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METHODS OF STRESS CALCULATION IN ROTATING DISKS*

By S. Tumarkin

SUMMARY

The paper describes methods of computing the stresses in disks of a given profile as well as methods of choosing the disk profiles for a given stress distribution for turbines, turbo blowers, and so forth. A new method of integrating the differential equations of Stodola leads to a simplification of the computation for disks of hyperbolic profile. It was found possible to apply to the equations a method analogous to the methods of Donath and Yanovsky for disks of constant thickness, the sum and difference of the stresses $S = \tau + \sigma$, $D = \tau - \sigma$ being replaced in the equations by the expressions $S = m\sigma + \tau$, $D = n\sigma + \tau$ where m and n are constants. There is investigated, for the first time apparently, the problem of the choice of profile for disks carrying lateral blades. In contrast to the case considered by Holzer of disks with blades attached at the rim, it is impossible in this case to assume arbitrarily the curve of radial stresses and the edge thickness of a disk. In a number of cases infinitely diverging and other unsuitable profiles occur. The dependence of the profile shape on the assumed stresses is investigated. An example of the improvement of a typical disk profile is analyzed showing considerable gain in material on approaching the condition of uniform strength. The method of Holzer, for disks with blades attached at the rim, is considerably simplified by dispensing with the necessity for graphical or mechanical integration. There is considered also the possible limit of tangential stresses for a given curve of radial stresses, a factor of much value in selecting a profile.

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INTRODUCTION

The determination of stresses in rotating disks is a problem that has received much attention. The problem was worked out at first chiefly in connection with steam turbines. The recent development of machines for moving gases (fans, blowers, superchargers for airplane engines) has led to an increased interest in the subject and has introduced specific requirements.

The papers by Yanovsky (reference 1), Volkov (in reference 2), Cherny and Baklanov (reference 3), Rees (reference 4), and the work conducted at CAHI are witness to Russia's heightened interest in the subject. The continually widening application of computed disks proves the importance of devising rational computation methods and explains the constant increase in the number of investigation papers in this field. The foreign literature on this problem is extensive though of unequal merit. At times, no use is made of important results already obtained - for example, the principle of Von Mises. Even such leading investigators as Kearton (reference 5) and Ostertag (reference 6) present extremely laborious and outmoded methods.

In the present paper methods are presented for the computation of the stresses in given disks and the selection of the profile of rotating disks for assumed stress distribution at various conditions of loading. The latter problem for disks carrying side blades is investigated apparently for the first time. The problem first considered is to compute the stresses for a given disk. The starting point here is an assumed approximation to the shape of the disk profile with the aid of hyperbolic curves.

The following method of integrating the differential stress equations leads to a solution in a form which is a direct generalization to hyperbolic disks of the method of Donath (reference 7) for disks of constant thickness. (The method of Donath is the basis of all subsequent work on flat disks.) A simple transition from the stresses at the inner radius of the ring to those at the outer radius are given in this method, and no recourse is had to the construction of tables or charts. In particular, for disks of constant thickness, this method leads to that of Yanovsky which appears the best development of the Donath method.

The second fundamental problem is that of selecting a profile for a given stress distribution. The problem was first formulated by Holzer (reference 8). The solution of this problem becomes difficult in the case of disks the load of which is distributed along the radius. The occurrence of singular points in the differential equation of the profile leads to the possibility of obtaining infinitely divergent profiles, and so forth. In the case of finite edge thickness the latter, it appears, cannot be arbitrarily assumed as in the case of the disk considered by Holzer, but is related in some manner with the chosen stresses and loads. In this connection a detailed investigation is made in the present paper of the problem of the dependence of the profile shape on the given stresses. As shown by examples, the application of these methods leads to a certain saving in material, the gain being most marked for a stress distribution giving uniform strength.

As regards the problem solved by Holzer, other authors such as Yanovsky (reference 1), Arrowsmith (reference 9), have departed from the direct path followed by Holzer in view of its technical complexity. The following shows how a suitable selection of the form of the functions which give the radial stress leads to the possibility of carrying out the quadratures so that the solution is obtained in the form of simple finite formulas. The computational work is many times reduced without impairing the accuracy. This method is particularly convenient for the disks of steam turbines.

In the present paper the questions of temperature stresses (reference 10) are not considered. The problems of the stresses due to unsymmetric disks or loadings are pressing and await full investigation.

The approximate equations of the stresses in disks of varying thickness were obtained by Stodola in 1903 (reference 11). Not counting disks of constant and ellipsoidal thickness, one attempt to obtain a more accurate solution may be noted - namely, that of Cornok (reference 12), who gives a general equation from which there is then derived the solution for hyperbolic disks. This solution is compared with that of Stodola. For conical disks a method is indicated in the form of an infinite series. The paper of Cornok contains errors, however, in the computation of the mean stresses because in integrating no account was taken of the dependence of the limits on the parameter.

It is noted that if the equations of Cornok were true, then they would provide for any profile a solution requiring only two quadratures. For conical disks there also would be obtained a simple finite solution instead of the infinite series used by Cornok.

The question of the effect of the assumptions made was subjected by Stodola to a theoretical analysis (reference 13). Moreover, he checked the assumption of the uniform distribution of the radial stresses in the cylindrical sections of the disks making use of the accurate solution of Cree for ellipsoids. (See Stodola, p. 896.) It is interesting to follow this comparison. The following conclusion may be drawn on the basis of the computation conducted by Stodola for various shapes of ellipsoids. In the method of Stodola no difference in stresses in these ellipsoids appears because all thicknesses are proportional and the equations of Stodola are unaffected by such variation in the thickness of the disks. The stress curves of Cree, however, show for the various ellipsoids a difference up to 30 percent, the maximum stress in the ellipsoids of flat shape being less than in more convex ellipsoids.

As is naturally to be expected and confirmed by this computation, the Stodola solution for thin ellipsoids practically does not differ from the accurate one. For ellipsoids approaching the spherical shape the equations of Stodola give stresses up to 30 percent less than the actual.

It also should be mentioned that the tests of Stodola on resinous models of disks (reference 13) have shown that in wide hubs the effect of bending begins to predominate.

REVIEW OF METHODS

The main paths followed in the development of the methods are indicated. (See also reference 10.)

The equations of Stodola are solvable in finite form only for certain particular shapes of profiles — for disks of constant thickness, uniform strength, and hyperbolic. On some of them more will be said. For trapezoidal sections (conical disks) the solution can be obtained only in the form of an infinite series. Practically applicable disks have complicated profiles, however, and for these the equations of Stodola permit obtaining approximate solutions.

To the first group the methods giving approximate solutions starting from the general form of these equations may be referred. Here belongs the graphical method of Stodola by which, assuming the curve of radial stresses, the profile of the disk is found; and the method of Kellar (in reference 1, p. 335), who substitutes small finite increments for the differentials. To this group also belongs the method of Holzer (reference 8) where a curve of radial stresses in graphical or analytical form is assumed and the work of Pöschl (reference 14) employing the method of Ritz. All these solutions are practically inconvenient. Much more simple are the methods using one of the previously mentioned types of profiles for which the integration does not present any special difficulties.

In regard to this, three methods based on disks of constant thickness, hyperbolic disks, or conical disks have been developed. Grüber (reference 15) was the first to follow this method. Recently there also has appeared an attempt to make use of certain exponential forms of profile (references 16 and 16a).

By substituting approximately for the profile of the disk a number of hyperbolic curves $y = \frac{c}{r^a}$, Grüber showed that by taking into account the boundary conditions of all such rings a sufficient number of equations is obtained for computing the stresses over the entire disk. It is true that in this case the method is still inconvenient. It has been adopted, however, by Kearton.

A new line of development was taken by Donath in 1912 (reference 7) whose method also was explained by Haerle (reference 17). Donath based his computation on constant thickness profiles from which a stepped disk is formed approximating the given disk. Between the individual steps the stresses undergo discontinuities which may be computed. The second main feature of the method of Donath lies in the fact that instead of the stresses themselves their sum and difference are used. This device has proved to be very convenient for reasons which will be explained and has been adapted for disks of constant thickness. In connection with this method Donath constructed a rather complicated chart with two families of curves.

The method of Donath was perfected by Grammel (reference 18) who substituted simple graphical constructions

for Donath's charts and by Yanovsky (reference 1) who proposed a convenient numerical method. Yanovsky gave much attention to the problem of Holzer. The methods of Donath and Keller required several trials for satisfying the boundary conditions. Von Mises showed how the boundary conditions could be satisfied by making use of a fundamental family of linear differential equations. Only two trials were found necessary, the second being facilitated by having the angular velocity equal to zero. This device received wide application and it was suitable for all methods - for steps of constant thickness as well as for hyperbolic or other steps. Driessen (reference 19), likewise developing the method of Donath, extended somewhat the application of the foregoing device. The method of Cherny and Baklanov is essentially contained in the method of Arrowsmith, but a fuller table is given for disks of constant thickness.

While they possess the advantage of simplicity, the methods based on constant thickness steps are inconvenient in that for a good approximation to the shape of the curve a considerable number of steps are required. In view of this fact, methods making use of hyperbolic steps continued to be developed. The ease thereby obtained of approximating to the shape of the disks is explained by the presence of two free parameters and also by the fact that in their construction applied disks approach the hyperbolic shape.

Martin (reference 20) constructed a family of curves to facilitate the computation of hyperbolic disks for certain values of the exponent: $\alpha = 0, 1, 2$. Recently, the charts of Martin have frequently been supplemented for other values of the exponent. (See Knight, reference 21, and Hodkinson, reference 22.) Arrowsmith (reference 9), dispensing with graphs, transformed the formulas of Stodola and constructed tables that permit finding the stresses in hyperbolic disks for the ratios of outer to inner radius of 1.02, 1.05, 1.1, and 1.2. This does not permit making use at all times of the fundamental advantage of the hyperbolic disks - namely, the small number of steps required. Ratios of the radii greater than 2 are encountered in the computation and for these, according to the method of Arrowsmith, no fewer than four steps would be required.

Volkov does not make use of the device of Von Mises, and the constants of integration are not excluded - a fact which complicates the computation.

The third group of methods is associated with conical disks. Fischer (reference 23) and Honegger (reference 24) present a large class of profiles, including conical, for which the stress equations lead to the hypergeometric series of the Gauss equation. Honegger computed tables of functions entering the solution. Martin (reference 25) arrived at analogous results making use of the principle of Castigliano in deriving his equations with the same assumptions as those of Stodola. In 1934 Malkin (reference 16) showed new forms of integrable profiles:

$$y = \alpha e^{-\beta r^{\frac{4}{3}}} \quad \text{and} \quad y = \alpha e^{-\beta r^{\frac{2}{3}}}$$

upon which a computation was based. Since for the latter profiles the equation may be transformed into a Bessel equation, it is possible that the corresponding tables are found among tables of Bessel functions. Some of the tables of Malkin have already been published (reference 16a).

Holzer (reference 8) approached the computation of disks from another direction. By assuming maximum stresses it was sought to obtain the profile requiring the minimum expenditure of material. This was supposed to be equivalent to finding the profile possessing the maximum energy of deformation possible at the given stresses. The solution of Holzer is unsuitable for disks not loaded at the rim (disks with side blades). A solution is given for this case. Holzer applies mechanical or graphical quadratures for solving the problem of the choice of profile. Other methods also were suggested by Yanovsky (reference 1) and Arrowsmith (reference 9). The method of Holzer becomes very simple if the forms of the functions for the radial stresses are suitably chosen.

It is of no value to raise the question as to which formulation of the problem - namely, to find the profile for given stresses or conversely - is the more correct. In practical construction both problems are equally important.

NOTATION

r	distance of element of disk from axis (cm)
y	thickness of disk at radius r (cm)
σ_r or σ	radial stress (kg/cm ²)
σ_t or τ	tangential stress (kg/cm ²)
ξ	radial displacement of disk element (cm)
ω	angular velocity of disk (sec ⁻¹)
ν	ratio of transverse compression to longitudinal extension (Poisson ratio)
E	elasticity modulus (kg/cm ²)
γ	density (kg/cm ³)
$c = \frac{\gamma \omega^2}{g} \left(\frac{\text{kg}}{\text{cm}^4} \right)$	

I. STRESSES IN GIVEN DISKS - FUNDAMENTAL EQUATIONS

The first equation is found from the condition of equilibrium of the disk element (fig. 1). The stresses σ and τ are assumed not to vary along the axial direction of the disk.

For the lower part of the element the radial force is equal to $rd\theta y \sigma$. The resultant of the radial stresses is equal to

$$d (ry\sigma d\theta)$$

The resultant of the tangential stresses is equal to

$$yrd\theta$$

The centrifugal force for the given element is equal to

$$\frac{\gamma \omega^2}{g} r^2 y dr d\theta$$

or

$$cr^2 y dr d\theta$$

From the equilibrium of the element there is obtained

$$\frac{d}{dr} (ry\sigma) - y\tau + cr^2 y = 0 \quad (1)$$

Owing to the centrifugal force of side blades, for example, the effect of loads distributed along the radius must be added to the foregoing forces. This is most simply effected by adding to the disk a certain nominal thickness η of the same material and which does not carry any stresses, but gives only an added centrifugal force. The thickness η is taken so that its centrifugal force at any radius is equivalent to the external load. Instead of the centrifugal forces of the element $cr^2 y dr d\theta$ in the equation there enters the total centrifugal force $cr^2 (y + \eta) dr d\theta$ and the equation becomes

$$\frac{d}{dr} (ry\sigma) - y\tau + cr^2 (y + \eta) = 0 \quad (2)$$

Thus one equation has been obtained for the two variables σ and τ . A relation can be obtained from Hooke's law connecting the stresses with the deformations. For the assumed condition of absence of axial stresses, the deformations are obtained

$$\left. \begin{aligned} \epsilon_r &= \frac{1}{E} (\sigma - \nu \tau) \\ \epsilon_t &= \frac{1}{E} (\tau - \nu \sigma) \end{aligned} \right\} \quad (3)$$

Both deformations can be expressed in terms of the radial displacement of the element

$$\left. \begin{aligned} \epsilon_t &= \frac{\xi}{r} \\ \epsilon_r &= \frac{d\xi}{dr} \end{aligned} \right\} \quad (4)$$

Thus

$$\left. \begin{aligned} \frac{d\xi}{dr} &= \frac{1}{E} (\sigma - \nu\tau) \\ \frac{\xi}{r} &= \frac{1}{E} (\tau - \nu\sigma) \end{aligned} \right\} \quad (5)$$

There are two ways in which to proceed: eliminating ξ there is obtained the condition for σ and τ :

$$\frac{d}{dr} [r (\tau - \nu\sigma)] = \sigma - \nu\tau \quad (6)$$

$$\frac{d\tau}{dr} - \nu \frac{d\sigma}{dr} = \frac{1 + \nu}{r} (\sigma - \tau) \quad (7)$$

This is the equation of the interrelated stresses. It is independent of the shape of the disk profile. Knowing one of the stresses, the other may be found from the equation with the aid of quadratures. It should be particularly noted that after the stresses σ and τ are computed the values of the radial displacements at the various radii of the disk are directly obtained from formulas (5).

By the second method, instead of this system of two equations of the first order there is obtained a single differential equation of the second order for the radial displacement ξ . For this purpose, σ and τ are eliminated from equation (2) with the aid of equations (5) to obtain

$$\frac{d^2\xi}{dr^2} + \left(\frac{d \ln y}{dr} + \frac{1}{r} \right) \frac{d\xi}{dr} + \left(\frac{\nu}{r} \frac{d \ln y}{dr} - \frac{1}{r^2} \right) \xi + \frac{1 - \nu^2}{E} \operatorname{cr} \left(1 + \frac{\eta}{y} \right) = 0 \quad (8)$$

If for a given profile this displacement equation can be solved, the stresses may then be found by equations (5) solved for the stresses:

$$\left. \begin{aligned} \sigma &= \frac{E}{1 - \nu^2} \left(\frac{df}{dr} + \nu \frac{f}{r} \right) \\ \tau &= \frac{E}{1 - \nu^2} \left(\nu \frac{df}{dr} + \frac{f}{r} \right) \end{aligned} \right\} \quad (9)$$

The system of equations (2) and (7) or equation (8) is the basis for computing the stresses in a given disk as well as for the choice of profile for given stresses. At the various parts of the disk it is generally assumed

that $\frac{\eta}{y} = \text{constant}$, which condition is applicable to a large number of the steps. This permits each step to be considered as free of external loads, but with density increased by $\frac{y + \eta}{y}$ times. For this reason disks without loads along the radius are considered.

Stresses in Disks for Load Applied at the Rim

In the absence of loads at the sides of the disk $\eta = 0$. The formulas of the stresses in the form given by Stodola for hyperbolic disks are first presented. Then it is shown that another method of integration leads to considerably simpler relations.

The equation of the hyperbolic profiles is

$$y = \frac{B}{r^\alpha} \quad (10)$$

Then $\frac{d \ln y}{dr} = -\frac{\alpha}{r}$ and equation (8) leads to the equation of Euler. By integrating it, Stodola finally arrives at the following form of the stress equations

$$\left. \begin{aligned} \sigma &= \frac{E}{1 - \nu^2} [(3 + \nu)ar^2 + (\psi_1 + \nu)b_1r^{\psi_1-1} + (\psi_2 + \nu)b_2r^{\psi_2-1}] \\ \tau &= \frac{E}{1 - \nu^2} [(1 + 3\nu)ar^2 + (1 + \psi_1\nu)b_1r^{\psi_1-1} + (1 + \psi_2\nu)b_2r^{\psi_2-1}] \end{aligned} \right\} \quad (11)$$

where

$$a = - \frac{(1 - \nu^2) \gamma \omega^2}{Eg[8 - (3 + \nu)\alpha]}$$

ψ_1 and ψ_2 are the roots of the quadratic equation

$$\psi^2 - \alpha\psi - (1 + \nu\alpha) = 0 \quad (12)$$

and b_1 and b_2 are constants of integration determined by the boundary conditions.

Another order of integration that leads to considerably simpler relations between the stresses σ and τ is given.

Transformation of the Linear System of Differential Equations of the First Order

Equations (1) and (7) form a nonhomogeneous system of two linear differential equations of the first order. A substitution may be found by which the solution of any such system leads to the integration of a Riccati equation and quadratures.

Let the given system be

$$\left. \begin{aligned} \frac{d\sigma}{dr} &= a\sigma + b\tau + c \\ \frac{d\tau}{dr} &= a'\sigma + b'\tau + c' \end{aligned} \right\} \quad (13)$$

where $a, b, c, a', b',$ and c' are functions of r . By introducing the new variable

$$z = k\sigma + \tau \quad (14)$$

where k may likewise depend on r

$$\frac{dz}{dr} = \sigma \frac{dk}{dr} + k \frac{d\sigma}{dr} + \frac{d\tau}{dr}$$

or

$$\frac{dz}{dr} = \left(\frac{dk}{dr} + ak + a' \right) \sigma + (bk + b') \tau + (ck + c') \quad (15)$$

The value of k may be chosen so that the right-hand side depends only on z without containing σ and τ explicitly. For this it is sufficient that the ratio between the coefficients before σ and τ be the same as in $z = k\sigma + \tau$ - that is,

$$\left(\frac{dk}{dr} + ak + a' \right) : (bk + b') = k : 1 \quad (16)$$

Then

$$\frac{dz}{dr} = (bk + b')z + (ck + c') \quad (17)$$

that is, instead of the system (13) there is obtained a single linear equation for z .

Requirement (16), however, which must be satisfied for k is no other than the equation of Riccati:

$$\frac{dk}{dr} = bk^2 + (b' - a)k - a' \quad (18)$$

If two particular solutions of this equation can be found - denote them by m and n - there is obtained for each of them the corresponding values of z - denote them by S and D - and the variables σ and τ may be expressed by the two equations arising in place of (14):

$$\left. \begin{aligned} m\sigma + \tau &= S \\ n\sigma + \tau &= D \end{aligned} \right\} \quad (19)$$

These equations contain two arbitrary constants. In

this case the system (13) on the basis of equations (1) and (7) is of the form

$$\left. \begin{aligned} \frac{d\sigma}{dr} &= - \left(\frac{1}{r} + \frac{d \ln y}{dr} \right) \sigma + \frac{\tau}{r} - c r \\ \frac{d\tau}{dr} &= \left(\frac{1}{r} - v \frac{d \ln y}{dr} \right) \sigma - \frac{\tau}{r} - v c r \end{aligned} \right\} \quad (20)$$

With $z = k\sigma + \tau$, equations (17) and (18) for z and k become

$$\frac{dz}{dr} = \frac{k-1}{r} z - c(k+v)r \quad (21)$$

and

$$\frac{dk}{dr} = \frac{1}{r} k^2 + \frac{d \ln y}{dr} k - \left(\frac{1}{r} - v \frac{d \ln y}{dr} \right) \quad (22)$$

Hyperbolic Profile

The preceding equations hold for any profile. The first does not at all depend on the shape of the disk and the second, for hyperbolic disks for which $y = \frac{B}{r^\alpha}$, assumes the form

$$\frac{dk}{dr} = \frac{1}{r} (k^2 - \alpha k - v\alpha - 1) \quad (23)$$

and evidently can be satisfied for constant values of k equal to the roots m and n of the quadratic equation:

$$k^2 - \alpha k - v\alpha - 1 = 0 \quad (24)$$

that is,

$$\left. \begin{aligned} m &= \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} + v\alpha + 1} \\ n &= \frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} + v\alpha + 1} \end{aligned} \right\} \quad (25)$$

These roots are real and different for any value of α . It is of interest to note that equation (24) is the same quadratic equation which figures in the solution of Stodola.

From the linear equation (21) the values of z are found. For $k = m$, for example,

$$z = m\sigma + \tau = e^{\int \frac{m-1}{r} dr} \left[A - \int c(m+v)re^{-\int \frac{m-1}{r} dr} dr \right]$$

or

$$m\sigma + \tau = r^{m-1} \left[A - (m+v)c \frac{r^3 - m}{3 - m} \right]$$

where A is the constant of integration.

Adding, by analogy, a similar equation for the other root $k = n$ and setting

$$\left. \begin{aligned} M &= c \frac{m+v}{3-m} \\ N &= c \frac{n+v}{3-n} \end{aligned} \right\} \quad (26)$$

finally, there is obtained for the relations between the stresses*

$$\left. \begin{aligned} \frac{m\sigma + \tau + Mr^2}{r^{m-1}} &= \text{constant} \\ \frac{n\sigma + \tau + Nr^2}{r^{n-1}} &= \text{constant} \end{aligned} \right\} \quad (27)$$

*If $m = 3$, which occurs for $\alpha = \frac{8}{3+v}$ (for steel 2.42) then a logarithmic function appears in the integration and the first of relations (27) assumes the form

$$\frac{3\sigma + \tau + c(3+v)r^2 \ln r}{r^2} = \text{constant}$$

A similar added note should be made to the solution of Stodola.

Equations (27) are much simpler than the solution in the form given by Stodola as represented by equations (11). It is immaterial for the computation that equations (27) are not solved for the stresses, since it is another factor that is essential: namely, that the equations contain only a single constant each and thus permit easy passage from the stresses at the inner radius of the disk to the stresses at the outer radius.

It is possible to express directly the final stresses in terms of the initial, as was done by Arrowsmith. Equations (27) readily permit finding these expressions in the form

$$\sigma_2 = A\sigma_1 + B\tau_1 - Ckr_1^2$$

$$\tau_2 = E\sigma_1 + F\tau_1 - Gkr_1^2$$

After such transformation, however, formulas (27) lose their simplicity and are less convenient for computations. The coefficients of the equations of Arrowsmith are cumbersome and further on it is found necessary to construct tables.

Generalization of the Method of Donath

A relation which exists between equations (27) and the method of Donath is observed.

What equations (27) become in the case of flat disks shall be considered. Setting $\alpha = 0$, there is found $m = 1$, $n = -1$; whence

$$\left. \begin{aligned} \tau + \sigma + Mr^2 &= \text{constant} \\ \tau - \sigma + Nr^2 &= \frac{\text{constant}}{r^2} \end{aligned} \right\} \quad (28)$$

according to which - leading to Donath's result for flat disks - operate with the sum and difference of the stresses in place of the stresses themselves. The expressions for the sum or difference of the stresses contain a single constant each. It is therefore necessary to know their values for one radius in order to determine their values

over the entire disk. Each stress by itself does not possess this property, since it involves two constants.

It can now be seen that in equations (27) for hyperbolic disks there should be considered not only the stresses themselves but also their combinations $m\sigma + \tau$ and $n\sigma + \tau$. Equations (27) thus generalize the method of Donath to hyperbolic disks.

Before proceeding to the description of the computation, the manner in which equations (27) may be derived from Stodola's solution will be observed. Each of equations (27) contains only a single constant. In order to obtain a similar kind of equation from Stodola's solution which contains both arbitrary constants, it is necessary to solve these equations for the constants. Bearing in mind that the roots of the quadratic equation (12) are connected by the relations

$$\psi_1 + \psi_2 = \alpha; \quad \psi_1 \psi_2 = -(1 + \nu\alpha); \quad \frac{1 + \nu\psi_1}{\nu + \psi_1} = -\psi_2; \quad \frac{1 + \nu\psi_2}{\nu + \psi_2} = -\psi_1,$$

the formulas thus derived agree accurately with equations (27).

The latter consideration permits a still further generalization of the method of Donath. For any profile the solution is expressed in the form

$$\sigma = C_1\varphi_1(r) + C_2\varphi_2(r) + \varphi_3(r)$$

$$\tau = C_1\psi_1(r) + C_2\psi_2(r) + \psi_3(r)$$

where C_1 and C_2 are the constants of integration. By solving these equations for the constants, equations similar to equations (27) are obtained. By this method, for example, the solution of Malkin for disks with profiles

$$y = \alpha e^{-\beta r^{1+\frac{1}{3}}}$$

might be simplified.

Computation of Hyperbolic Ring

A fundamental and repeated operation in the computation is the transition from the stresses at the inner radius of the ring to the stresses at the outer radius. Let r_1 , σ_1 , and so forth, be the values at the inner radius and r_2 , σ_2 , and so forth, the values at the outer radius.

$$\text{Setting} \quad \left. \begin{aligned} Mr^2 &= a \\ Nr^2 &= b \end{aligned} \right\} \quad (29)$$

Previously equation (19) was obtained

$$S = m\sigma + \tau$$

$$D = n\sigma + \tau$$

Equations (27) are rewritten as

$$\left. \begin{aligned} \frac{S + a}{r^{m-1}} &= \text{constant} \\ \frac{D + b}{r^{n-1}} &= \text{constant} \end{aligned} \right\} \quad (30)$$

whence

$$\left. \begin{aligned} S_2 + a_2 &= p (S_1 + a_1) \\ D_2 + b_2 &= q (D_1 + b_1) \end{aligned} \right\} \quad (30')$$

or

$$\left. \begin{aligned} S_2 &= p (S_1 + a_1) - a_2 \\ D_2 &= q (D_1 + b_1) - b_2 \end{aligned} \right\} \quad (31)$$

where

$$p = \left(\frac{r_2}{r_1} \right)^{m-1}; \quad q = \left(\frac{r_2}{r_1} \right)^{n-1} \quad (32)$$

Equations (31) are the fundamental equations giving the transition from radius r_1 to r_2 .

Conversely, knowing S and D , σ and τ are obtained from equation (19):

$$\left. \begin{aligned} \sigma &= \frac{S - D}{m - n} \\ \tau &= D - n\sigma \end{aligned} \right\} \quad (33)$$

The computation procedure is as follows: when the values σ_1 and τ_1 at the first radius are known, the values S_1 and D_1 are obtained by formulas (19):

$$S_1 = m\sigma_1 + \tau_1$$

$$D_1 = n\sigma_1 + \tau_1$$

Next, pass to the second radius by formulas (31) and finally return to the stresses by formulas (33):

$$\sigma_2 = \frac{S_2 - D_2}{m - n}$$

$$\tau_2 = D_2 - n\sigma_2$$

Computation of Ring of Constant Thickness

In this case, since $\alpha = 0$, $m = 1$, $n = -1$

$$\left. \begin{aligned} S &= \tau + \sigma \\ D &= \tau - \sigma \end{aligned} \right\} \quad (34)$$

$$\left. \begin{aligned} M &= \frac{1 + \nu}{2} c \\ N &= - \frac{1 - \nu}{4} c \end{aligned} \right\} \quad (34a)$$

$$\left. \begin{aligned} S_2 &= S_1 + a_1 - a_2 \\ D_2 &= q(D_1 + b_1) - b_2 \end{aligned} \right\} \quad (35)$$

$$\left. \begin{aligned} p &= 1 \\ q &= \left(\frac{r_1}{r_2} \right)^2 \end{aligned} \right\} \quad (36)$$

$$\left. \begin{aligned} \sigma_2 &= \frac{S_2 - D_2}{2} \\ \tau_2 &= D_2 + \sigma_2 \end{aligned} \right\} \quad (37)$$

These formulas lead essentially to the scheme of Yanovsky for flat disks.

Computation of Disks of Arbitrary Profile

(See following example and collection of formulas.)

Division of the profile into successive rings.— By taking the radius along one axis and the disk thickness along the other the profile to logarithmic scale is plotted. (On ordinary graph paper the values of $\lg r$ and $\lg y$ may be plotted.) The obtained curve is replaced by a broken line the sections of which correspond to the hyperbolic parts of the profile since a straight line in the logarithmic plot corresponds to the equation

$$y = \frac{B}{r^\alpha}$$

where the exponent α is the negative of the slope of the straight line. The coefficient B for a hyperbolic section may be found by the formula

$$\alpha = \frac{\lg y_1 - \lg y_2}{\lg r_2 - \lg r_1} \quad (38)$$

where y_1 and y_2 are the thickness of the profile at radii r_1 and r_2 , respectively.

In choosing the broken line, in order to decrease the number of steps, it is sometimes useful to disjoin the ends of the sections of the broken line, a procedure which corresponds to discontinuities in the thickness of the disk (although the given disk does not contain such discontinuities).

When the value of α is known for each section, the auxiliary magnitudes* m , n , p , q , M , N , a , and b are found from formulas (25), (26), (29), and (32). The value of m may also be taken from table I and since m and n are the roots of the quadratic equation (24), then

$$n = \alpha - m \quad (38')$$

Stresses. Principle of Von Mises. Discontinuities in Thickness.— Usually the radial stresses at the inner and outer radius of the disk are given. At the inner radius the stress is taken either equal to zero or to the bearing pressure on the shaft and in the latter case it should be considered as negative. (Kearton and Ostertag erroneously take that pressure to be positive.) At the outer radius in the absence of loads the stress is also taken as zero. In the presence of blades the centrifugal force of the latter creates a radial stress which must be computed in advance.

Not knowing the values of the tangential stress T_0 at the inner radius the computation, following Von Mises, is conducted twice. For the first computation any arbitrary value T_0 is taken. With the aid of computation experience of the given type of machine this value can be chosen to lie near the true value by choosing suitable

coefficients in the expression $\frac{\gamma \omega^2 R^2}{g}$, which gives the stresses in a rotating ring. The stresses in the successive rings of the disk are then computed. The stresses at the end of one step are taken for the initial stresses in the following step. An exception is made in the case where the thickness of the ring changes discontinuously at the boundary of the two steps.

*To compute p and q slide rules with log-log scales which raise a number to any power are convenient.

Let y be the thickness of the disk before and $y + \Delta y$ the thickness after the point of discontinuity. The stresses σ and τ will then likewise receive discontinuous increments $\Delta\sigma$ and $\Delta\tau$ which are computed by the formulas

$$\left. \begin{aligned} \Delta\sigma &= - \frac{\Delta y}{y + \Delta y} \sigma \\ \Delta\tau &= \nu \Delta\sigma \end{aligned} \right\} \quad (39)$$

where σ is the stress immediately before the point of discontinuity. It is recalled that the derivation of these formulas is based on the assumption of uniform distribution of the stresses over a cylindrical section of the disk. The equilibrium condition of an element for $dr = 0$ in this case becomes

$$y\sigma = (y + \Delta y)(\sigma + \Delta\sigma)$$

whence there is also obtained the first of relations (39). The second may be written on the basis of equations (5) in view of the equality of the radial displacements before and after the discontinuity. There is obtained

$$\tau - \nu\sigma = (\tau + \Delta\tau) - \nu(\sigma + \Delta\sigma)$$

whence

$$\Delta\tau = \nu \Delta\sigma$$

The second computation differs from the first essentially in that this time the disk is assumed stationary: $\omega = 0$. Hence the magnitudes c , M , N , a , and b become zero. The transition formulas to the new radius are simplified. They are written:

$$\left. \begin{aligned} S_2 &= pS_1 \\ D_2 &= qD_1 \end{aligned} \right\} \quad (40)$$

The initial tangential stress in the second computation may likewise be chosen arbitrarily, but the initial radial stress must be taken equal to zero.

By completing the second computation similar to the first for the entire disk (what was said with regard to

the discontinuities holds likewise for the second computation) the true stresses are obtained by multiplying all stresses of the second computation by a certain constant coefficient k and adding to the stresses obtained by the first computation for corresponding radii:

$$\left. \begin{aligned} \sigma_{\text{true}} &= \sigma_I + k \sigma_{II} \\ \tau_{\text{true}} &= \tau_I + k \tau_{II} \end{aligned} \right\} \quad (42)$$

where the coefficient k is found from the equation

$$\sigma_I + k \sigma_{II} = \sigma_{\text{true}}$$

It may be noted that the first computation corresponds to finding a particular integral of a nonhomogeneous differential equation and the second to the solution of the corresponding homogeneous equation; the general solution being the sum of the first and the second multiplied by a constant and evidently satisfying the boundary conditions.

Computation Check

By checking, the following formulas are convenient:

1. For checking p and q :

$$pq = \left(\frac{r_2}{r_1} \right)^{\alpha-2} \quad (44)$$

2. For checking a and b :

$$\frac{a}{b} = \frac{m + \frac{1+3\nu}{3+\nu}}{n + \frac{1+3\nu}{3+\nu}} \quad (45)$$

3. For checking the stresses at the rings:

$$m\sigma_2 + \tau_2 + a_2 = p(m\sigma_1 + \tau_1 + a_1) \quad (46)$$

In the second computation since $a_1 = a_2 = 0$, the following formula is used

$$m\sigma_2 + \tau_2 = p(m\sigma_1 + \tau_1) \quad (47)$$

The check is made immediately after obtaining the corresponding values. The accuracy obtainable with the common slide rule giving three significant figures is in general sufficient for the computation. The entire procedure also is clear from inspection of table 2 which should be filled in vertically in sequence. The columns for S and D , however, are simultaneously filled in with the stresses σ and τ for each step.

Example

The data for the disk that was computed by Holzer is chosen, making certain changes introduced by Yanovsky (reference 1). It is required to compute the stresses in a steel disk (fig. 2) with outside diameter 132 centimeters, inside diameter of the rim 120 centimeters, outside diameter of the hub 24 centimeters, inside diameter of the hub 16 centimeters, thickness of hub 11 centimeters, and thickness of rim 2.8 centimeters. The disk is to run at 3000 rpm. The load due to the centrifugal force of the blades under these conditions is 400 kilograms per square centimeter of the outer area of the rim. The radial stress at the inner surface of the hub is equal to zero.

The thickness of the disks at the various radii is given as follows:

r (cm)	y (cm)	Section	r	y
8	} 11	1	8	11
12			12	11
14		2	12	11
18			19.2	3.2
22.5		3	19.2	3.2
27.5			40	2.1
32.5		4	40	2.1
37.5			60	1.24
42.5		5	60	2.8
47.5			66	2.8
52.5	} 2.8			
57.5				
60				
66				

These values are plotted on logarithmic paper (fig. 3), from which it is seen that with great accuracy the obtain curve is replaced by a broken line of five sections, and a profile composed of three hyperbolic steps and two of constant thickness is obtained. These data are in columns 1 and 2 of table 2.

The auxiliary magnitudes are computed. For the hyperbolic steps α is found either graphically (fig. 3) or by formula (38). Corresponding to the value of α , table 1 gives the value of m . Further

$$n = \alpha - m$$

With m and n known, p and q are found by formulas (32) and are checked by formulas (44). Find M and N by formulas (26), compute a and b by formulas (29), and check a and b by formula (45). Now proceed to the first computation of the stresses.

First Step

Here are given $\sigma_1 = 0$; τ_1 may be arbitrarily assumed. Assume $\tau_1 = 1000$ kilograms per square centimeter.

$$S_1 = m\sigma_1 + \tau_1 = 1 \times 0 + 1000 = 1000$$

$$D_1 = n\sigma_1 + \tau_1 = -1 \times 0 + 1000 = 1000$$

Pass to the next radius ($r = 12$):

$$S_2 = p(S_1 + a_1) - a_2 = 1(1000 + 32.6) - 73.5 = 959.1$$

$$D_2 = q(D_1 + b_1) - b_2 = 0.445(1000 - 8.8) + 19.8 = 460.8$$

The stresses are

$$\sigma_2 = \frac{S_2 - D_2}{m - n} = \frac{959.1 - 460.8}{1 - (-1)} = 249$$

$$\tau_2 = D_2 - n\sigma_2 = 460.8 - (-1) 249 = 710$$

The entire step is checked by formula (46).

Second Step

Since there is no discontinuity in thickness between the first and second steps, the final stresses of the first step are taken as the initial stresses of the second:

$$\sigma_1 = 249 \quad \tau_1 = 710$$

and

$$S_1 = m\sigma_1 + \tau_1 = 3.19 \times 249 + 710 = 1505, \text{ and so forth.}$$

Account of the discontinuity in thickness must be taken in the fifth step. Denoting the number of the step by a superscript (for example, $\sigma_1^{(4)}$) will result by equation (39):

$$\Delta\sigma = - \frac{\Delta y}{y + \Delta y} \sigma_2^{(4)} = - \frac{1.56}{2.8} \times 348 = -194$$

whence

$$\sigma_1^{(5)} = \sigma_2^{(4)} + \Delta\sigma = 348 - 194 = 154$$

Further

$$\Delta\tau = \nu \Delta\sigma = 0.3(-194) = -58.2$$

$$\tau_1^{(5)} = \tau_2^{(4)} + \Delta\tau = 617 - 58.2 = 558.8$$

Proceed next to the last radius to complete the computation. (See table 2.)

The second computation differs from the first only in that a and b are equal to zero.

First Step

In every case $\sigma_1 = 0$; τ_1 as before is arbitrary. Assume, for example, $\tau_1 = 1000$ kilograms per square centimeter

$$S_1 = m\sigma_1 + \tau_1 = 1000$$

$$D_1 = n\sigma_1 + \tau_1 = 1000$$

$$S_2 = pS_1 = 1 \times 1000 = 1000$$

$$D_2 = qD_1 = 0.445 \times 1000 = 445, \text{ and so forth.}$$

Step	r_{cm}	y_{cm}	a	m	n	$\frac{r_2}{r_1}$	p	q	M	N	r^2	a	b	S	D	σ_I	τ_I	S	D	σ_{II}	τ_{II}	$k\sigma_{II}$	$k\tau_{II}$	σ	τ
1	8 12	11 11	0	1	-1	1,5	1	0,445	0,51	-0,1375	64 144	32,6 73,5	- 8,8 - 19,8	1000 959,1	1000 460,8	0 249,2	1000 710	1000 1000	1000 445	0 278	1000 723	0 132	475 343	0 381	1417 1053
2	12 19,2	11 3,2	2,63	3,19	-0,56	1,6	2,8	0,48	-14,43	-0,0574	144 369	-2080 -5330	- 8,26 - 21,2	1505 3720	570,5 291,2	249,2 914	710 802	1610 4510	567 272	278 1130	723 904	132 537	343 429	381 1451	1053 1231
3	19,2 40	3,2 2,1	0,574	1,407	-0,833	2,08	1,347	0,26	0,841	-0,109	369 1600	311 1348	- 40,2 - 174,3	2087 1877	42 174,8	914 758	802 805,8	2494 3360	37 9,61	1130 1490	904 1250	537 708	429 594	1451 1466	1231 1400
4	40 60	2,1 1,24	1,3	1,995	-0,695	1,5	1,497	0,503	1,79	-0,084	1600 3600	2860 6440	-134,5 -302,0	2317 1310	279,8 375	758 348	805,8 617	4220 6310	215 108	1490 2310	1250 1713	708 1095	594 815	1466 1443	1400 1432
5	60 66	2,8 2,8	0	1	-1	1,1	1	0,826	0,51	-0,1375	3600 4350	1837 2220	-495 -598	712,8 329,8	404,8 523,5	154 -96,6	558,8 427	2346 2346	306 253	1020 1045	1326 1298	485 497	630 616	639 400	1189 1043

Table II.- Collection of formulas.

$$\alpha = \frac{\lg y_1 - \lg y_2}{\lg r_2 - \lg r_1} \quad (38)$$

$$m, n = \frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} + \nu\alpha + 1} \quad (25)$$

or from table I.

$$p = \left(\frac{r_2}{r_1}\right)^{m-1}; \quad q = \left(\frac{r_2}{r_1}\right)^{n-1} \quad (32)$$

$$M = c \frac{m+\nu}{3-m}; \quad N = c \frac{n+\nu}{3-n} \quad (26)$$

$$a = M r^2; \quad b = N r^2 \quad (29)$$

$$S = m\sigma + \tau; \quad D = n\sigma + \tau \quad (19)$$

$$\left. \begin{aligned} S_2 &= p(S_1 + a_1) - a_2 \\ D_2 &= q(D_1 + b_1) - b_2 \end{aligned} \right\} \text{For first computation} \quad (31)$$

$$\left. \begin{aligned} S_2 &= p S_1 \\ D_2 &= q D_1 \end{aligned} \right\} \text{For second computation} \quad (40)$$

$$\sigma_2 = \frac{S_2 - D_2}{m - n}; \quad \tau_2 = D_2 - n\sigma_2 \quad (33)$$

$$\Delta\sigma = -\frac{\Delta y}{y + \Delta y} \cdot \sigma; \quad \Delta\tau = \nu \cdot \Delta\sigma \quad (39)$$

$$\sigma_I + k\sigma_{II} = \sigma; \quad \tau_I + k\tau_{II} = \tau \quad (42)$$

Check

$$pq = \left(\frac{r_2}{r_1}\right)^{\alpha-2} \quad (44)$$

$$\frac{a}{b} = \frac{m + \frac{1+3\nu}{3+\nu}}{n + \frac{1+3\nu}{3+\nu}} \left(\text{For steel } \frac{a}{b} = \frac{m+0,576}{n+0,576} \right) \quad (45)$$

$$m\sigma_2 + \tau_2 + a_2 = p(m\sigma_1 + \tau_1 + a_1) \quad (46)$$

For steps of constant thickness:

$$\alpha = 0$$

$$m = 1$$

$$n = -1$$

$$\left. \begin{aligned} M &= c \frac{1+\nu}{2} \\ N &= -c \frac{1-\nu}{4} \end{aligned} \right\} \text{For steel } \begin{aligned} M &= 0,65 \cdot c \\ N &= -0,175 \cdot c \end{aligned}$$

$$p = 1$$

$$q = \left(\frac{r_1}{r_2}\right)^2$$

$$\left. \begin{aligned} S &= \tau + \sigma \\ D &= \tau - \sigma \end{aligned} \right\}$$

$$\left. \begin{aligned} S_2 &= S_1 + a_1 - a_2 \\ D_2 &= q(D_1 + b_1) - b_2 \end{aligned} \right\}$$

$$\sigma_2 = \frac{S_2 - D_2}{2}$$

$$\tau_2 = D_2 + \sigma_2$$

$$\sigma_I + k\sigma_{II} = \sigma$$

$$\tau_I + k\tau_{II} = \tau$$

Check

$$\frac{a}{b} = -2 \frac{1+\nu}{1-\nu}; \quad \text{For steel } \frac{a}{b} = -3,71$$

$$\sigma_2 + \tau_2 + a_2 = \sigma_1 + \tau_1 + a_1$$

True Stresses

The coefficient k is found from the condition that $\sigma_2^{(s)} = 400$ kilograms per square centimeter. The equation $\sigma_I + k\sigma_{II} = \sigma_{true}$ in the given case will be

$$-96.6 + k 1045 = 400$$

whence

$$k = 0.475$$

Multiplying all σ and τ of the second computation by this number and adding σ and τ of the first computation according to equation (42):

$$\sigma_I + k\sigma_{II} = \sigma_{true}$$

$$\tau_I + k\tau_{II} = \tau_{true}$$

yields the true stresses.

For comparison there are given on figure 2 the stresses according to Holzer and Yanovsky, the dots indicating the stresses according to table 2. Notwithstanding that there were only 3 hyperbolic steps instead of 10 which were chosen in computing with constant thickness steps almost complete agreement was obtained. It would have been possible to decrease the number of steps still further in the given computation without any great impairment of the accuracy. The computations were all made with a 25-centimeter slide rule.

Disks with Laterally Arranged Blades

As has been said, a lateral load on the disk is taken into account by a nominal increase in the thickness of the disk, the added centrifugal force being equivalent to the external load. In other words, the density is increased while maintaining the same loaded area. The increased density over each step is generally considered constant. This is permissible for steps that are not very wide. In this case the foregoing scheme requires no changes except

that in computing $c = \frac{\gamma \omega^2}{g}$ at each step it is necessary to take for γ

$$\gamma_{\text{equiv.}} = \gamma \left(1 + \frac{\eta}{\gamma} \right)$$

In the case where the lateral load corresponds to the full centrifugal force of the blades the added thickness of the disk η is expressed through the area of the cylindrical section of the blade f at radius r , thus

$$\eta = \frac{nf}{2\pi r} \quad (48)$$

where n is the number of blades,

If, in conducting the computation by hyperbolic steps very wide rings are used, the increase in density due to the lateral blades cannot always be considered constant over the step. A more accurate method for computing the load will be indicated. It is assumed that at each step the added thickness depends on the radius according to the law

$$\eta = \lambda r^\epsilon \quad (49)$$

With two parameters in the preceding formula available, they may be chosen so that η very closely approaches the added thickness required. The substitution of this expression in the differential equation of the stresses hardly affects the integration.

To determine the parameters λ and ϵ for the given step, the values given by formula (48) for the outer and inner radii of the step may be required to agree with the true values:

$$\begin{aligned} \lambda r_1^\epsilon &= \eta_1 \\ r_2^\epsilon &= \eta_2 \end{aligned}$$

whence

$$\epsilon = \frac{\lg \eta_2 - \lg \eta_1}{\lg r_2 - \lg r_1} \quad (50)$$

For the case under consideration this gives

$$\epsilon = \frac{\lg f_2 - \lg f_1}{\lg r_2 - \lg r_1} - 1$$

The value of λ further on is not directly used.

Now return to the differential equation of the stresses. In system (20) there are changed only the terms free from unknowns which are increased in the same ratio as the density - that is, by the factor $(1 + \frac{\eta}{y})$. With the same reasoning, and setting as before

$$z = k\sigma + \tau$$

it is noted that equation (23) for k not containing the density maintains its form. In equation (21), however, for z the factor $(1 + \frac{\eta}{y})$ appears in the free term containing the density. By substituting for η its value $\eta = \lambda r^\epsilon$ and integrating this linear equation for both values $k = m$ and $k = n$ two equations again are derived for the stresses analogous to equations (30):

$$\left. \begin{aligned} \frac{S + A}{r^{m-1}} &= \text{constant} \\ \frac{D + B}{r^{nb-1}} &= \text{constant} \end{aligned} \right\} \quad (51)$$

where as before $S = m\sigma + \tau$, $D = n\sigma + \tau$.

The values of A and B are obtained from the previous a and b :

$$\left. \begin{aligned} A &= a \left(1 + M' \frac{\eta}{y} \right) \\ B &= b \left(1 + N' \frac{\eta}{y} \right) \end{aligned} \right\} \quad (52)$$

where

$$\left. \begin{aligned} M' &= \frac{3 - m}{3 + \epsilon + n} \\ N' &= \frac{3 - n}{3 + \epsilon + m} \end{aligned} \right\} \quad (53)$$

The remainder of the computation is not changed. What has been said with regard to the choice of profile, the two computations according to Von Mises, and the discontinuity in the stresses remains true.

II. CONVERSE PROBLEM - CHOICE OF DISK PROFILE FOR GIVEN DISTRIBUTION OF STRESSES

This problem was considered by Holzer. (Holzer considered the case where the load of the disk is applied at the rim as corresponds to the conditions of steam turbines.) According to him, the profile should be chosen so that the radial and tangential stresses almost over the entire disk are near the maximum admissible for the given material. By approximating to the condition of uniform strength economy of material is obtained. The general procedure of Holzer's solution is the following: A curve of variation of radial stresses is chosen which satisfies the boundary conditions and rises rapidly to a maximum value which is maintained practically over the entire disk. (See, for example, fig. 5 or 6.) From equation (7) of the interrelated stresses:

$$\frac{d\tau}{dr} - \nu \frac{d\sigma}{dr} = \frac{1 + \nu}{r} (\sigma - \tau)$$

the value of the tangential stress is found, the initial value of the tangential stress (at the inner edge of the disk) being assumed a maximum, since in practical cases the tangential stresses usually attain a maximum at the inner side of the hub. (See under the section on estimating the maximum tangential stresses.) The thickness of the profile may after this be found from differential equation (2):

$$\frac{d(ry\sigma)}{dr} - y\tau + cr^2(y + \eta) = 0$$

where σ and τ are now known.

Holzer considers disks for which the entire load is concentrated at the rim. In this case the equation permits separation of the variables. The initial values of the thickness of the disk are generally given by structural considerations.

The foregoing indicated solution of the problem is suitable for disks for which the radial stress at the outer radius does not become equal to zero. Such, for example, are disks of steam turbines carrying blades on the rim. (Zero stress at the inner edge of the disk has no significance. Over a certain distance it is here, in general, not necessary to choose a profile because the hub is assumed of constant thickness.) Such disks were considered by Holzer and Yanovsky.

The case, otherwise pertains to disks having zero radial stress at the outer radius. Such, for example, are the disks of blowers carrying blades at the sides or disks without external load. For these the method of Holzer is not directly applicable. (The reason for this is the occurrence of singular points in the differential equation of the profile. For $\sigma = 0$ the coefficients of this equation become infinite.) In these cases the shape of the radial stress curve is subject to additional restrictions which if not observed, lead to practically unsuitable profiles - such as disks of infinitely increasing thickness, disk with negative thickness, and so forth. Moreover, even for finite edge thickness of the disk the thickness cannot always be arbitrarily assumed as in the method of Holzer. In this case it is necessary to solve the problem of which conditions the chosen curve must satisfy in order that the disk have the required edge thickness.

By studying the differential equation (2) of the profile, criteria which determine the character of the profile near the edge are obtained. Practically, they give methods for the choice of profiles in the previously mentioned cases where the solution of Holzer is inapplicable.

As was mentioned previously, in practical cases the center hole of the disk is taken up by a hub of constant thickness. The investigation of the shape of the profile may therefore be limited to the outer part of the disk. Disks without holes will not be considered since for a load concentrated at the rim Laval disks solve the problem of uniform strength. In the case of loads distributed along the radius, the problem is solved almost in the same manner as for a disk with hole.

Case of Zero Radial Stresses at the Outer Edge of the Disk

In the absence of loads over the disk radius the differential equation of the profile becomes homogeneous. The more simple case will be considered first.

A. Disks without external load. - Differential equation (2) which defines the disk profile in the given case becomes

$$\frac{dy}{dr} + \frac{p}{\sigma} y = 0 \quad (54)$$

where

$$p = \frac{d\sigma}{dr} + \frac{\sigma - \tau}{r} + cr \quad (55)$$

At the free outer radius $r = R$, the radial stress σ is equal to zero; hence the coefficient p/σ before y here becomes infinite. If the case of finite slope of the σ curve at the edge is considered - that is, assume $\frac{d\sigma}{dr} \neq 0; \infty$ - setting

$$\frac{p}{\sigma} = \frac{p_0 + p_1(R - r) + p_2(R - r)^2 + \dots}{R - r}$$

which means that the factor p/σ has a simple pole at $r = R$.

From analytical theory of linear differential equations the behavior of the integral y near a singular point is judged and thus the character of the profile is obtained. Thus the theorem of Fuchs* applied to the simple case under consideration shows that the integral is "proper" and should be of the form

$$y = C(R - r)^k [1 + b_1(R - r) + b_2(R - r)^2 + \dots]$$

where C is an arbitrary constant and k is a certain exponent.

The shape of the profile depends on whether $k > 0$, $k = 0$, or $k < 0$. If $k < 0$, then evidently $y \rightarrow \infty$,

*See, for example, Goursat, Course in Mathematical Analysis.

when $r \rightarrow R$. The thickness of the disk increases infinitely. (See fig. 4.) If $k = 0$, the disk maintains a constant thickness. Finally, if $k > 0$ the thickness of the disk approaches zero. Which of the three cases occurs depends on the choice of the radial stress curve and on the initial value of the tangential stress.

In order to obtain quantitative criteria, as well as a general picture, the problem will be considered more in detail. For simplification of the computation, set

$$x = R - r$$

The equation of the profile becomes

$$\frac{dy}{dx} = \frac{p}{\sigma} y \quad (56)$$

where

$$\frac{p}{\sigma} = \frac{P(x)}{x} \quad (57)$$

and

$$P(x) = x \frac{p}{\sigma} = p_0 + p_1 x + p_2 x^2 + \dots \quad (58)$$

By integrating there is obtained

$$y = C e^{\int \frac{p}{x} dx} = C e^{\int \left(\frac{p_0}{x} + p_1 + p_2 x + \dots \right) dx} = C e^{p_0 \ln x + p_1 x + \frac{p_2}{2} x^2 + \dots}$$

or

$$y = C x^{p_0} (1 + h_1 x + h_2 x^2 + \dots) \quad (59)$$

an integral the form of which is that demanded by the theorem of Fuchs. The arbitrary constant C has only positive values.

The foregoing expressions for the profile may be written in a form suitable for computation:

$$y = C x^{p_0} e^{\int_0^x \frac{p - p_0}{x} dx} \quad (60)$$

The number p_0 determining the character of the profile is thus expressed:

$$p_0 = \lim_{x \rightarrow 0} \left(x \frac{p}{\sigma} \right) = \lim_{x \rightarrow 0} \frac{p}{\frac{\sigma}{x}} = \lim_{r \rightarrow R} \frac{\frac{d\sigma}{dr} + \frac{\sigma - \tau}{r} + cr}{-\frac{\sigma_r - \sigma_R}{r - R}}.$$

that is,

$$p_0 = - \frac{\left(\frac{d\sigma}{dr} \right)_R - \frac{\tau_R}{R} + cR}{-\left(\frac{d\sigma}{dr} \right)_R} \quad (61)$$

by which the dependence of the type of profile on the slope of the curve of radial stresses and on the initial value of the tangential stress is determined.

The criteria given previously may be expressed in the explicit form shown in figure 4 by using the expression for p_0 .

For a fuller characterization of the type of profile, consider the derivative

$$\frac{dy}{dr} = - \frac{dy}{dx} = -Cx^{p_0-1} (p_0 + g_1x + g_2x^2 + \dots)$$

There are three cases

1. $p_0 > 0$; $y \rightarrow 0$ for $r \rightarrow R$ (fig. 4, 1a, b, c)
2. $p_0 = 0$; $y \rightarrow C$ for $r \rightarrow R$ (fig. 4, 2)
3. $p_0 < 0$; $y \rightarrow +\infty$ for $r \rightarrow R$ (fig. 4, 3)

The first case contains several possibilities:

- 1a. $p_0 > 1$; $y \rightarrow 0$, $\frac{dy}{dr} \rightarrow 0$ for $r \rightarrow R$ (fig. 4, 1a)
- 1b. $p_0 = 1$; $y \rightarrow 0$, $\frac{dy}{dr} \rightarrow -C$ for $r \rightarrow R$ (fig. 4, 1b)
- 1c. $0 < p_0 < 1$; $y \rightarrow 0$, $\frac{dy}{dr} \rightarrow \infty$ for $r \rightarrow R$ (fig. 4, 1c)

The practically significant case is that for which the curve of radial stresses approaches zero from the positive side. The steeper the drop in the curve of radial stresses the greater the disk expands toward the edge for otherwise equal conditions.

Example 1. The condition that the thickness of the disk at the edge is finite and different from zero was obtained in the form:

$$\left| \frac{d\sigma}{dr} \right|_R = cR - \frac{\tau}{R}$$

It is not difficult to confirm that this condition is satisfied, for example, for a solid disk of constant thickness the stresses of which, as is known, are expressed by the formulas

$$\sigma = \frac{3 + \nu}{8} \frac{\gamma \omega^2}{g} (R^2 - r^2)$$

$$\tau = \frac{3 + \nu}{8} \frac{\gamma \omega^2}{g} \left(R^2 - \frac{1 + 3\nu}{3 + \nu} r^2 \right)$$

Example 2. To determine within what limits the slope of the radial stress curve must be taken in order to avoid infinite expansion of the disk profile. The disk is of steel - diameter 1 meter, rotations per minute 3000, initial value of the tangential stress at the outer radius about 500 kilograms per square centimeter.

The preceding formulas show that the slope of the radial stress curve must satisfy the condition

$$\left| \frac{d\sigma}{dr} \right| \leq cR - \frac{\tau}{R}$$

In this case $R = 50$ centimeters, $c = \frac{\gamma \omega^2}{g} = 0.785$ kilogram per centimeter⁴, and there is found

$$\left| \frac{d\sigma}{dr} \right|_R \leq 29.3 \text{ kilograms per square centimeter}$$

that is, the drop in the σ curve at the edge should not exceed 29.3 kilograms per square centimeter for 1 centimeter of radius.

B. Disks with Load Distributed along the Radius.-

Taking, as before:

$$x = R - r; \quad \left(\frac{d\sigma}{dr}\right)_R \neq 0; \quad \infty; \quad p = \frac{d\sigma}{dr} + \frac{\sigma - T}{r} + cr$$

there is found

$$\frac{dy}{dx} = \frac{p}{\sigma} y + \frac{c\eta(R - x)}{\sigma} \quad (62)$$

Assume as before that

$$P(x) = x \frac{p}{\sigma} = p_0 + p_1 x + p_2 x^2 + \dots$$

and similarly

$$Q(x) = x \frac{c\eta(R - x)}{\sigma} = q_0 + q_1 x + q_2 x^2 + \dots \quad (63)$$

Consider that the "added thickness," as is usually the case, is not zero: $\eta R > 0$. Integrating equation (62)

$$y = e^{\int \frac{p}{x} dx} \left(C + \int \frac{Q}{x} e^{-\int \frac{p}{x} dx} dx \right)$$

As in the previous case

$$e^{\int \frac{p}{x} dx} = x^{p_0} (1 + p_1 x + \dots)$$

whence

$$e^{-\int \frac{p}{x} dx} = x^{-p_0} (1 - p_1 x + \dots)$$

Setting

$$I = \int \frac{Q}{x} e^{-\int \frac{p}{x} dx} dx$$

There is obtained

$$I = \int \frac{q_0 + q_1 x + \dots}{x} x^{-p_0} (1 - p_1 x + \dots) dx$$

or

$$I = \int x^{-p_0-1} (q_0 + l_1 x + l_2 x^2 + \dots) dx$$

Consider the various cases:

$$1. \quad p_0 < 0$$

In this case the expression for I cannot contain logarithms and there is obtained

$$I = x^{-p_0} \left(-\frac{q_0}{p_0} + l_1' x + \dots \right)$$

By substituting in the expression for y

$$y = (1 + p_1 x + \dots) \left(C x^{p_0} - \frac{q_0}{p_0} + l_1' x + \dots \right)$$

Since $p_0 < 0$ for $r \rightarrow R$ or $x \rightarrow 0$ the thickness of the disk y will increase infinitely if $C \neq 0$. It is necessary to have the arbitrary constant $C > 0$ since when $C < 0$ the thickness of the disk approaches $-\infty$; that is, no disk is obtained. If $C = 0$ the thickness of the profile approaches $-q_0/p_0$. That this is a positive value will be shown. Consider the case $p_0 < 0$. Further, according to equation (63)

$$q_0 = \frac{c \eta_R R}{-\left(\frac{d\sigma}{dr}\right)_R}$$

Since $\eta_R > 0$ and assume as before that the curve of radial stresses drops to zero from the positive side, $q_0 > 0$. This means that the value of the thickness of the disk is positive.

$$2. \quad p_0 = 0$$

In this case there is obtained

$$I = q_0 \ln x + l_1 x + \dots$$

$$y = (1 + p_1 x + \dots)(C + q_0 \ln x + l_1 x + \dots)$$

It is noted that $q_0 > 0$. For this reason the thickness of the disk at the edge approaches negative infinity. This case is impossible - that is, the assumed stresses are not realized for any actual disk.

$$3. \quad p_0 > 0$$

The thickness of the disk is

$$y = x^{p_0}(1 + p_1 x + \dots)(C + I)$$

where

$$I = \int x^{-(p_0+1)}(q_0 + l_1 x + \dots)dx$$

The first term in I is equal to

$$- \frac{q_0}{p_0} x^{-p_0}$$

The remaining terms are of the form

$$\frac{L_i}{x^{p_0-i}}$$

where

$$i = 1, 2, 3, \dots$$

Moreover, if p_0 is an integer $\ln x$ may enter. The thickness of the disk will be

$$y = (1 + p_1 x + \dots) C x^{p_0} - \frac{q_0}{p_0} + \lambda x^{p_0} \ln x + L_1 x + \dots$$

where certain of the coefficients λ and L_i may vanish.

Since for $p_0 > 0$

$$\lim_{x \rightarrow 0} x^{p_0} \ln x = 0$$

and the thickness of the disk at the edge

$$y_R = - \frac{q_0}{p_0}$$

where p_0 and q_0 are positive; therefore the thickness of the disk is negative.

Summarizing, it is noted $\eta_R \neq 0$:

1. The case $p_0 \geq 0$ does not correspond to any real disk.

Explicitly this condition is

$$\left| \frac{d\sigma}{dr} \right|_R \leq cR - \frac{\tau}{R} \quad (64)$$

2. The case $p_0 < 0$ or

$$\left| \frac{d\sigma}{dr} \right|_R > cR - \frac{\tau}{R} \quad (65)$$

leads on the one hand to infinitely expanded disks and on the other to the only possible disk of finite edge thickness equal to

$$y_R = - \frac{q_0}{p_0}$$

or

$$y_R = \frac{c \eta R}{\frac{\tau}{R} - \left(\frac{d\sigma}{dr} \right)_R - cR} \quad (66)$$

In choosing a disk profile the slope of the radial stress curve at the edge should be greater than a certain

value given by the inequality equation (65). Moreover, the thickness of the disk at the edge cannot arbitrarily be assumed if the stresses are given. This constitutes the essential difference as compared with the method of Holzer.

Only the last case where the disk does not expand infinitely is of practical interest. For this case an expression for the thickness that is capable of giving numerical values can be readily obtained. It is found that

$$e^{\int \frac{p}{x} dx} = x^{p_0}(1 + h_1 x + \dots)$$

The expression in the parenthesis which is denoted by $f(x)$ is equal to:

$$f(x) = e^{\int_0^x \frac{p-p_0}{x} dx}$$

The solution for y , previously written under the assumption that the constant of integration C becomes zero, leads to the required expression:

$$y = x^{p_0} f(x) \int_0^x \frac{Q(x)}{x^{p_0+1} f(x)} dx \quad (67)$$

The integral exists, since $p_0 < 0$. This expression gives the thickness of the disk over its entire extent.

The practical computation is more conveniently based on the fact that the thickness of the disk at radius a not far from the edge may be considered equal to y_R . Taking for the new variable

$$z = r\sigma y \quad (68)$$

rewrite equation (2) thus:

$$\frac{dz}{dr} + \frac{cr - \frac{1}{r}}{\sigma} z + cr^2 \eta = 0 \quad (69)$$

By integrating this differential equation and setting

$$\left. \begin{aligned} p &= \frac{cr - \frac{1}{r}}{\sigma} \\ \varphi &= \int \frac{pdr}{r} \\ \psi &= e^{\varphi} \\ \theta &= \int_{r_c}^a \frac{\pi r^2}{\psi} dr \end{aligned} \right\} \quad (70)$$

there is obtained

$$z = \psi(z_a + c \theta) \quad (71)$$

Since the value of the radius a was taken near R , set as previously

$$z_a = r_a \sigma_a \nu R$$

and obtain the thickness of the disk at all radii by formula (71).

It will be shown with the aid of an example that these formulas are applicable for the choice of a disk profile loaded by lateral blades but first, however, two more problems are considered.

On the Limit of the Tangential Stresses

In selecting a profile both for the case considered by Holzer and that of disks with laterally attached blades, assume the distribution of the radial stresses as given and the initial value of the tangential stress at the inner edge of the disk equal to the maximum radial stress. An essential consideration in this connection is whether the tangential stresses may exceed the radial stresses if so and to what extent.

If $\sigma_0 = 0$, $\tau_0 = \sigma_{\max}$, and σ is everywhere greater than zero, then

$$0 < \tau < (1 + \nu) \sigma_{\max}$$

In selecting the profile these conditions may be considered to be satisfied since for steel disks - for example,

$$\tau < 1.3 \sigma_{\max}$$

This limit, in the general case, cannot be lowered because, for rapid increase in the radial stress from zero to the maximum, τ may approach this limit as nearly as is desired. If, however, as σ increases to the maximum the tangential stresses do not go beyond the limit $\sigma_{\max}$ then, as will be shown, they do not exceed this limit over the entire disk. Also an interesting remark by Holzer should be mentioned that at the parts where the radial stresses remain constant the tangential stress approaches the radial asymptotically.

All these statements are valid for disks independent of their profiles and of the character of the loads. They are based on the equation giving the relation between the stresses:

$$\frac{d\tau}{dr} = \nu \frac{d\sigma}{dr} + (1 + \nu) \frac{\sigma - \tau}{r}$$

which itself does not depend on the shape of profile or on the load.

Now proceed to the proof. Assuming in the above equation $\sigma(r)$ to be a known function, the tangential stress is obtained

$$\tau = \frac{1}{r^{1+\nu}} \left\{ C + \int r^{1+\nu} \left(\nu \frac{d\sigma}{dr} + \frac{1+\nu}{r} \sigma \right) dr \right\}$$

Integrating by parts and determining the constant of integration from the conditions, $\sigma = 0$; $\tau = \tau_0$, for $r = r_0$, results in

$$\tau = \left(\frac{r_0}{r} \right)^{1+\nu} \tau_0 + \nu \sigma + \frac{1 - \nu^2}{r^{1+\nu}} \int_{r_0}^r \sigma r^\nu dr \quad (72)$$

The tangential stresses always remain positive since all terms are positive.

Set

$$\theta = \left(\frac{r_0}{r}\right)^{1+\nu}$$

Evidently θ is always less than unity.

$$\frac{1}{r^{1+\nu}} \int_{r_0}^r \sigma r^{\nu} dr < \sigma_{\max} \frac{1 - \theta}{1 + \nu}$$

Since, moreover, $\tau_0 = \sigma_{\max}$, it is found from equation (72)

$$\tau < (1 + \nu \theta) \sigma_{\max}$$

and

$$\tau < (1 + \nu) \sigma_{\max} \quad (73)$$

Moreover, by equation (72);

$$\tau > \left(\frac{r_0}{r}\right)^{1+\nu} + \nu \sigma$$

If the maximum σ is attained at radius r_1 (fig. 5), then

$$\tau_1 > \left[\left(\frac{r_0}{r_1}\right)^{1+\nu} + \nu \right] \sigma_{\max}$$

If the increase in σ is such that r_1 only slightly differs from r_0 the tangential stress τ_1 will slightly differ from $(1 + \nu) \sigma_{\max}$. It is seen that the tangential stresses, in general, may exceed the radial and the limit set by the inequality (73) cannot be lowered without added restrictions. The correctness of the remark by Holzer follows from the fact that at the intervals with positive radial stress according to the equation of the interrelated stresses

$$\tau = \sigma = \frac{\text{constant}}{r^{1+\nu}}$$

With increasing radius the difference between the two stresses approaches zero, while the tangential stress remains at all times either greater than or less than the radial. Passing to the latter characteristic, it will be shown that if over the interval r_0, r_1 (fig. 5), where the radial stress increases from zero to the maximum, r does not exceed $\sigma_{\max}$ then the tangential stresses do not exceed $\sigma_{\max}$ over the entire disk.

From equation (72), having all of its terms positive, it may be seen that if, without varying the initial values τ_0 and $\sigma_0 = 0$, a new curve of radial stresses is taken which everywhere lies above or coincides with the first, the tangential stresses can only increase as a result of such substitution. The initial curve can now be changed so that starting from point B (fig. 5) it stays at the level of $\sigma_{\max}$ up to the end. The corresponding tangential stresses being at radius r_1 less than $\sigma_{\max}$ remain such to the end, since $\sigma = \text{constant}$. According to what was said this should all the more be true as regards the initial tangential stresses. Thus the tangential stresses over the entire disk do not exceed $\sigma_{\max}$.

Simplification of Holzer's Method

for Loads Applied at the Rim

Holzer assumes a graphically given σ curve or represents the parts of the curves by series:

$$\sigma = a_0 + a_1 r + a_2 r^2 + \dots$$

To find the profile in this case it is necessary to resort to one of the approximate methods for computing integrals. In view of this, the solution of Holzer is considered practically complicated.

The solution can, however, be made very simple. It is only necessary to seek a function to approximate the σ curve at various sections so that differential equations (7) and (1) determining the tangential stress are integrated exactly and not approximately.*

*This approach repeats the idea of the application of constant thickness disks or of hyperbolic disks to

The simplest method would be to represent the radial stresses by a broken line applying to each part the linear law;

$$\sigma = kr + \beta$$

For the tangential stresses, from differential equation (7), it is found in this case

$$\left(\tau - \sigma + \frac{1 - \nu}{2 + \nu} kr \right) r^{1+\nu} = \text{constant}$$

or, for practical computation:

$$\tau_2 - \sigma_2 + \mu r_2 = \frac{1}{Q} (\tau_1 - \sigma_1 + \mu r_1) \quad (74)$$

where

$$\mu = \frac{1 - \nu}{2 + \nu} k; \quad Q = \left(\frac{r_2}{r_1} \right)^{1+\nu} \quad (75)$$

Difficulties are encountered, however, in computing the profile.

If, however, there is assumed for the radial stresses at the various sections, instead of a linear, an exponential law likewise, with two parameters:

$$\sigma = \beta r^k$$

no difficulties are met with either in computing the tangential stresses or in finding the profile. For the tangential stresses there is obtained according to equation (7);

$$(\tau - \lambda \sigma) r^{1+\nu} = \text{constant} \quad (76)$$

compute the stresses in the given disk. In both cases it is easier to solve the differential equation relying on functions for which the integration is readily carried out than to make use of general approximation methods. This fact is not taken into account in the methods of Keller and Holzer.

where

$$\lambda = \frac{1 + \nu + k\nu}{1 + \nu + k} \quad (77)$$

For the practical computation of a step:

$$\tau_2 = \frac{\tau_1 - \lambda\sigma_1}{Q} + \lambda\sigma_2 \quad (78)$$

where

$$Q = \left(\frac{r_2}{r_1}\right)^{1+\nu} \quad (79)$$

In the same manner, it is readily found from equation (1) for the profile

$$\ln \frac{y_1}{y_2} = \frac{1}{1 + \nu + k} \Delta\left(\frac{\tau}{\sigma}\right) + \frac{k + 2}{1 + \nu + k} \Delta \ln \sigma - \frac{1}{k + 2} \Delta\left(\frac{cr^2}{\sigma}\right)$$

where Δ denotes the increment in passing from radius r_1 to r_2 : for example,

$$\Delta\left(\frac{\tau}{\sigma}\right) = \frac{\tau_2}{\sigma_2} - \frac{\tau_1}{\sigma_1}, \quad \text{and so forth}$$

Set for briefness

$$\left. \begin{aligned} A &= 2.3(1 + \nu + k) \\ B &= \frac{k + 2}{1 + \nu + k} \\ D &= 2.3(2 - k) \end{aligned} \right\} \quad (80)$$

where the number 2.3 is the conversion factor for passing to common logarithms. There is obtained finally,

$$\lg \frac{y_1}{y_2} = \frac{1}{A} \Delta\left(\frac{\tau}{\sigma}\right) + B \Delta \lg \sigma + \frac{1}{D} \Delta\left(\frac{cr^2}{\sigma}\right) \quad (81)$$

The above equation gives the change in the profile thickness over a section of the disk. Together with equation

(78) it solves the problem of the choice of profile. Both formulas are exact and there is thus no need for approximate integration. The computation is particularly simple for sections where $\sigma = \text{constant}$. In this case the exponent k is zero; hence $\lambda = 1$ and instead of equation (78), the result is

$$\tau = \frac{\tau_1 - \sigma_1}{Q} + \sigma_2 \quad (82)$$

In the formula for the profile thickness (81) the coefficients are simplified:

$$A = 2.3 (1 + \nu)$$

$$D = 4.6$$

and B need not be computed, since

$$B \Delta \lg \sigma = 0$$

The computational procedure is as follows. The given curve of radial stresses is replaced by a curve of the type $\sigma = \beta r^k$, passing through the ends of the section. The exponent k is found by the formula:

$$k = \lg \frac{\sigma_2}{\sigma_1} : \lg \frac{r_2}{r_1} \quad (83)$$

It is not necessary to compute β . Further, using the initial value τ_0 there is found by formula (78) the tangential stress over the entire disk. Finally, assuming the thickness given at any radius - usually at the rim - there is found the entire profile by formula (81). In the computation only a small number of steps need be used, since the presence in the formula $\sigma = \beta r^k$ of two arbitrary parameters permits close approximation to the given form with only four or five steps.

Examples of Selection of Disk Profiles

Take two examples: one for the case of loads applied at the rim, the second for a disk with laterally attached blades.

Example 1. (See table 3.) Consider the problem which was solved by Holzer, and in somewhat changed form also by Yanovsky. Keep the conditions in the same form as given by Holzer.

It is required to construct the profile of a disk for the following data: outside diameter 132 centimeters, inner diameter of hub 16 centimeters, outside diameter 24 centimeters, number of rotations per minute 3000, load due to the centrifugal forces of the blades produces on the outer surface of the disk a radial stress of 400 kilograms per square centimeter; at the inner side of the hub there is no radial stress. The permissible stress of the material is 1500 kilograms per square centimeter. At the rim the thickness of the disk must be 2.8 centimeters.

First of all, assume the value of the radial stress, taking it equal to 1500 kilograms per square centimeter over practically the entire disk. As regards the edges starting from the side of the shaft, the initial value of the tangential stress is given here, assuming for it, as indicated, the maximum value: $\tau_0 = 1500$ kilograms per square centimeter.

Now compute the stresses at radius $r = 12$ centimeters since the hub constitutes a step of constant though as yet unknown thickness. For computing the stresses a knowledge of this thickness is not necessary. Thus, there is found (see first pt. table 3) for the radius 12 centimeters.

$$\sigma = 388 \text{ kilograms per square centimeter;}$$

$$\tau = 1070 \text{ kilograms per square centimeter}$$

From this value of σ the radial stress must be increased to its maximum value $\sigma = 1500$ kilograms per square centimeter. (See fig. 6.) Let this maximum be reached at radius $r = 18$ centimeters. (Too steep an increase in σ leads to the same sharp drop in the thickness of the disk.) This increase in σ_1 , as already said, follows the law

$$\sigma = \beta r^k$$

By formula (83) for the given step there is found

A Hub.

Step	r	q	M	N	r ²	a	b	S	D	σ	τ
1	8	0,445	0,511	-0,1376	64	32,7	- 8,8	1500	1500	0	1500
	12				144	73,5	-19,8	1460	684	388	1072

B. Variable part of profile.

Step	r	σ	lg σ	K	λ	Q	τ	$\frac{\tau}{\sigma}$	$\Delta\left(\frac{\tau}{\sigma}\right)$	cr ²	$\frac{cr^2}{\sigma}$	$\Delta\left(\frac{cr^2}{\sigma}\right)$	A	B	D	$\lg \frac{y_2}{y_1}$	y
2	12	388	2,5888	3,33	0,496	1,694	1070	2,76	-1,916	113,3	0,292	-0,122	10,65	1,15	-3,06	0,5357	9,06
	18	1500	3,1761				1267	0,844		255,0	0,170						2,64
3	18	1500	3,1761	0	1	4,99	1267	0,844	0,125	255,0	0,170	1,85	2,99	—	4,6	0,4438	2,64
	62	1500	3,1761				1453	0,969		3030	2,02						0,95
4	62	1500	3,1761	-21,2	0,254	1,085	1453	0,969	1,751	3030	2,02	6,56	-45,8	0,964	53,4	-0,4691	0,95
	66	400	2,6021				1089	2,72		3430	8,58						2,80

Table III.- Collection of formulas.

$$k = \frac{\lg \sigma_2 - \lg \sigma_1}{\lg r_2 - \lg r_1} \quad (83)$$

$$\lambda = \frac{1 + v + kv}{1 + v + k} \quad (77)$$

$$Q = \left(\frac{r_2}{r_1}\right)^{1+v} \quad (79)$$

$$\tau_2 = \frac{\tau_1 - \lambda \sigma_1}{Q} + \lambda \sigma_2 \quad (78)$$

$$\left. \begin{aligned} A &= 2,3(1 + v + k) \\ B &= \frac{k + 2}{1 + v + k} \\ D &= 2,3(2 - k) \end{aligned} \right\} \quad (80)$$

$$\lg \frac{y_1}{y_2} = \frac{1}{A} \Delta\left(\frac{\tau}{\sigma}\right) + B \cdot \Delta \lg \sigma + \frac{1}{D} \Delta\left(\frac{cr^2}{\sigma}\right) \quad (81)$$

At hub where $\sigma = \text{const.}$

$$k = 0$$

$$\lambda = 1$$

$$\tau_2 = \frac{\tau_1 - \sigma_1}{Q} + \sigma_2$$

$$A = 2,3 \cdot (1 + v)$$

$$D = 4,6$$

B - not computed

$$k = \frac{\lg 1500 - \lg 388}{\lg 18 - \lg 12} = 3.33$$

(See table 3.)

For the following step (table 3, third step) σ is kept at its maximum value, 1500 kilograms per square centimeter, and only in the ring from $r = 62$ to $r = 66$ centimeters (fourth step) is the radial stress lowered from 1500 to 400 kilograms per square centimeter. This drop is likewise made according to the law $\sigma = \beta r^k$, the value, -21.2 being obtained for k .

Thus assuming this variation in σ over the entire disk, proceed to the computation of the tangential stress τ . For this purpose the auxiliary coefficients λ and Q are required (see table 3); and τ is computed by formula (78) for the second, third, and fourth steps. For the initial value of τ for the second step $\tau = 1070$, previously obtained for the end of the first step, and so forth. Thus, for example, there is obtained for the second step

$$\tau_2 = \frac{\tau_1 - \lambda \sigma_1}{Q} + \lambda \sigma_2 = \frac{1070 - 0.496 \times 388}{1.694} + 0.496 \times 1500 = 1267$$

To find the thickness of the disk formula (81) is applied, beginning this time with the outer edge of the disk since there is here given the thickness $y = 2.8$ centimeters. Having filled in the columns of the auxiliary coefficients in the table, there is found for the last step, by formula (81)

$$\begin{aligned} \lg \frac{y_1}{y_2} &= \frac{1}{A} \Delta \left(\frac{I}{U} \right) + B \Delta \lg \sigma + \frac{1}{D} \Delta \left(\frac{cr^2}{\sigma} \right) = \\ &= \frac{1}{-45.8} \times 1.751 - 0.964 \times 0.574 + \frac{1}{53.4} \times 6.56 = -0.4691 \end{aligned}$$

since $y_2 = 2.8$ centimeters, there is obtained $y_1 = 0.95$. Using this value as the final for the third step yields for the initial thickness of this step $y_1 = 2.64$ centimeters, and so forth. The problem of finding the profile of the disk is thus solved. From the curves of figure 6

it is seen that the obtained profile and the values of the tangential stresses agree in a very satisfactory manner with solution of Holzer. Thus, the width of the hub was obtained by the author as 9.06 centimeters as compared with 9.32 obtained by Holzer. The author required 5 steps. By the method of Yanovsky based on steps of constant thickness 12 steps were required for the solution of this problem.

Example 2. (See table 4.) Take the problem of the improvement of the disk profile of the fan (fig. 7) described by Ostertag (reference 6). The disk makes 4000 rotations per minute. Starting from radius $r = 20$ centimeters, the effect of the blades is taken into account by introducing an additional width of 0.7 centimeter at both sides of the disk. The outside diameter of the disk is 86 centimeters, the inside diameter of the hub 15 centimeters, and the outside diameter of the hub 19 centimeters.

For the disk considered by Ostertag, the maximum stress is the radial stress $\sigma_{\max} = 1128$ kilograms per square centimeter. The tangential stresses are considerably less than the radial. The maximum radial stress is, however, reached in the form of a peak. All this shows that the disk is far from being of uniform strength.

The disk profile will be improved, in the first place, by making the tangential stresses approach the radial, and in the second place by maintaining the maximum radial stress $\sigma = 1128$ kilograms per square centimeter over a considerable distance. At the outer edge of the disk, in lowering the radial stress to zero, it is necessary to take account of the previously derived condition for finite thickness of the disk (65):

$$\left| \frac{d\sigma}{dr} \right|_R > cR - \frac{T}{R}$$

If it is required that the thickness of the disk at the outer edge should, for example, be 1 centimeter - as in Ostertag's example - then from formula (66) the approximate value of the slope of the radial stress curve can be found;

$$\left| \frac{d\sigma}{dr} \right|_R = \frac{c \eta R}{y} + cR - \frac{T}{R} = \frac{1.39 \times 1.4 \times 43}{1} + 1.39 \times 43 - \frac{T_R}{43} = 143.6 - \frac{T_R}{43}$$

TABLE 4

A. Hub

r	q	M	N	r^2	a	b	S	D	σ	τ
7.5	0.623	0.904	-0.243	56.2	50.8	-13.65	1128	1128	0	1128
9.5				90.3	81.5	-21.9	1097	716	191	907

B. Computation of Tangential Stresses

r	σ	k	μ	$\frac{r_2}{r_1}$	Q	τ
9.5	191	144	43.8	1.342	1.466	907
12.75	660	144	43.8	1.255	1.343	874
16	1128	0	0	1.25	1.337	1001
20	1128	0	0	1.2	1.267	1033
24	1128	0	0	1.166	1.221	1053
28	1128	-30.8	-9.35	1.143	1.19	1067
32	1005	-67.5	-20.5	1.125	1.165	1033
36	735	-95	-28.9	1.111	1.147	933
40	355	-118.3	-36	1.075	1.10	778
43	0					624

C. Computation of Profile Thickness

r	σ	τ	$r\sigma$	p	φ	ψ	$\frac{\eta r^2}{\psi}$	θ	z	y
9.5	191	907	1815	-0.431	-0.328	0.721	-----	17980	21800	12
12.75	660	874	8430	-.0772	.343	1.41	-----	17980	42600	5.05
16	1128	1001	18080	-.0358	.513	1.671	-----	17980	50500	2.79
18	1128	1018	20300	-.0279	.578	1.783	-----	17980	53800	2.67
20	1128	1033	22400	-.0212	.625	1.869	0/300	17980	56400	2.58
24	1128	1053	27100	-.0093	.685	1.984	410	16550	55900	2.06
28	1128	1067	31600	.00062	.702	2.02	543	14670	51700	1.63
32	1005	1033	32200	.0121	.686	1.985	723	12300	44300	1.38
36	735	933	26500	.0328	.593	1.81	1003	8750	31400	1.19
38	560	865	21300	.0536	.505	1.657	1230	6550	23700	1.11
40	355	778	14200	.1017	.361	1.435	1560	3760	14950	1.05
41	244	730	10000	.1607	.250	1.284	1810	2050	10340	1.034
42	125	680	5250	.3380	0	1	2470	0	5200	.99

Assuming τ_R a magnitude of the order of 500 to 700 kilograms per square centimeter (for $\tau_0 = 1128 \text{ kg/cm}^2$) yields the slope at the edge:

$$\left| \frac{d\sigma}{dr} \right|_R = 130 \text{ kilograms per cubic centimeter}$$

Passing to the selection of the radial stress curve over the entire disk, first compute the hub as a step of constant thickness. (See table 4A.) The initial values are $\sigma_0 = 0$,* $\tau_0 = 1128$ kilograms per square centimeter. There is found for the end of the hub: $\sigma = 191$ kilograms per square centimeter, $\tau = 907$ kilograms per square centimeter.

Then increase σ linearly (see fig. 7) to the maximum 1128 kilograms per square centimeter at radius $r = 16$ centimeters and keep it at this level up to radius $r = 28$ centimeters. Then σ is lowered to zero, assuming at the edge a drop of 130 kilograms per square centimeter per centimeter of radius.

With the curve of radial stresses given, the tangential stress is found. For this purpose apply formula (74), assuming that σ at each step follows the linear law. For the steps for which $\sigma = \text{constant}$, $\mu = 0$ is obtained.

All computed tangential stresses are given in table 4B. Figure 7 shows the obtained curve of tangential stress. For comparison, the curves of radial and tangential stresses obtained by Ostertag are shown by dotted curves. The new curves evidently much more nearly approach the condition of uniform strength.

There still remains to be computed the thickness of the profile corresponding to the new stresses. The thickness at the outer edge is found by formula (66):

$$y = \frac{c \eta R}{\frac{1}{R} - \left(\frac{d\sigma}{dr} \right)_R - cR} = \frac{1.39 \times 1.4 \times 43}{\frac{624}{43} + 130 - 1.39 \times 43} = 0.99 \text{ centimeter}$$

*Ostertag assumes $\sigma_0 = 10 \text{ kg/cm}^2$, taking the bearing pressure on the shaft as positive. This pressure should be considered as a negative stress. The same error is repeated by Kearton.

Then compute the thickness of the profile by (71) (see table 4C), the integral for the individual steps being approximately computed by the Simpson formula:

$$\int_a^b y dx \approx \frac{b-a}{6} (y_0 + 4y_1 + y_2)$$

The obtained profile is shown on figure 7 where, for comparison, the profile obtained by Ostertag is indicated by the dotted curve. It is seen that by approaching the uniform strength condition a considerable saving in material results.

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TABLE 1

α	m	Difference
0.0	1.0	0.066
.1	1.066	.068
.2	1.134	.070
.3	1.205	.072
.4	1.277	.074
.5	1.351	.076
.6	1.427	.077
.7	1.504	.079
.8	1.583	.080
.9	1.663	.082
1.0	1.745	.083
1.1	1.828	.084
1.2	1.912	.085
1.3	1.996	.086
1.4	2.082	.087
1.5	2.169	.087
1.6	2.256	.088
1.7	2.344	.089
1.8	2.433	.089
1.9	2.522	.090
2.0	2.613	.090
2.1	2.703	.091
2.2	2.794	.092
2.3	2.886	.092
2.4	2.978	.092
2.5	3.070	.093
2.6	3.163	.093
2.7	3.256	.094
2.8	3.349	.094
2.9	3.443	.094
3.0	3.537	

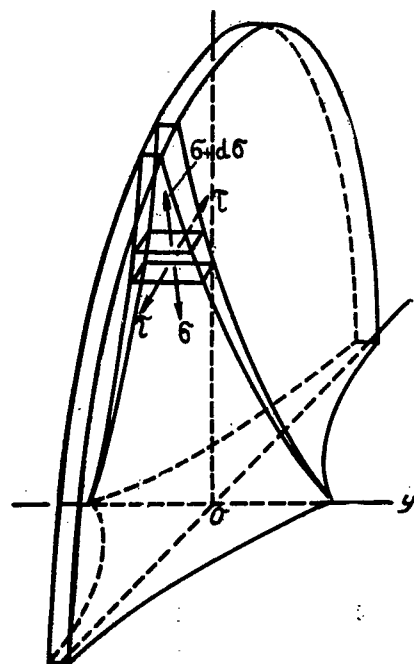


Figure 1.- Equilibrium of rotating disk element.

$\left \frac{d\sigma}{dr} \right _R < \frac{1}{2} \left(cR - \frac{\tau}{R} \right)$ $\text{for } r \rightarrow R \begin{cases} y \rightarrow 0 \\ \frac{dy}{dr} \rightarrow 0 \end{cases}$	
$\left \frac{d\sigma}{dr} \right _R = \frac{1}{2} \left(cR - \frac{\tau}{R} \right)$ $\text{for } r \rightarrow R \begin{cases} y \rightarrow 0 \\ \frac{dy}{dr} \rightarrow -C \end{cases}$	
$\frac{1}{2} \left(cR - \frac{\tau}{R} \right) < \left \frac{d\sigma}{dr} \right _R < cR - \frac{\tau}{R}$ $\text{for } r \rightarrow R \begin{cases} y \rightarrow 0 \\ \frac{dy}{dr} \rightarrow -\infty \end{cases}$	
$\left \frac{d\sigma}{dr} \right _R = cR - \frac{\tau}{R}$ $\text{for } r \rightarrow R \quad y \rightarrow C$	
$\left \frac{d\sigma}{dr} \right _R > cR - \frac{\tau}{R}$ $\text{for } r \rightarrow R \quad r \rightarrow \infty$	

Figure 4.- Types of profiles as functions of the stress distribution.

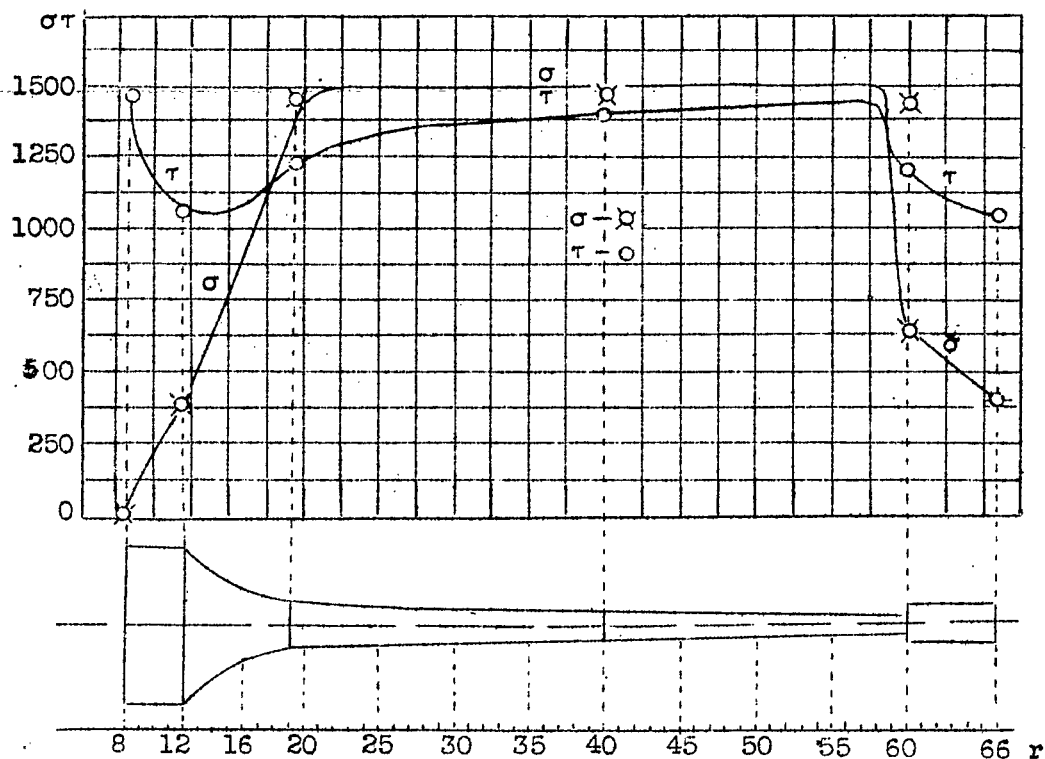


Figure 2.- Stress distribution in disk.

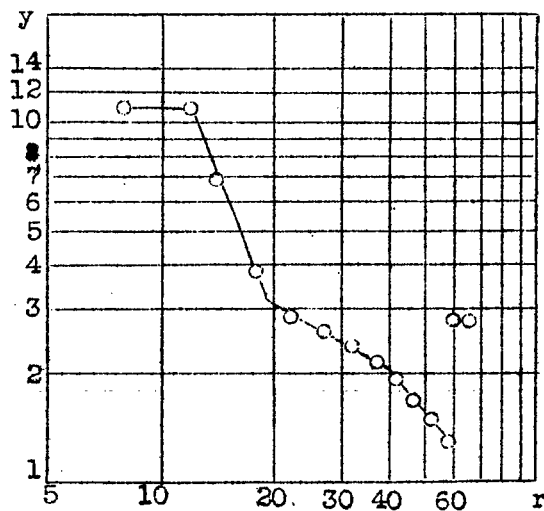


Figure 3.- Approximation to the profile with the aid of hyperbolic steps.

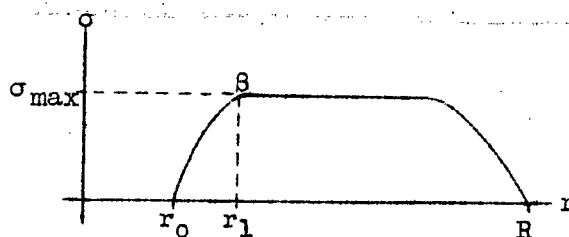


Figure 5.- General shape of radial stress curve.

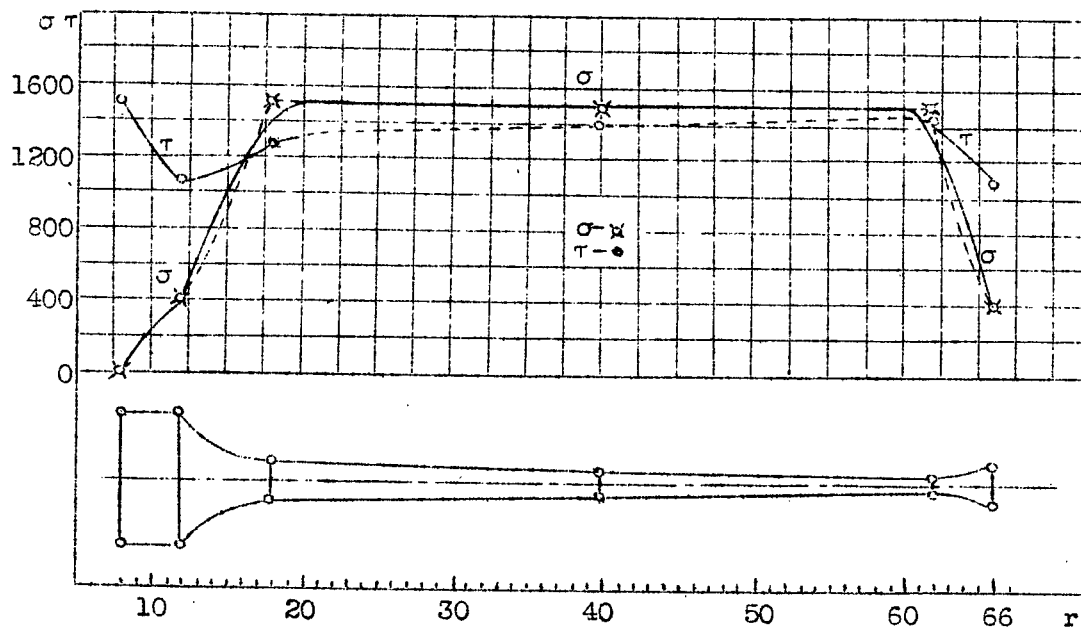


Figure 6.- Selection of profile. Solid curve gives the profile and stresses according to Holzer, dotted curve according to the method of stepped radial stresses.

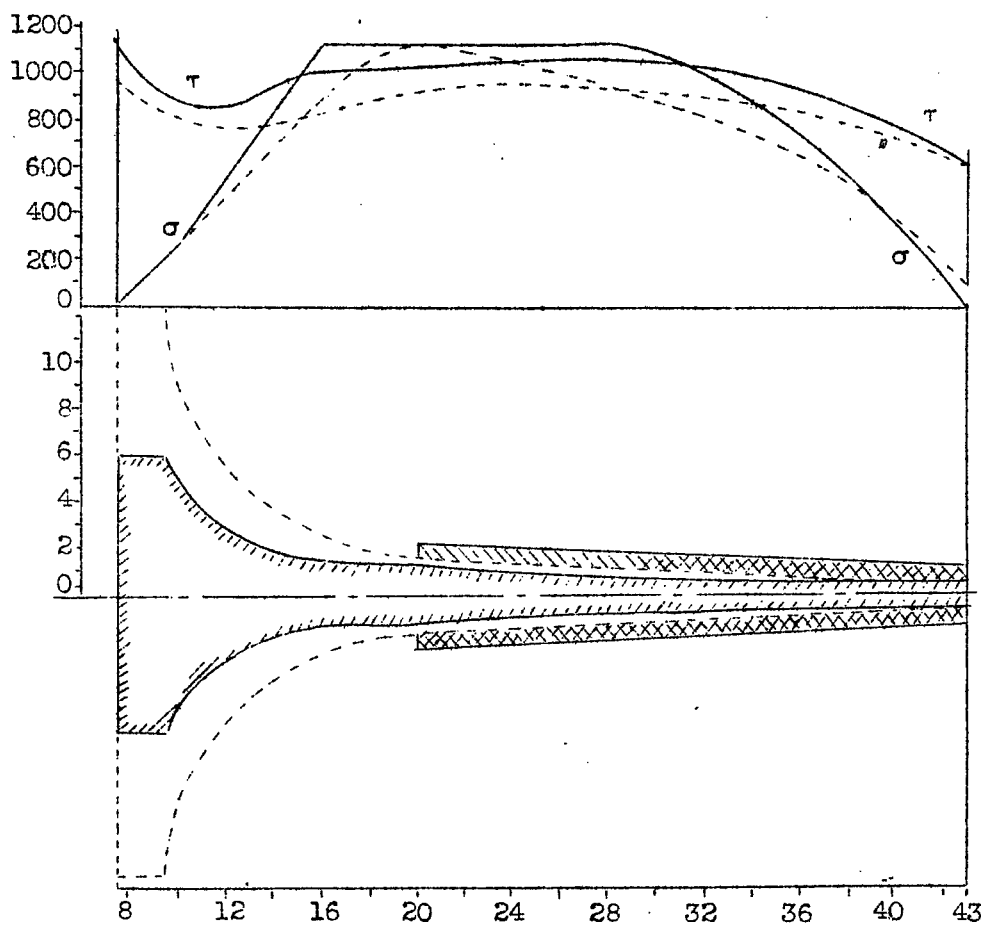


Figure 7.- Improvement of disk with lateral blades. Dotted curves give initial profile and stresses, improved profile and stresses given by solid curves,

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