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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

RESEARCH MEMORANDUM

SUPERSONIC WAVE DRAG OF SWEPTBACK TAPERED WINGS AT ZERO LIFT

By Kenneth Margolis

SUMMARY

On the basis of a recently developed theory for sweptback wings at supersonic velocities, equations are derived for the wave drag of sweptback tapered wings with thin symmetrical double-wedge sections at zero lift. Calculations of section wave-drag distributions and wing wave drag are presented for families of tapered plan forms.

Distributions of section wave drag along the span of tapered wings are, in general, very similar in shape to those of untapered plan forms. For a given taper ratio and aspect ratio, an appreciable reduction in wing wave-drag coefficient with increased sweepback is noted for the entire range of Mach number considered. For a given sweep and taper ratio, higher aspect ratios reduce the wing wave-drag coefficient at substantially subcritical supersonic Mach numbers. At Mach numbers approaching the critical value, that is, a value equal to the secant of the sweepback angle, the plan forms of low aspect ratio have lower drag coefficients.

Calculations for wings of equal root bending stress (and hence different aspect ratio) indicate that tapering the wing reduces the wing wave-drag coefficient at Mach numbers considerably less than the critical value but increases the drag coefficient at Mach numbers near the critical values. Comparisons on the basis of constant aspect ratio, however, indicate an increase of the wing wave-drag coefficient with taper at Mach numbers considerably less than the critical value and a decrease of the drag coefficient with taper at Mach numbers near the critical value.

INTRODUCTION

Recent developments in airfoil theory for supersonic speeds (references 1 and 2) indicate pronounced favorable effects of sweepback on the wave drag. In reference 1, a method is developed for calculating pressure drag at supersonic speeds for sweptback airfoils

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having thin sections at zero lift. Reference 3 applies this method to calculate the supersonic wave drag for a series of untapered wings with symmetrical biconvex airfoil sections.

The present paper applies the method of reference 1 to derive the generalized equations for the section wave drag and wing wave drag of sweptback tapered wings with thin symmetrical double-wedge sections at zero lift. Section wave-drag distributions and wing wave-drag calculations are presented for specific families of tapered plan forms. The airfoil sections and wing tips are chosen parallel to the direction of flight. The angle of sweepback is referred to that of the line of maximum thickness, and the range of Mach number considered is between 1 and the critical value corresponding to the condition where the Mach lines are parallel to the maximum-thickness line, that is, to a Mach number equal to the secant of the sweepback angle.

SYMBOLS

x, y, z	cartesian coordinates
V	velocity in flight direction
ρ	density of air
Δp	pressure increment
q	dynamic pressure $\left(\frac{1}{2}\rho V^2\right)$
φ	disturbance-velocity potential
M	Mach number
$\beta = \sqrt{M^2 - 1}$	
dz/dx	slope of airfoil surface
a	root semichord, measured in flight direction
c	chord length at spanwise station y, measured in flight direction
t	maximum thickness of section at spanwise station y

Λ	angle of sweep of the line of maximum thickness, degrees
m_0	slope of line of maximum thickness $(\cot \Lambda)$
m_1	slope of wing leading edge
m_2	slope of wing trailing edge $\left(\frac{m_1 m_0}{2m_1 - m_0}\right)$
b	span of wing
$d = \frac{b/2}{m_0}$	
S	wing area
A	aspect ratio $\left(\frac{b^2}{S}\right)$
λ	taper ratio, ratio of tip chord to root chord
c_{d_∞}	section wave-drag coefficient at spanwise station y exclusive of tip effect
$c_{d_{tip}}$	increment in section wave-drag coefficient at spanwise station y due to tip
c_d	section wave-drag coefficient at spanwise station y $\left(c_{d_\infty} + c_{d_{tip}}\right)$
C_{D_∞}	wing wave-drag coefficient exclusive of tip effect
$C_{D_{tip}}$	increment in wing wave-drag coefficient due to tip
C_D	wing wave-drag coefficient $\left(C_{D_\infty} + C_{D_{tip}}\right)$

Subscript s refers to conditions at root

ANALYSIS

The analysis is based on supersonic thin-airfoil theory and on the assumptions of small disturbances and a constant velocity of sound throughout the fluid. These assumptions lead to the linearized equation for the velocity potential φ (reference 4)

$$\left(1 - M^2\right)\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0 \tag{1}$$

where M is the Mach number of the flow and the derivatives are taken with respect to the variables x, y, and z of the cartesian-coordinate system. It should be noted that the linearized theory is not expected to be applicable near Mach number unity. On the basis of this linearized theory, a solution for a uniform sweptback line of sources in the pressure field is derived in reference 1. The pressure field associated with this solution corresponds to that over an airfoil of wedge section. The pressure coefficient $\Delta p/q$ at a spanwise station y and point x along the wedge is

$$\frac{\Delta p}{q} = \frac{2}{\pi}\frac{dz}{dx}\frac{m_1}{\sqrt{1 - \beta^2 m_1^2}}\cosh^{-1}\frac{x - \beta^2 m_1 y}{\beta\left|y - m_1 x\right|} \tag{2}$$

where m_1 is the slope of the leading edge of the wing, dz/dx is the tangent of the half-wedge angle (approx. equal to half-wedge angle since the angle is small), $\beta = \sqrt{M^2 - 1}$ and the origin of the line source is taken at $(0,0)$.

The distribution of pressure over sweptback wings of desired plan form and profile is obtained by superposition of wedge-type solutions. In order to satisfy the boundary conditions over the surface of a tapered wing of symmetrical double-wedge section, semi-infinite line sources are placed at the leading and trailing edge of the wing and a semi-infinite line sink of twice the strength is placed along the line of maximum thickness so that all three lines intersect at one point. At the tip where the wing is cut off in the flight direction, a reversed distribution of these lines of sinks and sources are placed so as to cancel exactly all effects of the original distribution farther spanwise than the tip. Figure 1 shows the distributions of sinks and sources for a tapered wing and identifies the system of axes and the symbols associated with the derivation of the drag equations.

The disturbances caused by the elementary line sources and sinks are limited to the regions enclosed by their Mach cones. Figure 2 shows the Mach line configuration for the tapered-wing plan form and indicates the regions of the wing affected by each line source and sink. For purposes of simplification the tapered wings considered were restricted to those with no tip effects other than the effects each tip exerts on its own half of the wing. For a wing of taper ratio 0, no tip effects need be considered since the Mach lines originating at the tip do not enclose any part of the wing.

The pressure coefficients obtained from superimposing solutions of the type shown in equation (2) are converted into drag coefficients by the following relations:

For section drag at a spanwise station y

$$c_d c = 2 \int_{\text{Leading edge}}^{\text{Trailing edge}} \frac{\Delta p}{q} \frac{dz}{dx} \, dx \tag{3}$$

where

$$c = \frac{y(m_1 - m_2) + 2am_1m_2}{m_1m_2}$$

is the chord length at y, and the integration is performed along the chord parallel to the flight direction.

The wing wave-drag coefficient is obtained by integrating the section drag along the span and dividing the result by the wing area.

$$C_D = \frac{2}{S}\int_{\text{Root}}^{\text{Tip}} c_d c \, dy = \frac{4}{S}\int_{\text{Root}}^{\text{Tip}}\int_{\text{L.E.}}^{\text{T.E.}} \frac{\Delta p}{q}\frac{dz}{dx}\, dx \, dy \qquad (4)$$

where S is the wing area, and the integration with respect to y is performed along the span.

DERIVATION OF GENERALIZED EQUATIONS

By superposition of wedge-type solutions (equation (2)), the pressure field is obtained for a tapered wing with leading edge, trailing edge, and line of maximum thickness sweptback. The drag equations are derived for half of the wing since the drag is distributed symmetrically over both halves. The induced effects of the opposite half-wing are represented by the conjugate terms in the integrands of the drag integrals.

For a symmetrical double-wedge profile,

$$\left|\frac{dz}{dx}\right| = \frac{t}{c}$$

where t/c is the section thickness ratio. The generalized equation, exclusive of tip effects, for the wing wave drag is obtained as follows: (See fig. 3 for information pertinent to integration limits.)

$$\frac{\pi S C_{D_\infty}}{8(t/c)^2} = \frac{\pi}{4(t/c)^2}\int_0^{dm_0} c_{d_\infty} c\,dy$$

$$= \frac{m_1}{\sqrt{1-\beta^2 m_1{}^2}}\left(\int_0^{dm_0}\int_{\frac{y-am_1}{m_1}}^{\frac{y}{m_0}} A\,dx\,dy - \int_0^{dm_0}\int_{\frac{y}{m_0}}^{\frac{y+am_2}{m_2}} A\,dx\,dy\right)$$

$$- \frac{2m_0}{\sqrt{1-\beta^2 m_0{}^2}}\left(\int_0^{\frac{am_1}{1-\beta m_1}}\int_{\beta y}^{\frac{y}{m_0}} B\,dx\,dy - \int_0^{dm_0}\int_{\frac{y}{m_0}}^{\frac{y+am_2}{m_2}} B\,dx\,dy\right.$$

$$\left. + \int_{\frac{am_1}{1-\beta m_1}}^{dm_0}\int_{\frac{y-am_1}{m_1}}^{\frac{y}{m_0}} B\,dx\,dy\right)$$

$$+ \frac{m_2}{\sqrt{1-\beta^2 m_2{}^2}}\left(-\int_0^{\frac{am_0}{1-\beta m_0}}\int_{\beta y+a}^{\frac{y+am_2}{m_2}} C\,dx\,dy\right.$$

$$+ \int_{\frac{am_0}{1-\beta m_0}}^{\frac{2am_1}{1-\beta m_1}}\int_{\beta y+a}^{\frac{y}{m_0}} C\,dx\,dy + \int_{\frac{2am_1}{1-\beta m_1}}^{dm_0}\int_{\frac{y-am_1}{m_1}}^{\frac{y}{m_0}} C\,dx\,dy$$

$$\left. - \int_{\frac{am_0}{1-\beta m_0}}^{dm_0}\int_{\frac{y}{m_0}}^{\frac{y+am_2}{m_2}} C\,dx\,dy\right) \tag{5}$$

where A, B, and C refer to the pressures resulting from the leading line sources, line sinks, and trailing line sources, respectively.

$$A = \cosh^{-1}\frac{x + a + m_1\beta^2 y}{\beta\left|y + m_1x + am_1\right|} + \cosh^{-1}\frac{x + a - m_1\beta^2 y}{\beta\left|y - m_1x - am_1\right|}$$

$$B = \cosh^{-1}\frac{x + m_0\beta^2 y}{\beta\left|y + m_0x\right|} + \cosh^{-1}\frac{x - m_0\beta^2 y}{\beta\left|y - m_0x\right|}$$

$$C = \cosh^{-1}\frac{x - a + m_2\beta^2 y}{\beta\left|y + m_2x - am_2\right|} + \cosh^{-1}\frac{x - a - m_2\beta^2 y}{\beta\left|y - m_2x + am_2\right|}$$

The limiting case (taper ratio 0) is obtained by letting $d = \frac{am_1}{m_0 - m_1}$, and the wing of constant chord (taper ratio 1.0) is obtained by equating $m_1 = m_0 = m_2$. The integrations in equation (5) are performed and the resulting formulas for the section wave drag and the wing wave drag for the complete range of conventional taper ($0 \leqq$ taper ratio $\leqq 1.0$) are presented in appendix A.

It was stated previously that the tapered wings considered have no tip effects other than those each tip exerts on its own half of the wing. This implies that the Mach lines from one tip do not enclose any part of the opposite half-wing. This condition is expressed mathematically as follows:

For $\beta m_2 \leqq 1$

$$\text{Aspect ratio} = \frac{2dm_0}{a(1 + \lambda)} \geqq \frac{2m_1m_0}{(1 + \lambda)(\beta m_1m_0 + m_0 - m_1)} \qquad (6a)$$

and for $\beta m_2 > 1$

$$\text{Aspect ratio} = \frac{2dm_0}{a(1 + \lambda)} \geqq \frac{4m_1}{(1 + \lambda)(1 + \beta m_1)} \qquad (6b)$$

where λ is the taper ratio $\left(\frac{\text{Tip chord}}{\text{Root chord}}\right)$. It can be seen from equations (6) that this simplification does not materially limit the range of Mach number that may be considered. For small taper ratios this limiting effect is negligible and for taper ratio 0 there is no limitation whatsoever since equations (6) reduce to expressions that are always valid.

The wave-drag contribution of the tip is (see fig. 3)

$$\frac{\pi S C_{D_{tip}}}{8(t/c)^2} = \frac{\pi}{4(t/c)^2} \int_{\frac{dm_0 m_2(1+\beta m_1)-2am_1m_2}{m_1(1+\beta m_2)}}^{dm_0} c_{d_{tip}} c \, dy$$

$$= \frac{m_1}{\sqrt{1 - \beta^2 m_1{}^2}} \left[- \int_{\frac{dm_0{}^2(1+\beta m_1)-am_1m_0}{m_1(1+\beta m_0)}}^{dm_0} \int_{\frac{dm_0(1+\beta m_1)-m_1(a+\beta y)}{m_1}}^{\frac{y}{m_0}} D \, dx \, dy \right.$$

$$+ \int_{\frac{dm_0{}^2(1+\beta m_1)-am_1m_0}{m_1(1+\beta m_0)}}^{dm_0} \int_{\frac{y}{m_0}}^{\frac{y+am_2}{m_2}} D \, dx \, dy$$

$$\left. + \int_{\frac{dm_0m_2(1+\beta m_1)-2am_1m_2}{m_1(1+\beta m_2)}}^{\frac{dm_0{}^2(1+\beta m_1)-am_1m_0}{m_1(1+\beta m_0)}} \int_{\frac{dm_0(1+\beta m_1)-m_1(a+\beta y)}{m_1}}^{\frac{y+am_2}{m_2}} D \, dx \, dy \right]$$

$$- \frac{2m_0}{\sqrt{1 - \beta^2 m_0{}^2}} \int_{\frac{dm_2(1+\beta m_0)-am_2}{1+\beta m_2}}^{dm_0} \int_{d(1+\beta m_0)-\beta y}^{\frac{y+am_2}{m_2}} E \, dx \, dy \qquad (7)$$

where D and E refer to the pressures resulting from the leading line sink and line source, respectively.

$$D = \cosh^{-1} \frac{m_1(x + a) - dm_0 - m_1{}^2\beta^2(y - dm_0)}{\beta m_1 \left| y - m_1(x + a) \right|}$$

$$E = \cosh^{-1} \frac{x - d - \beta^2 m_0(y - dm_0)}{\beta \left| y - m_0 x \right|}$$

The Mach cone from the trailing line sink at the tip does not enclose any part of the wing and, hence, has no effect on the wave drag.

Equation (7) is solved for section wave drag and wing wave drag for the complete range of taper and the results are presented in appendix B. The total wave-drag coefficients are then obtained by the following relations:

$$\left.\begin{aligned} c_d &= c_{d_\infty} + c_{d_{tip}} \\ C_D &= C_{D_\infty} + C_{D_{tip}} \end{aligned}\right\} \qquad (8)$$

It is found that $C_{D_{tip}}$ is identically equal to zero for all cases satisfying the aspect-ratio limitations expressed in equations (6) and, hence, $C_D = C_{D_\infty}$ for the tapered wings considered.

The conditions imposed in equations (6), although not materially limiting the range of Mach number for tapered wings, do limit to a certain extent the range of Mach number for low-aspect-ratio wings of constant chord. Equation (6a) for this case reduces to

$$\text{Aspect ratio} \geqq \frac{1}{\beta}$$

since $m_1 = m_0 = m_2$.

For untapered wings of aspect ratio 2, 1, and 0.5, the lowest Mach numbers that can be considered without taking into account additional tip effects are 1.118, 1.414, and 2.236, respectively. It is desirable, therefore, to take into consideration for untapered plan forms the induced effects of the opposite tip when the Mach lines from one tip enclose parts of the opposite half-wing. Figure 4 shows the Mach line configurations for these induced effects, and the drag equations are derived in appendix C. The wing wave-drag coefficient is then obtained from equation (8) where $C_{D_{tip}}$ for these cases includes the effects induced by the opposite tip.

RESULTS AND DISCUSSION

Calculations were made for families of tapered plan forms, each family characterized by a constant sweepback of the maximum-thickness line. The plan forms were obtained by considering the moment of the area about the root chord divided by the cube of the root chord to be constant for any given family. The aspect ratio varies with taper ratio because of this area-moment parameter.

For a constant thickness ratio the parameter, area moment divided by the product of the root chord and the square of the root thickness, is also constant. This condition is intended to imply that to a first approximation the root bending stress is the same for all members of any family having the same thickness ratio. A representative family of tapered plan forms and aspect-ratio variation with taper ratio is shown in figure 5.

Section wave drag.- Section wave-drag distributions for wings of taper ratio 0, 0.5, and 1.0 are presented in figures 6 to 10 for a Mach number of 1.414 and sweepback of 60°. The distributions of section wave drag of tapered wings are, in general, very similar to those of untapered plan forms. As a point of interest, the induced effects of the opposite half-wing and the tip-effect distribution are shown in figure 10 as separate curves. The total section wave-drag distribution is then obtained by adding the tip drag curve to the solid-line curve. The tip effect is placed correctly as shown for a wing of aspect ratio 1.0; for a wing of aspect ratio 2, this tip drag distribution should be shifted 1 semichord to the right. It is seen by reference to figure 5 that figures 6, 8, and 10 (fig. 10, A = 1) are section wave-drag distributions for one family of wings and that figures 7, 9, and 10 (fig. 10, A = 2) are for another family of wings whose aspect ratios are twice as large, respectively.

It is interesting to note at this point that for a given Mach number the section wave-drag coefficient at the root is a function of the sweep of the maximum-thickness line only; the terms involving leading-edge sweep adding up to zero. (See section drag equation in appendix A for $y = 0$.)

Wing wave drag.- Typical variations of wing wave-drag coefficient with Mach number for wings of taper ratio 0 and taper ratio 1.0 of the same family are shown in figures 11, 12, and 13 for 50°, 60°, and 70° sweepback, respectively. At some Mach number between 1.0 and the critical value $(M_{critical} = \sec \Lambda)$, the drag curve for the tapered wing has a discontinuous slope. This discontinuity occurs at that Mach number corresponding to the condition where the rear Mach line crosses the trailing edge of the wing, that is, where

$$\beta = \frac{1}{m_2} = \frac{2m_1 - m_0}{m_1 m_0}$$

In this region and near the critical Mach number $\left(\beta = \frac{1}{m_0}\right)$, the theory is not expected to be applicable because the assumption of small disturbances is violated, but the results are presented in order to give a more complete picture of the linearized theory.

It is seen from figures 11 to 13 that taper reduces the wing wave-drag coefficient at Mach numbers substantially below the critical value but increases the drag coefficient at Mach numbers approaching the critical value. This trend is similar to the one shown by the effect of high aspect ratio on the wave-drag coefficient of wings for a given taper ratio. It must be remembered that for the families of tapered wings considered in these calculations, however, the wings with greater taper have higher aspect ratios and, hence, the effects of aspect ratio as well as taper are included in this trend.

Variations of wing wave-drag coefficient with taper ratio for different sweepback angles at a Mach number of 1.2 are shown in figure 14. The untapered wing for this family has an aspect ratio of 1.0 and the variations of aspect ratio with taper ratio are presented in tabular form in the figure. For a given sweep angle, the wing of taper ratio 0 has the lowest drag coefficient and the untapered wing the highest. As the Mach number approaches the critical value $\left(\beta = \frac{1}{m_0}\right)$, this trend would reverse itself and the untapered wing would have the lowest drag coefficient as can be seen by reference to figures 11 to 13. It is also evident from figure 14 that for a given taper ratio and aspect ratio, an appreciable reduction in wing wave-drag coefficient is accomplished with increased sweepback.

Figure 15 presents variations of wing wave-drag coefficient with taper ratio for three families of wings based on untapered plan forms of aspect ratio 0.5, 1, and 2, respectively. The results are presented for 60° sweepback and a Mach number of 1.414. Pertinent details of the wings are presented in tabular form in the figure to facilitate interpretation of the plotted curves. The aforementioned trend of reduction in wing wave-drag coefficient associated with high aspect ratios at Mach numbers substantially below the critical Mach number for a given taper ratio is clearly seen in this figure. By choosing points along these curves corresponding to wings of the same aspect ratio, it is seen that for a constant aspect ratio tapering the wing increases the wing wave-drag coefficient. By a similar procedure it can be shown that for wings of constant aspect ratio, taper reduces the wing wave-drag coefficient at Mach numbers near the critical value. The increase in aspect ratio with taper ratio defined by the area-moment parameter thus has the effect of offsetting the adverse effects of taper at the lower Mach numbers.

CONCLUSIONS

1. Distributions of section wave drag along the span of tapered wings are, in general, very similar in shape to those of untapered plan forms.

2. The section wave-drag coefficient at the root is a function of the Mach number and the sweep of the maximum-thickness line and is independent of taper.

3. The increment in wing wave-drag coefficient caused by the tip is identically equal to zero for all tapered and untapered wings for which the Mach lines from one tip do not enclose any part of the opposite half-wing.

4. For wings of equal root bending stress, taper reduces the wing wave-drag coefficient at Mach numbers considerably less than the critical value - that is, a value equal to the secant of the sweepback angle - but increases the drag coefficient at Mach numbers near the critical value.

5. For wings of constant aspect ratio, taper increases the wing wave-drag coefficient at Mach numbers considerably below the critical value and decreases the wing wave-drag coefficient at Mach numbers near the critical value.

6. For a given taper ratio and aspect ratio, an appreciable reduction in wing wave-drag coefficient with increased sweepback is noted for the entire range of Mach number considered.

7. For a given sweep and taper ratio, higher aspect ratios reduce the wing wave-drag coefficient at substantially subcritical Mach numbers. At Mach numbers approaching the critical value, the plan forms of low aspect ratio have lower drag coefficients.

The generalized equations presented in the appendixes may be used to calculate the subcritical supersonic wave drag at zero lift for any conventionally tapered or untapered wing with symmetrical double-wedge airfoil sections and with leading edge, trailing edge, and line of maximum thickness sweptback.

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APPENDIX A

EVALUATION OF EQUATION (5) FOR SECTION WAVE DRAG AND WING WAVE DRAG EXCLUSIVE OF TIP EFFECTS $\left[\beta \leqq \frac{1}{m_0}\right]$

Section Drag for $0 \leqq$ Taper Ratio < 1

$$\frac{\pi c_{d_\infty} c}{4(t/c)^2} = A \qquad \text{for} \qquad 0 \leqq y \leqq \frac{am_1}{1 - \beta m_1}$$

$$= A + B \qquad \frac{am_1}{1 - \beta m_1} < y \leqq \frac{am_0}{1 - \beta m_0}$$

$$= A + B + C \qquad \frac{am_0}{1 - \beta m_0} < y \leqq \frac{2am_1}{1 - \beta m_1}$$

$$= A + B + C + D \qquad \frac{2am_1}{1 - \beta m_1} < y \leqq \frac{am_0 m_1}{m_0 - m_1}$$

where

$$A = \frac{2}{\sqrt{1 - \beta^2 m_1{}^2}} \left\{ \frac{y(m_0 + m_1) + am_0 m_1}{m_0} \cosh^{-1} \frac{y\left(1 + m_0 m_1 \beta^2\right) + am_0}{\beta\left[y(m_0 + m_1) + am_1 m_0\right]} \right.$$

$$\left. - y \cosh^{-1} \frac{1 + \beta^2 m_1{}^2}{2\beta m_1} + \frac{y(m_1 - m_0) + am_1 m_0}{m_0} \cosh^{-1} \frac{y\left(1 - m_0 m_1 \beta^2\right) + am_0}{\beta \left|y(m_0 - m_1) - am_1 m_0\right|} \right.$$

$$- \frac{y(m_2 + m_1) + 2am_1m_2}{2m_2} \cosh^{-1} \frac{y\left(1 + m_1m_2\beta^2\right) + 2am_2}{\beta\left[y(m_1 + m_2) + 2am_1m_2\right]}$$

$$- \frac{y(m_1 - m_2) + 2am_1m_2}{2m_2} \cosh^{-1} \frac{y\left(1 - m_1m_2\beta^2\right) + 2am_2}{\beta\left|y(m_2 - m_1) - 2am_1m_2\right|}\Bigg\}$$

$$- \frac{2y}{\sqrt{1 - \beta^2{m_2}^2}} \cosh^{-1} \frac{1 + {m_2}^2\beta^2}{2\beta m_2}$$

$$+ \frac{2}{\sqrt{1 - \beta^2{m_0}^2}} \left\{\frac{y(m_0 + m_2) + am_0m_2}{m_2} \cosh^{-1} \frac{y\left(1 + m_0m_2\beta^2\right) + am_2}{\beta\left[y(m_0 + m_2) + am_0m_2\right]}\right.$$

$$\left. - 4y \cosh^{-1} \frac{1 + {m_0}^2\beta^2}{2\beta m_0} + \frac{y(m_0 - m_2) + am_0m_2}{m_2} \cosh^{-1} \frac{y\left(1 - m_0m_2\beta^2\right) + am_2}{\beta\left|y(m_2 - m_0) - am_0m_2\right|}\right]\Bigg\}$$

$$B = \frac{2}{m_1\sqrt{1 - \beta^2 m_0{}^2}} \left\{ \left[y(m_0 + m_1) - am_0m_1\right] \cosh^{-1} \frac{y\left(1 + m_0m_1\beta^2\right) - am_1}{\beta\left|y(m_0 + m_1) - am_0m_1\right|} \right.$$

$$\left. + \left[y(m_0 - m_1) - am_0m_1\right] \cosh^{-1} \frac{y\left(1 - m_0m_1\beta^2\right) - am_1}{\beta\left|y(m_1 - m_0) + am_1m_0\right|} \right\}$$

$$C = \frac{2}{m_0\sqrt{1 - \beta^2 m_2{}^2}} \left\{ \left[y(m_0 + m_2) - am_2m_0\right] \cosh^{-1} \frac{y\left(1 + m_0m_2\beta^2\right) - am_0}{\beta\left|y(m_0 + m_2) - am_2m_0\right|} \right.$$

$$\left. + \left[y(m_2 - m_0) - am_2m_0\right] \cosh^{-1} \frac{y\left(1 - m_0m_2\beta^2\right) - am_0}{\beta\left|y(m_0 - m_2) + am_2m_0\right|} \right\}$$

and

$$D = \frac{1}{m_1\sqrt{1 - \beta^2 m_2{}^2}} \left\{ \left[y(m_1 - m_2) + 2am_1m_2\right] \cosh^{-1} \frac{y\left(1 - m_1m_2\beta^2\right) - 2am_1}{\beta\left|y(m_1 - m_2) + 2am_1m_2\right|} \right.$$

$$\left. - \left[y(m_1 + m_2) - 2am_1m_2\right] \cosh^{-1} \frac{y\left(1 + m_1m_2\beta^2\right) - 2am_1}{\beta\left|y(m_1 + m_2) - 2am_1m_2\right|} \right\}$$

Wing Drag for 0 < Taper Ratio < 1

$$\frac{\pi S C_{D_\infty}}{8(t/c)^2} = A \qquad \text{for} \qquad 0 \leqq dm_0 \leqq \frac{am_1}{1 - \beta m_1}$$

$$= A + B \qquad \frac{am_1}{1 - \beta m_1} < dm_0 \leqq \frac{am_0}{1 - \beta m_0}$$

$$= A + B + C \qquad \frac{am_0}{1 - \beta m_0} < dm_0 \leqq \frac{2am_1}{1 - \beta m_1}$$

$$= A + B + C + D \qquad \frac{2am_1}{1 - \beta m_1} < dm_0 \leqq \frac{am_0 m_1}{m_0 - m_1}$$

where

$$A = \frac{2}{\sqrt{1 - \beta^2 m_0^2}} \left\{ \frac{m_0^3 m_1 a^2}{m_0^2 - m_1^2} \cosh^{-1} \frac{d\left(1 - \beta^2 m_0^2\right) + a}{a\beta m_0} - 2d^2 m_0^2 \cosh^{-1} \frac{1 + m_0^2 \beta^2}{2\beta m_0} \right.$$

$$- \frac{m_0^2 \left[m_2(d - a) - dm_0\right]^2}{2m_2(m_2 - m_0)} \cosh^{-1} \frac{dm_0\left(1 - m_0 m_2 \beta^2\right) + am_2}{\beta m_0 \left|m_2(d - a) - dm_0\right|}$$

$$\left. + \frac{m_0^2 \left[dm_0 + m_2(d + a)\right]^2}{2m_2(m_0 + m_2)} \cosh^{-1} \frac{dm_0\left(1 + m_0 m_2 \beta^2\right) + am_2}{\beta m_0 \left[dm_0 + m_2(d + a)\right]} \right\}$$

$$+ \frac{2}{\sqrt{1 - \beta^2 m_1^2}} \left\{ \frac{m_0 \left[dm_0 + m_1(a + d)\right]^2}{2(m_0 + m_1)} \cosh^{-1} \frac{d\left(1 + \beta^2 m_0 m_1\right) + a}{\beta \left[dm_0 + m_1(d + a)\right]} \right.$$

$$- \frac{d^2 m_0^2}{2} \cosh^{-1} \frac{1 + m_1^2\beta^2}{2\beta m_1} - \frac{m_0\left[dm_0 - m_1(d+a)\right]^2}{2(m_0 - m_1)} \cosh^{-1} \frac{d\left(1 - \beta^2 m_0 m_1\right) + a}{\beta\left|dm_0 - m_1(d+a)\right|}$$

$$- \frac{\left[dm_0(m_1 + m_2) + 2am_1m_2\right]^2}{4m_2(m_1 + m_2)} \cosh^{-1} \frac{dm_0\left(1 + \beta^2 m_1 m_2\right) + 2am_2}{\beta\left[dm_0(m_1 + m_2) + 2am_1m_2\right]}$$

$$\left. + \frac{\left[dm_0(m_2 - m_1) - 2am_1m_2\right]^2}{4m_2(m_2 - m_1)} \cosh^{-1} \frac{dm_0\left(1 - m_1m_2\beta^2\right) + 2am_2}{\beta\left|dm_0(m_2 - m_1) - 2am_1m_2\right|} \right\}$$

$$+ \frac{1}{\sqrt{1 - \beta^2 m_2^2}} \left[{}^{*}-d^2 m_0^2 \cosh^{-1} \frac{1 + m_2^2\beta^2}{2\beta m_2} \right.$$

$$- \frac{4a^2 m_2^3 m_1}{m_2^2 - m_1^2} \cosh^{-1} \frac{dm_0\left(1 - m_2^2\beta^2\right) + 2am_2}{2\beta a m_2^2}$$

$$\left. + \frac{2a^2 m_2^3 m_0}{m_2^2 - m_0^2} \cosh^{-1} \frac{dm_0\left(1 - m_2^2\beta^2\right) + am_2}{a\beta m_2^2} \right]$$

$$+ a^2 m_0^2 m_1 \left[\frac{1}{(m_0 + m_1)\sqrt{1 - \beta^2 m_1^2}} \cosh^{-1} \frac{1}{\beta m_1} + \frac{1}{(3m_1 - m_0)\sqrt{1 - \beta^2 m_2^2}} \cosh^{-1} \frac{1}{\beta m_2} \right.$$

$$\left. - \frac{4m_1}{(3m_1 - m_0)(m_0 + m_1)\sqrt{1 - \beta^2 m_0^2}} \cosh^{-1} \frac{1}{\beta m_0} \right]$$

---

*When $1 - m_2^2\beta^2$ is negative neglect term marked with asterisk $\left(-d^2 m_0^2 \cosh^{-1} \frac{1 + m_2^2\beta^2}{2\beta m_2}\right)$ in values for A and use the relationship $\cos^{-1}x = -i \cosh^{-1}x$ for all terms involving $\frac{1}{\sqrt{1 - m_2^2\beta^2}}$ as multiplication factor.

$$B = \frac{2m_0}{\sqrt{1 - \beta^2 m_0^2}} \left\{ \frac{a^2 m_1^3 \sqrt{1 - m_0^2\beta^2}}{\left(m_1^2 - m_0^2\right)\sqrt{1 - m_1^2\beta^2}} \cosh^{-1} \frac{dm_0\left(1 - m_1^2\beta^2\right) - am_1}{a\beta m_1^2} \right.$$

$$+ \frac{m_0\left[dm_0 + m_1(d - a)\right]^2}{2m_1(m_0 + m_1)} \cosh^{-1} \frac{dm_0\left(1 + m_0 m_1\beta^2\right) - am_1}{\beta m_0 \left|d(m_0 + m_1) - am_1\right|}$$

$$\left. - \frac{m_0\left[m_1(d + a) - dm_0\right]^2}{2m_1(m_1 - m_0)} \cosh^{-1} \frac{dm_0\left(1 - m_0 m_1\beta^2\right) - am_1}{\beta m_0 \left|d(m_1 - m_0) + am_1\right|} \right\}$$

$$C = \frac{2m_0}{\sqrt{1 - \beta^2 m_2^2}} \left\{ \frac{a^2 m_0^2 m_2 \sqrt{1 - m_2^2\beta^2}}{\left(m_0^2 - m_2^2\right)\sqrt{1 - m_0^2\beta^2}} \cosh^{-1} \frac{d\left(1 - m_0^2\beta^2\right) - a}{a\beta m_0} \right.$$

$$+ \frac{\left[dm_0 + m_2(d - a)\right]^2}{2(m_0 + m_2)} \cosh^{-1} \frac{d\left(1 + m_0 m_2\beta^2\right) - a}{\beta \left|dm_0 + m_2(d - a)\right|}$$

$$\left. - \frac{\left[dm_0 + m_2(a - d)\right]^2}{2(m_0 - m_2)} \cosh^{-1} \frac{d\left(1 - m_0 m_2\beta^2\right) - a}{\beta \left|dm_0 + m_2(a - d)\right|} \right\}$$

and

$$D = \frac{1}{\sqrt{1 - m_2^2\beta^2}} \left\{ \frac{4a^2 m_1^3 m_2 \sqrt{1 - m_2^2\beta^2}}{\left(m_2^2 - m_1^2\right)\sqrt{1 - m_1^2\beta^2}} \cosh^{-1} \frac{dm_0\left(1 - m_1^2\beta^2\right) - 2am_1}{2a\beta m_1^2} \right.$$

$$+ \frac{\left[dm_0(m_1 - m_2) + 2am_1 m_2\right]^2}{2m_1(m_1 - m_2)} \cosh^{-1} \frac{dm_0\left(1 - m_1 m_2\beta^2\right) - 2am_1}{\beta \left|dm_0(m_1 - m_2) + 2am_1 m_2\right|}$$

$$\left. - \frac{\left[dm_0(m_1 + m_2) - 2am_1 m_2\right]^2}{2m_1(m_1 + m_2)} \cosh^{-1} \frac{dm_0\left(1 + m_1 m_2\beta^2\right) - 2am_1}{\beta \left|dm_0(m_1 + m_2) - 2am_1 m_2\right|} \right\}$$

Wing Drag for Taper Ratio 0

For $0 < \beta < \dfrac{1}{m_2}$

$$\frac{\pi S C_{D_\infty}}{8(t/c)^2} = \frac{a^2 m_0 m_1}{\left(m_0{}^2 - m_1{}^2\right)\sqrt{1 - m_1{}^2\beta^2}}\left[(m_0 + m_1)(2m_1 - m_0)\cosh^{-1}\frac{m_0\left(1 - m_1{}^2\beta^2\right) + 2(m_1 - m_0)}{2\beta m_1(m_0 - m_1)}\right.$$

$$\left. - 2m_1{}^2\cosh^{-1}\frac{1 - m_0 m_1\beta^2}{\beta(m_0 - m_1)} + m_0(m_0 - m_1)\log\left(1 + \sqrt{1 - m_1{}^2\beta^2}\right)\right]$$

$$+ \frac{2a^2 m_0{}^2 m_1}{\left(m_0{}^2 - m_1{}^2\right)(3m_1 - m_0)\sqrt{1 - m_0{}^2\beta^2}}\left[m_0(3m_1 - m_0)\cosh^{-1}\frac{1 - m_0 m_1\beta^2}{\beta(m_0 - m_1)}\right.$$

$$\left. - (2m_1 - m_0)(m_0 + m_1)\cosh^{-1}\frac{m_1\left(1 - m_0{}^2\beta^2\right) + m_1 - m_0}{\beta m_0(m_0 - m_1)} - 2m_1(m_0 - m_1)\log\left(1 + \sqrt{1 - m_0{}^2\beta^2}\right)\right]$$

$$+ \frac{a^2 m_1 m_0{}^2}{\sqrt{1 - m_2{}^2\beta^2}(2m_1 - m_0)(m_0 - m_1)(3m_1 - m_0)}\left\{2m_1{}^2\cosh^{-1}\frac{m_1\left(1 - m_0{}^2\beta^2\right) + m_1 - m_0}{\beta m_0(m_0 - m_1)}\right.$$

$$\left. - m_0(3m_1 - m_0)\cosh^{-1}\frac{m_0\left(1 - m_1{}^2\beta^2\right) + 2(m_1 - m_0)}{2\beta m_1(m_0 - m_1)} + (2m_1 - m_0)(m_0 - m_1)\log\left[1 + (2m_1 - m_0)\sqrt{1 - m_2{}^2\beta^2}\right]\right\}$$

For $\frac{1}{m_2} < \beta < \frac{1}{m_0}$, use $A + B + C$ where

$$A = \frac{2a^2 m_0{}^2 m_1}{\sqrt{1 - m_0{}^2\beta^2}(3m_1 - m_0)\left(m_0{}^2 - m_1{}^2\right)}\left[m_0(3m_1 - m_0)\cosh^{-1}\frac{1 - m_0 m_1\beta^2}{\beta(m_0 - m_1)}\right.$$

$$\left. - 2m_1(m_0 - m_1)\log\left(1 + \sqrt{1 - m_0{}^2\beta^2}\right)\right]$$

$$B = \frac{a^2 m_1 m_0}{\sqrt{1 - m_1{}^2\beta^2}\left(m_0{}^2 - m_1{}^2\right)}\left[m_0(m_0 - m_1)\log\left(1 + \sqrt{1 - m_1{}^2\beta^2}\right)\right.$$

$$\left. - 2m_1{}^2\cosh^{-1}\frac{1 - m_0 m_1\beta^2}{\beta(m_0 - m_1)}\right]$$

and

$$C = \frac{a^2 m_1 m_0{}^2}{\sqrt{1 - m_2{}^2\beta^2}(2m_1 - m_0)(m_0 - m_1)(3m_1 - m_0)}\left[2m_1{}^2\cos^{-1}\frac{m_1\left(1 - m_0{}^2\beta^2\right) + m_1 - m_0}{\beta m_0(m_0 - m_1)}\right.$$

$$+ (2m_1 - m_0)(m_0 - m_1)\cos^{-1}\frac{1}{\beta m_2}$$

$$\left. - m_0(3m_1 - m_0)\cos^{-1}\frac{m_1\left(1 - m_1 m_0\beta^2\right) + m_1 - m_0}{2\beta m_1(m_0 - m_1)}\right]$$

For $\beta = \frac{1}{m_2}$, use $A + B$ plus

$$a^2 m_1 m_0{}^2\left[\frac{1}{3m_1 - m_0} + \frac{\sqrt{m_1}}{(2m_1 - m_0)(m_0 - m_1)^{3/2}}\left(\frac{2m_1{}^{3/2}}{\sqrt{3m_1 - m_0}} - m_0\right)\right]$$

For $\beta = \frac{1}{m_0}$, use B + C plus

$$2a^2 m_0^2 m_1 \left[\frac{1}{(m_0 - m_1)\sqrt{1 - m_1^2\beta^2}} - \frac{2m_1}{(3m_1 - m_0)(m_0 + m_1)} \right]$$

Section Drag for Taper Ratio 1.0

$$\frac{c_{d_\infty} c\pi\sqrt{1 - \beta^2 m_0^2}}{4m_0(t/c)^2} = A \qquad \text{for} \qquad 0 \leqq y \leqq \frac{am_0}{1 - \beta m_0}$$

$$= A + B \qquad \frac{am_0}{1 - \beta m_0} < y \leqq \frac{2am_0}{1 - \beta m_0}$$

$$= A + B + C \qquad \frac{2am_0}{1 - \beta m_0} < y < \infty$$

where

$$A = \frac{4(2y + am_0)}{m_0} \cosh^{-1} \frac{y\left(1 + \beta^2 m_0^2\right) + am_0}{\beta m_0(2y + am_0)} - \frac{12y}{m_0} \cosh^{-1} \frac{1 + \beta^2 m_0^2}{2\beta m_0}$$

$$+ 4a \cosh^{-1} \frac{y\left(1 - \beta^2 m_0^2\right) + am_0}{a\beta m_0^2} - \frac{2(y + am_0)}{m_0} \cosh^{-1} \frac{y\left(1 + \beta^2 m_0^2\right) + 2am_0}{2\beta m_0(y + am_0)}$$

$$- 2a \cosh^{-1} \frac{y\left(1 - \beta^2 m_0^2\right) + 2am_0}{2a\beta m_0^2}$$

$$B = \frac{4(2y - am_0)}{m_0} \cosh^{-1} \frac{y\left(1 + \beta^2 m_0^2\right) - am_0}{\beta m_0(2y - am_0)} - 4a \cosh^{-1} \frac{y\left(1 - \beta^2 m_0^2\right) - am_0}{a\beta m_0^2}$$

and

$$C = 2a \cosh^{-1} \frac{y\left(1 - \beta^2 m_0{}^2\right) - 2am_0}{2a\beta m_0{}^2} - \frac{2(y - am_0)}{m_0} \cosh^{-1} \frac{y\left(1 + \beta^2 m_0{}^2\right) - 2am_0}{2\beta m_0(y - am_0)}$$

Wing Drag for Taper Ratio 1.0

$$\frac{C_{D_\infty}\pi S\sqrt{1 - \beta^2 m_0{}^2}}{8m_0(t/c)^2} = A \qquad \text{for} \qquad 0 \leqq dm_0 \leqq \frac{am_0}{1 - \beta m_0}$$

$$= A + B \qquad \frac{am_0}{1 - \beta m_0} < dm_0 \leqq \frac{2am_0}{1 - \beta m_0}$$

$$= A + B + C \qquad \frac{2am_0}{1 - \beta m_0} < dm_0 < \infty$$

where

$$A = \frac{am_0\left[4d\left(1 - \beta^2 m_0{}^2\right) + a\left(3 - \beta^2 m_0{}^2\right)\right]}{1 - \beta^2 m_0{}^2} \cosh^{-1} \frac{\left(1 - \beta^2 m_0{}^2\right)d + a}{a\beta m_0}$$

$$- \frac{2am_0}{\sqrt{1 - \beta^2 m_0{}^2}} \sqrt{\left(1 - \beta^2 m_0{}^2\right)d^2 + 2ad + a^2} + m_0(2d + a)^2 \cosh^{-1} \frac{\left(1 + \beta^2 m_0{}^2\right)d + a}{\beta m_0(2d + a)}$$

$$- 6d^2 m_0 \cosh^{-1} \frac{1 + \beta^2 m_0{}^2}{2\beta m_0}$$

$$- \frac{am_0\left[a\left(3 - \beta^2 m_0{}^2\right) + 2d\left(1 - \beta^2 m_0{}^2\right)\right]}{1 - \beta^2 m_0{}^2} \cosh^{-1} \frac{\left(1 - \beta^2 m_0{}^2\right)d + 2a}{2a\beta m_0}$$

$$+ \frac{am_0}{\sqrt{1 - \beta^2 m_0{}^2}} \sqrt{\left(1 - \beta^2 m_0{}^2\right)d^2 + 4ad + 4a^2} - m_0(d + a)^2 \cosh^{-1} \frac{\left(1 + \beta^2 m_0{}^2\right)d + 2a}{2\beta m_0(d + a)}$$

$$B = \frac{am_0\left[a\left(3 - \beta^2 m_0^2\right) - 4d\left(1 - \beta^2 m_0^2\right)\right]}{1 - \beta^2 m_0^2} \cosh^{-1} \frac{\left(1 - \beta^2 m_0^2\right)d - a}{a\beta m_0}$$

$$+ \frac{2am_0}{\sqrt{1 - \beta^2 m_0^2}} \sqrt{\left(1 - \beta^2 m_0^2\right)d^2 - 2ad + a^2} + m_0(2d - a)^2 \cosh^{-1} \frac{\left(1 + \beta^2 m_0^2\right)d - a}{\beta m_0(2d - a)}$$

and

$$C = -\frac{am_0\left[a\left(3 - \beta^2 m_0^2\right) - 2d\left(1 - \beta^2 m_0^2\right)\right]}{1 - \beta^2 m_0^2} \cosh^{-1} \frac{\left(1 - \beta^2 m_0^2\right)d - 2a}{2a\beta m_0}$$

$$- \frac{am_0}{\sqrt{1 - \beta^2 m_0^2}} \sqrt{\left(1 - \beta^2 m_0^2\right)d^2 - 4ad + 4a^2}$$

$$- m_0(d - a)^2 \cosh^{-1} \frac{\left(1 + \beta^2 m_0^2\right)d - 2a}{2\beta m_0(d - a)}$$

When $\beta = \frac{1}{m_0}$, use the following expression

$$\frac{C_{D_\infty}}{(t/c)^2} = \frac{16m_0}{3\pi d\sqrt{a}}\left[(2d + a)^{3/2} - (d + a)^{3/2}\right]$$

APPENDIX B

EVALUATION OF EQUATION (7) FOR TIP EFFECTS $\left[\beta \leqq \frac{1}{m_0}\right]$

Section Drag Increment for 0 < Taper Ratio < 1

For aspect-ratio limitations, see equations (6).

$$\frac{c_{d_{tip}} c\pi}{4(t/c)^2} = A \left(\text{for } \frac{dm_0^2(1 + \beta m_1) - am_1m_0}{m_1(1 + \beta m_0)} \leqq y \leqq dm_0\right)$$

$$+ B \left(\text{for } \frac{dm_0m_2(1 + \beta m_1) - 2am_1m_2}{m_1(1 + \beta m_2)} \leqq y \leqq dm_0\right)$$

$$+ C \left(\text{for } \frac{dm_2(1 + \beta m_0) - am_2}{1 + \beta m_2} \leqq y \leqq dm_0\right)$$

where

$$A = -2(y - dm_0) \cosh^{-1} \frac{ym_1 + am_1m_0 - dm_0^2}{\beta m_1m_0(dm_0 - y)}$$

$$+ \frac{2\left[y(m_0 - m_1) - am_1m_0\right]}{m_0\sqrt{1 - m_1^2\beta^2}} \cosh^{-1} \frac{ym_1\left(1 - m_0m_1\beta^2\right) - dm_0^2\left(1 - m_1^2\beta^2\right) + am_1m_0}{\beta m_1\left|y(m_0 - m_1) - am_1m_0\right|}$$

$$B = \left(y - dm_0\right) \cosh^{-1} \frac{y\left(2m_1 - m_0\right) + m_0\left(2am_1 - dm_0\right)}{\beta m_1 m_0 \left(dm_0 - y\right)}$$

$$- \frac{2\left[\left(m_0 - m_1\right)y - am_0 m_1\right]}{m_0\sqrt{1 - m_1^2\beta^2}} \cosh^{-1} \frac{y\left(2m_1 - m_0 - m_1^2 m_0 \beta^2\right) + 2am_0 m_1 - dm_0^2\left(1 - m_1^2\beta^2\right)}{2\beta m_1 \left|y\left(m_0 - m_1\right) - am_0 m_1\right|}$$

and

$$C = -2\left(y - dm_0\right) \cosh^{-1} \frac{y + m_2(a - d)}{\beta m_2 \left(dm_0 - y\right)}$$

$$+ \frac{2\left[y\left(m_2 - m_0\right) - am_2 m_0\right]}{m_2\sqrt{1 - m_0^2\beta^2}} \cosh^{-1} \frac{y\left(1 - m_0 m_2 \beta^2\right) - dm_2\left(1 - m_0^2\beta^2\right) + am_2}{\beta \left|y\left(m_2 - m_0\right) - am_0 m_2\right|}$$

Section Drag Increment for Taper Ratio 1.0; Aspect Ratio $\geqq \frac{1}{\beta}$

$$\frac{c_{d_{tip}} c\pi\sqrt{1 - \beta^2 m_0^2}}{4m_0 (t/c)^2} = A\left(\text{for } \frac{dm_0\left(1 + \beta m_0\right) - 2am_0}{1 + \beta m_0} \leqq y \leqq dm_0\right)$$

$$+ B\left(\text{for } \frac{dm_0\left(1 + \beta m_0\right) - am_0}{1 + \beta m_0} \leqq y \leqq dm_0\right)$$

where

$$A = \frac{\sqrt{1 - \beta^2 m_0^2}}{m_0}(y - dm_0) \cosh^{-1} \frac{y + m_0(2a - d)}{\beta m_0(dm_0 - y)}$$

$$+ 2a \cosh^{-1} \frac{\left(1 - m_0^2\beta^2\right)(y - dm_0) + 2am_0}{2a\beta m_0^2}$$

and

$$B = \frac{-4\sqrt{1 - \beta^2 m_0^2}}{m_0}\left(y - dm_0\right) \cosh^{-1} \frac{y - m_0(d - a)}{\beta m_0(dm_0 - y)}$$

$$- 4a \cosh^{-1} \frac{\left(1 - m_0^2\beta^2\right)(y - dm_0) + am_0}{a\beta m_0^2}$$

There is no tip effect whatsoever for the wing of taper ratio 0, and the increment in wing wave drag caused by the tip is identically equal to zero for all cases satisfying these aspect-ratio limitations.

APPENDIX C

EQUATIONS FOR ADDITIONAL TIP EFFECTS FOR WINGS OF CONSTANT CHORD [See fig. 4]

Case I

$$\frac{1}{2\beta} \leqq A < \frac{1}{\beta} \quad \text{and} \quad A \geqq \frac{2m_0}{1 + \beta m_0}$$

$$\frac{C_{D_{tip}} \pi S\sqrt{1 - \beta^2 m_0{}^2}}{8m_0 (t/c)^2} = \int_{\frac{m_0(dm_0\beta + d - 2a)}{1 - \beta m_0}}^{dm_0} \int_{(y+dm_0)\beta+d-a}^{\frac{y+am_0}{m_0}} \cosh^{-1} \frac{x + a - d + m_0\beta^2\left(y + dm_0\right)}{\beta\left(y + m_0 x + am_0\right)} \, dx \, dy$$

$$= m_0(a - d)^2 \cosh^{-1} \frac{a - dm_0{}^2\beta^2}{\beta m_0 |a - d|} + m_0(d + a)^2 \cosh^{-1} \frac{a + dm_0{}^2\beta^2}{\beta m_0(d + a)}$$

$$- 2d^2 m_0\sqrt{1 - \beta^2 m_0{}^2} \cosh^{-1} \frac{a}{\beta m_0 d}$$

Case II

$$\frac{1}{2\beta} \leqq A < \frac{1}{\beta} \quad \text{and} \quad A < \frac{2m_0}{1 + \beta m_0}$$

$$\frac{C_{D_{tip}} \pi S \sqrt{1 - m_0{}^2\beta^2}}{8m_0(t/c)^2} = \int_0^{dm_0} \int_{(y+dm_0)\beta+d-a}^{\frac{y+am_0}{m_0}} \cosh^{-1} \frac{x + a - d + m_0\beta^2(y + dm_0)}{\beta(y + m_0x + am_0)} \, dx \, dy$$

$$= m_0(a - d)^2 \left[\cosh^{-1} \frac{a - dm_0{}^2\beta^2}{\beta m_0 |a - d|} - \cosh^{-1} \frac{2a - d\left(1 + m_0{}^2\beta^2\right)}{2\beta m_0 |a - d|}\right]$$

$$- a^2 m_0 \cosh^{-1} \frac{2a - d\left(1 - m_0{}^2\beta^2\right)}{2a\beta m_0}$$

$$+ m_0(d + a)^2 \cosh^{-1} \frac{a + dm_0{}^2\beta^2}{\beta m_0(d + a)}$$

$$+ \frac{d^2 m_0}{2}\sqrt{1 - \beta^2 m_0{}^2}\left(\cosh^{-1} \frac{2a - d}{\beta m_0 d} - 4 \cosh^{-1} \frac{a}{\beta m_0 d}\right)$$

Case III

$$0 < A < \frac{1}{2\beta} \quad \text{and} \quad A \geqq \frac{2m_0}{1 + \beta m_0}$$

$$\frac{C_{D_{tip}} \pi S\sqrt{1 - \beta^2 m_0{}^2}}{8m_0(t/c)^2} = \int_{\frac{m_0(dm_0\beta+d-2a)}{1-\beta m_0}}^{\frac{m_0(dm_0\beta+d-a)}{1-\beta m_0}} \int_{(y+dm_0)\beta+d-a}^{\frac{y+am_0}{m_0}} \cosh^{-1} \frac{x + a - d + m_0\beta^2(y + dm_0)}{\beta(y + m_0x + am_0)} \, dx \, dy$$

$$+ \int_{\frac{m_0(dm_0\beta+d-a)}{1-\beta m_0}}^{dm_0} \int_{\frac{y}{m_0}}^{\frac{y+am_0}{m_0}} \cosh^{-1} \frac{x + a - d + m_0\beta^2(y + dm_0)}{\beta(y + m_0x + am_0)} \, dx \, dy$$

$$- \int_{\frac{m_0(dm_0\beta+d-a)}{1-\beta m_0}}^{dm_0} \int_{\beta(y+dm_0)+d-a}^{\frac{y}{m_0}} \cosh^{-1} \frac{x + a - d + m_0\beta^2(y + dm_0)}{\beta(y + m_0x + am_0)} \, dx \, dy$$

$$- 2 \int_{\frac{m_0(dm_0\beta+d-a)}{1-\beta m_0}}^{dm_0} \int_{\beta(y+dm_0)+d}^{\frac{y+am_0}{m_0}} \cosh^{-1} \frac{x - d + m_0\beta^2(y + dm_0)}{\beta(y + m_0x)} \, dx \, dy$$

$$= m_0\left[(a-d)^2\cosh^{-1}\frac{a-dm_0{}^2\beta^2}{\beta m_0|a-d|}-(a-2d)^2\cosh^{-1}\frac{a-2dm_0{}^2\beta^2}{\beta m_0|a-2d|}\right.$$

$$+(d+a)^2\cosh^{-1}\frac{a+dm_0{}^2\beta^2}{\beta m_0(d+a)}-(2d+a)^2\cosh^{-1}\frac{a+2dm_0{}^2\beta^2}{\beta m_0(2d+a)}$$

$$\left.-2d^2\sqrt{1-\beta^2 m_0{}^2}\left(\cosh^{-1}\frac{a}{\beta m_0 d}-4\cosh^{-1}\frac{a}{2\beta m_0 d}\right)\right]$$

Case IV

$$0 < A < \frac{1}{2\beta} \quad \text{and} \quad \frac{m_0}{1+\beta m_0} \leqq A < \frac{2m_0}{1+\beta m_0}$$

The lower limit for y is changed to 0 in the first integral of case III and the resultant expression for the drag is

$$\frac{c_{D_{tip}}\pi S\sqrt{1-\beta^2 m_0{}^2}}{8m_0(t/c)^2} = m_0\left\{(a-d)^2\left[\cosh^{-1}\frac{a-dm_0{}^2\beta^2}{\beta m_0|a-d|}-\cosh^{-1}\frac{2a-d\left(1+m_0{}^2\beta^2\right)}{2\beta m_0|a-d|}\right]\right.$$

$$-(a-2d)^2\cosh^{-1}\frac{a-2dm_0{}^2\beta^2}{\beta m_0|a-2d|}+(d+a)^2\cosh^{-1}\frac{a+dm_0{}^2\beta^2}{\beta m_0(d+a)}$$

$$-(2d+a)^2\cosh^{-1}\frac{a+2dm_0{}^2\beta^2}{\beta m_0(2d+a)}-a^2\cosh^{-1}\frac{2a-d\left(1-\beta^2 m_0{}^2\right)}{2a\beta m_0}$$

$$-d^2\sqrt{1-\beta^2 m_0{}^2}\left(2\cosh^{-1}\frac{a}{\beta m_0 d}-8\cosh^{-1}\frac{a}{2\beta m_0 d}\right.$$

$$\left.\left.-\frac{1}{2}\cosh^{-1}\frac{2a-d}{\beta m_0 d}\right)\right\}$$

Case V

$$0 < A < \frac{1}{2\beta} \quad \text{and} \quad A < \frac{m_0}{1 + \beta m_0}$$

All four lower limits for y are changed to 0 in case III.

$$\frac{C_{D_{tip}}\pi S\sqrt{1 - \beta^2 m_0{}^2}}{8m_0(t/c)^2} = m_0 \Biggl\{(a - d)^2 \left[\cosh^{-1} \frac{a - dm_0{}^2\beta^2}{\beta m_0|a - d|}\right.$$

$$\left. - 2\cosh^{-1} \frac{2a - d\left(1 + m_0{}^2\beta^2\right)}{2\beta m_0|a - d|}\right] - (a - 2d)^2 \left[\cosh^{-1} \frac{a - 2dm_0{}^2\beta^2}{\beta m_0|a - 2d|}\right.$$

$$\left. - \cosh^{-1} \frac{a - d\left(1 + \beta^2 m_0{}^2\right)}{\beta m_0|a - 2d|}\right] + (d + a)^2 \cosh^{-1} \frac{a + dm_0{}^2\beta^2}{\beta m_0(d + a)}$$

$$- (2d + a)^2 \cosh^{-1} \frac{a + 2dm_0{}^2\beta^2}{\beta m_0(2d + a)} + a^2 \left[\cosh^{-1} \frac{a - d\left(1 - m_0{}^2\beta^2\right)}{a\beta m_0}\right.$$

$$\left. - 2\cosh^{-1} \frac{2a - d\left(1 - \beta^2 m_0{}^2\right)}{2a\beta m_0}\right] - d^2\sqrt{1 - \beta^2 m_0{}^2}\left(2\cosh^{-1} \frac{a - d}{\beta m_0 d}\right.$$

$$\left. - \cosh^{-1} \frac{2a - d}{\beta m_0 d} + 2\cosh^{-1} \frac{a}{\beta m_0 d} - 8\cosh^{-1} \frac{a}{2d\beta m_0}\right)\Biggr\}$$

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2. Puckett, Allen E.: Supersonic Wave Drag of Thin Airfoils. Jour. Aero. Sci., vol. 13, no. 9, Sept. 1946, pp. 475-484.

3. Harmon, Sidney M., and Swanson, Margaret D.: Calculations of the Supersonic Wave Drag of Nonlifting Wings with Arbitrary Sweepback and Aspect Ratio. Wings Swept behind the Mach Lines. NACA TN No. 1319 , 1947.

4. Prandtl, L.: General Considerations on the Flow of Compressible Fluids. NACA TM No. 805, 1936.

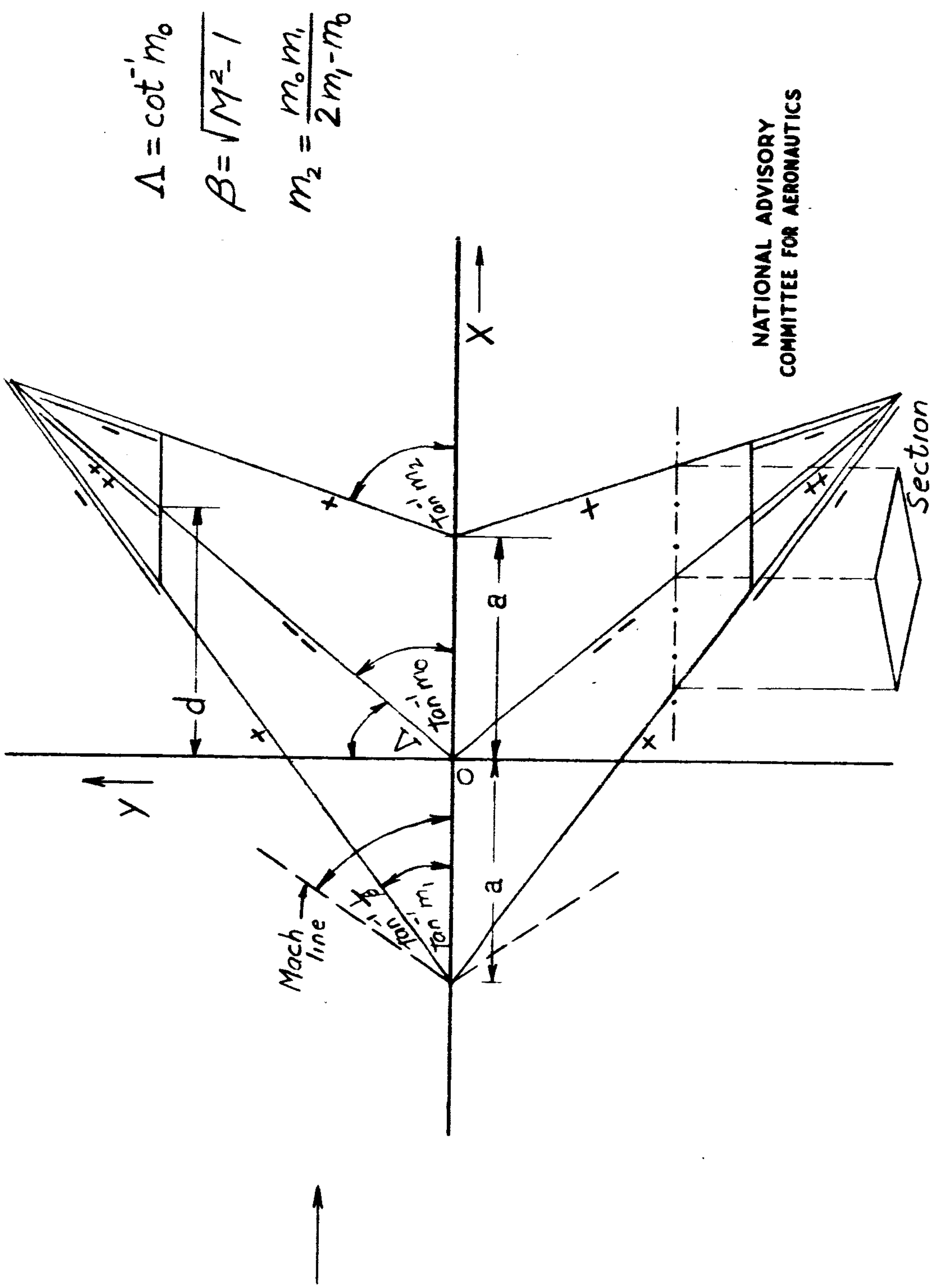


Figure 1.- Symbols and distributions of sinks and sources for a tapered wing.

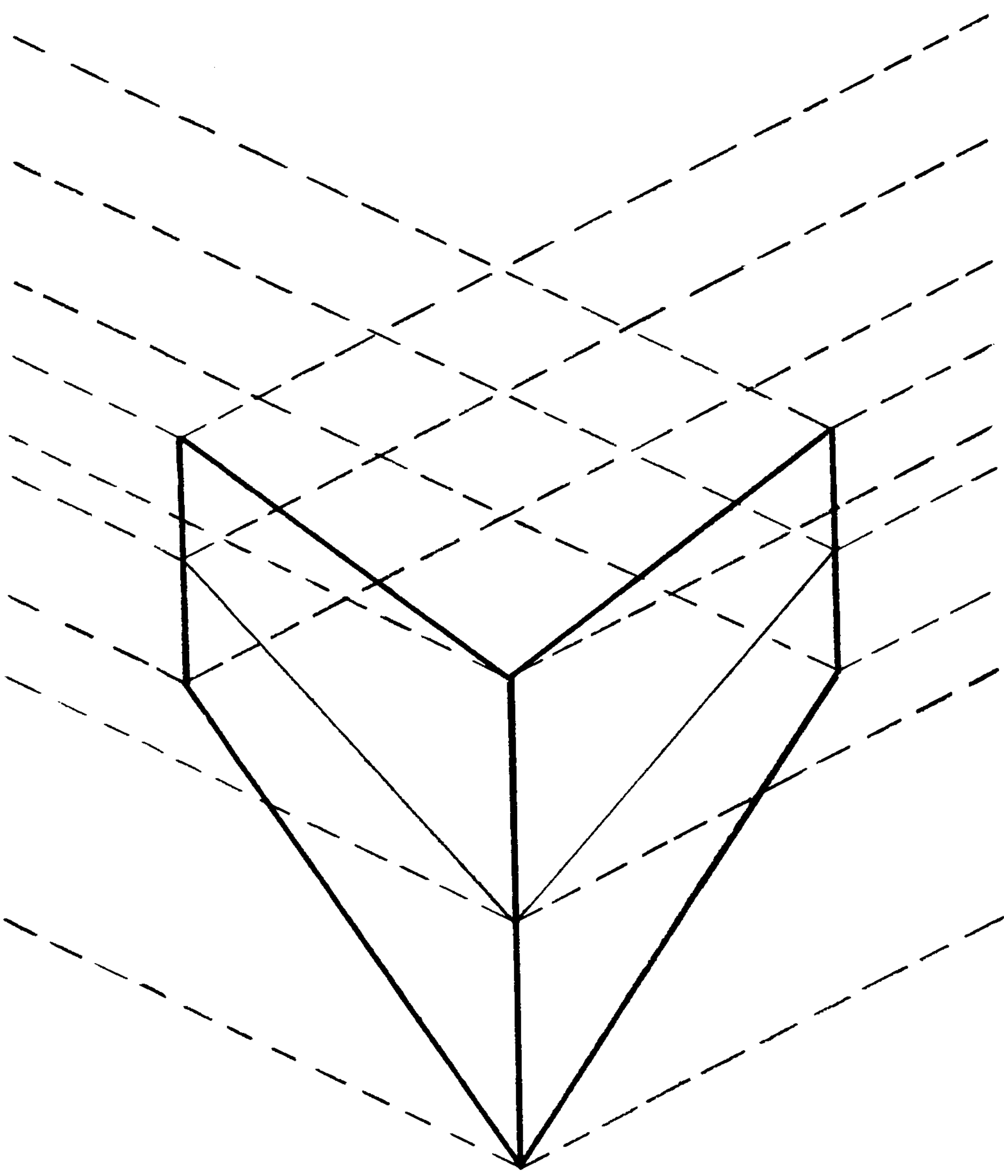


Figure 2.- Mach line configuration for a tapered plan form.

P	x-value	y-value
P_1	$\frac{\beta a m_1}{1-\beta m_1}$	$\frac{a m_1}{1-\beta m_1}$
P_2	$\frac{a(1+\beta m_1)}{1-\beta m_1}$	$\frac{2a m_1}{1-\beta m_1}$
P_3	$\frac{a}{1-\beta m_0}$	$\frac{a m_0}{1-\beta m_0}$
	y-value	
P_4	$\frac{dm_0 m_2(1+m_1\beta)-2am_1m_2}{m_1(1+\beta m_2)}$	
P_5	$\frac{dm_2(1+\beta m_0)-am_2}{1+\beta m_2}$	
P_6	$\frac{dm_0^2(1+\beta m_1)-am_1m_0}{m_1(1+\beta m_0)}$	

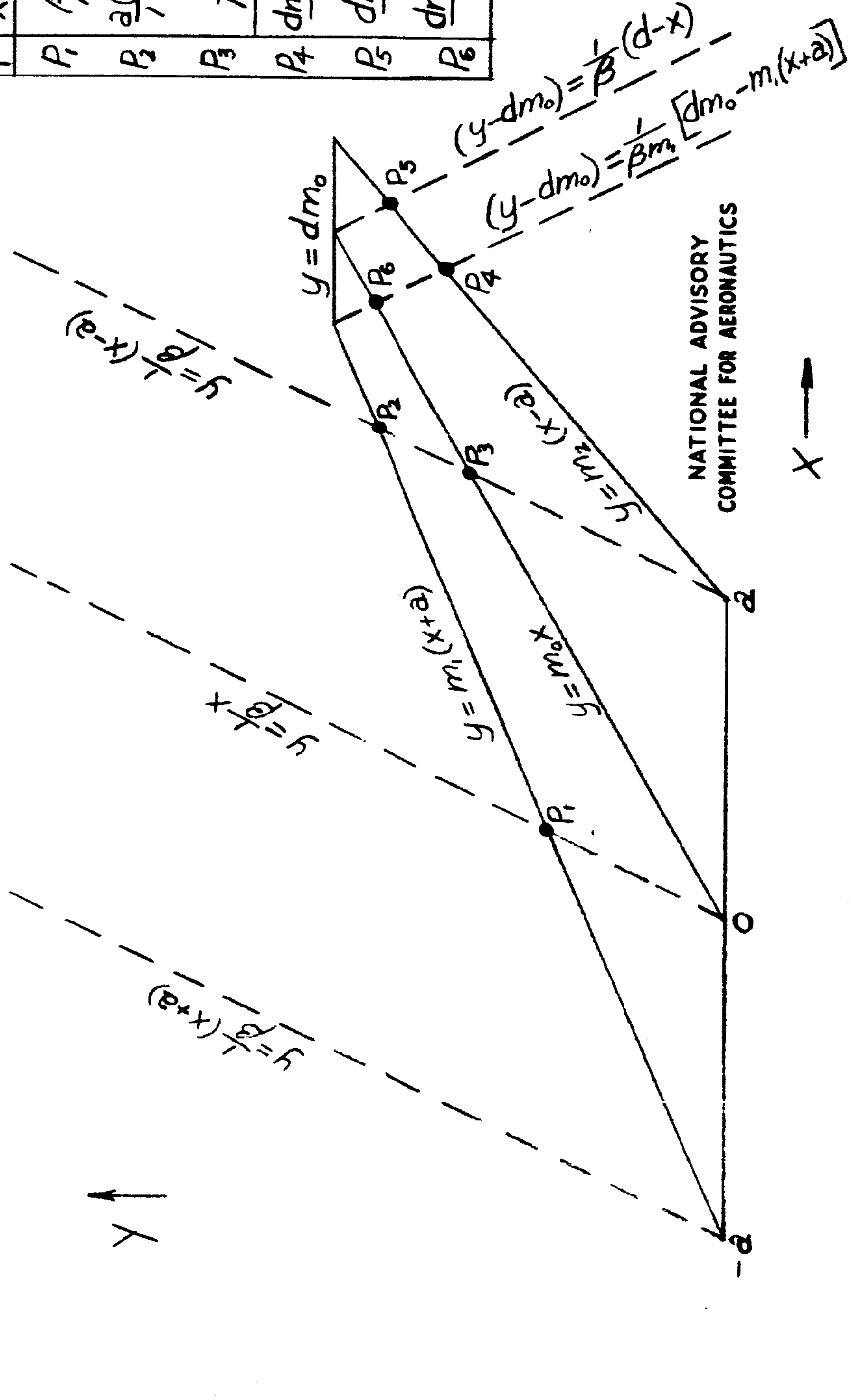


Figure 3.- Information pertinent to integration limits in equations (5) and (7).

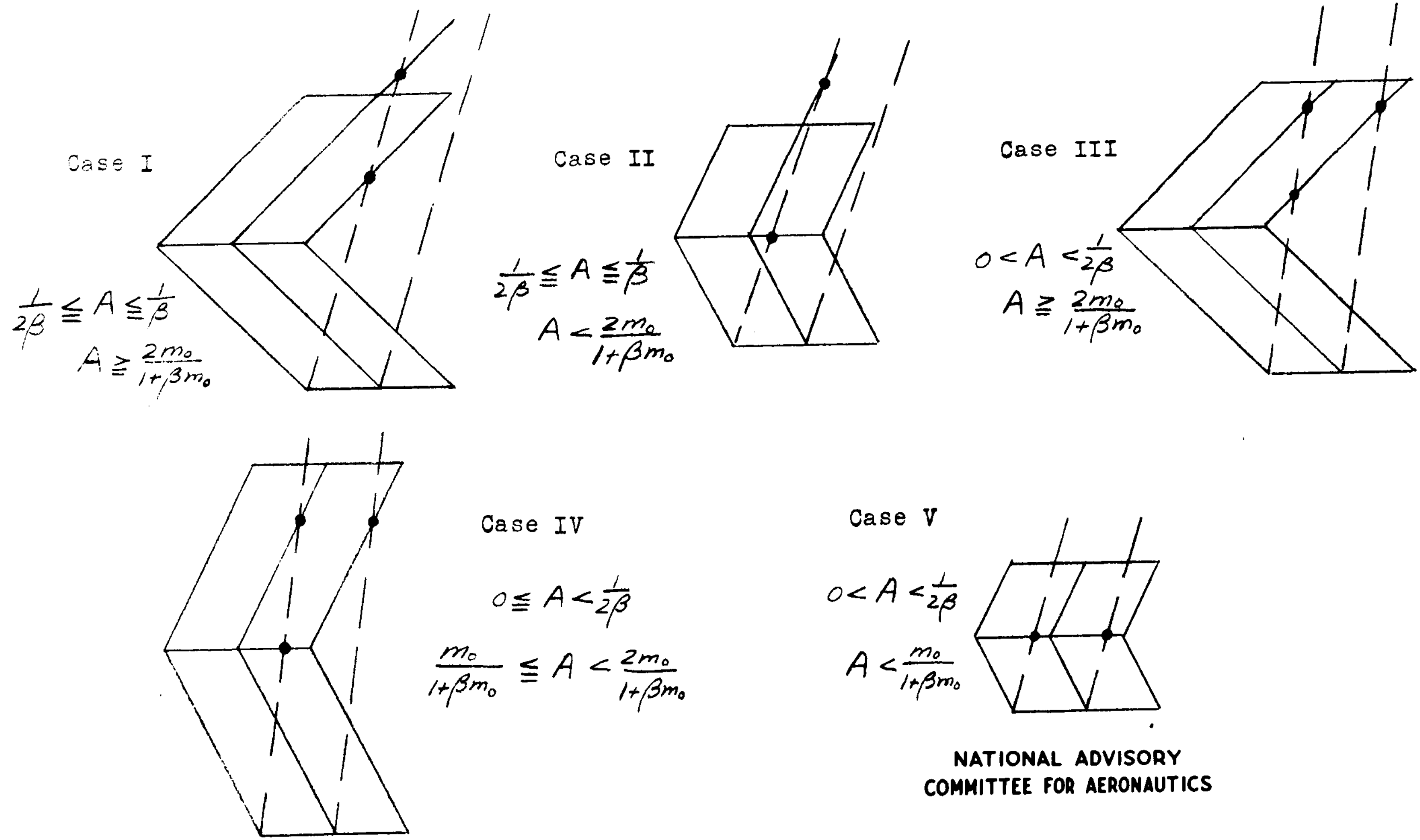
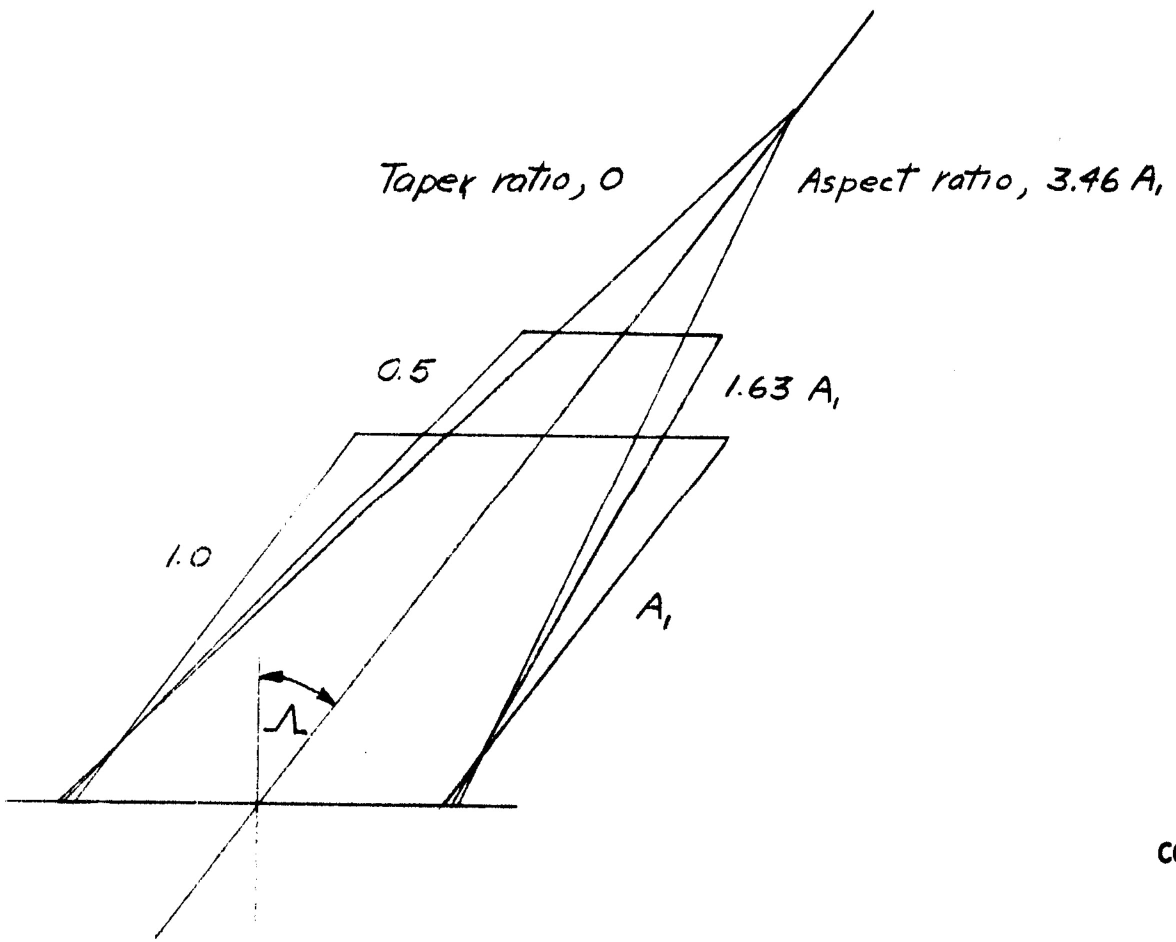


Figure 4.- Additional tip effects for wing of constant chord (taper ratio 1.0).



$$\left.\begin{array}{c}\dfrac{\text{Area moment}}{c_s t_s^2} \\ \dfrac{t}{c}\end{array}\right\}\text{Constant}$$

Taper ratio	Aspect ratio
1.0	1.00 A_1
.8	1.19
.6	1.46
.5	1.63
.4	1.84
.2	2.44
0	3.46

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Figure 5.- Family of tapered plan forms used for calculations. Plan forms shown have same area.

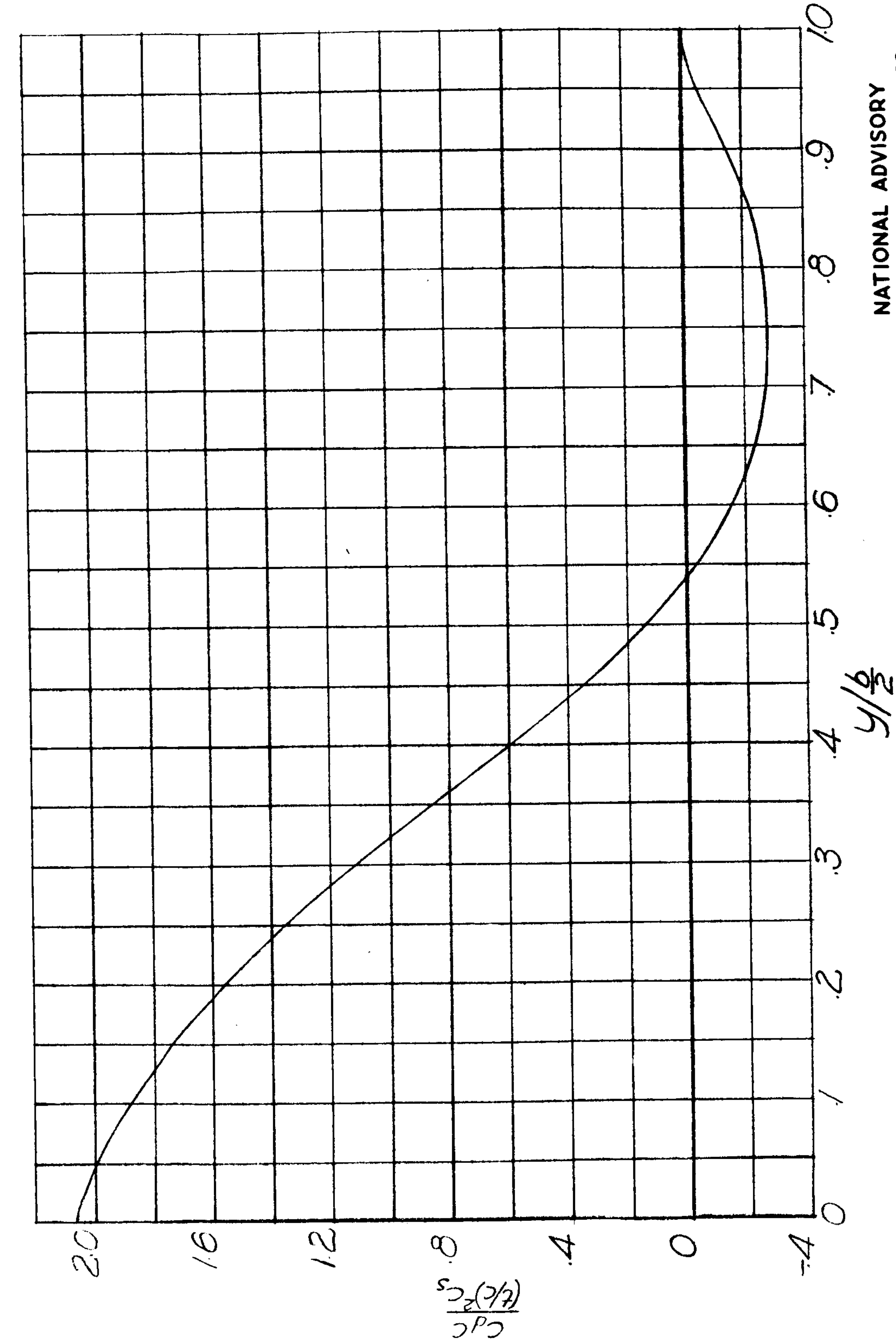

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Figure 6.- Section wave-drag distribution for wing of taper ratio 0. Mach number, 1.414; aspect ratio, 3.46; sweepback angle, 60°.

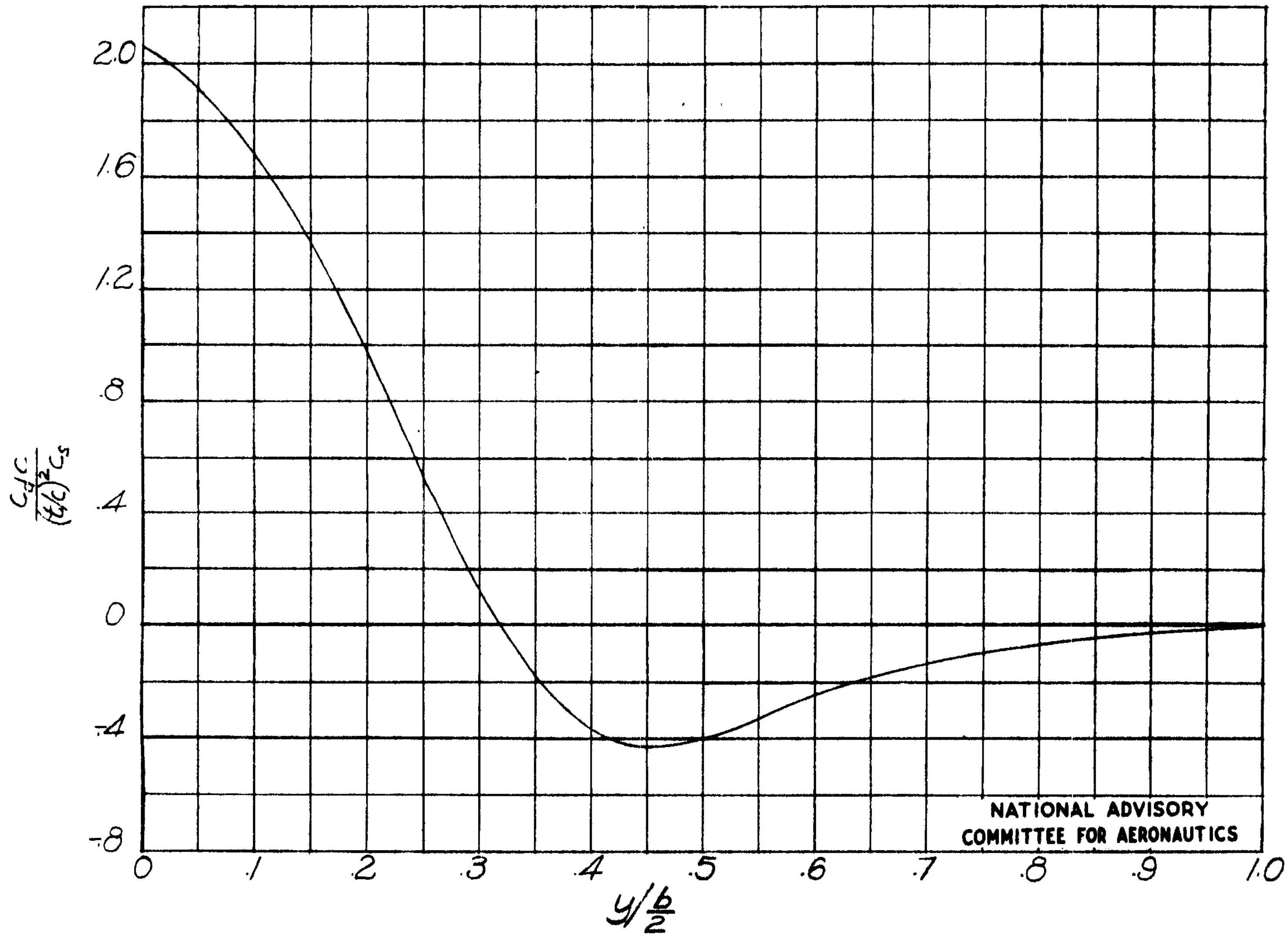

Figure 7.- Section wave-drag distribution for wing of taper ratio 0. Mach number, 1.414; aspect ratio, 6.92; sweepback angle, 60°.

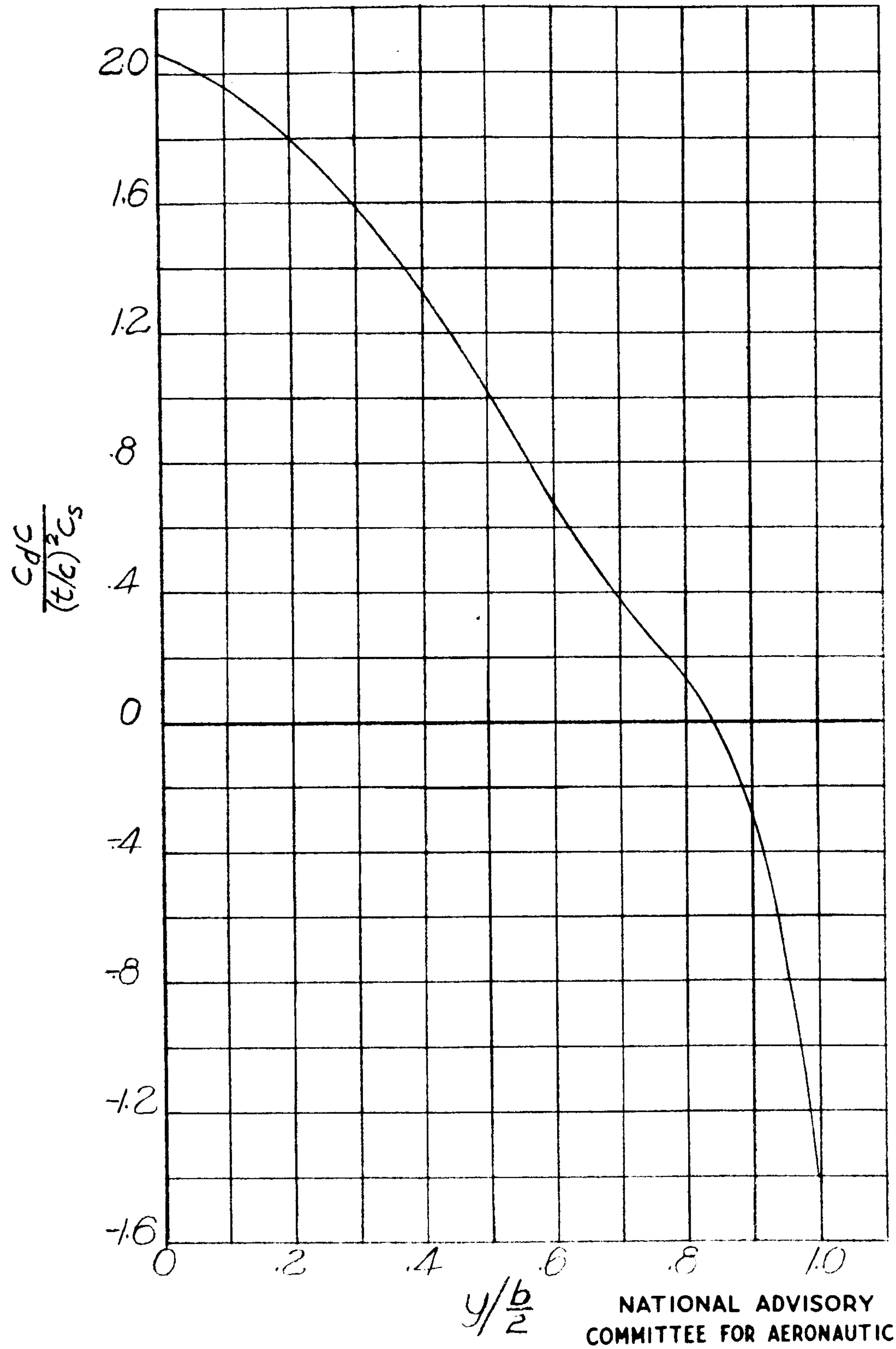

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Figure 8.- Section wave-drag distribution for wing of taper ratio 0.5. Mach number, 1.414; aspect ratio, 1.63; sweepback angle, 60°.

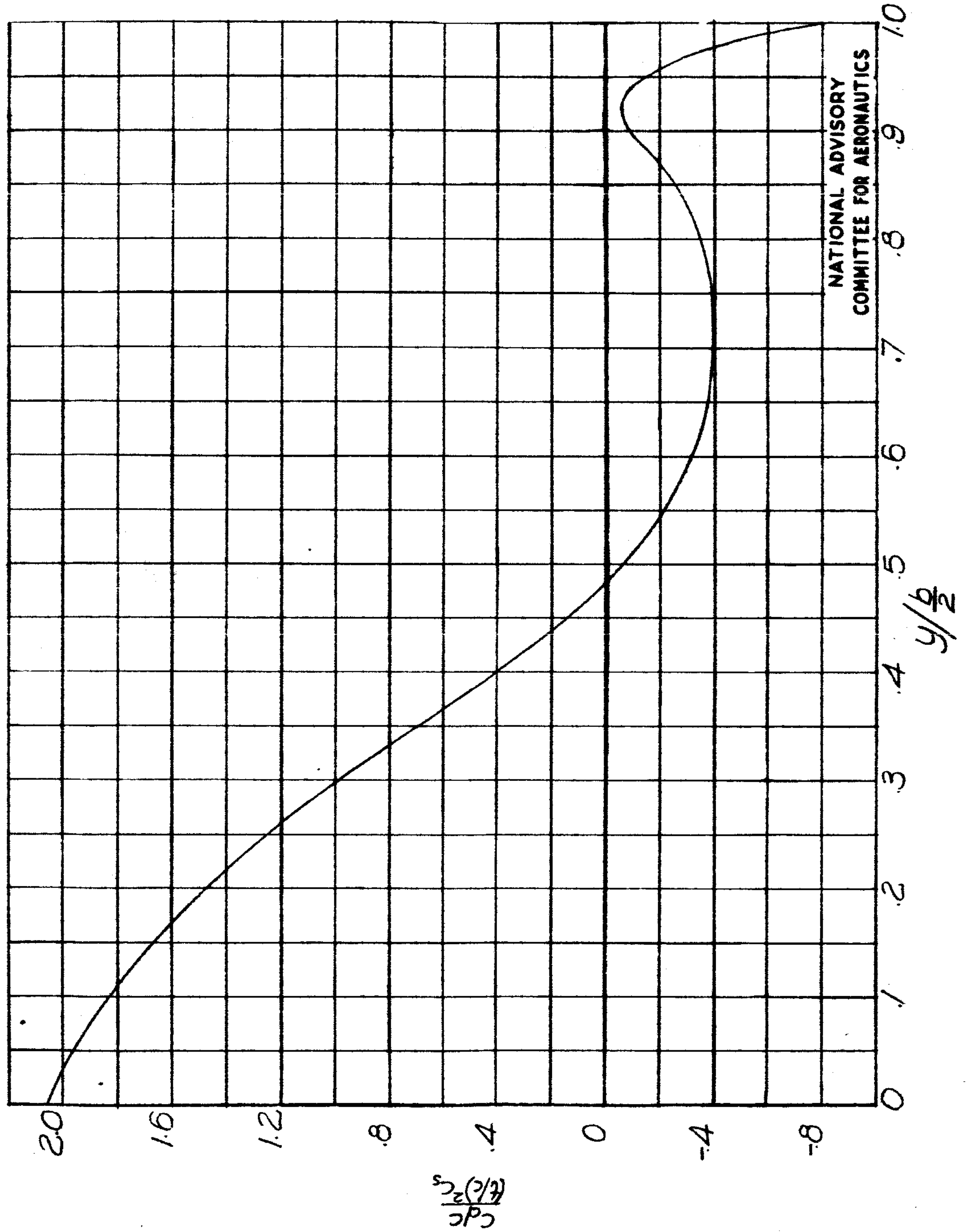


Figure 9.- Section wave-drag distribution for taper ratio 0.5. Mach number, 1.414; aspect ratio, 3.26; sweepback angle, 60°.

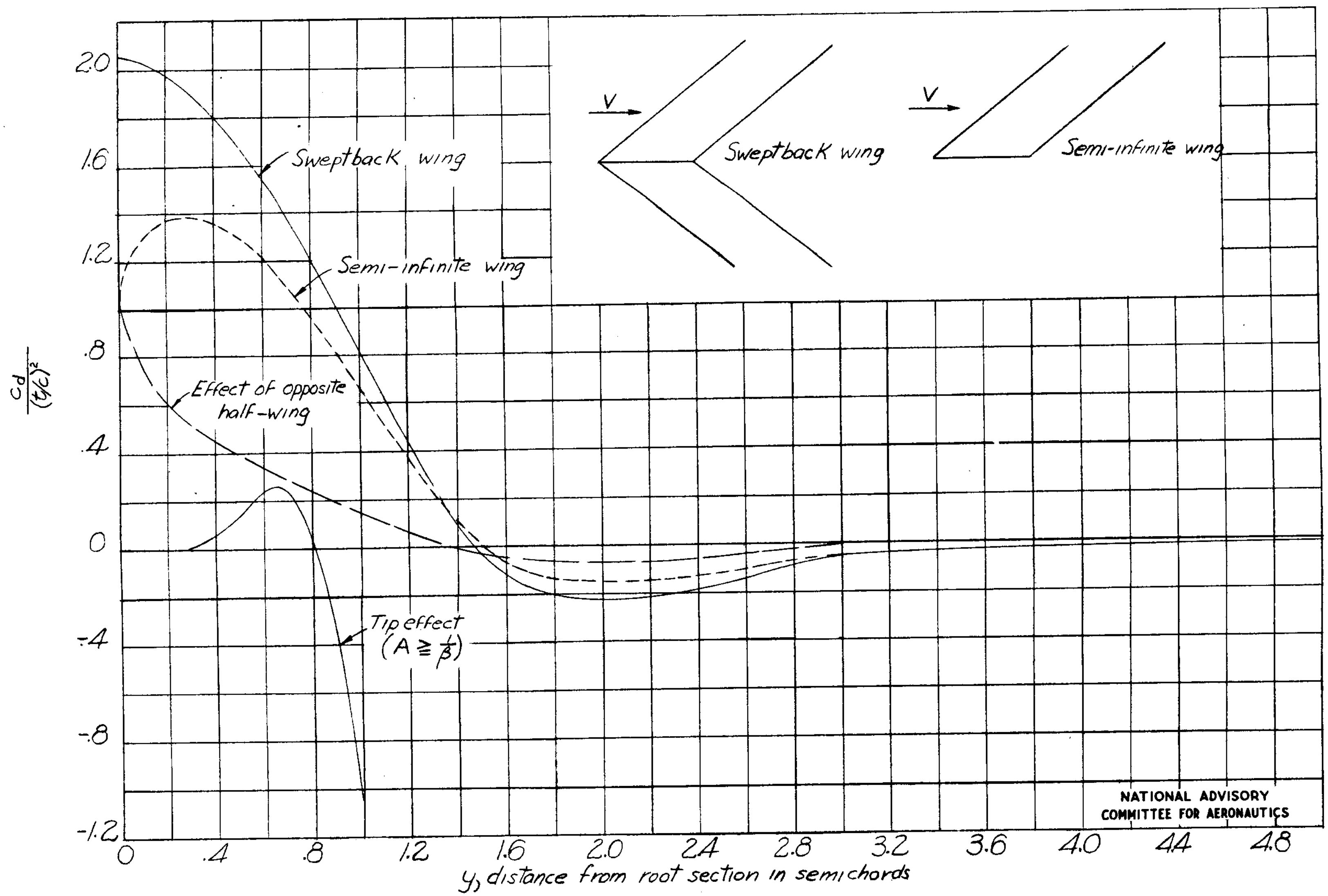


Figure 10.- Section wave-drag distribution for wing of constant chord (taper ratio 1.0). Mach number, 1.414; sweepback angle, 60°.

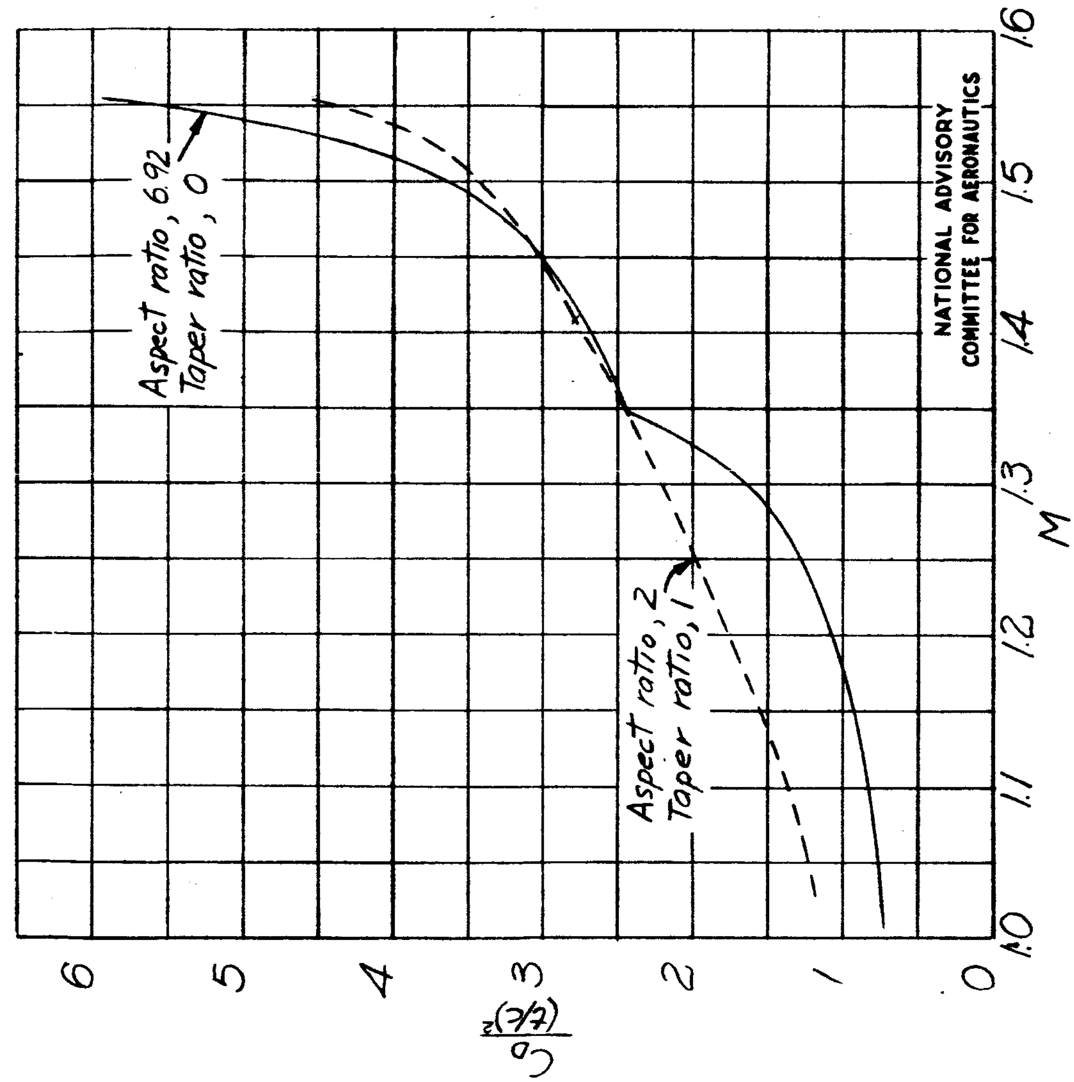


Figure 11.- Variation of wing wave-drag coefficient with Mach number. Sweepback angle, 50°.

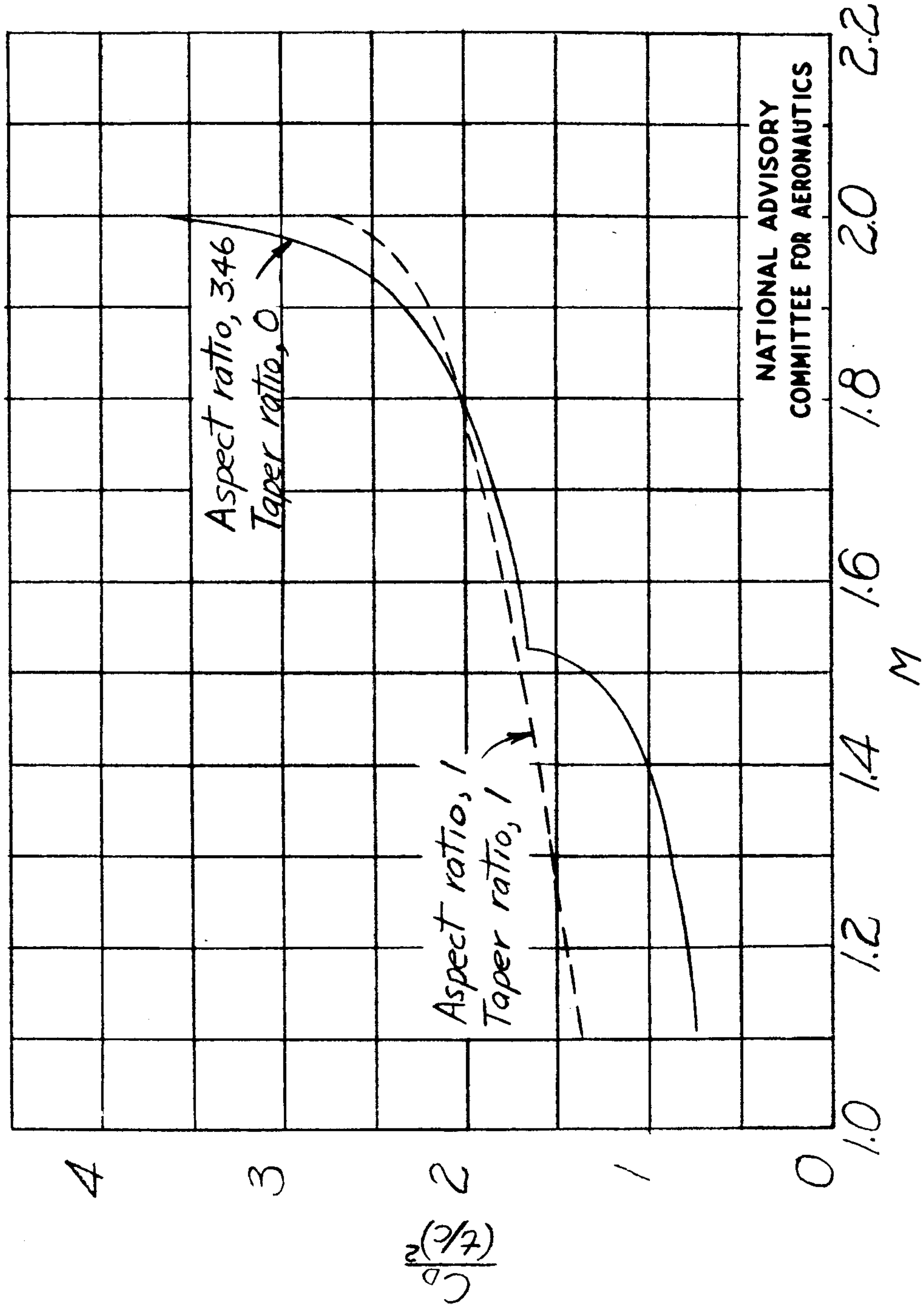


Figure 12.- Variation of wing wave-drag coefficient with Mach number. Sweepback angle, 60°.

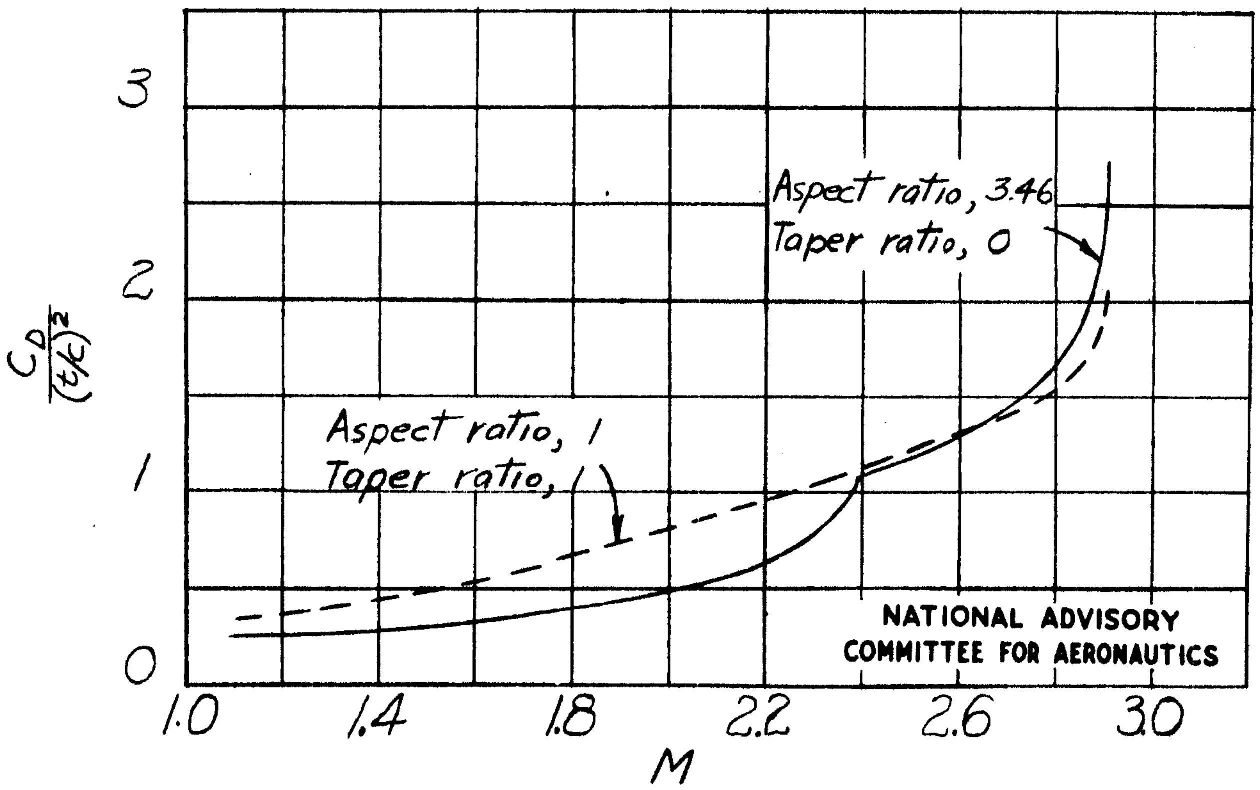
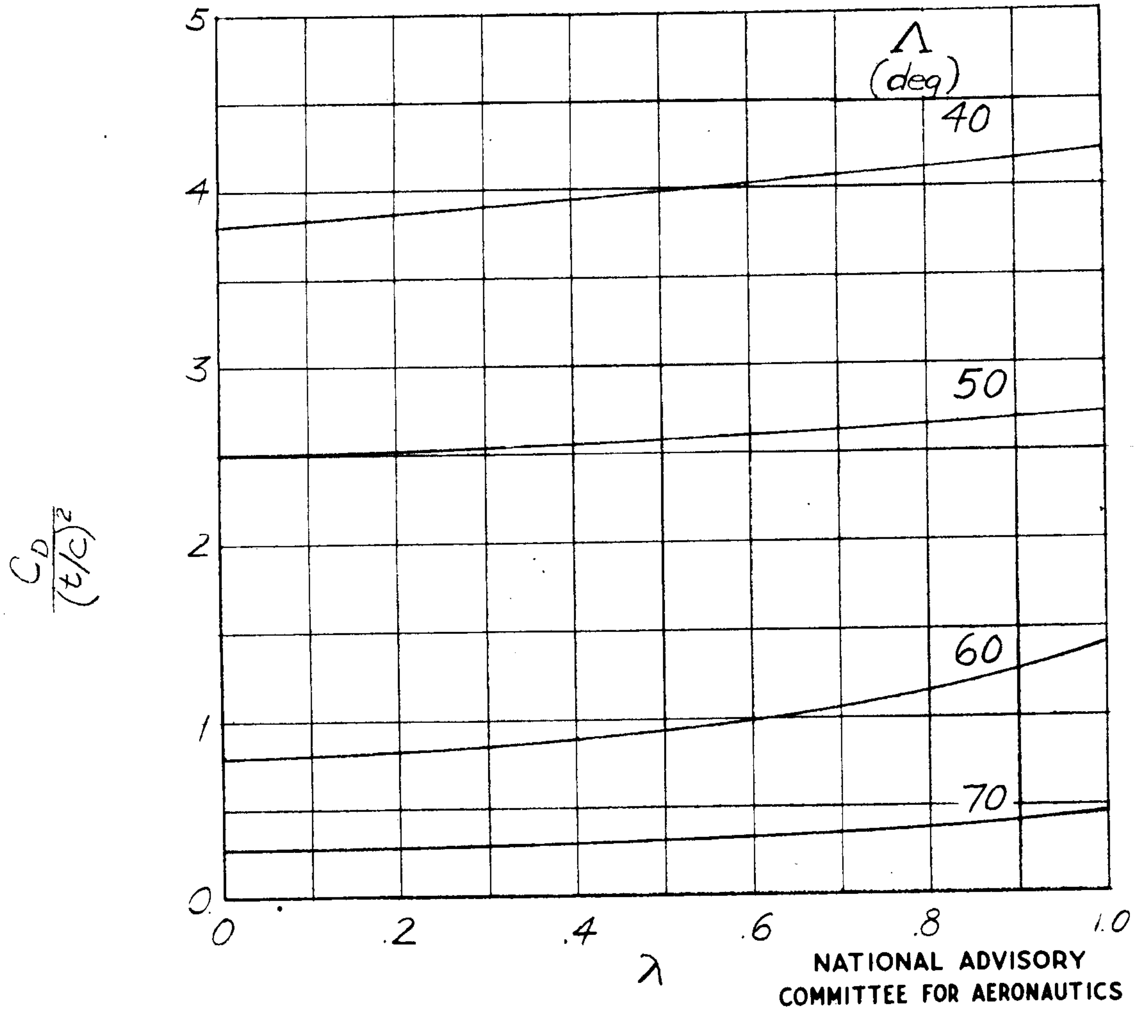
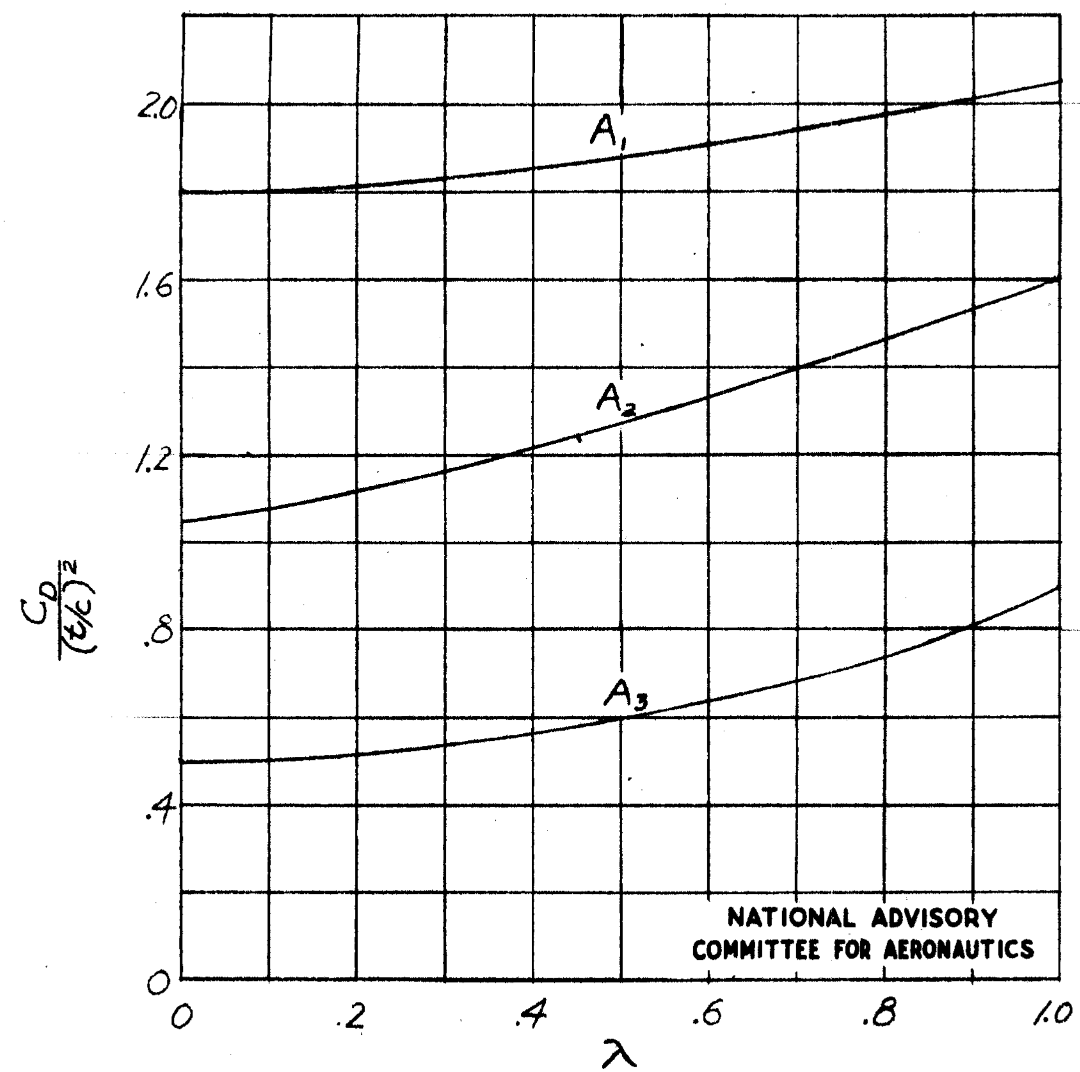


Figure 13.- Variation of wing wave-drag coefficient with Mach number. Sweepback angle, 70°.



Taper ratio	Aspect ratio
0	3.46
.2	2.44
.4	1.84
.6	1.46
.8	1.19
1.0	1.00

Figure 14.- Variation of wing wave-drag coefficient with taper ratio for different sweepback angles. Mach number, 1.2.



λ	A_1	A_2	A_3
0	1.73	3.46	6.92
.2	1.22	2.44	4.88
.4	.92	1.84	3.68
.6	.73	1.46	2.92
.8	.60	1.19	2.38
1.0	.50	1.00	2.00

Figure 15.- Variation of wing wave-drag coefficient with taper ratio for three families of wing plan forms. Mach number, 1.414; sweepback angle, 60°.