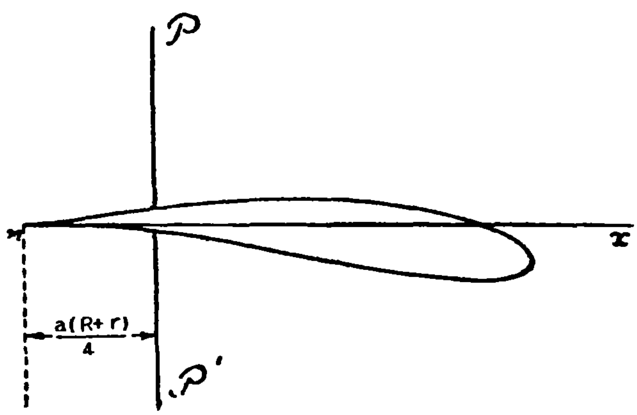


Fig. 6

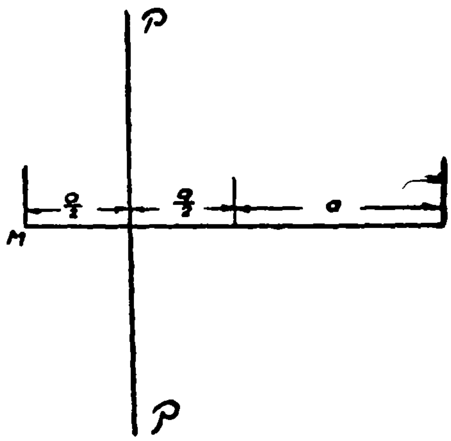
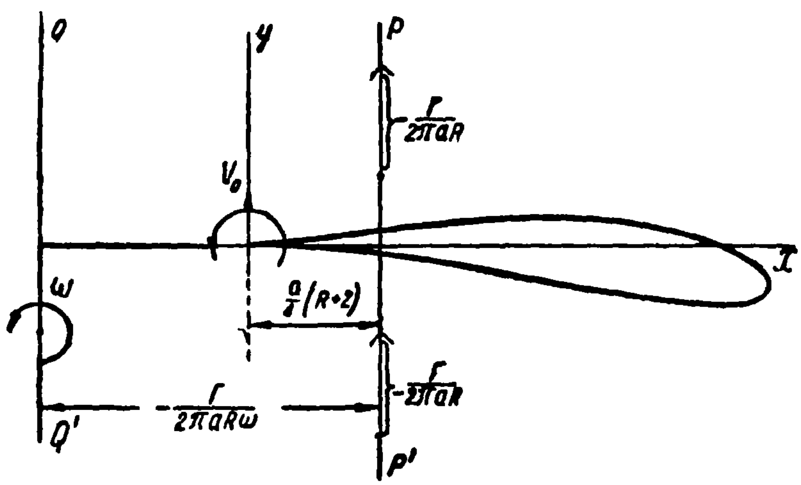


Fig. 7



For definiteness $V_0>0$, $\omega>0$, therefore $\Gamma<0$

Fig. 8

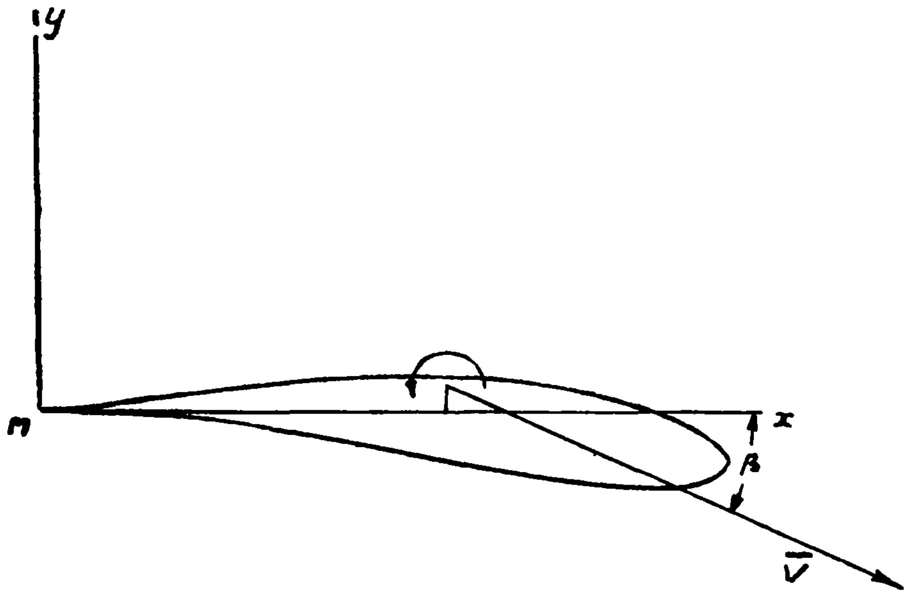


Fig. 9

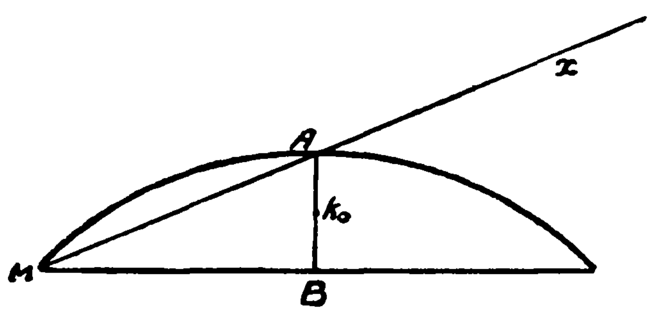


Fig. 10

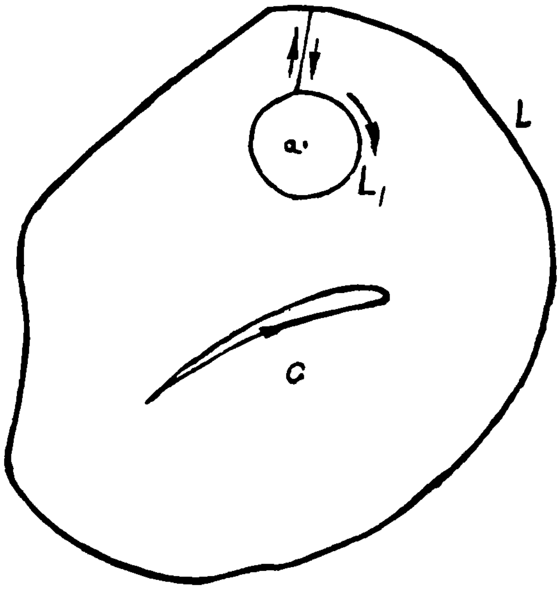


Fig. 11

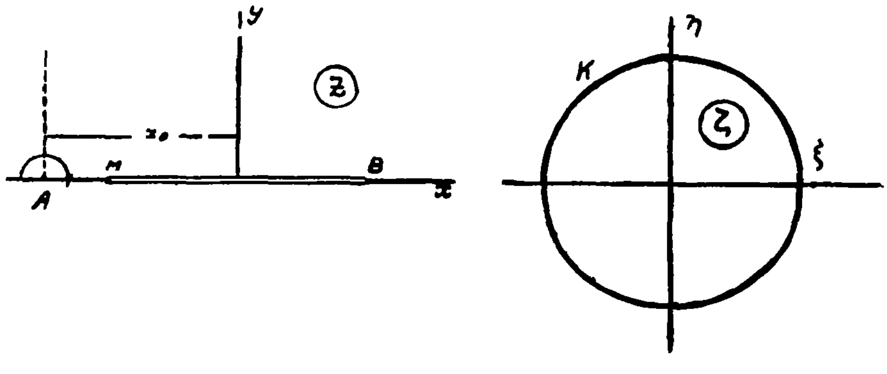


Fig. 12