TECH LIBRARY KAFB, NM
0144695

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1260

EXACT SOLUTIONS OF EQUATIONS OF GAS DYNAMICS

By I. A. Kiebel

Translation

"Primer Tochnogo Reshenia Ploskoi Vikhrevoi Zadachi Gazovoi Dinamiki." Prikladnaya Matematika i Mekhanica. Vol. XI, 1947.

NACA
Washington
June 1950

AFMDC
TECHNICAL LIBRARY

TECH LIBRARY KAFB, NM
0144695

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1260

EXACT SOLUTIONS OF EQUATIONS OF GAS DYNAMICS*

By I. A. Kiebel

The equations of the two-dimensional stationary problem of gas dynamics are of the form

$$\left.\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \\ \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} &= 0 \\ u\frac{\partial}{\partial x}\left(\frac{1}{\rho}p^{\frac{1}{\kappa}}\right) + v\frac{\partial}{\partial y}\left(\frac{1}{\rho}p^{\frac{1}{\kappa}}\right) &= 0 \end{aligned}\right\} \qquad (1)$$

where u and v are the components of the velocity along the coordinate axes x and y, respectively, p is the pressure, ρ the density, and $\kappa = c_p/c_v$, the ratio of specific heats. The equation of continuity permits the introduction of the stream function from the equations

$$\left.\begin{aligned} \rho u &= \frac{\partial \psi}{\partial y} \\ \rho v &= -\frac{\partial \psi}{\partial x} \end{aligned}\right\} \qquad (2)$$

The last of equations (1) gives

$$p^{\frac{1}{\kappa}} = \rho\vartheta(\psi) \qquad (3)$$

*"Primer Tochnogo Reshenia Ploskoi Vikhrevoi Zadachi Gazovoi Dinamiki." Prikladnaya Matematika i Mekhanica. Vol. XI, 1947, pp. 193-198.

where ϑ is a certain function only of ψ. The first two of equations (1) give the Bernoulli law

$$\frac{u^2+v^2}{2} + \frac{\kappa}{\kappa-1}\vartheta p^{\frac{\kappa-1}{\kappa}} = i_o\ (\psi) \tag{4}$$

where i_o is a function of ψ, and the equation for the vorticity is

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\rho\left(\frac{di_o}{d\psi} - \frac{\kappa}{\kappa-1} p^{\frac{\kappa-1}{\kappa}} \frac{d\vartheta}{d\psi}\right) \tag{5}$$

For the solution of the vortex problem in which at least one of the derivatives $di_o/d\psi$ and $d\vartheta/d\psi$ is different from zero, it is convenient to pass from the variables x and y to the variables $x^* = x$ and ψ. Equations (2) and (5) then assume the form (See, for example, reference 1.)

$$\left.\begin{aligned} \frac{\partial y}{\partial x^*} &= \frac{v}{u} \\ \frac{\partial y}{\partial \psi} &= \frac{1}{\rho u} \\ \frac{\partial p}{\partial \psi} &= -\frac{\partial v}{\partial x^*} \end{aligned}\right\} \tag{6}$$

The problem reduces to the determination of the five functions u, v, p, ρ, and y of x^* and ψ from equations (3), (4), and (6).

By using the last of equations (6), the function χ is introduced with the aid of the equations

$$\left.\begin{aligned} p &= \frac{\partial \chi}{\partial x^*} \\ v &= -\frac{\partial \chi}{\partial \psi} \end{aligned}\right\} \tag{7}$$

The first two equations of equation (6), on the basis of equations (3) and (4), then assume the forms

$$\frac{\partial y}{\partial x} = -\frac{\partial \chi}{\partial \psi}\left[2i_o - \frac{2\kappa}{\kappa-1}\vartheta\left(\frac{\partial \chi}{\partial x}\right)^{\frac{\kappa-1}{\kappa}} - \left(\frac{\partial \chi}{\partial \psi}\right)^2\right]^{-\frac{1}{2}} \tag{8}$$

$$\frac{\partial y}{\partial \psi} = \vartheta\left(\frac{\partial \chi}{\partial x}\right)^{-\frac{1}{\kappa}}\left[2i - \frac{2\kappa}{\kappa-1}\vartheta\left(\frac{\partial \chi}{\partial x}\right)^{\frac{\kappa-1}{\kappa}} - \left(\frac{\partial \chi}{\partial \psi}\right)^2\right]^{-\frac{1}{2}} \tag{9}$$

where the asterisk on x has been dropped.

By differentiating equation (8) with respect to ψ and equation (9) with respect to x, y can be eliminated and a single equation for the function χ obtained.

In order to obtain an example of the exact solution of the system of equations, i_o is set equal to a constant:

$$\chi = -H(\psi)\, x^{-\frac{\kappa-1}{\kappa+1}} \tag{10}$$

where H is a certain function of ψ to be determined. Equations (8) and (9) then give

$$\frac{\partial y}{\partial x} = H'\, x^{-\frac{\kappa-1}{\kappa+1}}\left\{2i_o - \left[\frac{2\kappa}{\kappa-1}\vartheta\left(\frac{\kappa-1}{\kappa+1}H\right)^{\frac{\kappa-1}{\kappa}} + H'^2\right] x^{-2\frac{\kappa-1}{\kappa+1}}\right\}^{-\frac{1}{2}} \tag{11}$$

$$\frac{\partial y}{\partial \psi} = \vartheta\left(\frac{\kappa-1}{\kappa+1}H\right)^{-\frac{1}{\kappa}} x^{\frac{2}{\kappa+1}}\left\{2i_o - \left[\frac{2\kappa}{\kappa-1}\vartheta\left(\frac{\kappa-1}{\kappa+1}H\right)^{\frac{\kappa-1}{\kappa}} + H'^2\right] x^{-2\frac{\kappa-1}{\kappa+1}}\right\}^{-\frac{1}{2}} \tag{12}$$

When equation (11) is differentiated with respect to ψ and equation (12) is differentiated with respect to x, terms with the same degree of x are collected and after simple transformations there is obtained

$$\left.\begin{aligned} \vartheta\left(\frac{\kappa-1}{\kappa+1}H\right)^{-\frac{1}{\kappa}} &= \frac{\kappa+1}{2}H'' \\ H'\left(H'^2+\kappa H H''\right)' &= -(\kappa-1)H''\left(H'^2+\kappa H H''\right) \end{aligned}\right\} \qquad (13)$$

Successive integration of the second of equations (13) gives

$$\left.\begin{aligned} H'^2+\kappa H H'' &= C_1 H'^{1-\kappa} \\ H &= C_2\left(C_1-H'^{1+\kappa}\right)^{-\frac{\kappa}{1+\kappa}} \end{aligned}\right\} \qquad (14)$$

From the last equation, the relation between ψ and H can be found with the aid of quadratures. It is more convenient, however, to introduce a numbering of the streamlines directly with the aid of H' and not ψ.

Thus, equations (14) give H as a function of H' and, using equations (13), permit finding ϑ in terms of H' in the form

$$\vartheta = \frac{\kappa-1}{2\kappa}\left(C_2\frac{\kappa-1}{\kappa+1}\right)^{\frac{1-\kappa}{\kappa}} H'^{1-\kappa}\left(C_1-H'^{1+\kappa}\right)^{\frac{2\kappa}{\kappa+1}} \qquad (15)$$

Substituting in equation (11) and integrating yields, for the streamlines,

$$y = \int H' x^{-\frac{\kappa-1}{\kappa+1}}\left[2i_0 - C_1 H'^{1-\kappa} x^{-2\frac{\kappa-1}{\kappa+1}}\right]^{-\frac{1}{2}} dx + F(H') \qquad (16)$$

Comparison with equation (12) shows that F = constant and without loss of generality can be taken as $F \equiv 0$.

For the determination of u, v, p, and ρ the following relations are written:

$$\left.\begin{aligned} u^2 &= 2i_o - C_1 H'^{1-\kappa} x^{2\frac{1-\kappa}{1+\kappa}} \\ v &= H' x^{-\frac{\kappa-1}{\kappa+1}} \end{aligned}\right\} \tag{17}$$

$$p = C_2 \left(C_1 - H'^{1+\kappa}\right)^{-\frac{\kappa}{1+\kappa}} x^{\frac{2\kappa}{1+\kappa}} \tag{18}$$

$$\rho = \frac{2\kappa}{\kappa-1} C_2 \left(C_1 - H'^{1+\kappa}\right)^{-\frac{2\kappa+1}{\kappa+1}} H'^{\kappa-1} x^{-\frac{2}{1+\kappa}} \tag{19}$$

For simplification it is convenient to introduce the nondimensional variables $\tilde{x}$ and $\tilde{y}$ and in place of H' introduce η with the aid of the following equations:

$$\left.\begin{aligned} x &= L\tilde{x} \\ y &= L\tilde{y} \\ H'^{1+\kappa} &= C_1 \frac{1}{\eta^2} \end{aligned}\right\} \tag{20}$$

where

$$L = (2i_o)^{-\frac{1}{2}\frac{\kappa+1}{\kappa-1}} C_1^{\frac{1}{\kappa-1}} \tag{21}$$

Equations (16) and (17) then assume the following simple form:

$$\left.\begin{aligned}\tilde{y} &= \int \left[m^{2\frac{\kappa-1}{\kappa+1}} - 1\right]^{-\frac{1}{2}} dm \\ \tilde{x} &= \eta m\,(\tilde{y})\end{aligned}\right\} \qquad (22)$$

$$\left.\begin{aligned}u^2 &= 2i_o\left(1-m^{-2\frac{\kappa-1}{\kappa+1}}\right) \\ v &= \sqrt{2i_o}\,\frac{1}{\eta}\,m^{-\frac{\kappa-1}{\kappa+1}} = \sqrt{2i_o}\,\frac{1}{\tilde{x}}\,m^{\frac{2}{\kappa+1}}\end{aligned}\right\} \qquad (23)$$

If $\kappa = 1.4$, the integral in equations (22) is evaluated and gives

$$y = 6\sqrt{\sqrt[3]{m} - 1}\left[1 + \frac{2}{3}\left(\sqrt[3]{m}-1\right) + \frac{1}{5}\left(\sqrt[3]{m}-1\right)^2\right] \qquad (24)$$

and the streamlines can therefore be easily constructed.

Figure 1 shows the streamlines for $\eta = 1, 2, 3, \ldots, 19$.

This motion possesses both supersonic and subsonic velocities, for the line of transition (shown dotted in fig. 1) is obtained if

$$u^2 + v^2 = 2\,\frac{\kappa-1}{\kappa+1}\,i_o$$

that is,

$$\left.\begin{aligned}1 - m^{-2\frac{\kappa-1}{\kappa+1}} + \frac{1}{\eta^2}\,m^{-2\frac{\kappa-1}{\kappa+1}} &= \frac{\kappa-1}{\kappa+1} \\ \text{or} \qquad m^{2\frac{\kappa-1}{\kappa+1}} &= \frac{\kappa+1}{2}\left(1 - \frac{1}{\eta^2}\right)\end{aligned}\right\} \qquad (25)$$

It is possible, without difficulty, to construct the characteristics in the x,y- and u,v-planes. Instead of the equation of the epicycloid in the u,v-plane, in the case considered a more complicated equation arises.

In the vorticity problem, along the characteristic there occurs

$$d\beta \mp \frac{\operatorname{ctg}\alpha}{w}\,dw = \pm\frac{\sin\alpha\cos\alpha}{\kappa-1}\,d\log\vartheta \tag{26}$$

where α is the Mach angle, w the magnitude of the velocity, and β the angle of inclination of the velocity to the x-axis. Equations (23) yield

$$\eta^2 = \frac{2i_o-u^2}{i^2} = \frac{2i_o-w^2\cos^2\beta}{w^2\sin^2\beta} \tag{27}$$

But ϑ depends only on H', that is, on η^2. According to equation (15), the following is obtained:

$$\vartheta = \frac{\kappa-1}{2\kappa}\left(C_2\frac{\kappa-1}{\kappa+1}\right)^{\frac{1-\kappa}{\kappa}} C_1\,\frac{1}{\eta^2}\,(\eta^2-1)^{\frac{2\kappa}{\kappa+1}}$$

$$= \frac{\kappa-1}{2\kappa}\left(C_2\frac{\kappa-1}{\kappa+1}\right)^{\frac{1-\kappa}{\kappa}} \frac{C_1}{2i_o-w^2\cos^2\beta}\,(w^2\sin^2\beta)^{-\frac{\kappa-1}{\kappa+1}}\,(2i_o-w^2)^{\frac{2\kappa}{\kappa+1}} \tag{28}$$

Substituting this value of ϑ in the right side of equation (26) yields the differential equation for the characteristics in the β,w-plane.

Finally, the question arises whether it is possible to obtain such vortex motion by transition through a surface of strong discontinuity of some other kind but with irrotational motion. This problem may be answered in the affirmative. For on a surface of discontinuity there must, among others, be satisfied the relations

$$\left.\begin{aligned} \frac{\rho_+}{\rho_-} &= \frac{\kappa-1}{\kappa+1} + \frac{2\kappa}{\kappa+1}\,\frac{p_+}{\rho_+{\theta_+}^2} \\ \frac{p_-}{p_+} &= 1 + \frac{2}{\kappa+1}\left(1 - \frac{\kappa p_+}{\rho_+{\theta_+}^2}\right)\frac{\rho_+{\vartheta_+}^2}{p_+} \end{aligned}\right\} \qquad (29)$$

where p_+, ρ_+, and θ_+ are the pressure, density, and velocity of propagation of the surface of discontinuity on one side of this surface and p_- and ρ_- are the pressure and density on the other side. The magnitudes p_+ and ρ_+ may be taken from the vortex motion. The magnitude θ_+ is found in terms of the elements describing the motion and in terms of the inclination of the surface of discontinuity. Finally, p_- and ρ_- can be connected by the relation ${p_-}^{1/\kappa} = \vartheta_1\rho_-$, where ϑ_1 is a constant (up to a certain degree of arbitrariness) magnitude. (At the left of the surface of discontinuity the motion is irrotational.)

Inasmuch as

$$\left(\frac{p_-}{p_+}\right)^{\frac{1}{\kappa}}\frac{\rho_+}{\rho_-} = \frac{\vartheta_1}{\vartheta_+} \qquad (30)$$

then

$$\left(M - \frac{\kappa-1}{2}\right)^{\frac{1}{\kappa}}\left(\frac{\kappa-1}{2\kappa} + \frac{1}{M}\right) = G\eta^2\,(\eta^2-1)^{-\frac{2\kappa}{\kappa+1}} \qquad (31)$$

where

$$\left.\begin{aligned} G &= \frac{\vartheta_1}{C_1}\,G_2^{\frac{1}{\kappa}-1}\left(\frac{\kappa+1}{2}\right)^{\frac{2}{\kappa}}\left(\frac{2}{\kappa-1}\right)^{\frac{1}{\kappa}} \\ M &= \frac{\rho_+{\theta_+}^2}{p_+} \end{aligned}\right\} \qquad (32)$$

Inasmuch as

$$-\theta_+ = \nu_+\cos\delta + \nu_+\sin\delta$$

where δ is the angle between the normal to the surface of discontinuity and the x-axis (the normal is directed toward the "positive" region); M may be expressed in terms of known magnitudes and δ.

By expressing $\tan \delta$ in terms of the derivative of $\tilde{x}$ with respect to η along the surface of discontinuity and using equations (22), (18), and (19), the differential equation for determining the surface of discontinuity ($\kappa = 1.4$) is obtained after simple transformations:

$$\frac{dm}{d\eta} = \frac{m\sqrt{\sqrt[3]{m}-1}}{1+\eta^2\left(\sqrt[3]{m}-1\right)}\left\{-\eta\sqrt{\sqrt[3]{m}-1} \pm \sqrt{\frac{7}{6}\,\frac{1+\eta^2\left(\sqrt[3]{m}-1\right)}{M\left(\eta^2-1\right)}-1}\right\} \tag{33}$$

where M is expressed in terms of η^2 from the transcendental equation (31) in which the constant G is, to a great extent, arbitrary. The velocity in the "negative" region will be determined first on the surface of discontinuity and then extended on the negative region by the usual graphical method of Busemann. A model of the motion about a contour having an angle is obtained.

The surface of discontinuity extends out from the angle and on passing through the angle the motion reconverts to the rotational motion herein considered.

A solution analogous to that developed can also be obtained for the problem with axial symmetry. Taking for the independent variable the distance $r^* = r$ from the axis of symmetry z and the stream function ψ yields the relations

$$\left.\begin{aligned}\frac{\partial z}{\partial \psi} &= \frac{v_z}{v_r}\\ \frac{\partial z}{\partial r^*} &= \frac{1}{r^*\rho v_r}\end{aligned}\right\} \tag{34}$$

where v_r and v_z are the velocity components. As before,

$$p^{\frac{1}{\kappa}} = \vartheta(\psi)\rho$$

Bernoulli's law will have the form

$$\frac{1}{2}\left(v_r^{2}+v_z^{2}\right)+\frac{\kappa\vartheta}{\kappa-1}\,p^{\frac{\kappa-1}{\kappa}}=i_o$$

The equations of Euler give

$$\frac{\partial v_z}{\partial r^*}=\frac{\partial r^* p}{\partial \psi}$$

The function $\chi(r^*,\psi)$ can therefore be introduced from the conditions

$$\left.\begin{aligned} v_z &= -\frac{\partial \chi}{\partial \psi} \\ p &= \frac{1}{r^*}\frac{\partial \chi}{\partial r^*} \end{aligned}\right\} \qquad (35)$$

Equations (34) and (35) now permit writing (the asterisks on r are dropped)

$$\left.\begin{aligned} \frac{\partial z}{\partial \psi} &= -\frac{\partial \chi}{\partial \psi}\frac{1}{v_r} \\ \frac{\partial z}{\partial r} &= r^{-\frac{\kappa+1}{\kappa}}\,\vartheta(\psi)\left(\frac{\partial \chi}{\partial r}\right)^{-\frac{1}{\kappa}}\frac{1}{v_r} \end{aligned}\right\} \qquad (36)$$

where

$$v_r=\left[2i_o-\frac{2\kappa}{\kappa-1}\,\vartheta(\psi)\left(\frac{1}{r}\frac{\partial \chi}{\partial r}\right)^{\frac{\kappa-1}{\kappa}}-\left(\frac{\partial \chi}{\partial \psi}\right)^{2}\right]^{\frac{1}{2}}$$

By eliminating z from equations (36), a single equation for the determination of the function χ is obtained. Particular solutions, analogous to the solutions in the first case, can be constructed by seeking χ in the form

$$\chi = - r^{-2\frac{\kappa-1}{\kappa+1}} H(\psi)$$

Translated by S. Reiss
National Advisory Committee
for Aeronautics.

REFERENCE

1. Kochin, N. E., Kiebel, I. A., and Rose, N. V.: Theoretical Hydromechanics. Pt. II, 1941, p. 80.

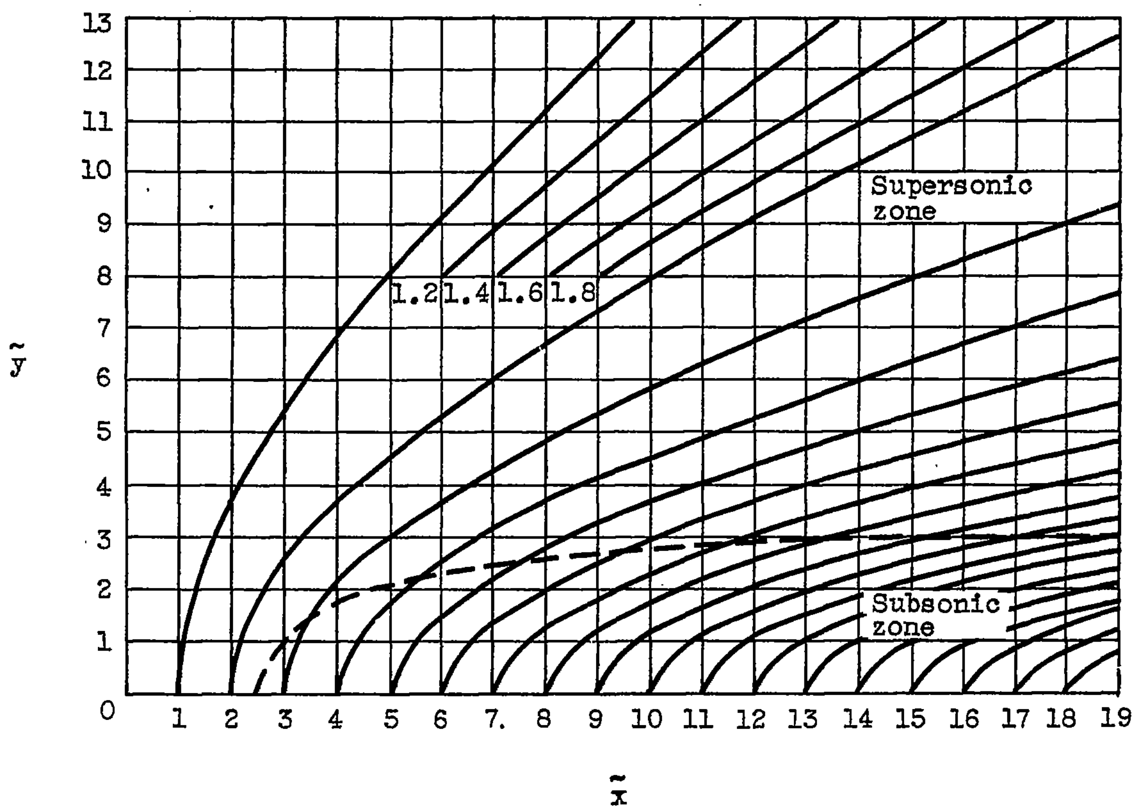

Figure 1.

NACA-Langley - 6-27-50 - 950