NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1231

REPORT ON INVESTIGATION OF DEVELOPED TURBULENCE

By L. Prandtl

Translation of "Bericht über Untersuchungen zur ausgebildeten Turbulenz." Zeitschrift für angewandte Mathematik und Mechanik, vol. 5, no. 2, April 1925.

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The recent experiments by Jakob and Erk (reference 1) on the resistance of flowing water in smooth pipes, which are in good agreement with earlier measurements by Stanton and Pannell (reference 2), have caused me to change my opinion that the empirical Blasius law (resistance proportional to the 7/4 power of the mean velocity) was applicable up to arbitrarily high Reynolds numbers. According to the new tests the exponent approaches 2 with increasing Reynolds number, where it remains an open question whether or not a specific finite limiting value of the resistance factor λ is obtained at $R = \infty$.

With the collapse of Blasius' law the requirements which produced the relation that the velocity in the proximity of the wall varied in proportion to the 7th root of the wall distance must also become void (reference 3). However, it is found that the fundamental assumption that led to this relationship can be generalized so as to furnish a velocity distribution for any empirical resistance law. These fundamental assumptions can be so expressed that for the law of velocity distribution in proximity of the wall as well as for that of friction at the wall, a form can be found in which the pipe diameter no longer occurs, or in other words, that the processes in proximity of a wall are not dependent upon the distance of the opposite wall.

For the velocity u (time average value) at y distance from the wall only one nondimensional can then be formed, namely, $\frac{uy}{\nu}$ (ν = kinematic viscosity), giving for u a formula of the form

$$u = C\varphi\left(\frac{uy}{\nu}\right) \tag{1}$$

where C is a velocity and φ an arbitrary function. The shearing stress at the wall then must be

$$\tau = \zeta\rho C^2 \tag{2}$$

where ζ is a constant.

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*"Bericht über Untersuchungen zur ausgebildeten Turbulenz." Zeitschrift für angewandte Mathematik und Mechanik, vol. 5, no. 2, April 1925, pp. 136-139.

To obtain the function φ we proceed from the value $\frac{d\ ln\ y}{d\ ln\ u}$, which by theory would be constant = 7 and put

$$\frac{d\ ln\ y}{d\ ln\ u} = f(\sigma) \tag{3}$$

where $\sigma = ln\frac{uy}{\nu}$. With $ln\ u = \eta$ and $ln\ y = \sigma - \eta + ln\nu$, formula (3) becomes

$$\frac{d\sigma}{d\eta} = f(\sigma) + 1,$$

and, after integration and removal of logarithms,

$$u = Ce^{\int^{\sigma}\frac{d\sigma}{f(\sigma) + 1}} \tag{4}$$

which, after including

$$y = \frac{\nu e^{\sigma}}{u} \tag{5}$$

gives the velocity profile in parameter representation.

The empirical law for wall friction reads $\tau = \lambda\rho\bar{u}^2$, with λ a function of $\frac{\bar{u}a}{\nu}$ ($\bar{u}$ = mean velocity, a = pipe diameter).

Discounting for simplicity the difference between the mean velocity and the velocity in the center u_1 and assuming for it the value from formula (4), although it is no longer exactly true in the pipe center, we put with $\sigma_1 = ln\frac{u_1 a}{\nu}$

$$-\ ln\lambda = g(\sigma_1) \tag{6}$$

By (2) and (4) we get

$$\zeta = \frac{\tau}{\rho C^2} = \frac{\tau}{\rho u_1{}^2}\frac{u_1{}^2}{C^2} = \lambda e^{2\int^{\sigma_1}\frac{d\sigma}{f(\sigma) + 1}}$$

or

$$\ln \zeta = - g(\sigma_1) + 2\int^{\sigma_1} \frac{d\sigma}{f(\sigma) + 1} = \text{const.} \tag{7}$$

hence after differentiation with respect to σ_1

$$g'(\sigma_1) = \frac{2}{1 + f(\sigma_1)} \quad \text{or} \quad f(\sigma) = \frac{2 - g'(\sigma)}{g'(\sigma)} \tag{8}$$

With this the problem is solved and it is readily seen that $f(\sigma)$, which for $g'(\sigma) = \frac{1}{4}$ assumes the value 7 as before, increases with decreasing $g'(\sigma)$. A more accurate experimental check is awaited, but even so it is plainly seen that at Reynolds numbers of about 200,000 the 8th root of the wall distance is definitely better than the 7th root. For $g'(\sigma) = 1$, $f(\sigma) = 1$, which corresponds to the laminar boundary zone.

Furthermore I would like to speak of a formula intended for a hydrodynamic calculation of the distribution of the base flow of a turbulent motion under the most varied conditions.

After various fruitless attempts gratifying success was attained. In addition it was found that the formula for the apparent shearing stress τ produced by the interchange of momentum, lends itself to a very clear explanation.

In the Boussinesq formula $\tau = \rho\epsilon\frac{du}{dy}$, ϵ is a measure for the turbulent "exchange" and in its dimension, which is the same as that of ν, it is the product of a length and a velocity. The velocity is the transverse velocity w at which on the average the fluid bodies advancing from both sides pass through the layer with the time average value of the velocity = u.

The liquid bodies coming from the side of the greater velocity entertain higher values of velocity u, those from the side of smaller velocities, smaller values, with the result that more and more momentum is transported in one direction than in the other (excepting the point of u_{max}). The desired length l is characterized by the fact that it indicates the distance of the particular layer, in which the average u velocities, which the liquid bodies have at their passage, are found as the time average value of the flow velocity. Approximated these velocities are $u + l\frac{du}{dy}$ and $u - l\frac{du}{dy}$. (Incidentally l is in agreement in order of magnitude with the diameter of the fluid bodies (more accurately it is the decelerating path of the fluid bodies in the remaining fluid, which is, however, proportional to the diameter).) As to the length l it can, for the present, only be stated that it must approach zero at the wall, where only bodies of smaller diameter than the wall distance can move as discussed. Elsewhere l is to have a very regular distribution. If β is the average proportional share of the surface occupied by the fluid bodies entering from one side, a momentum $\beta \rho w l \frac{du}{dy}$ per second passes at this side through the unit surface, and approximately the same amount from the other side. This confirms the Boussinesq theorem, so we can put $\epsilon = 2\beta w l$.

The next problem is to find a practical formula for the mixing speed w. This mixing speed is rapidly reduced and must be continuously renewed. Hence the assumption that it is produced at the concurrence of two bodies of different velocity u and therefore proportional to the velocity difference, that is, the magnitude of $l\frac{du}{dy}$. With this, however, if all unknown numerical factors are thrown on the not more accurately known length l, the apparent shearing stress becomes

$$\tau = \rho l^2 \left|\frac{du}{dy}\right| \frac{du}{dy} \tag{9}$$

This formula necessitates a correction for the case that $\frac{du}{dy} = 0$.

For the creation of velocity w the neighborhood in a certain width cooperates; it does not become zero when $\frac{du}{dy} = 0$; it rather can be put proportional to a statistical average value of $\left|\frac{du}{dy}\right|$ that is proportional to $\sqrt{\left(\frac{du}{dy}\right)^2}$. If the velocity profile varies in flow direction, as in convergent divergent channels, the points over which the averages are made must be shifted upstream for a certain amount, since the process of development of the velocity w takes time.

Formula (7) has already proved itself in many respects. In a pipe, for instance, the shearing stress is according to the equilibrium conditions proportional to the distance r from the center, hence

$$\tau = \mp \rho l^2 \left(\frac{du}{dy}\right)^2 = cr$$

Assuming l as constant gives

$$u = A - Br^{3/2} \tag{10}$$

for $r > 0$, the reflection for $r < 0$ (fig. 1). At $r = 0$ the radius of curvature is then equal to 0; taking instead of $\frac{du}{dy}$ the previously discussed statistical value that can be approximately written as $\sqrt{\left(\frac{du}{dy}\right)^2 + l'^2\left(\frac{d^2u}{dy^2}\right)^2}$ the curvature at $r = 0$ is finite. The actual velocity distributions (reference 1, p. 21) exhibit the singular behavior of the center very plainly (fig. 1). The refinement just mentioned needs to be applied only where increased accuracy requirements are involved. The total velocity distribution in the pipe is fairly accurately obtained when (in the range of the 1/7 power law) l is put proportional to $\left[(a - r)(a + r)\right]^{6/7}$, a = pipe radius.

The foregoing formulas have also been applied to the case of the "free turbulence," that is, flows without confining walls, as for example, of a fluid jet diffusing in a chamber and to the intermingling of a homogeneous air stream with the adjacent air at rest, which is entrained by it. For stationary flows of this type the formula $l = cx$ is appropriate, x being the distance from the point where the mixing starts. The case reproduced in figure 2 leads to the differential equation for the stream function $F\left(\frac{y}{x}\right)$

$$2cF''F''' \pm FF'' = 0$$

which is solved by $F'' = 0$ or by $2cF''' \pm F = 0$. The two solutions of uniform velocity and variable velocity abut in F''' with discontinuity. This and other calculations were numerically carried out by Tollmien, who is to publish an article on it. The agreement of his calculations with experimental data is excellent.

Further experimental studies included the velocity distributions in channels of other than circular sections with very unusual results and for which the explanation has not yet been found.[1] Slightly

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[1]It is approximately so that u^7 plotted against the cross section of the channel gives a sloping surface.

divergent and convergent smooth-walled channels have also been investigated, rough-walled channels are in preparation. It is hoped that our formula will prove itself here also.

Translated by J. Vanier
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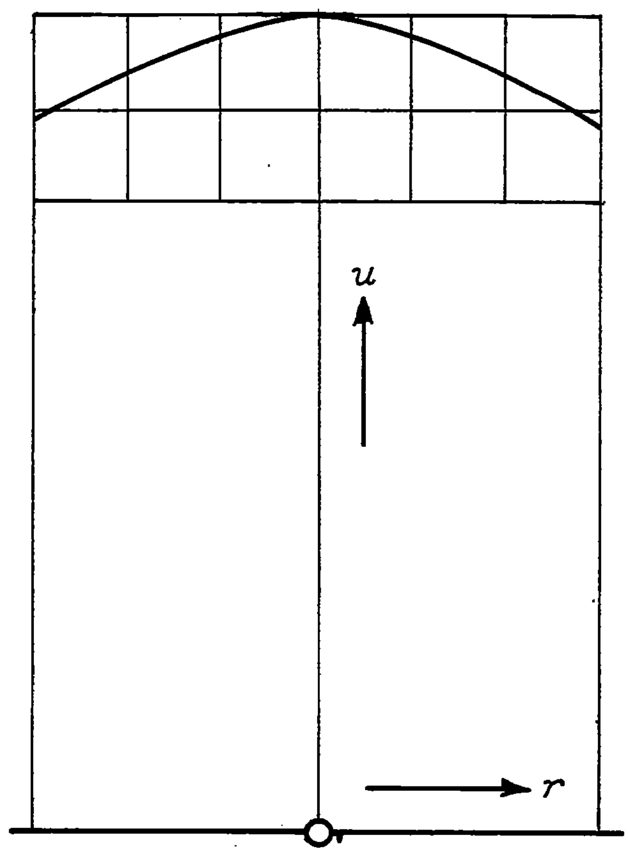

Figure 1

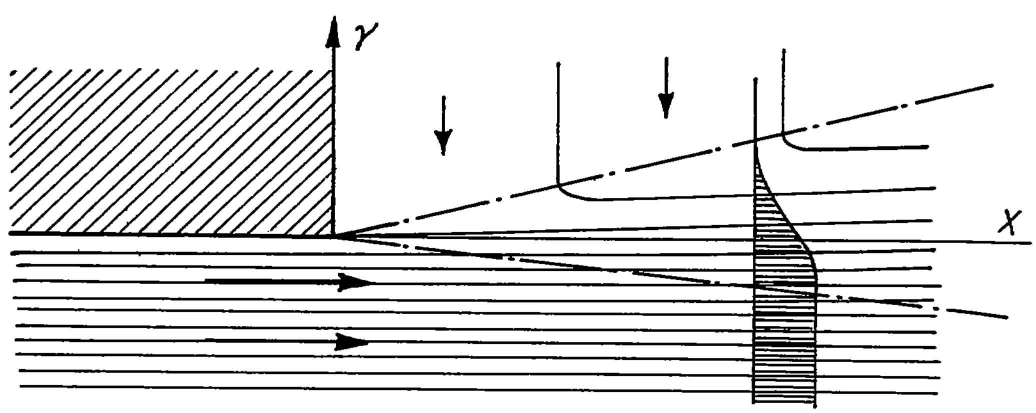

Figure 2