Frequency integrated radiation models for absorbing and scattering media

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1. Motivation and objectives

The objective of this work is to contribute to the simplification of existing radiation models used in complex emitting, absorbing, scattering media. The application in view is the the computation of flows occuring in such complex media, such as certain stellar interiors or combusting gases. In these problems, especially when scattering is present, the complexity of the radiative transfer leads to a high numerical cost, which is often avoided by simply neglecting it. This complexity lies partly in the strong dependence of the spectral coefficients on frequency (Modest 2003; Siegel & Howell 2001). Models are then needed to capture the effects of the radiation when one cannot afford to directly solve for it. In this work, the frequency dependence will be modeled and integrated out in order to retain only the average effects. A frequency-integrated radiative transfer equation (RTE) will be derived. In it, the absorption and scattering will be treated through the use of mean coefficients (Siegel & Howell 2001 and references in it). To obtain these coefficients, it is needed to assume a form for the intensity, which we take to be the maximum entropy closure (Minerbo 1978). Such an intensity is a function of the macroscopic radiative energy and flux and accounts for the variations of radiation in the considered medium. Models for mean absorption, mean isotropic scattering, and mean non-isotropic scattering coefficients will be proposed in the case where the various spectral coefficients can be written as polynomial functions of the frequency. Some of these models have already been derived and tested for non-scattering media in (Ripoll *et al.* 2001; Ripoll & Wray 2004a). They are here extended to the general case of emitting, absorbing, and scattering media. A direct application will be given for soot, which follows a linear frequency law for absorption; isotropic and incoming scattering spectral models are also here roughly approximated by a linear law. Macroscopic radiation models will also be derived with absorption and scattering coefficients since they constitute an alternative to the use of the RTE in cases where this equation is too costly to solve, as in many coupled problems. Finally, we believe another application of these models could be for radiating flows occuring in dusty media.

2. A frequency integrated RTE with mean coefficients

2.1. *Generalities*

The radiative transfer equation (RTE) describes the evolution of the radiative intensity within a emitting, absorbing, and scattering medium and is given by

$$\frac{1}{c}\partial_t I+\mathbf{\Omega}\cdot\nabla I = \sigma^a(\nu)\mathcal{B}(\nu,T)-\sigma^a(\nu)I-\sigma^{is}(\nu)I+\frac{\sigma^s(\nu)}{4\pi}\int_{\mathbf{\Omega}'} I(\mathbf{\Omega}')\Phi(\nu,\mathbf{\Omega}\to\mathbf{\Omega}')d\mathbf{\Omega}' \quad (2.1)$$

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where the intensity $I = I(t, \mathbf{r}, \mathbf{\Omega}, \nu)$ is a function of the time t, the position $\mathbf{r}$, the direction of propagation $\mathbf{\Omega}$, and the frequency ν. Here c is the velocity of light, $\sigma^a(\nu)$ is the spectral absorption coefficient, $\sigma^{is}(\nu)$ the spectral isotropic scattering coefficient, $\sigma^s(\nu)$ the spectral incoming scattering coefficient, and $\mathbf{\Omega}'$ the original direction of radiation scattered into $\mathbf{\Omega}$. The Planck radiative intensity $\mathcal{B}$ describes the isotropic emission of the medium at the frequency ν and temperature T by

$$\mathcal{B}(\nu, T) = \frac{2h\nu^3}{c^2}\left[\exp(\frac{h\nu}{kT}) - 1\right]^{-1} \tag{2.2}$$

where h is the Planck constant, k the Boltzmann constant, and ν the frequency. We assume in this paper that the phase function of scattering $\Phi(\nu, \mathbf{\Omega} \to \mathbf{\Omega}')$ can be expressed as

$$\Phi(\nu, \mathbf{\Omega} \to \mathbf{\Omega}') = \alpha(\nu) + \beta(\nu)\mathbf{\Omega} \cdot \mathbf{\Omega}' + \gamma(\nu)(\mathbf{\Omega} \cdot \mathbf{\Omega}')^2 + \eta(\nu)\delta(\mathbf{\Omega} - \mathbf{\Omega}') \tag{2.3}$$

where $\delta(\cdot)$ is the Dirac delta function, and the coefficients, α, β, γ, and η are all in general functions of the frequency. They must be defined such that the following normalization property holds:

$$\frac{\sigma^s(\nu)}{4\pi} \int_{\mathbf{\Omega}} \Phi(\nu, \mathbf{\Omega} \to \mathbf{\Omega}') d\mathbf{\Omega} = \sigma^{is}(\nu). \tag{2.4}$$

2.2. *Derivation of the RTE with mean coefficients*

By integrating the RTE over frequency and introducing the quantity† $J(t, \mathbf{r}, \mathbf{\Omega}) = \int_0^\infty I(t, \mathbf{r}, \mathbf{\Omega}, \nu)\, d\nu \equiv \langle I \rangle_\nu$, we obtain

$$\begin{aligned} \frac{1}{c}\partial_t J + \mathbf{\Omega} \cdot \nabla J = \langle \sigma^a(\nu)\mathcal{B}(\nu, T)\rangle_\nu - \langle \sigma^a(\nu) I \rangle_\nu - \langle \sigma^s(\nu) I \rangle_\nu \\ + \langle \frac{\sigma^s(\nu)}{4\pi} \int_{\mathbf{\Omega}'} I(\mathbf{\Omega}')\Phi(\nu, \mathbf{\Omega} \to \mathbf{\Omega}') d\mathbf{\Omega}' \rangle_\nu. \end{aligned} \tag{2.5}$$

We now introduce the following mean absorption and scattering coefficients:

$$\sigma_P^{e,\nu}(T) = \frac{\langle \sigma^a(\nu)\mathcal{B}(\nu, T)\rangle_\nu}{\langle \mathcal{B}(\nu, T)\rangle_\nu} \tag{2.6}$$

$$\sigma_E^{a,\nu}(t, \mathbf{r}, \mathbf{\Omega}) = \frac{\langle \sigma^a(\nu) I(t, \mathbf{r}, \mathbf{\Omega}, \nu)\rangle_\nu}{\langle I(t, \mathbf{r}, \mathbf{\Omega}, \nu)\rangle_\nu} \tag{2.7}$$

$$\sigma_E^{is,\nu}(t, \mathbf{r}, \mathbf{\Omega}) = \frac{\langle \left[\sigma^{is}(\nu) - \sigma^s(\nu)\eta(\nu)/4\pi\right] I(t, \mathbf{r}, \mathbf{\Omega}, \nu)\rangle_\nu}{\langle I(t, \mathbf{r}, \mathbf{\Omega}, \nu)\rangle_\nu} \tag{2.8}$$

$$\sigma_E^{s}(t, \mathbf{r}) = \frac{\langle \sigma^s(\nu)\alpha(\nu) I(t, \mathbf{r}, \mathbf{\Omega}', \nu)\rangle_{\nu,\mathbf{\Omega}'}}{\langle I(t, \mathbf{r}, \mathbf{\Omega}', \nu)\rangle_{\nu,\mathbf{\Omega}'}} \tag{2.9}$$

$$\sigma_F^{s}(t, \mathbf{r}, \mathbf{\Omega}) = \frac{\langle \sigma^s(\nu)\beta(\nu)\mathbf{\Omega} \cdot \mathbf{\Omega}' I(t, \mathbf{r}, \mathbf{\Omega}', \nu)\rangle_{\nu,\mathbf{\Omega}'}}{\langle \mathbf{\Omega} \cdot \mathbf{\Omega}' I(t, \mathbf{r}, \mathbf{\Omega}', \nu)\rangle_{\nu,\mathbf{\Omega}'}} \tag{2.10}$$

$$\sigma_G^{s}(t, \mathbf{r}, \mathbf{\Omega}) = \frac{\langle \sigma^s(\nu)\gamma(\nu)(\mathbf{\Omega} \cdot \mathbf{\Omega}')^2 I(t, \mathbf{r}, \mathbf{\Omega}'\nu)\rangle_{\nu,\mathbf{\Omega}'}}{\langle (\mathbf{\Omega} \cdot \mathbf{\Omega}')^2 I(t, \mathbf{r}, \mathbf{\Omega}', \nu)\rangle_{\nu,\mathbf{\Omega}'}} \tag{2.11}$$

where the superscripts *e, a, is,* and *s* designate respectively emission, absorption, isotropic scattering, and scattering. We define $\chi(\nu) \equiv \sigma^s(\nu)\eta(\nu)/(4\pi\sigma^{is}(\nu))$. Using these definitions and without any assumptions other than the form of the phase function (2.3), the

† We will denote the integration of a function f over the variables X, Y, Z as $\langle f \rangle_{X,Y,Z}$

frequency-integrated RTE becomes

$$\frac{1}{c}\partial_t J+\mathbf{\Omega}\cdot\nabla J=\frac{\sigma_P^{e,\nu}}{4\pi}aT^4-\sigma_E^{a,\nu}J-\sigma_E^{is,\nu}J+\frac{1}{4\pi}\int_{\mathbf{\Omega}'}J(\mathbf{\Omega}')(\sigma_E^s+\sigma_F^s\mathbf{\Omega}\cdot\mathbf{\Omega}'+\sigma_G^s(\mathbf{\Omega}\cdot\mathbf{\Omega}')^2)\,d\mathbf{\Omega}' \tag{2.12}$$

where the constant $a = 8\pi^5k^4/(15h^3c^3)$. It should be noted that the mean scattering coefficients in the integral do not depend on $\mathbf{\Omega}$, unlike $\sigma_E^{a,\nu}$ and $\sigma_E^{is,\nu}$.

As a simplifying approximation, we now eliminate the $\mathbf{\Omega}$ dependence of the two mean coefficients σ_E^a and σ_E^{is}. This can be done by simply approximating the numerator and denominator in (2.7) and (2.8) by their $\mathbf{\Omega}$-integrated forms. Alternately, we integrate 2.12 indefinitely over $\mathbf{\Omega}$, using the polar variables $\mu = \cos\theta$ and ϕ, to obtain

$$\begin{aligned}\int \frac{1}{c}\partial_t J+\mathbf{\Omega}\cdot\nabla J\,d\mu\,d\phi = {}& \frac{\sigma_P^{e,\nu}}{4\pi}aT^4\mu\phi-\frac{\int\langle\sigma^a(\nu)I(t,\mathbf{r},\mathbf{\Omega},\nu)\rangle_\nu\,d\mu\,d\phi}{\int\langle I(t,\mathbf{r},\mathbf{\Omega},\nu)\rangle_\nu\,d\mu\,d\phi}\int J\,d\mu\,d\phi\\ &-\frac{\int\langle\sigma^{is}(\nu)\left[1-\chi(\nu)\right]I(t,\mathbf{r},\mathbf{\Omega},\nu)\rangle_\nu\,d\mu\,d\phi}{\int\langle I(t,\mathbf{r},\mathbf{\Omega},\nu)\rangle_\nu\,d\mu\,d\phi}\int J\,d\mu\,d\phi\\ &+\int\frac{1}{4\pi}\int_{\mathbf{\Omega}'}J(\mathbf{\Omega}')(\sigma_E^s+\sigma_F^s\mathbf{\Omega}\cdot\mathbf{\Omega}'+\sigma_G^s(\mathbf{\Omega}\cdot\mathbf{\Omega}')^2)\,d\mathbf{\Omega}'\,d\mu\,d\phi\end{aligned} \tag{2.13}$$

We now replace the indefinite integrals in the numerator and denominator of the second and third terms on the rhs of (2.13) with definite integrals over the full 4π of $\mathbf{\Omega}$ and then differentiate the resulting equation with respect to μ and ϕ.

This approximation allows us to define the following new mean absorption and scattering coefficients σ_P^e, σ_E^a, and σ_E^{is}, the last two being approximations of $\sigma_E^{a,\nu}$ and $\sigma_E^{is,\nu}$ respectively:

$$\sigma_P^e(T)=\frac{\langle\sigma^a(\nu)\mathcal{B}(\nu,T)\rangle_{\nu,\mathbf{\Omega}}}{\langle\mathcal{B}(\nu,T)\rangle_{\nu,\mathbf{\Omega}}}=\sigma_P^{e,\nu}(T) \tag{2.14}$$

$$\sigma_E^a(t,\mathbf{r})=\frac{\langle\sigma^a(\nu)I(t,\mathbf{r},\mathbf{\Omega},\nu)\rangle_{\nu,\mathbf{\Omega}}}{\langle I(t,\mathbf{r},\mathbf{\Omega},\nu)\rangle_{\nu,\mathbf{\Omega}}}\simeq\frac{\langle\sigma^a(\nu)I(t,\mathbf{r},\mathbf{\Omega},\nu)\rangle_{\nu,4\pi}}{\langle I(t,\mathbf{r},\mathbf{\Omega},\nu)\rangle_{\nu,4\pi}}=\sigma_E^{a,\nu} \tag{2.15}$$

$$\begin{aligned}\sigma_E^{is}(t,\mathbf{r})&=\frac{\langle\sigma^{is}(\nu)\left[1-\chi(\nu)\right]I(t,\mathbf{r},\mathbf{\Omega},\nu)\rangle_{\nu,\mathbf{\Omega}}}{\langle I(t,\mathbf{r},\mathbf{\Omega},\nu)\rangle_{\nu,\mathbf{\Omega}}}\\ &\simeq\frac{\langle\sigma^{is}(\nu)\left[1-\chi(\nu)\right]I(t,\mathbf{r},\mathbf{\Omega},\nu)\rangle_{\nu,4\pi}}{\langle I(t,\mathbf{r},\mathbf{\Omega},\nu)\rangle_{\nu,4\pi}}=\sigma_E^{is,\nu}.\end{aligned} \tag{2.16}$$

The elimination of the $\mathbf{\Omega}$ dependence of the mean absorption coefficients constitutes our first approximation. It is not strictly necessary to the closure: one could choose to not make it. The three mean incoming scattering coefficients are unchanged from (2.9)-(2.11). The frequency-integrated RTE with these approximations becomes

$$\begin{aligned}\frac{1}{c}\partial_t J+\mathbf{\Omega}\cdot\nabla J={}&\frac{\sigma_P^e}{4\pi}aT^4-\sigma_E^aJ-\sigma_E^{is}J\\ &+\frac{1}{4\pi}\int_{\mathbf{\Omega}'}J(\mathbf{\Omega}')(\sigma_E^s+\sigma_F^s\mathbf{\Omega}\cdot\mathbf{\Omega}'+\sigma_G^s(\mathbf{\Omega}\cdot\mathbf{\Omega}')^2)d\mathbf{\Omega}'\,.\end{aligned} \tag{2.17}$$

We now introduce the following macroscopic quantities: the radiative energy given by

$$E_R(t,\mathbf{r})=\frac{1}{c}\left\langle I(t,\mathbf{r},\mathbf{\Omega},\nu)\right\rangle_{\nu,\Omega}=\frac{1}{c}\left\langle J(t,\mathbf{r}\mathbf{\Omega})\right\rangle_\Omega \tag{2.18}$$

the radiative flux

$$\mathbf{F}_R(t,\mathbf{r}) = \langle \mathbf{\Omega}\, I(t,\mathbf{r},\mathbf{\Omega},\nu)\rangle_{\nu,\Omega} = \langle \mathbf{\Omega}\, J(t,\mathbf{r},\mathbf{\Omega})\rangle_{\Omega} \tag{2.19}$$

and finally the radiative pressure

$$\mathbf{P}_R(t,\mathbf{r}) = \frac{1}{c}\langle \mathbf{\Omega}\otimes\mathbf{\Omega}\, I(t,\mathbf{r},\mathbf{\Omega},\nu)\rangle_{\nu,\Omega} = \frac{1}{c}\langle \mathbf{\Omega}\otimes\mathbf{\Omega}\, J(t,\mathbf{r},\mathbf{\Omega})\rangle_{\Omega} \tag{2.20}$$

It should be noted that in (2.17) these moments appear in the integral term of the scattering. This term can then be absorbed in the definition of the moments and the integrated RTE written as

$$\begin{aligned}\frac{1}{c}\partial_t J + \mathbf{\Omega}\cdot\nabla J = \frac{\sigma_P^e}{4\pi}aT^4 - \sigma_E^a J - \sigma_E^{is} J \\ + \frac{1}{4\pi}\big(c\sigma_E^s E_R + \sigma_F^s \mathbf{F}_R\cdot\mathbf{\Omega} + c\sigma_G^s \sum_{i,j=1..3} P_R^{i,j}\Omega^i\Omega^j\big).\end{aligned} \tag{2.21}$$

For this equation four remarks should be given. First, the introduction of the mean coefficients allows expressing the scattering term as a function of the moments. Integrations of the intensity are still needed in order to compute the moments, and in that sense the formulations (2.17) and (2.21) are equivalent. Second, if the phase function has moments of order higher than 2, this will introduce moments of order higher than the pressure in the scattering term. Third, if the intensity used in the mean incoming scattering coefficients is the exact one, then the incoming scattering term is exact. Finally, this equation has been derived with two assumptions relative to the absorption and isotropic scattering terms, namely (2.15) and (2.16).

The frequency-integrated RTE (2.21) will be closed in the next section by proposing expressions for the mean coefficients σ_P^e, σ_E^a, σ_E^{is}, σ_E^s, σ_F^s, and σ_G^s in terms of the microscopic spectral coefficients σ^a, σ^{is}, σ^s, α, β, γ, and η.

2.3. *Closure of the radiative equation with mean coefficients*

The mean coefficients (2.15), (2.16), (2.9), (2.10), and (2.11) are closed in this section by assuming a particular functional form for the intensity used in their definitions. For σ_E^a for instance, it is assumed that

$$\sigma_E^a = \frac{\langle \sigma^a(\nu) I(t,\mathbf{r},\mathbf{\Omega},\nu)\rangle_{\nu},4\pi}{\langle I(t,\mathbf{r},\mathbf{\Omega},\nu)\rangle_{\nu,4\pi}} \simeq \frac{\langle \sigma^a(\nu) I^*(t,\mathbf{r},\mathbf{\Omega},\nu)\rangle_{\nu,4\pi}}{\langle I^*(t,\mathbf{r},\mathbf{\Omega},\nu)\rangle_{\nu,4\pi}} = \sigma_E^{a\,*} \tag{2.22}$$

where the assumed intensity is denoted I^*. Similarly we assume that

$$\sigma_E^{is} \simeq \sigma_E^{is\,*}, \quad \sigma_E^s \simeq \sigma_E^{s\,*}, \quad \sigma_F^s \simeq \sigma_F^{s\,*}, \quad \sigma_G^s \simeq \sigma_G^{s\,*} \tag{2.23}$$

where the designation * for the mean coefficients, $\sigma_E^{a\,*}$, $\sigma_E^{is\,*}$, $\sigma_E^{s\,*}$, $\sigma_F^{s\,*}$, and $\sigma_G^{s\,*}$, indicates that the intensity I has been replaced by the pseudo-intensity I^*.

If we now assume that, at a microscopic level, the spectral absorption coefficient σ^a can be approximated as a sum of polynomial functions of frequency: $\sigma^a(\nu) = \sum_{i=1}^N C_i^a \nu^{i-3}$, it follows that

$$\sigma_P^e = \sum_{i=1}^N \sigma_{Pi}(C_i^a, T) \quad \text{and} \quad \sigma_E^{a\,*} = \sum_{i=1}^N \sigma_{Ei}^*(C_i^a, T_R, \mathbf{f}) \tag{2.24}$$

where σ_{Pi} and σ_{Ei} will be derived in the next section (Eqs. (2.32) and (2.38)). In particular, it will be shown that they depend on the radiative temperature T_R and on the anisotropic factor $\mathbf{f}$ defined below.

If the spectral scattering coefficient can also be written as $\sigma^{is}(\nu)\,[1-\chi(\nu)] = \sum_{i=1}^{N} C_i^{is}\nu^{i-3}$, then the isotropic mean scattering coefficient can be then written in terms of σ_{Ei} (see Eq. (2.38) in the next section) and is given by

$$\sigma_E^{is\,*} = \sum_{i=1}^{N} \sigma_{Ei}^{*}(C_i^{is}, T_R, \mathbf{f}) \tag{2.25}$$

Now let us assume that the incoming spectral scattering coefficients can also be approximated as sums of polynomial functions:

$$\sigma^s(\nu)\alpha(\nu) = \sum_{i=1}^{N} C_i^{\alpha,s}\nu^{i-3},\ \sigma^s(\nu)\beta(\nu) = \sum_{i=1}^{N} C_i^{\beta,s}\nu^{i-3},\ \text{and}\ \sigma^s(\nu)\gamma(\nu) = \sum_{i=1}^{N} C_i^{\gamma,s}\nu^{i-3} \tag{2.26}$$

then we have

$$\sigma_E^{s\,*} = \sum_{i=1}^{N} \sigma_{Ei}^{*}(C_i^{\alpha,s}, T_R, \mathbf{f}),\quad \sigma_F^{s\,*} = \sum_{i=1}^{N} \sigma_{Fi}^{*}(C_i^{\beta,s}, T_R, \mathbf{f}),\quad \sigma_G^{s\,*} = \sum_{i=1}^{N} \sigma_{Gi}^{*}(C_i^{\beta,s}, T_R, \mathbf{f}). \tag{2.27}$$

Finally, using these models for the mean absorption coefficients, the frequency-integrated RTE in its closed form is given by

$$\begin{aligned}\frac{1}{c}\partial_t J + \mathbf{\Omega}\cdot\nabla J = {} & \frac{\sigma_P^e}{4\pi}aT^4 - \sigma_E^{a\,*}J - \sigma_E^{is\,*}J + \\ & \frac{1}{4\pi}(c\sigma_E^{s\,*}E_R + \sigma_F^{s\,*}\mathbf{F}_R\cdot\mathbf{\Omega} + c\sigma_G^{s\,*}\sum_{i,j=1..3} P_R^{i,j}\Omega^i\Omega^j)\end{aligned} \tag{2.28}$$

where all the mean coefficients are defined in terms of the functions σ_{Ei}^*, σ_{Fi}^* , and σ_{Gi}^* defined in the following section.

2.4. *Computation of the mean coefficients*

The pseudo-intensity which is used in the definition of the mean coefficients is obtained from the maximization of the radiative entropy (Minerbo 1978; Fort 1997) and is given by

$$I^*(t,\mathbf{r},\mathbf{\Omega},\nu) = \frac{2h\nu^3}{c^2}\left[\exp(\frac{h\nu}{kT^*(\mathbf{\Omega})}) - 1\right]^{-1} \tag{2.29}$$

with $T^*(\mathbf{\Omega}) = 1/(B(1-\mathbf{A}\cdot\mathbf{\Omega}))$ and $\mathbf{A}$ and B defined by

$$\mathbf{A} = \frac{2-\sqrt{4-3||\mathbf{f}||^2}}{||\mathbf{f}||^2}\mathbf{f},\quad B = \frac{1}{T_R}\left[\frac{3+||\mathbf{A}||^2}{3(1-||\mathbf{A}||^2)^3}\right]^{\frac{1}{4}}. \tag{2.30}$$

These two coefficients are defined from the macroscopic quantities T_R, the radiative temperature, and $\mathbf{f}$, the anisotropic factor. The radiative temperature is defined in terms of the radiative energy by $E_R = aT_R^4$, and the anisotropic factor is given by $\mathbf{f} = \mathbf{F}_R/(cE_R)$.

Let us assume that a polynomial approximation in frequency of the spectral absorption and scattering coefficients can be done. Such a coefficient is chosen to have the following form

$$\sigma(\nu) = \sum_{i=1}^{N} C_i\nu^{i-3} \tag{2.31}$$

where C_i are constants which depend on the gaseous medium considered, the volume fraction of the main species, the pressure, etc.

First, the well-known Planck mean absorption coefficient is computed from the Planck function (see Ripoll *et al.* 2001):

$$\sigma_P(T) = \frac{\langle \sigma(\nu)\mathcal{B}(\nu,T)\rangle_{\nu,\Omega}}{\langle \mathcal{B}(\nu,T)\rangle_{\nu,\Omega}} = \sum_{i=1}^{N} \sigma_{P\,i}(C_i,T) = \frac{15}{\pi^4} \sum_{i=1}^{N} i!\; C_i\; \zeta(i+1) \left(\frac{kT}{h}\right)^{i-3} \tag{2.32}$$

$$\tag{2.33}$$

Second, the three mean coefficients σ_E^*, σ_F^*, and σ_G^* are defined by

$$\sigma_E^*(t,\mathbf{r}) = \frac{\langle \sigma(\nu) I^*(t,\mathbf{r},\mathbf{\Omega},\nu)\rangle_{\nu,\mathbf{\Omega}}}{\langle I^*(t,\mathbf{r},\mathbf{\Omega},\nu)\rangle_{\nu,\mathbf{\Omega}}} \tag{2.34}$$

$$\sigma_F^*(t,\mathbf{r},\mathbf{\Omega}) = \frac{\langle \sigma(\nu)\mathbf{\Omega}\cdot\mathbf{\Omega}' I^*(t,\mathbf{r},\mathbf{\Omega}',\nu)\rangle_{\nu,\mathbf{\Omega}'}}{\langle \mathbf{\Omega}\cdot\mathbf{\Omega}' I^*(t,\mathbf{r},\mathbf{\Omega}',\nu)\rangle_{\nu,\mathbf{\Omega}'}} = \frac{\mathbf{\Omega}\cdot\langle \sigma(\nu)\mathbf{\Omega}' I^*(t,\mathbf{r},\mathbf{\Omega}',\nu)\rangle_{\nu,\mathbf{\Omega}'}}{\mathbf{\Omega}\cdot\langle \mathbf{\Omega}' I^*(t,\mathbf{r},\mathbf{\Omega}',\nu)\rangle_{\nu,\mathbf{\Omega}'}} \tag{2.35}$$

$$\sigma_G^*(t,\mathbf{r},\mathbf{\Omega}) = \frac{\langle \sigma(\nu)(\mathbf{\Omega}\cdot\mathbf{\Omega}')^2 I^*(t,\mathbf{r},\mathbf{\Omega}',\nu)\rangle_{\nu,\mathbf{\Omega}'}}{\langle (\mathbf{\Omega}\cdot\mathbf{\Omega}')^2 I^*(t,\mathbf{r},\mathbf{\Omega}',\nu)\rangle_{\nu,\mathbf{\Omega}'}}. \tag{2.36}$$

It should be noticed here that in I^* the $\mathbf{\Omega}$ dependence is only present through a scalar product with the vector $\mathbf{A}$. This greatly simplifies σ_F as follows. The vector $\langle \mathbf{\Omega}' I^*(t,\mathbf{r},\mathbf{\Omega}',\nu)\rangle_{\mathbf{\Omega}'}$ necessarily has the form $X(t,\mathbf{r},\nu)\,\mathbf{A}(t,\mathbf{r})$, where X is a scalar, which leads to

$$\sigma_F^*(t,\mathbf{r}) = \frac{\langle \sigma(\nu) X(t,\mathbf{r},\nu)\rangle_\nu}{\langle X(t,\mathbf{r},\nu)\rangle_\nu} \tag{2.37}$$

Thus, σ_F^* does not depend on $\mathbf{\Omega}$.

Moreover, the simple form of I^* leads to analytical expressions for σ_E and σ_F, though the computation is not detailed here (see Ripoll *et al.* 2001). These coefficients have also been tested and validated for simple 1D problems (see Ripoll & Wray 2004a). They are given by:

$$\begin{aligned}\sigma_E^*(T_R,\mathbf{f}) &= \sum_{i=1}^{N} \sigma_{E\,i}^*(C_i,T_R,\mathbf{f}) \\ &= \frac{45}{2\pi^4} \sum_{i=1}^{N} (i-1)!\; C_i\; \zeta(i+1) \left(\frac{k}{hB(1-\|\mathbf{A}\|^2)}\right)^{i-3} \frac{P_E^i(\|\mathbf{A}\|)}{\|\mathbf{A}\|}\end{aligned} \tag{2.38}$$

$$\begin{aligned}\sigma_F^*(T_R,\mathbf{f}) &= \sum_{i=1}^{N} \sigma_{F\,i}^*(C_i,T_R,\mathbf{f}) \\ &= \frac{45}{8\pi^4} \sum_{i=1}^{N} (i-2)!\; C_i\; \zeta(i+1) \left(\frac{k}{hB(1-\|\mathbf{A}\|^2)}\right)^{i-3} \frac{P_F^i(\|\mathbf{A}\|)}{\|\mathbf{A}\|^3}\end{aligned} \tag{2.39}$$

with

$$P_E^i(\|\mathbf{A}\|) = ((1+\|\mathbf{A}\|)^i - (1-\|\mathbf{A}\|)^i)/(3+\|\mathbf{A}\|^2) \tag{2.40}$$

$$P_F^i(\|\mathbf{A}\|) = (1-\|\mathbf{A}\|)^i(i\|\mathbf{A}\|+1) + (1+\|\mathbf{A}\|)^i(i\|\mathbf{A}\|-1) \tag{2.41}$$

with ζ the Riemann Zeta function and $\mathbf{A}$ and B given in (2.30)†. The derivation of σ_G

† for i real but non-integer, these expressions are valid provided $i!$ is replaced with $\Gamma(i+1)$

involves the pressure tensor and is much more complex. We have

$$\sigma_G^s = \frac{\langle \sigma(\nu) \sum_{j,k} \Omega_j \Omega_j' \Omega_k \Omega_k' I^* \rangle_{\nu,\Omega'}}{\langle \sum_{j,k} \Omega_j \Omega_j' \Omega_k \Omega_k' I^* \rangle_{\nu,\Omega'}} \quad j,k = 1..3 \tag{2.42}$$

Following the same developments as in Ripoll *et al.* 2001, it is found that

$$\sigma_G^*(T_R, \mathbf{f}) = \sum_{i=1}^{N} \sigma_{Gi}^*(C_i, T_R, \mathbf{f}) \tag{2.43}$$

$$= \frac{45}{2\pi^4} \sum_{i=1}^{N} (i-3)!\, C_i\, \zeta(i+1) \left(\frac{k}{hB(1-\|\mathbf{A}\|^2)} \right)^{i-3} \frac{P_G^i(\mathbf{\Omega}, \|\mathbf{A}\|)}{\|\mathbf{A}\|^5} \tag{2.44}$$

where $P_G^i(\mathbf{A}, \|\mathbf{\Omega}\|)$ is given by

$$\begin{aligned} P_G^i = & \frac{(1-\|\mathbf{A}\|)^i}{1-\|\mathbf{A}\|^2+4(\mathbf{A}\cdot\mathbf{\Omega})^2} \left[(1-\|\mathbf{A}\|^2)\|\mathbf{A}\|^2 + (\|\mathbf{A}\|^2-3)(\mathbf{A}\cdot\mathbf{\Omega})^2 \right. \\ & \left. + i\|\mathbf{A}\| \left[(1+\|\mathbf{A}\|)\|\mathbf{A}\|^2 - (\|\mathbf{A}\| i + 3)(\mathbf{A}\cdot\mathbf{\Omega})^2 \right] \right] \\ + & \frac{(1+\|\mathbf{A}\|)^i}{1-\|\mathbf{A}\|^2+4(\mathbf{A}\cdot\mathbf{\Omega})^2} \left[(\|\mathbf{A}\|^2-1)\|\mathbf{A}\|^2 + (3-\|\mathbf{A}\|^2)(\mathbf{A}\cdot\mathbf{\Omega})^2) \right. \\ & \left. + i\|\mathbf{A}\| \left[(1-\|\mathbf{A}\|)\|\mathbf{A}\|^2 + (\|\mathbf{A}\| i - 3)(\mathbf{A}\cdot\mathbf{\Omega})^2 \right] \right] \end{aligned} \tag{2.45}$$

It should be noticed here that the Planck mean σ_P is a function of the volume fractions C_i and of the temperature, while the effective mean coefficients σ_E^*, σ_F^* are functions of C_i and of both the radiative temperature and anisotropic factor. In addition, σ_G^* is dependent on the scalar product between a vector parallel to the flux and the direction $\mathbf{\Omega}$.

The mean coefficients have been introduced to avoid the cost of the frequency integration. Thus, to be useful, the numerical cost of evaluating the mean coefficients and of the iterations needed to solve for J must be lower than that of these time integrations. The coefficients which are proposed here are analytic and should have low computational cost. Furthermore, the three coefficients $\sigma_{E,F,G}^*$ have common parts (see (2.38)-(2.44)). For absorbing media, it has been found that fewer than 6 iterations to solve the RTE were needed in the cases studied (Ripoll & Wray 2004a), which indicates that this method does not have a high numerical cost.

3. The linear case: application to soot

The apparent complexity of the coefficients in the previous section is only due to the polynomial expansion in frequency. For sooty media the absorption dependence is linear, and in this case the coefficients turn out to be quite simple.

The linear spectral absorption and isotropic scattering coefficients are written as

$$\sigma^e(\nu) = C^e \nu, \quad \sigma^a(\nu) = C^a \nu, \quad \text{and} \quad \sigma^{is}(\nu)\,[1-\chi(\nu)] = C^{is}\nu \tag{3.1}$$

As a rough approximation, we here also assume that the scattering follows a linear law†. The spectral scattering coefficients are then written as

$$\sigma^s(\nu)\alpha(\nu) = C^{\alpha,s}\nu, \quad \sigma^s(\nu)\beta(\nu) = C^{\beta,s}\nu, \quad \text{and} \quad \sigma^s(\nu)\gamma(\nu) = C^{\gamma,s}\nu. \tag{3.2}$$

† More complex models, such as in Houf 1999, could also be treated but would need to introduce higher order terms in frequency

We define for convenience the following coefficients:

$$C_P^x = 360\frac{k\zeta_5}{\pi^4 h}C^x \tag{3.3}$$

where x is a variable which will alternatively take the value a, e, is, (α,s), (β,s), (γ,s) below. The Planck absorption coefficient becomes then

$$\sigma_P^e(T) = \sigma_{P4}(C^e,T) = 360\ C^e\frac{k\zeta_5}{\pi^4 h}T = C_P^e\,T \tag{3.4}$$

Using (2.38), (2.39), (2.44) for $i=4$, and the definition (3.1), the general form for the mean coefficients is given by

$$\sigma_{E4}^*(C^x,T_R,\mathbf{f}) = 3C_P^x\frac{1+\|\mathbf{A}\|^2}{B(1-\|\mathbf{A}\|^2)(3+\|\mathbf{A}\|^2)} = C_P^x\,T_R\,G_E(\mathbf{f}) \tag{3.5}$$

$$\sigma_{F4}^*(C^x,T_R,\mathbf{f}) = \frac{C_P^x}{4}\frac{5+\|\mathbf{A}\|^2}{B(1-\|\mathbf{A}\|^2)} = C_P^x\,T_R\,G_F(\mathbf{f}) \tag{3.6}$$

$$\sigma_{G4}^*(C^x,T_R,\mathbf{f}) = C_P^x\frac{(1-\|\mathbf{A}\|^2+6(\mathbf{A}\cdot\mathbf{\Omega})^2)}{B(1-\|\mathbf{A}\|^2)(1-\|\mathbf{A}\|^2+4(\mathbf{A}\cdot\mathbf{\Omega})^2)} = C_P^x\,T_R\,G_G(\mathbf{f},\mathbf{\Omega}) \tag{3.7}$$

where the functions $G_{E,F,G}$ are obtained by using the definitions (2.30) in (3.5)-(3.7). Hence, the mean absorption and isotropic scattering coefficients are given in the linear case by

$$\sigma_E^{a\,*} = \sigma_{E4}^*(C^a,T_R,\mathbf{f}) = C_P^a\,T_R\,G_E(\mathbf{f}) \tag{3.8}$$

$$\sigma_E^{is\,*} = \sigma_{E4}^*(C^{is},T_R,\mathbf{f}) = C_P^{is}\,T_R\,G_E(\mathbf{f}) \tag{3.9}$$

and the incoming mean scattering coefficients are given by

$$\sigma_E^{s\,*} = \sigma_{E4}^*(C^{\alpha,s},T_R,\mathbf{f}) = C_P^{\alpha,s}\,T_R\,G_E(\mathbf{f}) \tag{3.10}$$

$$\sigma_F^{s\,*} = \sigma_{F4}^*(C^{\beta,s},T_R,\mathbf{f}) = C_P^{\beta,s}\,T_R\,G_F(\mathbf{f}) \tag{3.11}$$

$$\sigma_G^{s\,*} = \sigma_{G4}^*(C^{\gamma,s},T_R,\mathbf{f},\mathbf{\Omega}) = C_P^{\gamma,s}\,T_R\,G_G(\mathbf{f},\mathbf{\Omega}). \tag{3.12}$$

The integrated RTE,

$$\frac{1}{c}\partial_t J+\mathbf{\Omega}\cdot\nabla J = \frac{\sigma_P^e}{4\pi}aT^4-\sigma_E^{a\,*}J-\sigma_E^{is\,*}J+\frac{1}{4\pi}(c\sigma_E^{s\,*}E_R+\sigma_F^{s\,*}\mathbf{F}_R\cdot\mathbf{\Omega}+c\sigma_G^{s\,*}\sum_{i,j=1..3}P_R^{i,j}\Omega^i\Omega^j) \tag{3.13}$$

becomes using the previous definitions :

$$\begin{aligned}\frac{1}{c}\partial_t J+\mathbf{\Omega}\cdot\nabla J = {} & \frac{C_P^a}{4\pi}aT^5 - C_P^a T_R G_E(\mathbf{f})J - C_P^{is}T_R G_E(\mathbf{f})J \\ & + \frac{T_R}{4\pi}(cC_P^\alpha G_E(\mathbf{f})E_R + C_P^\beta G_F(\mathbf{f})\mathbf{F}_R\cdot\mathbf{\Omega} + cC_P^\gamma G_G(\mathbf{f},\mathbf{\Omega})\sum_{i,j=1..3}P_R^{i,j}\Omega^i\Omega^j).\end{aligned} \tag{3.14}$$

If isotropy is assumed for the functions $G_{E,F,G}$, the following simpler form is obtained:

$$\begin{aligned}\frac{1}{c}\partial_t J+\mathbf{\Omega}\cdot\nabla J = {} & \frac{C_P^a}{4\pi}aT^5 - C_P^a T_R J - C_P^{is}T_R J \\ & + \frac{T_R}{4\pi}(cC_P^\alpha E_R + C_P^{\beta\,\prime}\mathbf{F}_R\cdot\mathbf{\Omega} + cC_P^\gamma\sum_{i,j=1..3}P_R^{i,j}\Omega^i\Omega^j)\end{aligned} \tag{3.15}$$

with ${C_P^\beta}' = 5/4C_P^\beta$. In Ripoll & Wray 2004a, it was found that the function G_E played an important role. Eq. 3.15 should hence not be used for radiating flows where the anisotropy $||f||$ is larger than 0.3.

4. Macroscopic radiation models for absorbing and scattering media

We show in this section that the mean coefficients can be easily included in macroscopic moment models. We define the three first moments with respect to direction as

$$E_R^{\mathbf{\Omega}}(t,\mathbf{r},\nu) = \frac{1}{c}\langle I(t,\mathbf{r},\mathbf{\Omega},\nu)\rangle_\Omega \tag{4.1}$$

$$\mathbf{F}_R^{\mathbf{\Omega}}(t,\mathbf{r},\nu) = \langle \mathbf{\Omega}\, I(t,\mathbf{r},\mathbf{\Omega},\nu)\rangle_\Omega \tag{4.2}$$

$$\mathbf{P}_R^{\mathbf{\Omega}}(t,\mathbf{r},\nu) = \frac{1}{c}\langle \mathbf{\Omega}\otimes\mathbf{\Omega}\, I(t,\mathbf{r},\mathbf{\Omega},\nu)\rangle_\Omega \tag{4.3}$$

The first moment equation is obtained by integrating Eq. (2.1) with respect to $\mathbf{\Omega}$. We obtain

$$\partial_t E_R^{\mathbf{\Omega}} + \nabla\cdot\mathbf{F}_R^{\mathbf{\Omega}} = 4\pi c\sigma^a(\nu)\mathcal{B} - c\sigma^a(\nu)E_R^{\mathbf{\Omega}} \tag{4.4}$$

where the incoming and isotropic scattering have canceled using the normalization property (2.4).

Multiplying (2.1) by $\mathbf{\Omega}$, and integrating with respect to it, using the phase function definition (2.3), we obtain

$$\frac{1}{c}\partial_t\mathbf{F}_R^{\mathbf{\Omega}} + c\nabla\cdot\mathbf{P}_R^{\mathbf{\Omega}} = -(\sigma^a(\nu) + \sigma^{is}(\nu)(1-\chi(\nu)) - \sigma^s(\nu)\frac{\beta(\nu)}{3})\mathbf{F}_R^{\mathbf{\Omega}}. \tag{4.5}$$

The following three equalities have been used:

$$\int\Omega_i\, d\mathbf{\Omega} = 0, \quad \int\Omega_i\Omega_j\, d\mathbf{\Omega} = 4\pi/3\delta_{ij}, \quad \int\Omega_i\Omega_j\Omega_k\, d\mathbf{\Omega} = 0. \tag{4.6}$$

Using definitions (2.18), (2.19), and (2.20), the integration of (4.4) over frequency gives

$$\partial_t E_R + \nabla\cdot\mathbf{F}_R = ca\left(\sigma_P^e T^4 - \sigma_E^a E_R\right) \tag{4.7}$$

where models σ_P^e and $\sigma_E^a \simeq {\sigma_E^a}^*$ were given previously in (2.24).

Integrating (4.5) over frequency leads to the second moment equation

$$\frac{1}{c}\partial_t\mathbf{F}_R + c\nabla\cdot\mathbf{P}_R = -(\sigma_F^a + \sigma_F^{is} - \frac{1}{3}\sigma_F^\beta)\mathbf{F}_R, \tag{4.8}$$

for which the models for the mean coefficients are

$$\sigma_F^a \simeq {\sigma_F^a}^* = \sum_{i=1}^N \sigma_{F\,i}(C_i^a, T_R, \mathbf{f}) \tag{4.9}$$

$$\sigma_F^{is} \simeq {\sigma_F^{is}}^* = \sum_{i=1}^N \sigma_{F\,i}(C_i^{is}, T_R, \mathbf{f}) \tag{4.10}$$

$$\sigma_F^\beta \simeq {\sigma_F^\beta}^* = \sum_{i=1}^N \sigma_{F\,i}(C_i^{\beta,s}, T_R, \mathbf{f}) \tag{4.11}$$

where $\sigma_{F\,i}$ is defined in (2.39).

Closure of the macroscopic model (4.7)-(4.8) is achieved by modeling the radiative pressure in (4.8). In many different closures the pressure is written as $\mathbf{P_R} = \mathbf{D_R}(\mathbf{f})E_R$, where $\mathbf{D_R}(\mathbf{f})$ is the Eddington tensor (Ripoll & Wray 2004b).

Combining the steady forms of (4.8) and (4.7) to eliminate the flux $\mathbf{F}_R$ leads to the general Milne-Eddington equations

$$-\nabla \cdot (\frac{1}{(\sigma_F^a + \sigma_F^{is} - \frac{1}{3}\sigma_F^{\beta})}\nabla \cdot (\mathbf{D}_R E_R)) = \sigma_P^e a T^4 - \sigma_E^a E_R \ . \qquad (4.12)$$

Or, similarly, by eliminating the energy E_R, one obtains

$$-\nabla \cdot (\frac{\mathbf{D}_R}{\sigma_E}\ \nabla \cdot \mathbf{F}_R) + (\sigma_F^a + \sigma_F^{is} - \frac{1}{3}\sigma_F^{\beta})\mathbf{F}_R = -c\vec{\nabla} \cdot (\frac{\sigma_P^e}{\sigma_E^a}\mathbf{D}_R a T^4) \ . \qquad (4.13)$$

We have two main comments on the model (4.7)-(4.8); they also apply to the formulation (4.12)-(4.13). First, the normalization property (2.4) eliminates the pressure term coming from the scattering in the first moment equation (4.7). Second, the contribution of the incoming scattering integrated over direction only enters through the first order and delta-function parts of the scattering function in (4.13), since the zero and second order terms vanish.

5. On the use of these models and their numerical costs

The models presented here will be useful when the solution for frequency dependent intensities cannot be done due to its cost. This is usually the case for coupled problems. We now give a discussion on reducing the computational cost of the models presented here.

First, when using mean coefficients, it is possible to reduce the computational cost by noticing that when the radiation is isotropic and close to equilibrium, the coefficients $\sigma_{E,G}$ are equal to the Planck mean and σ_F to $(i + 1/4)\sigma_P$. There is hence no need to evaluate the complex expressions for $\sigma_{E,F,G}$, and they should be simply replaced by their limits. More generally, these limits can be extended to $\|f\| < 0.1$ when $T \simeq T_R$. In the case where the radiation is isotropic but $T \neq T_R$, the limit that should be taken is also the Planck mean but evaluated at T_R instead of T†. It should also be noticed that the form of the mean coefficients proposed here, in terms of $\mathbf{A}$ and B, should be retained. As a matter of fact, this choice allows checking the different limits and avoids introducing singularities. For instance, (2.30) can directly be replaced by their limits, respectivelly $A = 0$ and $B = 1/T$, for $\|f\| < 0.1$.

The solution of the RTE with mean coefficients requires iteration since the mean coefficients are nonlinear functions of I. The required number of iterations has been found to be small in many simple cases (Ripoll & Wray 2004a), but one could be tempted to reduce it further. In the case of using such an equation for coupled problems, the previous time step provides excellent starting values for the iterative solution so that the required number of iterations should be lower, perhaps only one or two. Another alternative in this case could be not to iterate the RTE at all, i.e., to lag it in time, assuming that a small difference between the opacity and the intensity due to their non-synchronization in time with the fluid motion will not strongly affect the solution. This is likely when a global convergence process of the hydrodynamics and radiation to a steady state leads to synchronizing all the variables at the end. For the very first iteration, the Planck limit seems to be a good initial condition. It should also be noticed here that the moment model

† as was shown in (3.15) in the linear case

(4.7)-(4.8) can similarly be solved by using the mean coefficients from the previous time step of the radiation loop or of the hydrodynamics loop in coupled problems.

6. Conclusion

The objective of this work was to propose a simple model accounting for radiation in complex emitting, absorbing, and scattering media. To do that, models for mean absorption and mean isotropic and incoming scattering coefficients have been proposed in the case where the various spectral coefficients can be written in terms of polynomial functions. Some of these models were previously derived and validated (Ripoll *et al.* 2001, Ripoll & Wray 2004a) for non-scattering media; they have been extended here to the general case. An integrated RTE which uses these coefficients has been derived where the integral scattering term has been absorbed into the modeling. Such a form of the RTE is much less costly to solve than the RTE in its non-modeled form. Macroscopic moment radiation models, written in their hyperbolic or diffusive forms, have also been derived using these coefficients. The particular case where the spectral coefficients are linear in frequency has been treated. This case is particularly important for soot and hence for combustion applications. It has also been explained how such models can be used at a lower cost by reducing the number of iterations needed. We believe the formulations proposed here could be used in many complex or coupled problems, like flows radiating in dusty media, where, for instance, isotropic and non-isotropic scattering are usually disregarded and neglected due to their computational cost.

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