Route Monopolies and Optimal Nonlinear Pricing

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Abstract

To cope with air traffic growth and congested airports, two solutions are apparent on the supply side: 1) use larger aircraft in the hub and spoke system; or 2) develop new routes through secondary airports. An enlarged route system through secondary airports may increase the proportion of route monopolies in the air transport market. The monopoly optimal non linear pricing policy is well known in the case of one dimension (one instrument, one characteristic) but not in the case of several dimensions. This paper explores the robustness of the one dimensional screening model with respect to increasing the number of instruments and the number of characteristics. The objective of this paper is then to link and fil the gap in both literatures. One of the merits of the screening model has been to show that a great variety of economic questions (non linear pricing, product line choice, auction design, income taxation, regulation...) could be handled within the same framework. We study a case of non linear pricing (2 instruments (2 routes on which the airline provides customers with services), 2 characteristics (demand of services on these routes) and two values per characteristic (low and high demand of services on these routes)) and we show that none of the conclusions of the one dimensional analysis remain valid. In particular, upward incentive compatibility constraint may be binding at the optimum. As a consequence, they may be distortion at the top of the distribution. In addition to this, we show that the optimal solution often requires a kind of form of bundling, we explain explicitly distortions and show that it is sometimes optimal for the monopolist to only produce one good (instead of two) or to exclude some buyers from the market. Actually, this means that the monopolist cannot fully apply his monopoly power and is better off selling both goods independently. We then define all the possible solutions in the case of a quadratic cost function for a uniform distribution of agent types and explain the implications for airlines in terms of service diffenrentiation.

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1 Introduction

To cope with air traffic growth and congested airports, two solutions are apparent on the supply side: 1) use larger aircraft in the hub and spoke system; or 2) develop new routes through secondary airports. An enlarged route system through secondary airports may increase the proportion of route monopolies in the air transport market. Other solutions exist (slot auctions for example) are out of the scope of this paper (see for example, M Raffarin (2003)).

Because large aircraft took years to develop, required enormous up-front investment and has useful lives of more than 30 years, Airbus and Boeing had to generate long-term demand projections. To do so, they prepared 20-year forecasts for large commercial aircraft: Airbus published its Global Market Forecast (GMF hereafter) while Boeing published its Current Market Outlook (CMO hereafter). Eventhough these forecasts are produced according to 2 different models, both manufacturers agreed that there would be a significant growth in the air transportation industry (worldwide passenger traffic will almost triple by 2019; Airbus forecast an average annual growth rate of 4.9% while Boeing estimates an average growth rate of 4.8%). They also agreed that Asia would register the world's highest growth rates over the next 20 years (See Ref 1).

To produce GMF, Airbus predicted annual demand for new aircraft on each of the 10000 passenger routes linking almost 2000 airports assuming that passenger and cargo demand would track the Growth Domestic Product (GDP hereafter) growth as they had over the past 50 years. For each airline, on each route pair, the model estimated the need for specific aircraft and compared that number against the then existing supply of aircraft. The model calculated maximum feasible frequency limits for each route based on assumptions about airport capacity, airplane speed, distance and some other factors. It assumed that airlines would attempt to keep their market share by adding capacity as demand increased and by increasing aircraft size when it was no longer feasible to increase flight frequencies. Airbus forecast demand for 14661 new passenger aircraft and 703 new are freighter over the next 20-year period by 2019. It forecast demand for 727 new aircraft seating from 400 to 500 passengers (the mainstay of the 747 market) and 1550 new aircraft seating more than 500 passengers (1235 passenger aircraft and 315 cargo aircraft). The GMF also predicted that by 2019 , Asia-Pacific airlines would hold half of the very large aircraft (VLA hereafter) passenger fleet and that 6 of the top 10 airports served by VLA would be in Asia (See Ref 1).

In contrast, Boeing forecast economic growth in 12 regions in its CMO. It then used these growth assumptions to forecast regional traffic flows in 51 intra- and inter-regional markets. For example, travel within China would grow at an average annual rate of 9% compared to 2.8% in North America. The CMO concluded there would be demand for 22315 new aircraft through 2019. One reason for this difference between the two forecasts was that the CMO included for more than 4000 regional jets. Despite general agreement on overall growth, Boeing forecast a much smaller VLA market. The CMO stated bluntly: "The demand for VLA is small" (See Ref 2). It forecast total demand for only 1010 new aircraft seating 400 passengers or more, 40% of which would be 747-400's (410 new aircraft). Of the remaining 600 planes, 270 would be cargo planes, leaving demand for only 330 aircraft seating 500 passengers or more. More importantly, most of the demand for the larger planes would not materialize for at least 10 years (See Ref 2).

The disparity between these two forecasts could be traced to conflicting assumptions regarding the relative importance of flight frequency, new route development and aircraft size. Airbus believed that increased flight frequencies and new routes would provide only short term solutions

to the problem of growing demand. Airport curfews, gate and runaway capacity and passenger arrival preferences would limit the ability to increase flight frequencies at many airports including some of the world's busiest airports like London Heathrow, Tokyo Narito, Singapore and LA International. As Airbus EVP John Leahy said "the trouble is that on these long distance flights, nobody wants to arrive at 3.00AM and nobody wants to drive at the airport for a 2.30AM departure so that they can have more flights" (See Ref 3). At the same time Airbus did not believe that development of new routes would provide a long term solution. Adam Brown, VP for strategic planning and Forecasting, noted "the pace of new route development has slowed sharply...between 1990 and 1995, the total number of route grew by less than 700, an average increase of 1.7% per year. Part of the problem was the difficulty of opening new airports. In fact, only 10 major airports where scheduled to open within the next 10 years and only 18 airports had approved plans to grow (See Ref 3). An even bigger concern was the fact that new routes would not solve the problem of growth at largest population centers, especially in Asia. While Boeing and others cited the expansion of new routes in transatlantic markets as a model for growth in transpacific travel, Airbus pointed out that Asia lacked secondary urban centers to support new destinations(See Ref 4). Thus, hub-to-hub transport would remain the industry standard in these markets.

Like Airbus, Boeing assumed that increasing flight frequency at existing airports would absorb certain amount of growth, but that congestion at the largest hubs would require an alternative solution, which according to Boeing would be new point-to-point routes using medium sized, long-range aircraft like 777 or 340. In support of this view, a Boeing executive claimed "...60% of the airlines bought 1000 or so 747s we've sold bought them for their range, not for their capacity" (See Ref 5). To the extent, there was demand for VLA, the 747-400 would be sufficient for most airlines.

Recent development in the airline industry supported this assertion. In the USA, following airline deregulation in 1985, southwest airlines had prospered by introducing new services at secondary airports such as Providence, Rhode Island and Islip, NY. In Europe Ireland based Ryanair was copying Southwest's model and had achieved 25% annual passenger growth since 1989 by offering no-frills, economy service between secondary airports (See Ref 6). More recently start-ups and buzz and easyjet had adopted similar business models. Transatlantic and to a much lesser extent transpacific travel reflected this trend towards segmentation. Randy Base, a Boeing Vice President explained "back in 1987, the only daily flight between Chicago and Europe was a TWA to London. In those days, 60% of American carriers' transatlantic flights were in 747s operated by Pan Am and TWA in and out of big East cost airports. In 2000, United and American Airlines were operating from Chicago to 11 European destinations using smaller 767 and 777 aircraft (See Ref 7). More recently, both Delta Airlines and American Airlines had introduced new point-to-point service across the Pacific, the former from Portland, Oregon , to Nagoya, Japan and the latter from San Jose, California, to Tokyo (See Ref 8). Related to the opening of new routes was a decline in the average seating capacity in many airline fleets. In fact, Boeing predicted that smaller jets such as its 777 or Airbus A330 would provide 160 fragmentation across the Pacific(See Ref 7). Boeing assumed these trends would continue because people seemed to favor timely and direct service over minimum cost as they became wealthier (See Ref 2). Another factor would contribute to further fragmentation, though one that was exceedingly difficult to predict, was entry by new airlines. Nevertheless Boeing

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believed entry was likely.

Industry analysts made projections that felt somewhere in between the two. The Airline Monitor, a leading industry journal, assumed that airlines would buy the A380 for its operating advantages and passenger appeal. Based on this assumption, the journal forecast total demand for 735 A380 through 2019 (including Cargo planes) (See Ref 9). It is also predicted that the hub-and-spoke system will remain the industry standard. Even if, new routes development through secondary airports will not solve entirely the air traffic growth and congested airports issues, it will enable airlines to lower their cost (low airport fees, city-, region- or government-support...) and at the same time to have monopoly positions on these routes allowing positive profits (if there is enough demand of course).

This paper does not intend to show that Airbus's or Boeing's predictions are right or false. The above comments show that the hub-and-spoke system will remain the industry standard and also that new route development through secondary airports will be one of the key feature of the 21st century air transport industry. Assuming the above statement, new route development through secondary airports will also give birth to route monopolies (as it is often the case for low cost airlines). In addition to being new market shares for airlines, these new point-to-point services through secondary airports will also enable airlines, at least for a certain period of time, to have monopoly positions on these routes and as a consequence, to make positive profits. We assume monopolistic positions for the carrier as the products (point-to-point routes) are so differentiated from the other products of the market (routes through the hub-and spoke system) that they are assume to be different. Example: direct flight from Toulouse (France) to Vancouver (Canada) is different from flying from Toulouse to Amsterdam, then from Amsterdam to Seattle and then from Seattle to Vancouver.

The objective of this paper is to study the optimal pricing policy of a monopolist facing a population of heterogenous buyers. In the air transport literature, on the one hand, after airline industry deregulation, pricing policy in the hub-and-spoke system, airport dominance (...) has been largely reviewed (for examples, Graham et al. (1983), Berry (1990), Borenstein (1989, 1990), Bruekner et al. (1992)) but on the other hand, the monopolist's optimal pricing policy has been put aside even if 60% of the routes in the world are operated through a monopolistic position. In Economics, the analysis of optimal screening has been subject of a large literature, partly justified by the great variety of contexts to which such an analysis can be applied: non linear pricing, product lines design, optimal taxation, regulations, auctions. Although it is often recognized that agents have several characteristics and principal have several instruments, this problem is often studied under the assumption of a unique characteristic and a unique instrument. The case of several characteristics and several instruments has been studied and the qualitative features of the optimal pricing policy have been established (Wilson (1993), Armstrong (1996), Rochet-Choné (1998). Nevertheless, only few examples were solved (Wilson (1993), Rochet (1995)).

The objective of this paper is to link and fill the gap in both literatures. This paper studies the optimal pricing policy of a monopolist (airline) who produces two goods (operates 2 routes) and faces a heterogenous population of buyers. The buyers (travel agencies) have two characteristics (one for each good), two possible values per characteristic and therefore four possible types of buyer. After a brief survey in section 2, where we recall the qualitative features of the solution to the monopolist's problem where agents are represented by a single characteristic and the

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principal has only one instrument, section 3 focuses on the specific problem of non linear pricing by a monopolist in the simplest possible multidimensional model. Section 4 gives the solution and shows that results obtain in one-dimension model are not robust to increases in numbers of characteristics and instruments (some upward incentive compatibility constraints are binding at the optimum). For example, it is sometimes optimal for the airline to provide services above the first best level to some of its customers. Section 5 studies the problem in a different but equivalent way and enables to explain distortions in resource allocations. Section 5 studies the general solutions and section 6 defines several sets of solutions and implications for airline in terms of service differentiation. Section 7 concludes.

2 A brief Survey

2.1 Airline pricing policy

Airline deregulation has led to profound changes in the structure of the industry. In addition to giving airlines the freedom to set fares, deregulation removed restrictions on entry and exit, allowing the carriers to expand and rationalize their route structures. This flexibility led in the 1980s to a dramatic expansion of hub-and-spoke networks, where passengers change planes at a hub airport on the way to their eventual destination. By funneling all passengers into a hub, such a system generates high traffic densities on its "spoke" routes. This allows the airline to exploit economies of density, yielding lower cost per passenger. These economies arise partly because high traffic density on a route allows the airline to use larger and more efficient aircraft and to operate this equipment more intensively (at a higher load factor).

Restructuring of the industry in response to deregulation has also led to renewed interest among economists in the determinants of airfares in individual city-pair markets. This line of research contains notable contributions by Graham-Kaplan-Sibley (1983), Borenstein (1989), Morisson-Winston (1990). These studies typically explore the connection between airfares and market-specific measures of demand (city populations and incomes, tourism potential...), cost (flight distance, load factors...)and competition (number of competitors, market share...) without paying much attention to network characteristics. Bruekner et al. (1992) shows the impact of network characteristics in airfares determination by providing first evidence linking airfares to the structure of airline hub-and-spoke networks. When a hub-and-spoke successfully raises traffic density, ticket prices are likely to reflect the resulting lower cost per passenger. Any force that increases the traffic volume on the spokes of a network will reduce fares in the market it serves. This effect arises because of economies of density on the spokes. For example, since a large network (as measured by the number of city pairs it connects) offers many potential destinations to the residents of an endpoint city, its spoke should have higher traffic densities than the spokes of a smaller network. With costs correspondingly lower, fares in the individual markets served should be lower in the large network, other things equal. Similarly, holding size fixed, a network that connects large cities should have higher traffic densities on its spoke (and thus lower fares in individual markets) than one serving small cities.

The case of fare determination in the case of route monopolies has received little attention. With these hard times and cut-throat competition in the hub-and-spoke system, this could be serious omission.

In addition to being new markets shares for airlines, new routes through secondary airports

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(secondary population centers), point-to-point service, can also be seen as a business model. "Low cost" airlines often use this model. Under the "low cost" label, we can easily say that these carriers have monopolistic position on some routes (Ryanair in Europe, SouthWest in the United States of America for examples). Even if they are "low cost" airlines and are able to provide low fare tickets, this does not means that do not use their monopoly on their market niches (they probably do to reach such operating margin). What we propose in this paper is to assume that an airline may have incentives to use a part of the "low cost" business model (secondary airports) but on another business model. Thanks to low "airport costs", point-to-point service the airline is able to propose attractive prices and also to apply its monopoly power on its market niche. Moreover, the airline may also be supported by cities, regions, or governments for the impact of a route opening on the economic development of a population center. To do so, we assume that there exists city pairs, population centers where the macro-economic environment allows an airline to operate successfully a new route. Instead of providing customers with single class flights, the airline proposes several levels of services (economy, business and first for example) to apply price discrimination through its monopoly power. If an airline can operate one route on such a city pair, the optimal non linear pricing policy is described in section as the one-dimension case. If an airline can operate two routes on two such city pairs or through three population centers, the optimal pricing policy is described and explained in the remaining part of the paper.

2.2 Mechanism design and non linear pricing

The analysis of optimal screening has been the subject of a large literature, partly justified by the great variety of contexts to which such an analysis can be applied: nonlinear pricing, product lines design, optimal taxation, regulation, auctions...

Although it is often recognized that agents have several characteristics and principals have several instruments, this problem is often studied under the assumption of a unique characteristic and a unique instrument. In this case we obtain the following results:

- the set of self selection constraints that are binding at the optimum is typically the same: the Individual Rationality Constraints (IRC hereafter) at the lower extreme of the distribution of types, and downward local Incentive Compatibility Constraints (ICC hereafter) everywhere.

- we decompose the problem in two parts: first, find the system of transfers that minimizes the expected informational rent for a given allocation of goods; second, find the allocation of goods that maximizes the profit of the principal.

- because of the informational rent, second best allocation is distorted below, except at the upper extreme of the distribution of types where it coincides with the first best allocation.

Considering the multiplicity of characteristics and instruments in most applications, it is important to know if those results are still valid in the multidimensional context.
In the screening literature, the question has been essentially considered in two polar cases:

- one instrument-several characteristics: in this case, it is impossible to obtain perfect screening among agents. The same level of the instrument is chosen by many different agents

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(in Lewis-Sappington (88), this corresponds to firms having different costs and demands). However Laffont-Maskin-Rochet (87) and Lewis-Sappington (88) show it is possible to aggregate these types and reason in terms of the average cost function of all firms having chosen the same level of production. Nevertheless, several qualitative properties persist: for instance, Lewis-Sappington (88) show that price exceeds expected marginal cost, except at the top of the distribution where they are equal.

- several instruments-one characteristic: the situation is very different. Matthews-Moore (87) extend the Mussa-Rosen's model by allowing the monopolist to offer different levels of warranties as well as qualities. One of their most striking result is that the allocation of qualities is not necessarily monotonic with respect to types. As a consequence, non local ICC may be binding at the optimum.

The most interesting case is the one with several characteristics and several instruments, because it is the most realistic but also the most difficult.

Seade (79) has studied the optimal taxation problem for multidimensional consumers and has shown that it was equivalent to a variation calculus problem with several variables. McAfee-McMillan (88) make a decisive step in the study of monopoly pricing under multidimensional uncertainty. They introduce a General Single Crossing Condition under which ICC can be replaced by the local conditions of the agents' decision problems. They also show that the optimal screening mechanism can be obtained as the solution of a variations calculus problem. In 92, Armstrong provides a very clear treatment of the difficulties involved in the multidimensional extension of the Mussa-Rosen's problem of optimal product lines design by a monopolist. Wilson (93) contains a very original and almost exhaustive treatment of nonlinear pricing models: several multidimensional examples are solved.

Focus can now be made on bidimensional problems: Rochet (84) studies an extension of Baron-Myerson regulation problem in a bidimensional context where both marginal and fixed costs are unknown to the regulator. Contrarily to Lewis-Sappington, Rochet allows for stochastic mechanisms (as Baron-Myerson did): this provides a second instrument to the regulator. On a particular example, Rochet (84) shows that the optimal mechanism can indeed be stochastic, as conjectured by Baron-Myerson. Following Spence's early(80) and deep contribution, Dana (93) gives in a different context a partial solution of the discrete screening model with two types and two attributes, and finds two solutions depending on the correlation of types' characteristics. Armstrong-Rochet (98) study a bidimensional screening model in which four types of agents, who are discretely distributed, can undertake two kinds of activities. Each activity can be undertaken at a high or low level. They consider two ways to solve the problem:

- The relaxed problem: they only consider the five downward ICC. They show that three types of solution can occur (cases C and D are in fact identical if one permutes activity A and B). Those solutions change according to the correlation between activities. The binding constraints are in those cases are the IRC of the lowest type and a set of downward ICC. The highest type always gets his first allocation and there is no distortion at the top. This can be explained by the considered ICC: when only downward ICC are considered, rents do not depend on the high type activity levels and therefore do not need to be distorted.

- The fully constrained problem: they consider the whole set of ICC. This gives rise to a fourth solution (actually there are two, but they are symmetric): one upward ICC and the transverse ICC are binding. As a consequence, an "intermediary type" gets his first best allocation and the highest gets an allocation above the first best level.

2.3 A Brief Recall of the Unidimensional Case

Consider an economy in which a monopolist, an airline, sells his production (travel on this route) to a heterogenous population of two types ($t = \{t^l, t^h\}$), customers having low (l) or high (h) demand of services on this route, who are represented in proportion π^l and π^h ($\pi^l + \pi^h = 1$). As the airline operates through secondary airports, it can lower its airport fees and thus proposes point-to-point service, attractive prices and can fully apply its monopoly power. The monopolist wants to maximize his profit:

$$max_q \sum_i [p^i q^i - c(q^i)],$$

$$= max_q \sum_i [u^i(t, q) - U^i(t, q) - C(q^i)],$$

where $u^i(t^i, q^i) = t^i q^i$ and $U^i(t^i, q^i) = u(t^i, q^i) - p^i q^i$. This maximization problem is subject to the Individual Rationality and the Incentive Compatibility constraints:

$$IRC : U(t^i, q^i) \geq 0,$$
$$ICC : U(t^i, q^i) - U(t^i, q^j) \geq u(t^i, q^j) - u(t^j, q^j).$$

We then obtain:

$$U^l = u(t^l, q^l) - p^l q^l,$$
$$U^h - U^l = (t^h - t^l)q^l,$$
$$U^l - U^h > (t^l - t^h)q^h.$$

As a consequence buyers of type h get their first best allocation whereas types l receive a downward distorted allocation.

Writing the Lagrangian and computing the solution gives:

$$L = \sum_i [u^i(t, q) - U^i(t, q) - C(q^i)] - \sum_{i,j} \mu_{ij}(U(t^i, q^i) - U(t^i, q^j) - u(t^i, q^j) + u(t^j, q^j))$$
$$- \sum_i \lambda^i U^i.$$

The first order conditions give:

$$\lambda^l = -1,$$
$$\mu_{hl} = -\pi^h,$$
$$p^l = C'(q^l) + \frac{\pi^h}{\pi^l}(t^h - t^l),$$
$$p^h = C'(q^h).$$

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This means that the airline is able to find the optimal pricing mechanism and service allocation that enables its profit maximization. Customers with high demand of services will receive their first best allocation (price equals marginal cost) whereas customers with low demand of services will receive a downward distorted allocation of services (below the first best level of course). Airlines provide services below the first best level on low fare tickets to give incentives to customers with high demand of services to buy high fare tickets instead of buying low fare tickets. They just propose "minimum" services to "poor customers" and offer "superfluity" to "rich travellers".

Focus can now be made on the bidimensional case.

3   Statement of the Problem

We focus here on the specific problem of nonlinear pricing by a monopolist in the simplest possible model (two characteristics, two possible values per characteristic and therefore four possible types), and first give the solution which has the same qualitative features than in dimension one, in the case of a linear quadratic cost function(in a previous version of his paper with Choné, Rochet (95), alone, studies this problem, where production is a level of quality, and we study the case of quantity with a non zero cross cost) whereas Armstrong-Rochet (98) study the case of constant marginal costs. Such a solution appears when the correlation between characteristics is near zero and the cross cost parameter is not too big: in such a case, only downward local ICC and IRC of the lowest type are binding but according to the value of the cross cost parameter, the quantities can be distorted downwards or upwards. This differs from dimension one (except when there is no cross cost parameter) and the highest type receives his first best allocation of goods. We call it the regular case. Matthews and Moore (87) study a standard model (Mussa-Rosen (78)), taking three types into account, the IRC is only effective for type 3. In this case type 1 faces undistorted quality while the other two receive degraded quality (unidirectional downward distortion), but allocations are not monotonic. In Donnenfeld-White (88) there is unidirectional upward distortion. But Srinagesh-Bradburg (89) find bidirectional distortions(one upwards, one downwards). We then define the complete solution in three different situations, first, where there is no cross cost and a discrete uniform distribution of types, where we obtain the same four solutions as in Rochet (95), second, where there is no cross cost and a general discrete distribution of types and also in the case of a discrete uniform distribution, with a non zero cross cost parameter, where we obtain ten other solutions.

3.1   The Model

We consider a natural bidimensional extension of the Mussa-Rosen's (1978) model, in which a multiproduct monopolist sells two goods to a heterogenous population of four types of discretely distributed buyers. The model is different from Mussa-Rosen's because production represents quantities and not qualities. The airline operates two routes, route 1 and route 2 and provides her customers with quantities of services on route 1, q_1, and on route 2, q_2. Customers are characterized by their type, $t = (t_1, t_2)$, their demand for services on route 1 and 2 (distance, departure time, arrival time, number of possible changes, on-board space, ...).

Buyers utilities are quasilinear:

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$$U^i = u(t^i, q_1, q_2) - p(q_1, q_2) - E,$$
$$u(t^i, q_1, q_2) = t_1^i q_1^i + t_2^i q_2^i,$$
$$p^i q^i = p_1^i q_1^i + p_2^i q_2^i,$$

where t^i is a bidimensional vector of characteristics of the consumers (his type), q is a bidimensional vector of the quantities of the goods and p is the unit price of the goods. $u(t, q)$ represents the monetary equivalent of the goods, with characteristic vector q for a consumer of type t.

Notice that q and t have the same dimensionality so that perfect screening (in the sense that consumers with different types always buy different quantities of goods) is possible, i.e., we assume that the (full information) optimal resource allocation $\hat{q}(\cdot)$ involves perfect screening. $\hat{q}(\cdot)$ is defined as follows:

$\forall t \in T, \hat{q}(t) = \arg\max_q [S(t, q)]$,

where $S(t, q) = u(t, q) - C(q)$, and T represents the set of possible types and $C(q)$ is the cost function of the monopolist:

$$C(q_1, q_2) = \frac{1}{2}(q_1^2 + q_2^2) + \rho q_1 q_2.$$

We use such a cost function, as we assume that for a low quantity of services, the necessary services are served first (the flight itself) and as the number of services increases, the services become more and more superfluous and as a consequence, more and expansive. ρ is used for positive or negative production synergies. We also assume that the surplus function $S(t, q)$ is strictly concave, differentiable in q and has, for all t, a unique maximum $\hat{q}(t)$. The set of $\hat{q}(\cdot)$ will be used as a benchmark for evaluating the distortions of resource allocations entailed by the monopoly power under bidimensional adverse selection.

3.2 The Monopolist's Problem

We describe the strategy of the monopolist as a set of price-quantity allocations (p^i, q^i) satisfying ICC and IRC :

IR:$U^i \geq 0$, for all i,

IC: $u(t^i, q^i) - p^i q^i - E^i \geq u(t^i, q^j) - p^j q^j - E^j$, for all i, j.

The airline has to define the price of a ticket and the associated quantity of services for each type of agent that satisfy his or her IRC and the ICC.

They are equivalent to:

IR:$U^i \geq 0$,

IC: $U^i - U^j \geq u(t^i, q^j) - u(t^j, q^j)$.

Finally, π^i denotes for all i the proportion of agents of type i in the population of potential buyers. The monopolist's optimal strategy is obtained by maximizing his profit:

$Max \sum_i \pi^i (p^i q^i - C(q^i) + E^i)$,

which is equivalent to:

$Max \sum_i \pi^i (u(t^i, q^i) - U^i - C(q^i))$ under the above constraints.

Note: we write this expected profit as:

$$Max \sum_i \pi^i (p^i q^i - C(q^i) + E^i),$$

instead of the commonly used expression:

$$Max \sum_i \pi^i(p^i q^i + E^i) - C(\sum_i q^i).$$

We choose the first formulation because $C(q)$ is convex. As a consequence, $C(\sum_i q^i) \geq \sum_i C(q^i)$. With a technology that generates such a cost function, it is less costly for the monopolist to use "just in time production" and to maximize profit as previously mentioned. This is the case for services.

There are two characteristics and two possible values for each characteristic. Therefore there are four possible types of consumers . We denote these types by letter $i = A, B, C, D$.

$$
\begin{aligned}
t^A &= (t_1^l, t_2^l), &\quad (\pi^A), \\
t^B &= (t_1^h, t_2^l), &\quad (\pi^B), \\
t^C &= (t_1^h, t_2^h), &\quad (\pi^C), \\
t^D &= (t_1^l, t_2^h), &\quad (\pi^D).
\end{aligned}
$$

To avoid symmetric problem and to better differentiate B and D, we assume:

$$(t_1^h - t_1^l)\pi^B \geq (t_2^h - t_2^l)\pi^D,$$

which means for example that route 2 is shorter that route 1.

The specified a quadratic cost function is:

$$C(q_1, q_2) = \frac{1}{2}(q_1^2 + q_2^2) + \rho q_1 q_2, \quad \rho \in \,]-1,1[. \,\dot{}$$

This implies that $\hat{q}$ is linear:
$\hat{q}(t_1, t_2) = (\frac{t_1 - \rho t_2}{1 - \rho^2}, \frac{t_2 - \rho t_1}{1 - \rho^2})$.

The Lagrangian of the problem :
$L(U, q) = \sum_i \pi^i(u(t^i, q^i) - U^i - C(q^i)) - \sum_{i,j} \mu_{ij}(U^i - U^j - u(t^i, q^j) + u(t^j, q^j)) - \sum_i \lambda^i U^i$
where μ_{ij} and λ^i represent respectively the (non positive) multipliers associated to $ICC(i,j)$ and $IRC(i)$. The first order conditions (which are necessary and sufficient for the linear program) give:$\forall i, \delta L/\delta U^i = -\pi^i - \sum_j(\mu_{ij} - \mu_{ji}) - \lambda^i = 0$,
$\leftrightarrow \pi^i + \lambda^i = \sum_j(\mu_{ji} - \mu_{ij})$ and adding these equations on i gives:
$\sum_i \pi^i = -\sum \lambda^i = 1$,
$\leftrightarrow \forall i, \pi^i \leq \sum_j(\mu_{ji} - \mu_{ij})$, with $=$ if $U^i > 0$ (equivalent to $\lambda^i = 0$).
This is equivalent to say that at least for one i, the IRC is binding. This is economically clear since adding a uniform fee to a price schedule does not alter ICC, and increases the monopolist's profit.

Following Border-Sobel (87), we say that j attracts i when $IC(ij)$ is binding. These relations (who attracts whom?) are crucial for determining distortions in resource allocation. Therefore if i attracts no one ($\mu_{ij} = 0, \forall j$), i receives his first best allocation.

When $U^i > 0$: $\pi^i = \sum_j(\mu_{ji} - \mu_{ij})$. This condition explains the trade off faced by the monopolist in the choice of unit price p^i, price paid by types t^i. A small increase for p^i to $p^i + \epsilon$

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has a direct effect on profit: it increases it by $\epsilon \pi^i$ per unit, where π^i is the proportion of types t^i. However this marginal increase also has a complex, indirect effect on profit: for all js who are attracted by i ($-\mu_{ji} > 0$), prices have to be decreased by ϵ, otherwise $IC(ij)$ would no longer be satisfied anymore, which decreases the profit per unit by $\epsilon \mu_{ij}$. Symmetrically, for all js such that i is attracted to j ($\mu_{ij} < 0$) prices can be increased by ϵ so that $IC(ij)$ still binds. This increases the profit per unit by $-\epsilon \mu_{ij}$. This condition shows that these effects are exactly compensated when prices are optimally chosen.

4 The Solution (1)

To solve the problem, the question is to guess what constraints are binding and then check that is indeed the case. Intuitively for t^l large enough (the lowest level of service that the airline offers is the travel itself) and ρ not too big, all quantities served will be strictly positive, which implies that $IR(A)$ will be the IRC that binds. In this case:
$\forall i \neq A, \pi^i = \sum_j (\mu_{ji} - \mu_{ij}).$

4.1 The regular symmetric case

By analogy with the one dimensional case, it is tempting to assume that only downward or leftward ICC will be binding. This is what we call the Regular Case (Regular because only downward or leftward ICC can be binding). If, moreover, all such constraints are binding, we obtain what we call the Regular Symmetric Case, represented on diagram 1 (see diagram 1), where an arrow between i and j means that $IC(i,j)$ is binding (j attracts i), and a circle around point i means that $IR(i)$ is binding ($U^i = 0$).

We assume now that: $\Delta t_1 \pi^B > \Delta t_2 \pi^D$. This assumption allows to better differentiate the market of good 1 from the market of good 2.

We look for the regular symmetric solution of the monopolist's problem.

Applying the first order conditions, we get:

$$\mu_{BA} + \mu_{DA} = \pi^A - 1, \quad (\lambda^A = -1)$$
$$-\mu_{BA} + \mu_{CB} = \pi^B,$$
$$-\mu_{CB} - \mu_{CD} = \pi^C,$$
$$\mu_{CD} - \mu_{DA} = \pi^D,$$

and the other Lagrange multipliers are equal to 0.

We then obtain the optimal quantities (substituting q_1^i in q_2^i and vice and versa) as functions of μ_{BA}:

$$q_1^A = \frac{1}{(1-\rho^2)}[t_1^A + \Delta t_1 \frac{\mu_{BA}}{\pi^A} - \rho(t_2^A + \Delta t_2 \frac{(\pi^A - 1 - \mu_{BA})}{\pi^A})],$$

$$q_1^B = \frac{1}{(1-\rho^2)}[t_1^B - \rho(t_2^B + \Delta t_2 \frac{(\pi^B + \mu_{BA})}{\pi^B})],$$

$$q_1^C = \frac{1}{(1-\rho^2)}[t_1^C - \rho(t_2^C)],$$

$$q_1^D = \frac{1}{(1-\rho^2)}[t_1^D + \Delta t_1 \frac{(-\mu_{BA} - \pi^B - \pi^C)}{\pi^D} - \rho(t_2^D)],$$

$$q_2^A = \frac{1}{(1-\rho^2)}[t_2^A + \Delta t_2 \frac{(\pi^A - 1 - \mu_{BA})}{\pi^A} - \rho(t_1^A + \Delta t_1 \frac{(\mu_{BA})}{\pi^A})],$$

$$q_2^B = \frac{1}{(1-\rho^2)}[t_2^B + \Delta t_2 \frac{(\pi^B + \mu_{BA})}{\pi^B} - \rho(t_1^B)],$$

$$q_2^C = \frac{1}{(1-\rho^2)}[t_2^C - \rho(t_1^C)],$$

$$q_2^D = \frac{1}{(1-\rho^2)}[t_2^D - \rho(t_1^D + \Delta t_1 \frac{(-\mu_{BA} - \pi^B - \pi^C)}{\pi^D})].$$

In this case, agents utilities are:

$$U^A = 0,$$
$$U^B = \Delta t_1 q_1^A,$$
$$U^D = \Delta t_2 q_2^A,$$
$$U^C = \Delta t_1 q_1^A + \Delta t_2 q_2^B = \Delta t_1 q_1^D + \Delta t_2 q_2^A,$$

where $\Delta t_i = t_i^h - t_i^l$.

We have 4 unknowns (μ_{ij}s) and 3 equations. The missing equation is obtained by examining diagram 1. The difference of utility between C and A can be computed along two paths (through B or through D):

$$U^C - U^A = \Delta t_1 q_1^A + \Delta t_2 q_2^B = \Delta t_1 q_1^D + \Delta t_2 q_2^A.$$

Replacing the expressions of q^is in the above equation and solving the system of four equations with four unknowns, we obtain:

$$\mu_{BA} = -\frac{[\Delta t_1^2((\pi^B + \pi^C)/\pi^D) + \Delta t_2^2(1/\pi^A) - \rho \Delta t_1 \Delta t_2((\pi^A - 1)/\pi^A)]}{[((1/\pi^A) + (1/\pi^D))\Delta t_1^2 + ((1/\pi^A) + (1/\pi^B))\Delta t_2^2 + 2\rho \Delta t_1 \Delta t_2/\pi^A]}.$$

This defines a solution for the problem only if all μ_{ij}s are non positive, which is equivalent to:

RSC0:$\mu_{BA} \leq -\pi^B$

$\Leftrightarrow \pi^A \pi^C - \pi^B \pi^D \geq \frac{-\Delta t_2^2(\pi^D(\pi^C + \pi^D)) + \rho \pi^D(1 - \pi^A - 2\pi^B)\Delta t_1 \Delta t_2}{\Delta t_1^2}$

RSC1:$\mu_{BA} \geq -\pi^B - \pi^C$

$\Leftrightarrow \pi^A \pi^C - \pi^B \pi^D \geq \frac{-\Delta t_1^2(\pi^B(\pi^C + \pi^B)) + \rho \pi^B(1 - \pi^A - 2\pi^B - 2\pi^C)\Delta t_1 \Delta t_2}{\Delta t_2^2}$

We then look for the other conditions that must be checked by the parameters of the model for the RSC to be solution (see appendix 1) and we obtain conditions on q_α^is and μ_{ij}s.

These conditions can be written as functions of μ_{BA}:

RSC0:$\mu_{BA} \leq -\pi^B$,

RSC1:$\mu_{BA} \geq -\pi^B - \pi^C$,

RSC2:$q_1^A \leq q_1^D$,

$\hookrightarrow \mu_{BA}[-\Delta t_1(\frac{\pi^A + \pi^D}{\pi^A \pi^D}) - \rho \frac{\Delta t_2}{\pi^A}] - \rho \frac{\Delta t_2}{\pi^A} - \Delta t_1(\frac{\pi^B + \pi^C}{\pi^D}) \geq 0.$

RSC3:$q_1^D \leq q_1^B$,

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$\hookrightarrow \mu_{BA}[\frac{\Delta t_1}{\pi^D} - \rho\frac{\Delta t_2}{\pi^B}] + \frac{\Delta t_1(1-\pi^A)}{\pi^D} \geq 0.$

RSC4:$q_1^A \leq q_1^B$,

$\hookrightarrow \mu_{BA}[-\frac{\Delta t_1}{\pi^A} - \rho\Delta t_2(\frac{\pi^A+\pi^B}{\pi^A\pi^B})] + \Delta t_1 - \rho\frac{\Delta t_2}{\pi^A} \geq 0.$

RSC5:$q_2^A \leq q_2^B$,

$\hookrightarrow \mu_{BA}[\Delta t_2(\frac{\pi^A+\pi^B}{\pi^A\pi^B}) + \rho\frac{\Delta t_1}{\pi^A}] + \frac{\Delta t_2}{\pi^A} - \rho\Delta t_1 \geq 0.$

RSC6:$q_2^B \leq q_2^D$,

$\hookrightarrow \mu_{BA}[-\frac{\Delta t_2}{\pi^B} + \rho\frac{\Delta t_1}{\pi^D}] + \rho\frac{\Delta t_1(1-\pi^A)}{\pi^D} \geq 0.$

RSC7:$q_2^A \leq q_2^D$,

$\hookrightarrow \mu_{BA}[\frac{\Delta t_2}{\pi^A} + \rho\Delta t_1(\frac{\pi^A+\pi^D}{\pi^A\pi^D})] + \frac{\Delta t_2}{\pi^A} + \rho\frac{\Delta t_1(\pi^B+\pi^C)}{\pi^D} \geq 0.$

RSC8:$q_2^B \leq q_2^C$,

$\hookrightarrow \mu_{BA} \leq 0.$

RSC9:$q_1^D \leq q_1^C$,

$\hookrightarrow \mu_{BA} + 1 - \pi^A \geq 0.$

For $\varrho = 0$, notice that the solution exists only when:

$(\pi^A\pi^C - \pi^B\pi^D) \in \left]-\left(\frac{\Delta t_2}{\Delta t_1}\right)^2 \pi^D\left(\pi^C + \pi^D\right), \frac{\pi^B\pi^D}{\pi^A}\right[$ (see conditions RSC1, and RSC2 or RSC5).

The optimal strategy of the monopolist is defined as a set of price-quantity allocations satisfying IRC and ICC. With fixed quantities as defined previously, optimal prices are determined by the following equation:

$p^i = \frac{\partial u}{\partial q}(t^i, q^i) = \frac{\partial C}{\partial q}(q^i) - \sum_j \frac{\mu_{ji}}{\pi^i}(\frac{\partial u}{\partial q}(t^j, q^i) - \frac{\partial u}{\partial q}(t^i, q^i)).$

We can notice that the unit prices paid by buyers of type i are a convex combination of the marginal cost and of the marginal utilities of buyers that are attracted to i (his quantity vector q^i). In our case, with linear preferences, we get:

$p^i = \frac{\partial C}{\partial q}(q^i) - \sum_j \frac{\mu_{ji}}{\pi^i}(t^j - t^i).$

When i attracts no one ($\mu_{ji} = 0$ for all i, j), i pays prices equal to the marginal costs and gets his first best allocation. In the other case, ($\mu_{ji} < 0$) prices are always superior to marginal costs, because only downwards local ICC bind: as a consequence, in the R.S.C, $\mu_{ji} < 0$ implies that t^j is superior to t^i .

Proposition 1 *In the regular symmetric case, prices are superior to marginal costs except for the highest type who is charged with a price equal to marginal cost and gets his first best allocation in quantity. The price charged on a good depends on the consumed quantity of the other good: p_α^i depends on q_β^i. The optimal nonlinear pricing requires a kind of form of bundling.*

To show the second part of the proposition, we consider types A and D for the good 1: we know that $t_1^A = t_1^D$ and the solution gives the following result: $p_1^A > p_1^D$ and $q_1^A < q_1^D$ and this comes from the fact that $q_2^A < q_2^D$ as $t_2^A < t_2^D$. Symmetrically we obtain the same result on good 2 for types A and B.

When $\rho = 0$, we have the following solution for prices in the R.S.C:

$$t_\alpha^i = t_\alpha^h \iff p_\alpha^i = \frac{\partial C}{\partial q}(q_\alpha^i) \iff q_\alpha^i = \hat{q},$$
$$t_\alpha^i = t_\alpha^l \iff p_\alpha^i > \frac{\partial C}{\partial q}(q_\alpha^i) \iff q_\alpha^i < \hat{q},$$
$$\Rightarrow \text{no upward distortion.}$$

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As in Lewis-Sappington (88), prices always exceed marginal costs, except for the highest type where they are equal.

4.2 Complete solution: the case of a general discrete distribution and $\rho = 0$

4.2.1 The Regular Symmetric Case

We have previously seen that the regular symmetric case does not exist for all distributions. This solution is defined by its two frontiers: $\mu_{CB} \leq 0$, and $q_1^D \geq q_1^A$ (or $q_2^B \geq q_2^A$, see figures 1 and 2). These two frontiers give the following conditions:

$$\mu_{CB} \leq 0 \iff \pi^A \pi^C - \pi^B \pi^D \geq -(\frac{\Delta t_2}{\Delta t_1})^2 \pi^D (\pi^C + \pi^D),$$

$$q_1^D \geq q_1^A \iff \pi^A \pi^C - \pi^B \pi^D \leq \frac{\pi^B \pi^D}{\pi^A}.$$

Proposition 2 *If $\pi^A \pi^C - \pi^B \pi^D \in [-(\frac{\Delta t_2}{\Delta t_1})^2 \pi^D (\pi^C + \pi^D), \frac{\pi^B \pi^D}{\pi^A}]$ the solution is regular symmetric. Types C get their first best allocation and $q_1^D, q_1^A, q_2^B, q_2^A$ are downwards distorted. There is no upward distortion.*

4.2.2 The Separable Case

Now we assume that one of these two conditions is no longer satisfied anymore: if $\pi^A \pi^C - \pi^B \pi^D \geq \frac{\pi^B \pi^D}{\pi^A}$ this means that C is also directly attracted by A. Thus, the non local downward ICC is binding. As a consequence, $q_1^D = q_1^A$, $q_2^B = q_2^A$ and the solution is now separable (see diagram 2). $q_1^C, q_1^B, q_2^C, q_2^D$ are equal to the first best allocation.

Diagram 2 enables to compute the indirect utility levels and we get:

$$U^A = 0,$$
$$U^B = \Delta t_1 q_1^A,$$
$$U^D = \Delta t_2 q_2^A,$$
$$U^C = \Delta t_1 q_1^A + \Delta t_2 q_2^B = \Delta t_1 q_1^D + \Delta t_2 q_2^A = \Delta t_1 q_1^A + \Delta t_2 q_2^A.$$

The F.O.C and the conditions on quantities give:

$$\mu_{BA} + \mu_{DA} = \pi^A - 1, (\lambda^A = -1),$$
$$-\mu_{BA} + \mu_{CB} = \pi^B,$$
$$-\mu_{CA} - \mu_{CB} - \mu_{CD} = \pi^C,$$
$$\mu_{CD} - \mu_{DA} = \pi^D,$$
$$q_1^D = q_1^A,$$
$$q_2^B = q_2^A.$$

We get the following results:

$$\mu_{BA} = -\frac{\pi^B}{\pi^A + \pi^D}, \mu_{DA} = -\frac{\pi^D}{\pi^A + \pi^D},$$
$$\mu_{CB} = \pi^B + \mu_{BA} < 0, \mu_{CD} = \pi^D + \mu_{DA} < 0,$$
$$\mu_{CA} = \pi^B + \pi^C + \pi^D + \mu_{BA} + \mu_{DA} < 0.$$

Proposition 3 *When there is a strong positive correlation between types ($\pi^A \pi^C - \pi^B \pi^D \geq \frac{\pi^B \pi^D}{\pi^A}$), the non local downward ICC is binding, and the solution is separable: $q_1^D = q_1^A, q_2^B = q_2^A$, $q_1^C = q_1^B, q_2^C = q_2^D$. C gets his first best allocation and $q_1^D, q_1^A, q_2^B, q_2^A$ are downwards distorted.*

The optimal quantities are:

$$q_1^D = q_1^A = t_1^l - \frac{\pi^A + \pi^D}{\pi^B + \pi^C} \Delta t_1,$$
$$q_1^B = q_1^C = t_1^h,$$
$$q_2^B = q_2^A = t_2^l - \frac{\pi^A + \pi^B}{\pi^C + \pi^D} \Delta t_2,$$
$$q_2^D = q_2^C = t_2^h.$$

We have to check that all the Lagrange multipliers are negative under the condition

$$\pi^A \pi^C - \pi^B \pi^D \geq \frac{\pi^B \pi^D}{\pi^A}$$

and it is indeed the case. Conditions on quantities to satisfy the ICC are always true in this case.

For such a distribution, the optimal solution does not generate perfect screening and the allocations do not require the kind of form of bundling mentioned previously: for each type, the unit price charged on a good does not depend on the consumed quantity of the other good. Optimal quantities depend on the marginal distribution of types. But we still have the usual unidimensional result: prices are greater than marginal costs, except at the top of the distribution where they are equal.

4.2.3 The Regular Asymmetric Case

Now we assume that the other condition for R.S.C to be solution($\pi^A \pi^C - \pi^B \pi^D \geq -(\frac{\Delta t_2}{\Delta t_1})^2 \pi^D (\pi^C + \pi^D)$, i.e., $\mu_{CB} < 0$) is no longer satisfied. In this case, we obtain a solution in which C is not attracted to B ($\mu_{CB} = 0$). We call it the regular asymmetric case (Regular because only downward or leftward ICC can be binding and Asymmetric because there is no attraction between B and C, whereas D attracts C)(see diagram 3) .

Diagram 3 enables to compute the indirect utility levels and we get:

$$U^A = 0,$$
$$U^B = \Delta t_1 q_1^A,$$
$$U^D = \Delta t_2 q_2^A,$$
$$U^C = \Delta t_1 q_1^D + \Delta t_2 q_2^A.$$

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The F.O.C give:

$$\mu_{BA} = -\pi^B,$$
$$-\mu_{DA} + \mu_{CD} = \pi^D,$$
$$\mu_{CD} = -\pi^C.$$

We get the following results:

$$\mu_{BA} = -\pi^B$$
$$\mu_{DA} = -\pi^D - \pi^C$$
$$\mu_{CD} = -\pi^C$$

Now we have to check that B is not attracted by C. This is equivalent to:

$$\Delta t_1(q_1^D - q_1^A) \leq \Delta t_2(q_2^C - q_2^A),$$

and this condition implies after computations that:

$$\pi^A \pi^C - \pi^B \pi^D \geq -(\frac{\Delta t_2}{\Delta t_1})^2 \pi^D (\pi^A + \pi^C + \pi^D).$$

The same condition applies to show that B is not attracted to D.

Proposition 4 *When $\pi^A \pi^C - \pi^B \pi^D \in [-(\frac{\Delta t_2}{\Delta t_1})^2 \pi^D (\pi^A + \pi^C + \pi^D), -(\frac{\Delta t_2}{\Delta t_1})^2 \pi^D (\pi^C + \pi^D)]$ the solution is regular asymmetric: B and C receive their first best allocation, whereas q_1^D, q_1^A, q_2^A are distorted downwards.*

The optimal quantities are:

$$q_1^A = t_1^l - \frac{\pi^B}{\pi^A} \Delta t_1,$$

$$q_2^A = t_2^l - \frac{\pi^C + \pi^D}{\pi^A} \Delta t_2,$$

$$q_1^D = t_1^l - \frac{\pi^C}{\pi^D} \Delta t_1,$$

$$q_2^D = t_2^h,$$
$$q_1^B = t_1^l,$$
$$q_2^B = t_2^h,$$
$$q_1^C = t_1^h,$$
$$q_2^C = t_2^h.$$

Note that D is not attracted to B if:

$$\pi^A \pi^C - \pi^B \pi^D \geq -(\frac{\Delta t_1}{\Delta t_2})^2 \pi^B (\pi^A + \pi^C + \pi^D).$$

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If we had not assumed that $\Delta t_1 \pi^B > \Delta t_2 \pi^D$, we would have obtained the same condition for B and D to not attract each other.

In this case the unidimensional qualitative properties are almost satisfied. For the good 1, the usual results are satisfied in the sense that the downward local ICC are binding (C is attracted to D, and B is attracted to A). This is not the case on the market of good 2: D is attracted to A, but C is not attracted to B. We also have to notice that two different types get their first best allocation(these types pay prices equal to marginal costs) and perfect screening is obtained in this optimal solution.

Under this condition, $\pi^A \pi^C - \pi^B \pi^D \geq -(\frac{\Delta t_2}{\Delta t_1})^2 \pi^D(\pi^C + \pi^D)$, the Lagrange multipliers are all non positive, and conditions on quantities are all satisfied.

4.2.4 The Singular Case

Now we assume that $\pi^A \pi^C - \pi^B \pi^D \leq -(\frac{\Delta t_2}{\Delta t_1})^2 \pi^D(\pi^A + \pi^C + \pi^D)$, i.e., $(\Delta t_1(q_1^D - q_1^A) \leq \Delta t_2(q_2^C - q_2^A)$ is not satisfied anymore), and B is now attracted to C and D.This condition also means that

$$\pi^B \geq \frac{\Delta t_1}{\Delta t_1 + \Delta t_2} \frac{\pi^A \pi^C}{\pi^D} + \frac{\Delta t_2}{\Delta t_1 + \Delta t_2}.$$

In this case the solution is singular: surprisingly, the transverse and an upward ICC are binding (see diagram 4).

Diagram 4 enables to compute the indirect utility levels and we get:

$$U^A = 0,$$
$$U^B = \Delta t_1 q_1^A = \Delta t_1 q_1^D - \Delta t_1 q_2^D + U^D,$$
$$U^D = \Delta t_2 q_2^A,$$
$$U^C = \Delta t_1 q_1^A + \Delta t_2 q_2^C = \Delta t_1 q_1^D + \Delta t_2 q_2^A.$$

The F.O.C and the conditions on quantities give:

$$\mu_{BD} + \mu_{CD} - \mu_{DA} = \pi^D,$$
$$-\mu_{BA} - \mu_{BC} - \mu_{BD} = \pi^B,$$
$$\mu_{BC} - \mu_{CD} = \pi^C,$$
$$q_2^C = q_2^D,$$
$$\Delta t_1(q_1^D - q_1^A) = \Delta t_2(q_2^D - q_2^A).$$

We get the following results:

We check the sign of μ_{ij}s:

$$\frac{\mu_{BC}}{\pi^C} = \frac{\mu_{BD}}{\pi^D} \rightarrow \mu_{BC} = -\theta \pi^C \text{ and } \mu_{BD} = -\theta \pi^D,$$

$$\mu_{CD} = -(1+\theta)\pi^C \text{ and } \mu_{DA} = -(1+\theta)(\pi^C + \pi^D),$$

$$\text{and } \mu_{BA} = -\pi^B + \theta(\pi^C + \pi^D).$$

$$(\Delta t_1)^2[-\frac{\mu_{CD} + \mu_{BD}}{\pi^D} + \frac{\mu_{BA}}{\pi^A}] = (\Delta t_2)^2[\frac{\mu_{BD}}{\pi^D} + \frac{\mu_{DA}}{\pi^A}] \text{ gives the following for } \theta,$$

$$\theta = \frac{(\pi^B \pi^D - \pi^A \pi^C)\Delta t_1^2 - \Delta t_2^2 \pi^D(1 - \pi^B)}{\Delta t_1^2(\pi^C + \pi^D)(\pi^A + \pi^D) + \Delta t_2^2 \pi^D(1 - \pi^B)} \leq 0 \text{ by assumption.}$$

Proposition 5 *When $\pi^A\pi^C - \pi^B\pi^D \leq -(\frac{\Delta t_2}{\Delta t_1})^2\pi^D(\pi^A + \pi^C + \pi^D)$ the solution is singular. B gets his first best allocation and C attracts B. As a consequence q_2^C is distorted upwards. D also attracts B and consequently q_{12}^D is distorted upwards.*

All the conditions on the quantities are always satisfied for such distributions.

In this solution, it is optimal, for the monopolist, to price below the marginal cost on good 2 for types C and D and to give the first best allocation to an "intermediary" type, type B.

The intuition for upward distortions is as follows: proportion of type B is so high that it is in the seller's interest to extract maximum surplus from them and thus sell them their first best allocation. But the seller has also to prevent types B from choosing the allocations offered to other types, in particular type C. In order to reduce competition between q^C and q^D, it is therefore optimal to increase q_2^C above the first best level.

To sum up, here are the four solutions of the monopolist's problem:

- $\pi^A\pi^C - \pi^B\pi^D \in [\frac{-1}{4}, -(\frac{\Delta t_2}{\Delta t_1})^2\pi^D(\pi^A + \pi^C + \pi^D)]$: Singular Solution.

- $\pi^A\pi^C - \pi^B\pi^D \in [-(\frac{\Delta t_2}{\Delta t_1})^2\pi^D(\pi^A + \pi^C + \pi^D), -(\frac{\Delta t_2}{\Delta t_1})^2\pi^D(\pi^C + \pi^D)]$: Regular Asymmetric Solution.

- $\pi^A\pi^C - \pi^B\pi^D \in [-(\frac{\Delta t_2}{\Delta t_1})^2\pi^D(\pi^C + \pi^D), \frac{\pi^B\pi^D}{\pi^A}]$: Regular Symmetric Solution.

- $\pi^A\pi^C - \pi^B\pi^D \in [\frac{\pi^B\pi^D}{\pi^A}, \frac{1}{4}]$:Separable Solution.

We have seen that, in our problem, four possible solutions exist depending on the correlation of types. Dana (1993) also finds solutions depending on the correlation of types but only gets two since he assumes that $\Delta t_1 = \Delta t_2$ and $\pi^B = \pi^D$.

4.3 Complete Solution: the case of a Uniform Discrete Distribution and $\rho \in]-1, 1[$

We assume:

$$\pi^A = \pi^C = \frac{\epsilon}{2},$$

$$\pi^B = \pi^D = \frac{(1-\epsilon)}{2}.$$

See Appendix 2 and figure 4.

We have obtained four solutions in the previous section: we call respectively A,C,D,E the singular case, the regular asymmetric case, the regular symmetric case and the separable case. We obtain ten other solutions (see diagrams) depicted in figure 4. Two constraints are always binding: for all (ϵ, ρ) the individual rationality constraint of types A (not surprising) and $IC(AD)$ that is D is always attracted to A. Two other constraints never bind: $IC(CA)$ and $IC(CD)$.

There are two solutions (G and N) where the seller never offers a first best allocation. Whatever the type is, the allocation is distorted.

We go from D to C when μ_{CB} becomes positive, then $IC(BC)$ is not binding anymore.

We go from C to A when $IC(CB), IC(DB)$ fail, then they are binding.

We go from A to B when μ_{BD} becomes positive.

We go from D to E when $IC(AC)$ fails.

We go from E to F when μ_{CB} becomes positive.

We go from F to G when $IC(CB), IC(BC)$ fail.

We go from G to H when μ_{CB} becomes positive.

We go from H to M when μ_{BA} becomes positive.

We go from M to N when $IC(BC)$ fails.

We go from D to K when $IC(DB)$ fails.

We go from K to J when μ_{CD} becomes positive.

We go from J to I when μ_{BA} becomes positive.

4.3.1 Case B

An upward ICC is binding (B is attracted to C). As a consequence, q^C is distorted upwards for the same reasons than in case A. B gets his first best allocation.

4.3.2 Case F

B and C get their first best allocation. q_2^D is not distorted, whereas q_1^A, q_1^D, q_2^A are distorted downwards. The solution is separable on the market of good 1 (the solution is separable on the market of good α means: $t_\alpha^i = t_\alpha^j \rightarrow q_\alpha^i = q_\alpha^j$): $q_1^A = q_1^D < q_1^B = q_1^C$.

4.3.3 Case G

This solution is similar to the case E solution, but B is attracted to C. This means that $q_2^A = q_2^B = q_2^C$ whereas q_2^D is optimal. The solution is not globally separable anymore, but it is still separable on the market of good 1. In this case, the seller never offers a first best allocation.

4.3.4 Case H

B does not attract C, but is attracted to C. As a consequence q_2^C is distorted upwards and the solution is separable for the good 1. B gets his first best allocation even if π^B becomes relatively small (when ϵ increases).

4.3.5 Case I

The downward local ICC are only binding on the market of good 2: C is attracted to B and D is attracted to A. The transverse ICC is also binding (B is attracted to D). As a consequence we obtain this condition on the quantities obtained by types B and D :

$$\Delta t_1(q_1^B - q_1^D) > \Delta t_2(q_2^D - q_2^B).$$

C gets his first best allocation.

4.3.6 Case J

C gets his first best allocation and we obtain the following conditions on quantities:

$$\Delta t_1(q_1^B - q_1^A) = \Delta t_2(q_2^D - q_2^A),$$
$$\Delta t_1(q_1^D - q_1^A) > \Delta t_2(q_2^B - q_2^A).$$

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4.3.7 Case K

This solution is similar to the regular symmetric case but the transverse ICC is binding. This solution differs from the case D solution because we obtain:

$$q_1^A \le q_1^D \le q_1^B \le q_1^C,$$
$$q_2^A \le q_2^D = q_2^B \le q_2^C.$$

B and D receive the same quantity of good 2.

4.3.8 Case L

The solution differs from the case K one because $q_2^D > q_2^B$ and B gets his first best allocation. As in case C, the proportion of type B is so high that it is in the seller's interest to offer them their first best allocation to extract the maximum surplus from them.

4.3.9 Case M

B gets his first best allocation whereas C receives an upwards distorted allocation. It is surprising in this case, as in the case H (especially the South-East of the H area), because the proportion of buyers of type B is relatively small. The solution is separable for the good 1.

4.3.10 Case N

The seller does not offer any first best allocation. The solution is separable for the good 1. For the market of good 2, D receives his first best allocation and

$$q_2^D = q_2^B.$$

5 The Equivalent Approach

In our simple model (discrete distribution of four types of buyers, linear utilities in money), we defined the global solution to the monopolist's problem: we defined the allocations and rents for each solution. To do so, following Spence (80)we maximize the expected profit of the seller under Individual Rationality Constraints (IRC) and Incentive Compatibility Constraints (ICC); we decompose the problem into two subproblems:

- minimize the agents's expected utilities for fixed allocations

- Choose the allocation so as to maximize the expected surplus minus the expected utilities

Rents are defined so that IRC and ICC are satisfied and computed according to the binding constraints (or attractions, see Border-Sobel (87)).

These attractions allow us to define optimal paths from a type of agent to a dummy type (the "lowest type", type A in chapter 2) that enable to compute these rents. Thanks to Rochet (87) that shows that closed paths (from a type of agents to himself) do not increase expected utility (and are thus non optimal), we know that the rents defined by optimal paths from a type to the dummy type are optimal.

Once we know the solution and the optimal rents we can write the monopolist's problem as:

- Compute the optimal rents according to binding constraints.

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- maximize the expected profit of the seller under the constraints that each type of agents gets his optimal rent.

This new formulation of the seller's problem, the equivalent problem hereafter, which of course leads to the same solution, enables to explain distortions in resource allocation. Distortions depend on optimal paths and more especially of the length (the number of type involved in the path) and the number of these optimal paths that define a solution.

After some brief remarks on the above results, we study the equivalent problem and explain the way allocations are distorted. We confirm Rochet-Chone (1998) that defines the sets of agents who do participate as: i) a set of types who get no rent: this set is in fact reduced to a singleton, ii) a set of types who get strictly positive rents and iii) a set of types who get strictly positive rents and their first best allocations. We then study the Regular Symmetric Case (see case D in section 4) to show the equivalence of the two problems and to explain distortions in resource allocations. We also study the general solution (defined in section 4) and define several sets of solutions: ordered types, called **Pure Cases** (solutions that enables to define a complete type ordering implying uniqueness of rents), single optimum, called **Almost Pure Cases**(uniqueness of rents and partial type ordering), several optima, regular solutions (only downward binding ICC), singular solutions (some upward or transverse ICC are binding). We also exhibit two special cases in which the monopolist only produces one good and this is closed, but not equivalent, to Armstrong (96) who shows that it is sometimes optimal for the monopolist to exclude some customers from its products in order to extract more revenue from other higher value consumers.

5.1 Remarks

As defined in Border-Sobel (87), we say that j attracts i when $IC(i,j)$ is binding. These attractions (who attracts whom?) are crucial for determining distortions in resource allocation and also to compute rents. If j attracts i we can define U^i as a function of U^j and q^j. If j is attracted by another type, say k, we can define his rent as a function of U^k and q^k and so on. If j is not attracted by any other type, this means that j is the dummy type or lowest type and his IRC is binding (see section 4).

A path is defined as a way to reach the dummy type from another type (with one or several steps from type to type). An optimal path from one type to another is defined as a succession of attractions from one type to the dummy. As successions of attractions always lead to the dummy type (see section 4), we can compute the set of rents using backward induction. This means that the set of possible paths defines the set of possible rents and the set of optimal paths defines a solution and the optimal rents associated to this solution.

Example: i is attracted to j and j is attracted to the dummy type, say k. This defines the optimal paths $\gamma^i = \{i \to j \to k\}, \gamma^j = \{j \to k\}$. Then, we have:

$$U^i = U^j + u(t^i, q^j) - u(t^j, q^j)$$
$$U^j = U^k + u(t^j, q^k) - u(t^k, q^k)$$
$$U^k = 0$$

then, we obtain:

$$U^i = u(t^i, q^j) - u(t^j, q^j) + u(t^j, q^k) - u(t^k, q^k)$$
$$U^j = u(t^j, q^k) - u(t^k, q^k)$$
$$U^k = 0.$$

This enables to solve the first subproblem of the monopolist's problem: "minimize expected utilities". We can then define a set of rents $U(q)$ such that ICC are satisfied for all i, j and the rent of the lowest type is zero. According to these attractions, we can say that j is an immediate successor of i and also an immediate predecessor of j. This also means that for any optimal path we are able to define an ordered type subset where $\{i \rightarrow j \rightarrow k\}$ is equivalent to:

$$i \succeq j \succeq k.$$

Consequently, for any solution, we can reorganize T into a partition of ordered subsets Υ.

We define as closed paths, successions of attractions that leads one type to himself. Using Lemmas from Rochet (87): *considering arbitrary paths in the set of types T, a path from type t^i to t^j is denoted by a function γ. We denote the "length" of γ :$\{0, 1, ...l\} \rightarrow T$. Finally, we say that a path of length l is closed if: $\gamma(0) = \gamma(l)$.*

Lemma 1: $U(q)$ is non-empty if and only if for every closed path γ :

$$\sum_{k=0}^{l-1} u(t_{\gamma(k+1)}, q_{\gamma(k)}) - u(t_{\gamma(k)}, q_{\gamma(k)}) \leq 0. \qquad \cdot$$

Lemma 2: When the above condition is satisfied, $U(q)$ has a unique element U_-, characterized for all i by:

$$U^i_- = \sup_{\gamma} \sum_{k=0}^{l-1} u(t_{\gamma(k+1)}, q_{\gamma(k)}) - u(t_{\gamma(k)}, q_{\gamma(k)})$$

where the sup is taken over all the possible open paths from type t^i to the dummy type, whose utility is zero.

This means that rents computed according to optimal paths from a type to the dummy one are optimal. On the one hand, these rents do not depend on the distribution of types but only on the set of types. On the other hand, quantities and distortions, on which depends these rents, depend on the distribution of types (see section 4).

5.2 The Monopolist's Equivalent Problem

As we know how to compute the optimal rents according to optimal paths, we can now write the monopolist's problem as:

$$Max \sum_i \pi^i (u(t^i, q^i) - C(q^i) - U_-^i)$$

$$s/c : U_-^i = \sup_{\gamma^i} \sum_{k=0}^{l-1} u(t_{\gamma^i(k+1)}, q_{\gamma^i(k)}) - u(t_{\gamma^i(k)}, q_{\gamma^i(k)}) \text{ for all } i.$$

From section 4, we know that whatever the solution:

$$U\_^A = 0.$$

We can explain this as follows: at least for one type, A in our model, the IRC is binding. Since adding a uniform fee in the price schedule does alter the ICC and increases the seller's profit, it's optimal for the monopolist to give no rent to the dummy type.

We also know that there are usually several ways to compute the optimal rents of the other types. There exists as many ways to compute these rents as the number of possible optimal paths from one type to the dummy type.

For example, if there exists two possible optimal paths from type t^C to t^A, γ_1^C through B and γ_2^C through D, there are clearly two expressions for $U\_^C$. In this particular example, the Regular Symmetric Case, we have:

$$\gamma^C = \{\gamma_1^C, \gamma_2^C\}$$

$$U^C(\gamma_1^C) = \Delta t_2 q_2^B + \Delta t_1 q_1^A,$$
$$U^C(\gamma_2^C) = \Delta t_1 q_1^D + \Delta t_2 q_2^A.$$

This means that in this particular case, there are two constraints on type t^C : the constraint

$$U\_^C = \max_{\gamma^C} U^C(\gamma^C) \text{ for all } \gamma^C$$

is equivalent to:

$$U\_^C = U^C(\gamma_1^C),$$
$$U\_^C = U^C(\gamma_2^C),$$

or

$$U\_^C = \sigma_1^C U^C(\gamma_1^C) + \sigma_2^C U^C(\gamma_2^C),$$
$$\sigma_1^C + \sigma_2^C = 1,$$

considering σ_j^C like "the probability that type t^C chooses path γ_j^C to reach the dummy type".

Once we know the solutions, we are able to identify, for all types, all the possible expressions of the optimal rents in each solution. To solve the monopolist's problem, we have to define for all solutions, the set of possible optimal paths: for solution $s \in s^*$, we define $\gamma(s)$ as the set of possible paths. In $\gamma(s)$, we define for all t^i all the possible paths to reach t^A as $\gamma^i(s)$:

$\gamma^i(s) = \{\gamma_1^i, \gamma_2^i ... \gamma_n^i\}$ if there are n possible optimal paths from t^i to t^A.

$\gamma(s) = \{\gamma_p^i$, for all i, for all possible optimal paths $p\}$.

Notice that if type $t^j \neq t^i$ belongs to one of the optimal paths γ_p^i from t^i to t^A, this means that the truncation of γ_p^i between t^j and t^A defines an optimal path between these two types and also an expression of the optimal rent of type t^j. From the above example, this means that an expression of $U\_^B$ is:

$$U\_^B = \Delta t_1 q_1^A,$$

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since B belongs to γ_1^C and an expression of $U\_^D$ is:

$$U\_^D = \Delta t_2 q_2^A,$$

since D belongs to γ_2^C.

To refer to the above formulation of the monopolist's problem, we define a type ordering for each optimal path. γ_1^C defines the following type ordering:

$$C \succsim B \succsim A$$

this means that in this optimal paths B is the immediate successor of C and A is the immediate successor of B and this defines the unique γ^B; γ_2^C defines the following type ordering:

$$C \succsim D \succsim A$$

this means that in this optimal paths D is the immediate successor of C and A is the immediate successor of D and this defines the unique γ^D. We can reorganize T into a partition of ordered subsets Υ :

$$\Upsilon_1 = \{C, B, A\},$$
$$\Upsilon_2 = \{C, D, A\}.$$

As γ_1^C is of length 2 ($l = 2$), we can define:

$$U^C(\gamma_1^C) = \sum_{k=0}^{l-1} u(t_{\gamma_1^c(k+1)}, q_{\gamma_1^c(k)}) - u(t_{\gamma_1^c(k)}, q_{\gamma_1^c(k)(k)})$$

where $k = 2$ represents the highest type, C in this example, $k = 1$ represents the immediate successor to the highest type in the optimal path γ_1^C, B and $k = 0$ represents the dummy type, A. We also define:

$$U^C(\gamma_2^C) = \sum_{k=0}^{l-1} u(t_{\gamma_2^c(k+1)}, q_{\gamma_2^c(k)}) - u(t_{\gamma_2^c(k)}, q_{\gamma_2^c(k)(k)})$$

where $k = 2$ represents the highest type, C in this example, $k = 1$ represents the immediate successor to the highest type in the optimal path γ_1^C, D and $k = 0$ represents the dummy type, A.

The Lagrangian of the problem can now be written as:

$$L(U, q) = \sum_i \{\pi^i(u(t^i, q^i) - C(q^i) - U\_^i) - \sum_j \lambda_j^i(U\_^i - U^i(\gamma_j^i))\}$$

$$= \sum_i \{\pi^i(u(t^i, q^i) - C(q^i) - U\_^i)$$

$$- \sum_j \lambda_j^i [U\_^i - (\sum_{k=0}^{l-1} u(t_{\gamma_j^i(k+1)}, q_{\gamma_j^i(k)}) - u(t_{\gamma_2^i(k)}, q_{\gamma_2^i(k)(k)}))]\}.$$

where λ_j^i represents the non positive multipliers associated to the constraints on the optimal rent of type t^i computed along the optimal path γ_j^i.

The first order conditions (which are necessary and sufficient for this linear program) give:

$$\frac{\partial L}{\partial U^i} = -\pi^i - \sum_j \lambda^i_j = 0, \text{ for all } i,$$

$$\iff \pi^i = -\sum_j \lambda^i_j, \text{ for all } i,$$

and:

$$\frac{\partial L}{\partial q^i} = \pi^i \frac{\partial S(t^i, q^i)}{\partial q^i} + \sum_{k \neq i} \sum_j \lambda^k_j \frac{\partial U^k(\gamma^k_j)}{\partial q^i} = 0, \text{ for all } i,$$

$$\iff \frac{\partial S(t^i, q^i)}{\partial q^i} = -\frac{1}{\pi^i} \sum_{k \neq i} \sum_j \lambda^k_j \frac{\partial U^k(\gamma^k_j)}{\partial q^i} = \text{ for all } i.$$

We make the sum for all $k \neq i$ as closed paths are not optimal. This means that there exists no (classical) solution when two types attract each other (see special cases in section 6, in which the monopolist only produces good 1). We also have:

$$\frac{\partial U^k(\gamma^k_j)}{\partial q^i} = (t^{i+1} - t^i), \text{ for all } i.$$

We can also write the Lagrangian of the problem as:

$$L(U, q) = \sum_i \{\pi^i(u(t^i, q^i) - C(q^i) - U\_^i) - \lambda^i \sum_j \sigma^i_j (U\_^i - U^i(\gamma^i_j))\}$$

$$= \sum_i \{\pi^i(u(t^i, q^i) - C(q^i) - U\_^i)$$

$$- \lambda^i \sum_j \sigma^i_j [U\_^i - (\sum_{k=0}^{l-1} u(t_{\gamma^i_j(k+1)}, q_{\gamma^i_j(k)}) - u(t_{\gamma^i_2(k)}, q_{\gamma^i_j(k)}(k)))]\},$$

where λ^i is the Lagrange multiplier of the constraint: the optimal rent of type t^i is a combination of all the possible rents $U^i(\gamma^i_j)$ which can be chosen by types t^i who reach t^A along the optimal path γ^i_j with probability σ^i_j :

$$U\_^i = \sum_j \sigma^i_j (U\_^i - U^i(\gamma^i_j)), \text{ for all } i \text{ and } j,$$

$$\sum_j \sigma^i_j = 1.$$

In this case we obtain as first order conditions:

$$\frac{\partial L}{\partial U^i} = -\pi^i - \lambda^i \sum_j \sigma^i_j = 0, \text{ for all } i,$$

$$\iff \pi^i = -\lambda^i \text{for all } i,$$

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and:

$$\frac{\partial L}{\partial q^i} = \pi^i \frac{\partial S(t^i, q^i)}{\partial q^i} + \sum_{k \neq i} \sum_j \pi^k \sigma_j^k \frac{\partial U^k(\gamma_j^k)}{\partial q^i} = 0, \text{ for all } i,$$

$$\iff \frac{\partial S(t^i, q^i)}{\partial q^i} = -\frac{1}{\pi^i} \sum_{k \neq i} \sum_j \pi^k \sigma_j^k \frac{\partial U^k(\gamma_j^k)}{\partial q^i} = \text{ for all } i.$$

5.3 Optimal Resource Allocations

The first order conditions with respect to q^i give:

$$\frac{\partial S(t^i, q^i)}{\partial q^i} = -\frac{1}{\pi^i} \sum_{k \neq i} \sum_j \lambda_j^k \frac{\partial U^k(\gamma_j^k)}{\partial q^i}, \text{ for all } i,$$

where i belongs to one of the optimal paths that links types t^k to the dummy type. If types t^i do not belong to any optimal paths, this means that it does not attract any types. In this case,

$$\sum_{k \neq i} \sum_j \lambda_j^k \frac{\partial U^k(\gamma_j^k)}{\partial q^i} = 0$$

and the first order condition gives:

$$\frac{\partial S(t^i, q^i)}{\partial q^i} = 0$$

and types t^i obtain their first best allocation.

In our setting, when $\rho = 0$, the first order conditions can be written as:

$$\frac{\partial L}{\partial q_\alpha^i} = \pi^i \frac{\partial S(t^i, q^i)}{\partial q_\alpha^i} + \sum_{k \neq i} \sum_j \lambda_j^k \frac{\partial U^k(\gamma_j^k)}{\partial q_\alpha^i} = 0, \text{ for all } i \text{ and } \alpha = 1, 2,$$

$$\iff \pi^i(t_\alpha^i - q_\alpha^i) + \sum_{k \neq i} \sum_j \lambda_j^k \frac{\partial U^k(\gamma_j^k)}{\partial q_\alpha^i} = 0, \text{for all } i \text{ and } \alpha = 1, 2,$$

$$\iff \pi^i(t_\alpha^i - q_\alpha^i) + \sum_{k \neq i} \sum_j \lambda_j^k (t_\alpha^{i+1} - t_\alpha^i) = 0, \text{for all } i \text{ and } \alpha = 1, 2,$$

$$\iff q_\alpha^i = t_\alpha^i + \frac{1}{\pi^i} \sum_{k \neq i} \sum_j \lambda_j^k (t_\alpha^{i+1} - t_\alpha^i), \text{for all } i \text{ and } \alpha = 1, 2,$$

$$\iff q_\alpha^i = t_\alpha^i + \frac{1}{\pi^i} \sum_{k \neq i} \sum_j \pi^k \sigma_j^k (t_\alpha^{i+1} - t_\alpha^i), \text{for all } i \text{ and } \alpha = 1, 2$$

The surplus function is assumed to be convex, differentiable in q and has, for all t, a unique maximum $\hat{q}(t)$. The set of $\hat{q}(t)$ will be used as a benchmark for evaluating the distortions in resource allocations entailed by the monopoly power under bidimensional adverse selection; the first best allocation is:

$$\hat{q}(t_\alpha^i) = t_\alpha^i, \text{ for all } i \text{ and } \alpha = 1, 2,$$

and distortions:

$$q(t_\alpha^i) - \hat{q}(t_\alpha^i) = \frac{1}{\pi^i} \sum_{k \neq i} \sum_j \lambda_j^k (t_\alpha^{i+1} - t_\alpha^i), \text{for all } i \text{ and } \alpha = 1, 2,$$

$$\Longleftrightarrow q(t_\alpha^i) - \hat{q}(t_\alpha^i) = \frac{\lambda(q, t^i)}{\pi^i} (t_\alpha^{i+1} - t_\alpha^i), \text{for all } i \text{ and } \alpha = 1, 2$$

where:

$$\lambda(q, t^i) = \sum_{k \neq i} \sum_j \lambda_j^k \leq 0,$$

$$\lambda(q, t^i) = -\sum_{k \neq i} \sum_j \pi^k \sigma_j^k \leq 0$$

which depends on q (see section 4).

The distortion in $q(t_\alpha^i)$ can be seen as a ratio between the weighted sum of the differences between t^i's predecessor and t^i and the proportion of types t^i in the population. The weights are represented by λ_j^k, Langrange multiplier associated to the constraint:

$$U\_^k - U^k(\gamma_j^k) = 0.$$

Even if $\lambda(q, t^i)$ is always negative, the sign of the distortion depends on the sign of $[t^{i+1} - t^i]$: upwards distortion if it is positive, downwards otherwise. The sign of the distortion of type t^i's allocation only depends on its predecessor's type. If type $\bar{t}$ is attracted to type t^i (t^i is $\bar{t}$'s immediate successor) and if there exists a good α such $\bar{t}_\alpha - t_\alpha^i < 0$, then:

$$q(\bar{t}_\alpha) = \hat{q}(\bar{t}_\alpha),$$
$$q(t_\alpha^i) > \hat{q}(t_\alpha^i).$$

This generalizes the result of no distortion at the top of the distribution: the highest type $\bar{t}$ who attracts no other type, gets his first best allocation. The intuition for upward distortion: if the proportion of type $\bar{t}$ is high it might be interesting for the monopolist's to extract the maximum surplus from type t^i and thus sell them their first best allocation. In this case, the seller has also to prevent type $\bar{t}$ from choosing the other types's allocations and especially his immediate successor's allocation, $q(\bar{t}^{-1})$. To reduce competition between $q(\bar{t})$ and $q(\bar{t}^{-1})$ it might be therefore optimal for the seller to increase $q(\bar{t}_\alpha^{-1})$ above the first best level and this is the case when:

$$\bar{t}_\alpha^{-1} > \bar{t}_\alpha.$$

- We call **Pure Case** a solution such that there is for the highest type an optimal path which runs through all of the types (and also unique since no closed paths are optimal), this means that we can define a unique ordered subset $\Upsilon \in T$. We can then rank these types from t^0, the dummy type to $\bar{t}$, the highest type:

$$\Upsilon = \{t^0, t^1..., \bar{t}\}.$$

This also means that if there are T types, the longest optimal path, from $\bar{t}$ to t^0 is of length $T - 1$. And for all optimal path of length l we have:

$$\gamma(l - 1) \subset \gamma(l), \text{ for all } l = \{1, ..., T - 1\}.$$

In this particular case, the first order conditions give:

$$\frac{\partial L}{\partial U^i} = -\pi^i - \lambda^i = 0, \text{ for all } i,$$

$$\Longleftrightarrow \lambda^i = -\pi^i, \text{ for all } i,$$

$$\Rightarrow q(t_\alpha^i) - \hat{q}(t_\alpha^i) = \frac{1}{\pi^i} \sum_{k>i} \pi^k (t_\alpha^{i+1} - t_\alpha^i), \text{for all } i \text{ and } \alpha = 1, 2,$$

$$\Longleftrightarrow q(t_\alpha^i) - \hat{q}(t_\alpha^i) = \frac{1 - \sum_{k \leq i} \pi^k}{\pi^i} (t_\alpha^{i+1} - t_\alpha^i), \text{for all } i \text{ and } \alpha = 1, 2.$$

In this particular case, as in the classic one-dimensional case, $\lambda(q, t^i)$ does not depend on q,

$$\lambda(q, t^i) = -\frac{1 - \sum_{k \leq i} \pi^k}{\pi^i}, \text{for all } i.$$

As a consequence, the subproblem 2 can be solved by maximizing the virtual surplus $S_v(t^i, q^i)$:

$$S_v(t^i, q^i) = S(t^i, q^i) - \lambda(q, t^i)[u(t^{i+1}, q^i) - u(t^i, q^i)], \text{ for all } i,$$

$$S_v(t^i, q^i) = S(t^i, q^i) - \frac{1 - \sum_{k \leq i} \pi^k}{\pi^i}[t^{i+1} - t^i]q^i, \text{ for all } i$$

$$S_v(\bar{t}, \bar{q}) = S(\bar{t}, \bar{q}).$$

where $\bar{t}$ is the highest type, nobody's successor. This generalizes the result of no distortion at the top of the distribution. But this does not imply that allocations are only downward distorted.

- We call **Almost Pure Case** a solution such that there is a single optimal path to compute the rent of each type, this does not mean, as above, that there exists a single optimal path running through all the types. This means that we can define several ordered subsets, say J:

$$\Upsilon = \{\Upsilon_1, \Upsilon_2 ... \Upsilon_J\},$$

and for every Υ_j we can rank its types from t_j^0, the dummy type in Υ_j to $\bar{t}_j$, the highest type in Υ_j. $t_j^0 = t^0$ for all j. Υ is such that :

$$\Upsilon_1 \cap \Upsilon_2 \cap ... \cap \Upsilon_J = \{t^0\}.$$

This shows an important result (which is valid for all solutions, not only almost pure cases):

Proposition 6 *whatever the solution, the set of types that participate and get no rent is always a singleton, the dummy type, if all the quantities served by the monopolist are strictly positive.*

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To show this, imagine there are two such types, t_j^0 the lowest type in Υ_j and t_i^0 the lowest type in Υ_j, as nobody attracts them, we have (which is also equivalent to say that attract each other, i.e, a closed path exists between t_j^0 and t_i^0):

$$U_j^{\prime 0} = U_i^{\prime 0}.$$

We also have:

$$
\begin{aligned}
U_j^0 - U_i^0 &= (t_j^0 - t_i^0)q_i^0 = 0, \\
U_i^0 - U_j^0 &= (t_i^0 - t_j^0)q_j^0 = 0, \\
&\implies q_{j\alpha}^0 = q_{i\alpha}^0 = 0 \text{ for all } (t_{i\alpha}^0 - t_{j\alpha}^0) \neq 0.
\end{aligned}
$$

If $(t_{i\alpha}^0 - t_{j\alpha}^0) > 0$, this means that $q_{i\alpha}^0$ is upwards distorted and $q_{j\alpha}^0$ is downwards distorted, then we have:

$$q_{i\alpha}^0 > \hat{q}_{i\alpha}^0 > \hat{q}_{j\alpha}^0 > q_{j\alpha}^0,$$

which cannot be satisfied with:

$$q_{j\alpha}^0 = q_{i\alpha}^0.$$

As a consequence, $q_{j\alpha}^0 = q_{i\alpha}^0 = 0 (QED)$. Then we define by l_{ji} the length of the optimal path of type t^i in subset Υ_j , $\bar{l}_j$ the length of the longest one and by $\gamma(\bar{l}_j)$ this optimal path from $\bar{t}_j$ to t^0. This means that for all $i \neq 0$, there exists a unique $\gamma(\bar{l}_j)$ running through t^i. And for all optimal path of length l_j we have:

$$\gamma(l_j - 1) \subset \gamma(l_j), \text{ for all } l = \{1, ..., \bar{l}_j\},$$
$$\sum_j \bar{l}_j = T - 1.$$

In this case, the first order conditions give:

$$
\begin{aligned}
\frac{\partial L}{\partial U^i} &= -\pi^i - \lambda^i = 0, \text{ for all } i, \\
&\Longleftrightarrow \lambda^i = -\pi^i, \text{ for all } i, \\
&\Rightarrow q(t_\alpha^i) - \hat{q}(t_\alpha^i) = \frac{1}{\pi^i} \sum_{k>i} \pi^k (t_\alpha^{i+1} - t_\alpha^i),
\end{aligned}
$$

for all $i, i+1, k \in j$ and $\alpha = 1, 2,$

$$\Longleftrightarrow q(t_\alpha^i) - \hat{q}(t_\alpha^i) = \frac{1 - \sum_{k \leq i} \pi^k}{\pi^i} (t_\alpha^{i+1} - t_\alpha^i),$$

for all $i, i+1, k \in j$ and $\alpha = 1, 2.$

As above, $\lambda(q, t^i)$ does not depend on q

$$\lambda(q, t^i) = -(1 - \sum_{k \leq i} \pi^k), \text{ for all } i, i+1, k \in j.$$

As a consequence, the subproblem 2 can be solved by maximizing the virtual surplus:

$$S_v(t^i, q^i) = S(t^i, q^i) - \frac{1 - \sum_{k \leq i} \pi^k}{\pi^i} [u(t^{i+1}, q^i) - u(t^i, q^i)], \text{ for all } i \neq \bar{t}_j,$$
$$S_v(\bar{t}_j, \bar{q}_j) = S(\bar{t}_j, \bar{q}_j) \text{ for all } j.$$

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- In a much more complex solution, in which some types can reach the dummy type with several optimal paths, we can also define several ordered subsets $\Upsilon_1, \Upsilon_2 ... \Upsilon_J$ but

$$\Upsilon_1 \cap \Upsilon_2 \cap ... \cap \Upsilon_J \neq \{t^0\},$$

the intersection of all these ordered subsets is not reduced to a singleton. In this case,

$$\lambda(q, t^i) = \sum_{k \neq i} \sum_j \lambda_j^k$$

depends on q. We can notice that:

$$\pi^i = -\sum_j \lambda_j^i, \text{ for all } i,$$

$$= -\pi^i \sum_j \sigma_j^i, \text{ for all } i,$$

and

$$\sum_j \sigma_j^i = 1, \text{ for all } i$$

which means that:$\lambda_j^i \in [-\pi^i, 0]$, for all i, j. σ_j^i can be seen as the probability that type t^i chooses the optimal path j to reach the dummy type. These constraints on Lagrange multipliers are crucial to determine the general solution of the monopolist's problem. The study of the Regular Symmetric Case in the following subsection is an example. We can also notice that in this kind of solution:

$$\lambda(q, t^i) \geq -(1 - \sum_{k \leq i} \pi^k), \text{ for all } i, i+1, k \in m,$$

$$\lambda(q, t^i) \geq -\sum_{k \neq i} \sum_j \pi^k \sigma_j^k.$$

Then the part of the distortions in resource allocations due to optimal paths starting from t^k is $-\pi^k$ ($\lambda^k = -\pi^k$). Then the part of the distortions in q^i due to type t^k is:

$$-\pi^k \sigma_j^k \in [-\pi^k, 0].$$

As a consequence, we can write distortions as:

$$q(t_\alpha^i) - \hat{q}(t_\alpha^i) = -\sum_{k \neq i} \sum_j \pi^k \sigma_j^k (t_\alpha^{i+1} - t_\alpha^i)$$

where $\pi^k \sigma_j^k \in]0, \pi^k[$ if there exists at least two optimal paths from t^k going through t^i, $\pi^k \sigma_j^k = \pi^k$ if there is only one such path, and $\pi^k \sigma_j^k = 0$ if no paths runs trough t^i (in this case there exists at least one J such that $t^i = t_J$). This generalizes the result of no distortion at the top of the distribution in each Υ_J.

This confirms a result of Rochet-Chone (98) saying that for any solution we can define several ordered subsets Υ_j defining several sets of agents (in our simple model we do not obtain as they do the set of agents who do no participate and get nothing):

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- a set T^0 (reduced to a singleton):

$$T^0 = \{t_j^0, \text{ for all } j\} = \{t^0\}.$$

where t_j^0 is the lowest type in subset Υ_j. The element of T^0 gets a rent $U\_^0$:

$$U\_^0 = 0.$$

- a set $\tilde{T} = \{t^i\}$ for all i such that:

$$U\_^i > 0,$$
$$q(t^i) \neq \hat{q}(t^i).$$

- as set $\bar{T}$

$$\bar{T} = \{\bar{t}_j, \text{ for all } j\},$$

. where $\bar{t}_j$ is the highest type in subset Υ_j. Every element in $\bar{T}$ is such that:

$$U\_^{-j} > 0,$$
$$q(\bar{t}_j) = \hat{q}(\bar{t}_j).$$

5.4 The Regular Symmetric Case in the Equivalent Problem

The monopolist produces two goods with the following cost function:

$$C(q) = \frac{1}{2}(q_1^2 + q_2^2)$$

and wants to sells these two goods to a heterogenous population of four type: We denote these types by letter $i = A, B, C, D$.

$$t^A = (t_1^l, t_2^l), \qquad (\pi^A),$$
$$t^B = (t_1^h, t_2^l), \qquad (\pi^B),$$
$$t^C = (t_1^h, t_2^h), \qquad (\pi^C),$$
$$t^D = (t_1^l, t_2^h), \qquad (\pi^D),$$

and $t_\alpha^h - t_\alpha^l = \Delta t_\alpha$, for $\alpha = 1, 2$. Buyers utilities are quasilinear:

$$U^i = u(t^i, q_1, q_2) - p(q_1, q_2) - E,$$
$$u(t^i, q_1, q_2) = t_1^i q_1^i + t_2^i q_2^i,$$
$$p^i q^i = p_1^i q_1^i + p_2^i q_2^i.$$

This solution is such that, C is attracted to B and D and A attracts B and D. We can then define two optimal paths from C to A, one running through B, the other through D. This defines $\gamma_1^C = \{C \to B \to A\}$ and $\gamma_2^C = \{C \to D \to A\}$. Consequently, we can define two ordered subsets $\Upsilon_1 = \{t^A, t^B, t^C\}$ and $\Upsilon_2 = \{t^A, t^D, t^C\}$ and two expressions for $U\_^C$:

$$U^C(\gamma_1^C) = \Delta t_2 q_2^B U\_^B,$$
$$U^C(\gamma_2^C) = \Delta t_1 q_1^D + U\_^D.$$

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as B and D are immediate successor of C. As A is the immediate successor of B :

$$U\_{}^B = \Delta t_1 q_1^A,$$

and also the immediate successor of D :

$$U\_{}^D = \Delta t_2 q_2^A.$$

Then:

$$U^C(\gamma_1^C) = \Delta t_2 q_2^B + \Delta t_1 q_1^A,$$
$$U^C(\gamma_2^C) = \Delta t_1 q_1^D + \Delta t_2 q_2^A.$$

In this case the monopolist problem can be written as:

$$Max \sum_i \pi^i (u(t^i, q^i) - C(q^i) - U\_{}^i)$$

$$s/c : U\_{}^A = 0, (\lambda^A),$$
$$: U\_{}^B = \Delta t_1 q_1^A, (\lambda^B),$$
$$: U\_{}^D = \Delta t_2 q_2^A, (\lambda^D),$$
$$: U\_{}^C = \Delta t_2 q_2^B + \Delta t_1 q_1^A, (\lambda_1^C),$$
$$: U\_{}^C = \Delta t_1 q_1^D + \Delta t_2 q_2^A. (\lambda_2^C).$$

The constraint on $U\_{}^C$ can also be written as:

$$U\_{}^C = \sigma_1 U^C(\gamma_1^C) + \sigma_2 U^C(\gamma_2^C),$$
$$\sigma_1 + \sigma_2 = 1,$$
$$\sigma_1 = \sigma.$$

The first order conditions give:

$$\frac{\partial L}{\partial U^i} = -\pi^i - \sum_j \lambda_j^i = 0, \text{ for all } i,$$

$$\Longleftrightarrow \lambda^A = -\pi^A,$$
$$\Longleftrightarrow \lambda^B = -\pi^B,$$
$$\Longleftrightarrow \lambda^D = -\pi^D,$$
$$\Longleftrightarrow \lambda_1^C + \lambda_2^C = -\pi^C,$$

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and

$$\frac{\partial L}{\partial q_\alpha^i} = \pi^i \frac{\partial S(t^i, q^i)}{\partial q_\alpha^i} + \sum_{k \neq i} \sum_j \lambda_j^k \frac{\partial U^k(\gamma_j^k)}{\partial q_\alpha^i} = 0, \text{ for all } i \text{ and } \alpha = 1, 2$$

$$\Longleftrightarrow \frac{\partial L}{\partial q_1^A} = \pi^A(t_1^A - q_1^A) + (\lambda^B + \lambda_1^C)\Delta t_1 = 0,$$

$$\Longleftrightarrow \frac{\partial L}{\partial q_2^A} = \pi^A(t_2^A - q_2^A) + (\lambda^D + \lambda_2^C)\Delta t_2,$$

$$\Longleftrightarrow \frac{\partial L}{\partial q_1^B} = \pi^B(t_1^B - q_1^B) = 0,$$

$$\Longleftrightarrow \frac{\partial L}{\partial q_2^B} = \pi^B(t_2^B - q_2^B) + (\lambda_1^C)\Delta t_2 = 0,$$

$$\Longleftrightarrow \frac{\partial L}{\partial q_1^C} = \pi^C(t_1^C - q_1^C) = 0,$$

$$\Longleftrightarrow \frac{\partial L}{\partial q_2^C} = \pi^C(t_2^C - q_2^C) = 0,$$

$$\Longleftrightarrow \frac{\partial L}{\partial q_1^D} = \pi^D(t_1^D - q_1^D) + (\lambda_2^C)\Delta t_1 = 0,$$

$$\Longleftrightarrow \frac{\partial L}{\partial q_2^D} = \pi^D(t_2^D - q_2^D) = 0.$$

To define the solution, we have to determine the value of the non positive Lagrange multipliers. We still have two unknowns, λ_1^C, λ_2^C and only one equation $\lambda_1^C + \lambda_2^C = -\pi^C$. The missing equation is obtained by:

$$U^C(\gamma_1^C) = U^C(\gamma_2^C)$$
$$\Longleftrightarrow \Delta t_2 q_2^B + \Delta t_1 q_1^A = \Delta t_1 q_1^D + \Delta t_2 q_2^A,$$
$$\Longleftrightarrow \Delta t_1(q_1^D - q_1^A) = \Delta t_2(q_2^B - q_2^A).$$

We write quantity as functions of λ_1^C and define its value with the above equation: if paths γ_1^C and γ_2^C are both optimal they provide the same rent.

$$q_1^A = t_1^A + \frac{(-\pi^B + \lambda_1^C)}{\pi^A}\Delta t_1,$$

$$q_2^A = t_2^A + \frac{(-\pi^D - \pi^C - \lambda_1^C)}{\pi^A}\Delta t_2,$$

$$q_2^B = t_2^B + \frac{\lambda_1^C}{\pi^B}\Delta t_2,$$

$$q_1^D = t_1^D + \frac{(-\pi^C - \lambda_1^C)}{\pi^D}\Delta t_1,$$

Solving $\Delta t_1(q_1^D - q_1^A) = \Delta t_2(q_2^B - q_2^A)$ we obtain:

$$\lambda_1^C = -[\frac{\pi^B + \pi^C}{\pi^D}\Delta t_1^2 + \frac{1}{\pi^A}\Delta t_2^2]/[(\frac{1}{\pi^A} + \frac{1}{\pi^D})\Delta t_1^2 + (\frac{1}{\pi^A} + \frac{1}{\pi^B})\Delta t_2^2].$$

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This defines a solution if all λ_j^i are non positive. We obtain the condition already found in chapter 2 according to the non positivity of the Lagrange multipliers:

$$\pi^A \pi^C - \pi^B \pi^D > -\frac{\Delta t_2^2}{\Delta t_1^2} \pi^D (\pi^C + \pi^D).$$

The other conditions that must be checked by the model are such that: If type t^i is attracted by its immediate successor t^{i-1} we have:

$$U\_^i - U\_^{i-1} = (t^i - t^{i-1})q^{i-1},$$

and if t^{i-1} is not attracted to t^i we have:

$$U\_^{i-1} - U\_^i \geq (t^{i-1} - t^i)q^i,$$
$$\Longleftrightarrow U\_^i - U\_^{i-1} < (t^i - t^{i-1})q^i,$$
$$\Longleftrightarrow (t^i - t^{i-1})q^{i-1} < (t^i - t^{i-1})q^i,$$

and we obtain, as in chapter 2:

$$\pi^A \pi^C - \pi^B \pi^D < \frac{\pi^B \pi^D}{\pi^A}.$$

In this solution, $q_1^B, q_1^C, q_2^C, q_2^D$ are equal to the optimal resource allocation. $q_1^A, q_2^A, q_2^B, q_1^D$ are downward distorted. The distortions D_α^i are:

$$D_1^A = \frac{(-\pi^B + \lambda_1^C)}{\pi^A}\Delta t_1 \in \left] \frac{(-\pi^B - \pi^C)}{\pi^A}\Delta t_1, \frac{-\pi^B}{\pi^A}\Delta t_1 \right[,$$

$$D_2^A = \frac{(-\pi^D - \pi^C - \lambda_1^C)}{\pi^A}\Delta t_2 \in \left] \frac{(-\pi^D - \pi^C)}{\pi^A}\Delta t_2, \frac{-\pi^D}{\pi^A}\Delta t_2 \right[,$$

$$D_2^B = \frac{\lambda_1^C}{\pi^B}\Delta t_2 \in \left] -\frac{\pi^C}{\pi^B}\Delta t_2, 0 \right[,$$

$$D_1^D = \frac{(-\pi^C - \lambda_1^C)}{\pi^D}\Delta t_1 \in \left] \frac{-\pi^C}{\pi^D}\Delta t_1, 0 \right[.$$

and for all $i = \{A, B, C, D\}$ and $\alpha = \{1, 2\}$, $D_\alpha^i \leq 0$, since for every optimal path defining an ordered subset, we always have:

$$t_\alpha^{i+1} \geq t_\alpha^i, \text{ for all } i, \alpha.$$

We can explain distortions as follows:

- Type t^B is attracted to t^A, and only t^A,

$$t_1^B > t_1^A,$$
$$t_2^B = t_2^A,$$

then the weight of the distortion on q_1^A due to type t^B's attraction is

$$-\pi^B.$$

Type t^D is attracted to t^A, and only t^A,

$$t_1^D = t_1^A,$$
$$t_2^D > t_2^A,$$

then the weight of the distortion on q_2^A (due to type t^D's attraction is

$$-\pi^D.$$

- Type t^C is attracted to t^B, and also t^D. Then the part of the distortion due to type t^C's attraction is

$$-\pi^C.$$

This weight is split between distortions explained by γ_1^C and γ_2^C. This means that we can define distortions as:

$$D_1^A = \frac{(-\pi^B - \sigma\pi^C)}{\pi^A}\Delta t_1,$$

$$D_2^A = \frac{(-\pi^D - \pi^C(1-\sigma))}{\pi^A}\Delta t_2,$$

$$D_2^B = \frac{-\sigma\pi^C}{\pi^B}\Delta t_2,$$

$$D_1^D = \frac{-\pi^C(1-\sigma)}{\pi^D}\Delta t_1,$$

where $\sigma \in\ ]0,1[$ depends on q,

$$U\_^C = \sigma U^C(\gamma_1^C) + (1-\sigma)U^C(\gamma_2^C),$$

and can be seen as the "probability" that type t^C chooses paths $\gamma_1^C = \{C \rightarrow B \rightarrow A\}$ to reach the dummy type, $(1-\sigma)$ for γ_2^C. This means that a weight $-\sigma\pi^C$ is dedicated to D_2^B and D_1^A and a weight $-(1-\sigma)\pi^C$ is dedicated to D_1^D and D_2^A. Types t^B and t^D have only one path to reach t^A, so the probability that they choose these paths is one. Consequently, the weight $-\pi^B$ is totally dedicated to D_1^A and the weight $-\pi^D$ is totally dedicated to D_2^A.

Even if, on each route, there are only two levels of demand of services, the airline in this case has to propose 3 levels of services on each route. In this particular example, we have the following solution:

$$q_1^C = q_1^B > q_1^D > q_1^A$$
$$q_2^C = q_2^D > q_2^B > q_2^A$$

which means 3 levels of services on each route.

Customers C receive their first best allocation of services on each route. They fly first on routes 1 and 2. Customers B fly first on route 1 and fly "above economy" on route 2. Customers D fly first on route 2 and fly "above economy" on route 1. Customers A fly economy on both routes. We can also think about services that can be provided outside the aircraft to allow the airline to offer the same level of services on board to customers B and A on route 1, and D and A on route 2.

In the following section, we study the set of solutions $s^* = \{a, b...n\}$ already defined in section 4 as cases $\{A,B...N\}$.

6 The Solution (2) and implications for service allocation in the airline industry

In this section we use the set of solutions defined in section 4, in which characteristics are such that:

$$t_1^l = 4, t_1^h = 6,$$
$$\Delta t_1 = 2,$$
$$t_2^l = 4, t_2^h = 5,$$
$$\Delta t_2 = 1,$$

and types are distributed as follows:

$$\pi^A = \pi^C = \frac{\epsilon}{2},$$
$$\pi^B = \pi^D = \frac{(1 - \epsilon)}{2},$$

and

$$\rho \in] -1, 1[.$$

Even if there are only two levels of demand of services (high or low), as the optimal pricing policy requires a form of bundling, the optimal resource allocation often requires 3 levels of services, sometimes 4 on each route.

6.1 Pure Case

A **Pure Case** is a solution such that there is a unique optimal path of length which runs through all of the types defining a unique ordered partition Υ of T. As a consequence, this also defines a unique optimal path for each type and consequently a unique expression for each rent.

Solution i (case I in section 4) is the only Pure Case in the set of solutions. The attractions are as follows: C is attracted to B, who is attracted to D, who is attracted to A. We can then say that $C \succ B \succ D \succ A$. We can thus define:

$$\Upsilon = \{A, D, B, C\}.$$

In this case, we have:

$$U\_^C = \Delta t_2 q_2^B + \Delta t_1 q_1^D - \Delta t_2 q_2^D + \Delta t_2 q_2^A,$$
$$U\_^B = \Delta t_1 q_1^D - \Delta t_2 q_2^D + \Delta t_2 q_2^A,$$
$$U\_^D = \Delta t_2 q_2^A,$$
$$U\_^A = 0.$$

and quantities are:

$$q_1^A = t_1^A,$$

$$q_2^A = t_2^A + \frac{(-\pi^D - \pi^B - \pi^C)}{\pi^A} \Delta t_2$$

$$q_1^D = t_1^D + \frac{(-\pi^B - \pi^C)}{\pi^D} \Delta t_1,$$

$$q_2^D = t_2^D + \frac{\pi^C + \pi^B}{\pi^D} \Delta t_2,$$

$$q_1^B = t_1^B,$$

$$q_2^B = t_2^B + \frac{-\pi^C}{\pi^B} \Delta t_2,$$

$$q_1^C = t_1^C,$$

$$q_2^C = t_2^C.$$

This solution is singular as a quantity is upward distorted. As distortions depend on the predecessor's type, q_2^D is upward distorted: B is attracted to D so distortions in q_α^D are positive when $t_\alpha^D > t_\alpha^B$ and negative otherwise:

$$t_1^D < t_1^B \implies D_1^D < 0,$$

$$t_2^D > t_2^B \implies D_2^D > 0.$$

As a consequence,

$$q_1^C = q_1^B > q_1^A > q_1^D,$$

$$q_2^D > q_2^C > q_2^B > q_2^A.$$

Again in this case, customers C fly first on both routes. Surprisingly, customers D fly "above first" on route 2 and economy on route 1. Customers B fly first on route one and "above economy" on route 2. Customer A fly economy on route 2 and "above economy" on route 2.

6.2   Almost Pure Case

An **Almost Pure Case** is a solution such that there is a single optimal paths to compute the rent of each type. There exist several distinct optimal paths such that every type has one and only optimal path running through it.

Solution c (case C, Regular Asymmetric Case in section 4) is the only Almost Pure Case in the set of solutions. The attractions are as follows: C is attracted to D, who is attracted to A, and B is attracted to A. We can then say that $C \succ D \succ A$ and $B \succ A$. We can thus define: $\Upsilon_1 = \{A, D, C\}$ and $\Upsilon_2 = \{A, B\}$ defining: $\gamma^C = \{C \to D \to A\}, \gamma^D = \{D \to A\}$ and $\gamma^B = \{B \to A\}$.

The set of optimal rents is accordingly as follows:

$$U\_{}^C = \Delta t_1 q_1^D + \Delta t_2 q_2^A,$$

$$U\_{}^B = \Delta t_2 q_1^A,$$

$$U\_{}^D = \Delta t_2 q_2^A,$$

$$U\_{}^A = 0.$$

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This solution is regular and quantities are:

$$q_1^A = t_1^A + \frac{-\pi^B}{\pi^A} \Delta t_2,$$

$$q_2^A = t_2^A + \frac{(-\pi^D - \pi^C)}{\pi^A} \Delta t_2,$$

$$q_1^D = t_1^D + \frac{-\pi^C}{\pi^D} \Delta t_1,$$

$$q_2^D = t_2^D,$$

$$q_1^B = t_1^B,$$

$$q_2^B = t_2^B,$$

$$q_1^C = t_1^C,$$

$$q_2^C = t_2^C.$$

C and B are the highest type in the respective ordered subsets Υ_1 and Υ_2. As a consequence, C and B obtain their first best allocations.

we have the following solution:

$$q_1^C = q_1^B > q_1^D > q_1^A$$

$$q_2^C = q_2^D > q_2^B > q_2^A$$

which means 3 levels of services on each route.

Customers C and B receive their first best allocation of services on each route. Customers C fly first on routes 1 and 2. Customers B fly first on route 1 and fly "above economy" (at his first best level) on route 2. Customers D fly first on route 2 and fly "above economy" on route 1. Customers A fly economy on both routes.

6.3 Regular and Completely Ordered

Such a solution does not exist since we have to rank t^B and t^D. To be able to do so, whether there exists an attraction between them or they can reach each other through t^C without involving any transverse binding ICC. In both cases there is upward distortions: whether t^B or t^D attracts the other, quantities allocated to the attracted one are distorted, one upwards and one downwards as:

$$t_1^B > t_1^D,$$

$$t_2^D > t_2^B;$$

if t^B or t^D can reach the other one through t^C implies that q^C is upward distorted as:

$$t_1^C > t_1^D,$$

$$t_2^C > t_2^B.$$

6.4 Regular and Partially Ordered

Solutions d, e and f, respectively Regular Symmetric Case and the Separable Case, are regular and allow partially type ordering.

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For solution d, see the previous Section.

In solution e, C is attracted to B and D and A attracts B, D and C. We can then define three optimal paths from C to A, one running through B, the other through D, and a third one going directly to A. This defines $\gamma_1^C = \{C \to B \to A\}$, $\gamma_2^C = \{C \to D \to A\}$ and $\gamma_3^C = \{C \to A\}, \gamma^D = \{D \to A\}, \gamma^B = \{B \to A\}$. Consequently, we can define three ordered subsets $\Upsilon_1 = \{t^A, t^B, t^C\}, \Upsilon_2 = \{t^A, t^D, t^C\}$ and $\Upsilon_3 = \{t^A, t^C\}$ and three expressions for $U\_{}^C$.

In solution f, C is attracted to D and A, and A attracts B, D and C. We can then define three optimal paths from C to A, one running through B, the other through D, and a third one going directly to A. This defines $\gamma_1^C = \{C \to D \to A\}$, $\gamma_2^C = \{C \to A\}, \gamma^D = \{D \to A\}, \gamma^B = \{B \to A\}$. Consequently, we can define two ordered subsets $\Upsilon_1 = \{t^A, t^D, t^C\}, \Upsilon_2 = \{t^B, t^C\}$ and two expressions for $U\_{}^C$.

Solutions d,e and f defines a type ordering for each good: for the good 1, $C = B \succ D \succeq A$ and for good 2: $C = D \succ B \succeq A$.

Again, we obtain the following solution:

$$q_1^C = q_1^B > q_1^D > q_1^A$$
$$q_2^C = q_2^D > q_2^B > q_2^A$$

which means 3 levels of services on each route.

Customers C receive their first best allocation of services on each route. They fly first on routes 1 and 2. Customers B fly first on route 1 and fly "above economy" on route 2. Customers D fly first on route 2 and fly "above economy" on route 1. Customers A fly economy on both routes. We can also think about services that can be provided outside the aircraft to allow the airline to offer the same level of services on board to customers B and A on route 1, and D and A on route 2.

6.5 Singular and Completely Ordered

A pure case belongs to this class of solutions.

Solutions a, b, h, i, j, k, m are singular (upward distortions) and completely ordered (there exists an ordered subset Υ_j of four types).

In this set of solutions there are two possible type ordering:

$$C \succ B \succ D \succ A \text{ for solutions } i, j, k,$$
$$B \succ C \succ D \succ A \text{ for solutions } a, b, h.$$

This can be explained as follows:

- solutions a, b, h occur for low values of ϵ equivalent to low proportions of types t^C and t^A and high proportions of types t^B and t^D. It is in the monopolist's interest to extract maximum surplus from types t^B and thus sell them their first best allocations. Even if $\pi^B = \pi^D$ and whatever the solution, neither D is the highest type nor D is superior to B. This comes from the parameters of the model, $\Delta t_1 > \Delta t_2$.

In this case we have,

$$q_1^C = q_1^B > q_1^A > q_1^D,$$
$$q_2^D > q_2^C > q_2^B > q_2^A.$$

Again, customers C fly first on both routes. Surprisingly, customers D fly "above first" on route 2 and economy on route 1. Customers B fly first on route one and "above economy" (at their first best level) on route 2. Customer A fly economy on route 2 and "above economy" on route 2.

- solutions i, j, k are such that B is attracted to D. This enables to rank them and defines the unique and expected type ordering $C \succ B \succ D \succ A$, where C is the highest as $t_\alpha^C \geq t_\alpha^i$ for all α, i. Then B is bigger than D as B has more impact on distortions than D since $\Delta t_1 > \Delta t_2$. At last A is the lowest type as $t_\alpha^A \leq t_\alpha^i$ for all α, i.

In this case we have,

$$q_1^C = q_1^B > q_1^A > q_1^D,$$
$$q_2^D > q_2^C > q_2^B > q_2^A.$$

Again, customers C fly first on both routes. Surprisingly, customers D fly "above first" on route 2 and economy on route 1. Customers B fly first on route one and "above economy" on route 2. Customer A fly economy on route 2 and "above economy" on route 2.

6.6 Singular and Partially Ordered

Solution l is such that:

$$\gamma^C = \{C \to D \to A\},$$
$$\gamma^D = \{D \to A\},$$
$$\gamma_1^B = \{B \to D \to A\},$$
$$\gamma_2^B = \{B \to A\},$$

defining:

$$\Upsilon_1 = \{A, D, C\},$$
$$\Upsilon_2 = \{A, D, B\}.$$

B and C obtain their first best allocations while q_2^D is upwards distorted and q_1^D, q_1^A, q_2^A are downwards distorted. Even we can rank B and D we cannot rank C and D and define a unique type ordering. We have:

$$T^0 = \{t^A\},$$
$$\bar{T} = \{t^D\},$$
$$\bar{T} = \{t^C, t^B\}.$$

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In this case we have,

$$q_1^C = q_1^B > q_1^A > q_1^D,$$
$$q_2^D > q_2^C > q_2^B > q_2^A.$$

Again, customers C fly first on both routes. Surprisingly, customers D fly "above first" on route 2 and economy on route 1. Customers B fly first on route one and "above economy" (at their first best level) on route 2. Customer A fly economy on route 2 and "above economy" on route 2.

6.7 Special cases

Solutions n and g as two types attract each other, B and C.

This kind of solution means that there exists two types such that:

$$U\_^i - U\_^j = (t^i - t^j)q^i,$$
$$U\_^j - U\_^i = -(t^i - t^j)q^j,$$
$$\Longrightarrow q_\alpha^i = q_\alpha^j \text{ for all } (t_\alpha^i - t_\alpha^j) \neq 0.$$

If $(t_\alpha^i - t_\alpha^j) > 0$, this means that q_α^i is upwards distorted and q_α^j is downwards distorted, then we have:

$$q_\alpha^i > \hat{q}_\alpha^i > \hat{q}_\alpha^j > q_\alpha^j,$$

which cannot be satisfied with:

$$q_\alpha^i = q_\alpha^j.$$

As a consequence, $q_\alpha^i = q_\alpha^j = 0$.

In solution g we can define the following ordered subsets:

$$\Upsilon_1 = \{C, B, A\}$$
$$\Longrightarrow C \succ B \succ A,$$

$$\Upsilon_2 = \{B, C, D, A\}$$
$$\Longrightarrow B \succ C \succ D \succ A.$$

In this particular case where B is at the same time the immediate successor, in Υ_1, and the immediate predecessor, in Υ_2, of C . We have:

$$U\_^C - U\_^B = \Delta t_2 q_2^B,$$
$$U\_^B - U\_^C = -\Delta t_2 q_2^C,$$

and consequently:

$$q_2^C = q_2^B,$$
$$= q_2^A.$$

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In this solution optimal paths γ_j^i define the rents of B and C are as follows:

$$
\begin{aligned}
U\_^C &= \Delta t_2 q_2^B + \Delta t_1 q_1^A, \text{ with probability } \sigma_1^C, \\
&= \Delta t_2 q_2^A + \Delta t_1 q_1^A, \text{ with probability } \sigma_2^C, \\
&= \Delta t_2 q_2^A + \Delta t_1 q_1^D, \text{ with probability } \sigma_3^C, \\
U\_^B &= \Delta t_1 q_1^A, \text{ with probability } \sigma_1^B, \\
&= -\Delta t_2 q_2^C + \Delta t_2 q_2^A + \Delta t_1 q_1^A, \text{ with probability } \sigma_2^B, \\
&= -\Delta t_2 q_2^C + \Delta t_2 q_2^A + \Delta t_1 q_1^D, \text{ with probability } \sigma_3^B.
\end{aligned}
$$

As $\gamma_2^C \subset \gamma_2^B$ and $\gamma_3^C \subset \gamma_3^B$ we have:$\sigma_2^C = \sigma_2^B$ and $\sigma_3^C = \sigma_3^B$ and consequently: $\sigma_1^C = \sigma_1^B$. Then we have:

$$
\begin{aligned}
q_2^C &= t_2^C + \Delta t_2 \frac{(\sigma_2^B + \sigma_3^B)\pi^B}{\pi^C} + \Delta t_2 \frac{\sigma_1^B \pi^C}{\pi^C}, \\
q_2^B &= t_2^B - \Delta t_2 \frac{\sigma_1^B \pi^C}{\pi^B} - \Delta t_2 \frac{(\sigma_2^B + \sigma_3^B)\pi^B}{\pi^B}
\end{aligned}
$$

where the second term of each distortion is explained by mutual attractions, which means according to distortions that are usually computed as:

$$
q(t_\alpha^i) - \hat{q}(t_\alpha^i) = -\sum_{k \neq i}\sum_j \pi^k \sigma_j^k (t_\alpha^{i+1} - t_\alpha^i),
$$

are computed:

$$
q(t_\alpha^i) - \hat{q}(t_\alpha^i) = -\sum_{k}\sum_j \pi^k \sigma_j^k (t_\alpha^{i+1} - t_\alpha^i),
$$

that we make the sum for all k as $q_\alpha^k = q_\alpha^i$ when k and i attract each other and $t_\alpha^k - t_\alpha^i \neq 0$. $q_2^C = q_2^B$ implies:

$$
\frac{(\sigma_2^B + \sigma_3^B)\pi^B}{\pi^C} + \frac{\sigma_1^B \pi^C}{\pi^B} = 0
$$

which is impossible. If we neglect closed paths (and mutual attractions), using:

$$
q(t_\alpha^i) - \hat{q}(t_\alpha^i) = -\sum_{k \neq i}\sum_j \pi^k \sigma_j^k (t_\alpha^{i+1} - t_\alpha^i),
$$

this second part of the distortions does not appear and the condition becomes:

$$
\frac{(\sigma_2^B + \sigma_3^B)\pi^B}{\pi^C} + \frac{\sigma_1^B \pi^C}{\pi^B} = -1
$$

which is also impossible. This defines a solution in which the monopolist only sells good 1 as:

$$
q_2^C = q_2^B = q_2^A = 0.
$$

In this case:

$$
\begin{aligned}
T^0 &= \{t^A, t^D\} = \{t_1^l\}, \\
\tilde{T} &= \emptyset, \\
\bar{T} &= \{t^B, t^C\} = \{t_1^h\}
\end{aligned}
$$

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as in the classic one dimensional case of course since only one good is produced. This does not means that good 2 is not profitable. This means that there is no optimal mechanism enabling the monopolist to fully apply his monopoly power and he is better off selling both goods independently.

We then obtain the following solution (see separable case in section 4),

$$q_1^C = q_1^B > q_1^D = q_1^A,$$
$$q_2^C = q_2^D > q_2^B = q_2^A.$$

Again, customers C fly first on both routes. Customers D fly first on route 2 and economy on route 1. Customers B fly first on route one and economy on route 2. Customer A fly economy on both routes.

Solution n is equivalent as B and C attract each other implying:

$$q_2^C = q_2^B,$$

implying the sum of strictly positive terms to be negative,

$$\frac{\pi^C}{\pi^B} + \frac{\pi^B}{\pi^C} = -1$$

if we neglect mutual attraction, which is of course impossible. Computing utilities:

$$U\_^C = \Delta t_2 q_2^A + \Delta t_1 q_1^A, \text{with probability } \sigma_1^C,$$
$$= \Delta t_2 q_2^A + \Delta t_1 q_1^D, \text{with probability } \sigma_2^C,$$
$$U\_^B = -\Delta t_2 q_2^C + \Delta t_2 q_2^A + \Delta t_1 q_1^A, \text{with probability } \sigma_1^B,$$
$$= -\Delta t_2 q_2^C + \Delta t_2 q_2^A + \Delta t_1 q_1^D, \text{with probability } \sigma_2^B,$$

as C is attracted to B but cannot reach A as B is not attracted to A. This creates a closed paths, $C \rightarrow B \rightarrow C$. Again:

$$q_2^i = 0, \text{ for all } i.$$

We then obtain the same solution as in case g.

In both cases, there is no optimal selling mechanism enabling the monopolist to fully apply his monopoly power. In this case, as we assume that both routes are profitable, the airline will market these routes independantly.

7 Conclusion

In this paper we have derived the complete solution of a two-product monopolist in the 2*2 case with a quadratic cost function and linear preferences. We have seen that a solution with the same qualitative features than the one in dimension one does not exists for all parameters neither for all distributions.

In the case where there is no cross cost parameter, we have discovered four possible solutions depending on the correlation between types, as in Armstrong-Rochet (98), whereas Dana (93) in a different context, but a similar setting, finds only two solutions. One of the most striking

result is the solution for strongly negative correlation, the singular case: there is an upward binding ICC (B towards C and not D towards C because $\Delta t_1 \geq \Delta t_2$) and a transverse one (B towards D because $\Delta t_1 \geq \Delta t_2$). As a consequence, the highest type C receives an upwards distorted allocation, whereas an "intermediary" type, B, gets his first best allocation (and also in the regular asymmetric case). The type who has the highest indirect utility level does not always get his first best allocation(see type C in the singular case), and is not necessarily the only one to get it.

When we take ρ into account several other solutions appear each one having its own properties. We know that D is always attracted to A and never to B and C. The monopolist always captures all the surplus of the lowest type (A) and the others always obtain a strictly positive surplus.

We also show that the bidimensional monopolist's problem has solutions such that:

- the optimal pricing mechanism often requires a form of bundling

- it is sometimes impossible for the monopolist to apply his monopoly power, and he is sometimes better off selling both goods independently,

- if all the quantities served by the monopolist are positive, there is no closed path or mutual attraction and the set of types who get no rent is reduced to a singleton, the dummy type,

- distortions depend on the number of optimal paths and also on their lengths, especially the number of types involved in these optimal paths,

- the sign of the distortions in types t^is allocations only depends on their immediate predecessors,

- the rule of no distortion at the top of the distribution is always satisfied.

In this particular bidimensional setting we succeed in finding the global solution as there are only two arguments, two characteristics and four types. We are also able to solve the problem by maximizing the expected virtual surplus for Pure and Almost Pure cases. In a general multidimensional setting, solving the problem is always feasible as we have enough equations to determine all the unknowns. But to do so we have to be able to answer the question: What is the set of optimal paths that defines the solution?

On the airline side, empirical research is needed on route monopolies. Further research should also be undertaken on network monopolies connecting secondary airports.

8 Graphs and Figures

9 Appendices

9.1 Appendix 1

Conditions on quantities for the Regular Symmetric Case:

$$U^B - U^A = \Delta t_1 q_1^A,$$
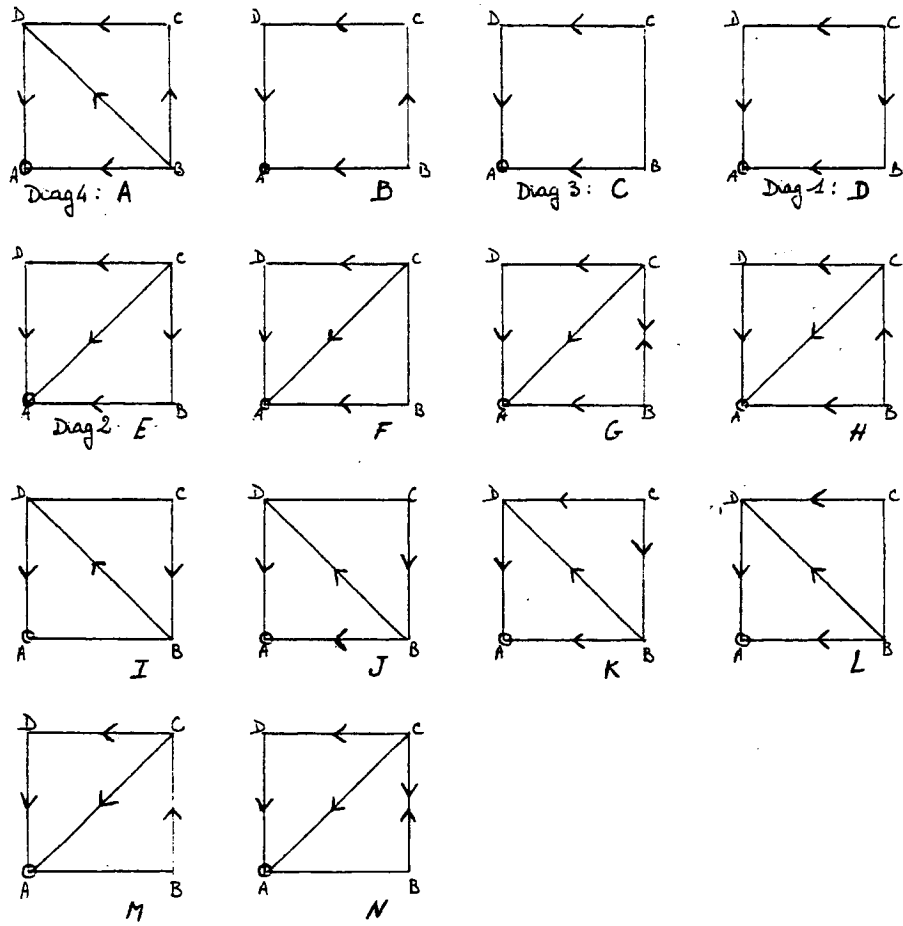$$U^A - U^B \geq -\Delta t_1 q_1^B,$$
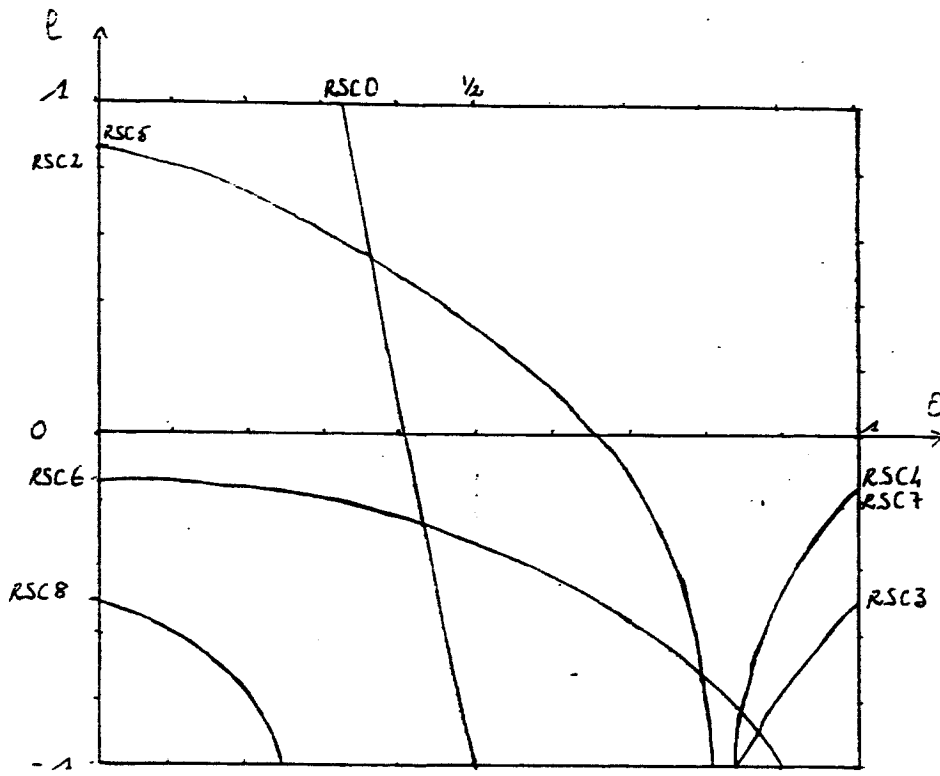$$\Leftrightarrow q_1^B \geq q_1^A.$$

Figure 1:

Figure 1
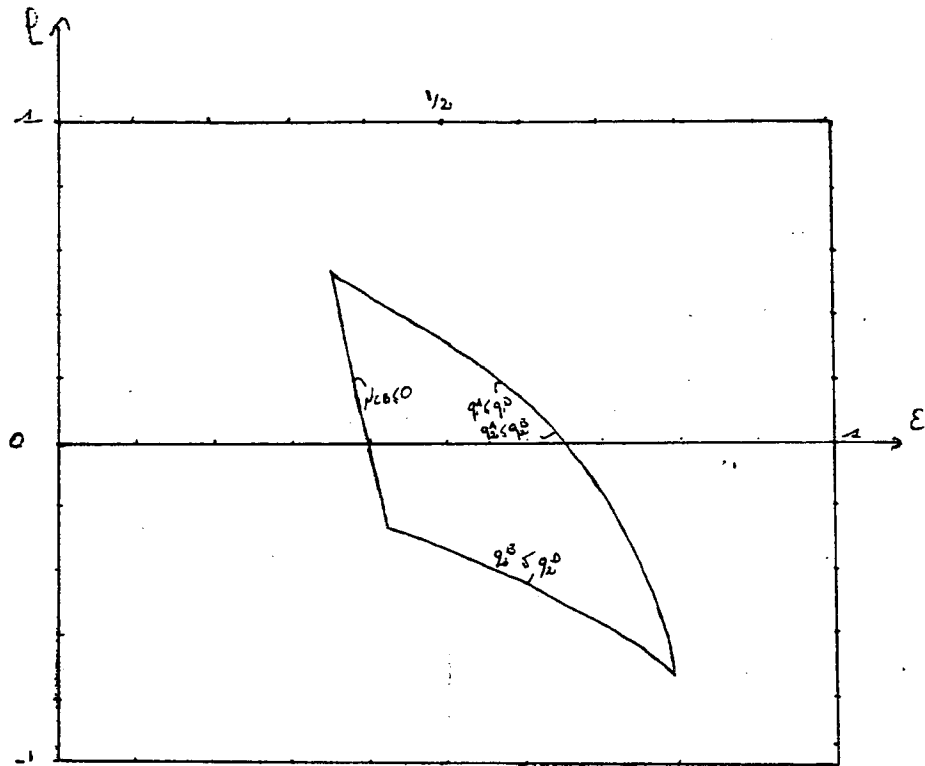

Figure 2:

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Figure 2

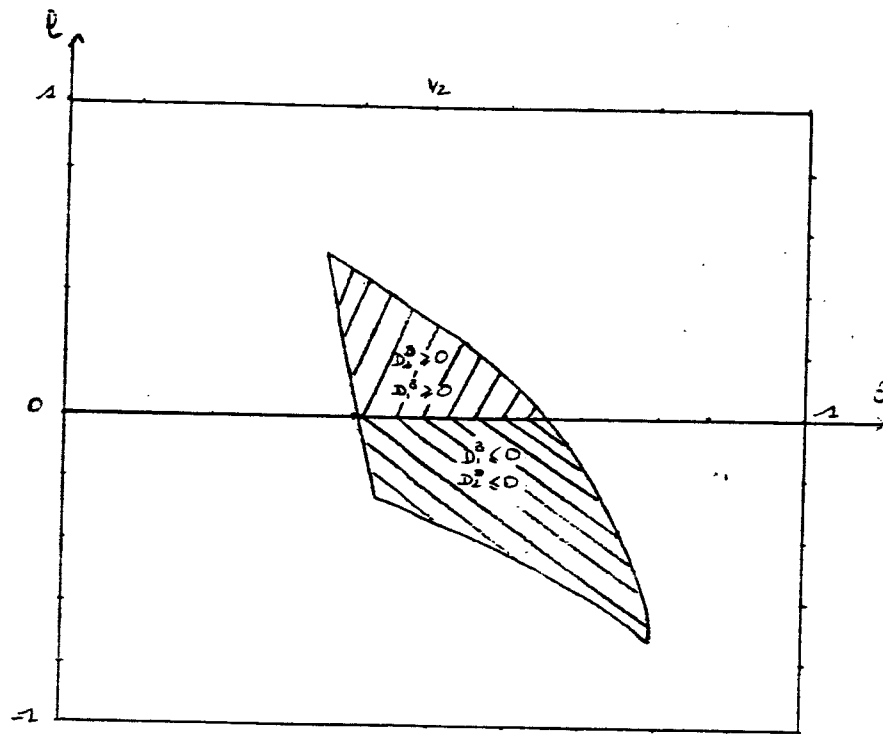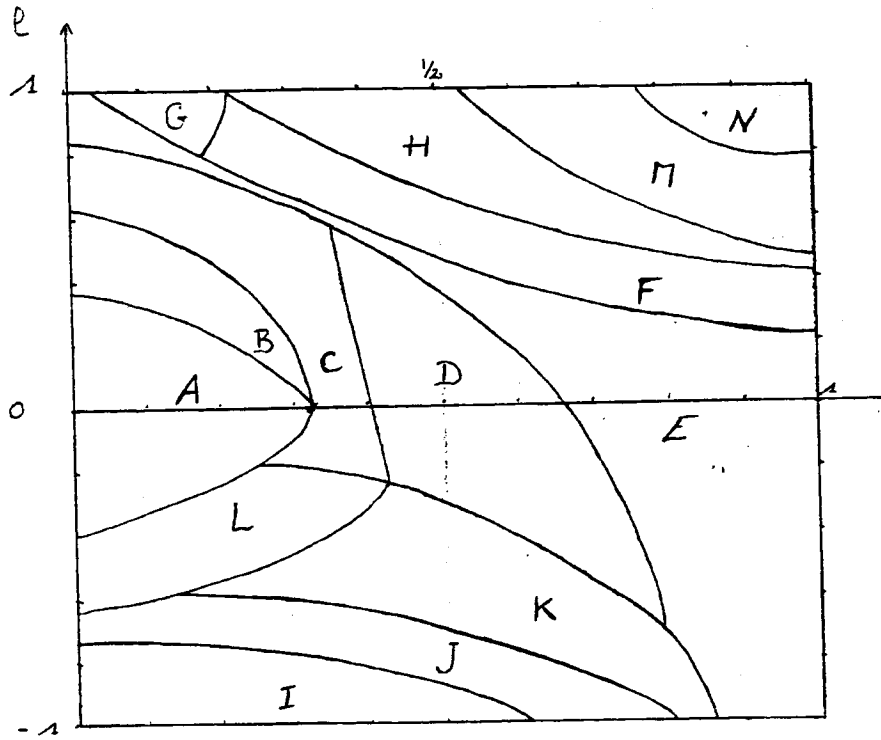

Figure 3:

Figure 3

Figure 4:

Figure 4

Figure 5:

$$U^D - U^A = \Delta t_2 q_2^A$$

,

$$U^A - U^D \geq -\Delta t_2 q_2^D$$

,

$$\Leftrightarrow q_2^D \geq q_2^A$$

.

$$U^C - U^A = \Delta t_1 q_1^D + \Delta t_2 q_2^A$$

$$U^C - U^A = \Delta t_1 q_1^A + \Delta t_2 q_2^B$$

$$U^C - U^A \geq \Delta t_1 q_1^A + \Delta t_2 q_2^A$$

$$\Leftrightarrow q_1^D \geq q_1^A, \text{et } q_2^B \geq q_2^A.$$

$$U^C - U^A \leq \Delta t_1 q_1^C + \Delta t_2 q_2^C$$

$$U^C - U^B = \Delta t_2 q_2^B$$

$$U^B - U^C \geq -\Delta t_2 q_2^C$$

$$\Leftrightarrow q_2^C \geq q_2^B$$

$$U^C - U^D = \Delta t_1 q_1^D$$

$$U^D - U^C \geq -\Delta t_1 q_1^C$$

$$\Leftrightarrow q_1^C \geq q_1^D$$

$$U^B - U^D = \Delta t_1 q_1^D - \Delta t_2 q_2^B$$

$$U^B - U^D \geq \Delta t_1 q_1^D - \Delta t_2 q_2^D$$

$$\Leftrightarrow q_2^D \geq q_2^B$$

$$U^D - U^B \geq -\Delta t_1 q_1^B + \Delta t_2 q_2^B$$

$$\Leftrightarrow q_1^B \geq q_1^D$$

9.2   Appendix 2

We write here the F.O.C of the problem in the 2*2 case when buyers are uniformly distributed. In this case we obtain the following equations for the optimal quantities:

$$q_1^A = \frac{1}{(1-\rho^2)}[t_1^A + 2\Delta t_1 \frac{\mu_{BA}+\mu_{CA}}{\epsilon} - \rho(t_2^A + 2\Delta t_2 \frac{-\mu_{CA}-\mu_{DA}}{\epsilon})]$$
$$q_1^B = \frac{1}{(1-\rho^2)}[t_1^B + 2\Delta t_1 \frac{-\mu_{AB}-\mu_{DB}}{1-\epsilon} - \rho(t_2^B + 2\Delta t_2 \frac{-\mu_{CB}-\mu_{DB}}{1-\epsilon})]$$
$$q_1^C = \frac{1}{(1-\rho^2)}[t_1^C + 2\Delta t_1 \frac{-\mu_{AC}-\mu_{DC}}{\epsilon} - \rho(t_2^C + 2\Delta t_2 \frac{\mu_{BC}+\mu_{AC}}{\epsilon})]$$
$$q_1^D = \frac{1}{(1-\rho^2)}[t_1^D + 2\Delta t_1 \frac{\mu_{BD}+\mu_{CD}}{1-\epsilon} - \rho(t_2^D + 2\Delta t_2 \frac{\mu_{AD}+\mu_{BD}}{1-\epsilon})]$$
$$q_2^A = \frac{1}{(1-\rho^2)}[t_2^A + 2\Delta t_2 \frac{\mu_{CA}+\mu_{DA}}{\epsilon} - \rho(t_1^A + 2\Delta t_1 \frac{-\mu_{BA}-\mu_{CA}}{\epsilon})]$$
$$q_2^B = \frac{1}{(1-\rho^2)}[t_2^B + 2\Delta t_2 \frac{\mu_{CB}+\mu_{DB}}{1-\epsilon} - \rho(t_1^B + 2\Delta t_1 \frac{\mu_{AB}+\mu_{DB}}{1-\epsilon})]$$
$$q_2^C = \frac{1}{(1-\rho^2)}[t_2^C + 2\Delta t_2 \frac{-\mu_{BC}-\mu_{AC}}{\epsilon} - \rho(t_1^C + 2\Delta t_1 \frac{\mu_{AC}+\mu_{DC}}{\epsilon})]$$
$$q_2^D = \frac{1}{(1-\rho^2)}[t_2^D + 2\Delta t_2 \frac{-\mu_{AD}-\mu_{BD}}{1-\epsilon} - \rho(t_1^D + 2\Delta t_1 \frac{(-\mu_{BD}-\mu_{CD})}{1-\epsilon})]$$

We have the following equations for the Lagrange multipliers of the problem:

$$-\pi^A - \lambda^A - \mu_{AB} - \mu_{AC} - \mu_{AD} + \mu_{BA} + \mu_{CA} + \mu_{DA} = 0,$$
$$-\pi^B - \lambda^B - \mu_{BA} - \mu_{BC} - \mu_{BD} + \mu_{AB} + \mu_{CB} + \mu_{DB} = 0,$$
$$-\pi^C - \lambda^C - \mu_{CA} - \mu_{CB} - \mu_{CD} + \mu_{AC} + \mu_{BC} + \mu_{DC} = 0,$$
$$-\pi^D - \lambda^D - \mu_{DA} - \mu_{DB} - \mu_{DC} + \mu_{AD} + \mu_{BD} + \mu_{CD} = 0.$$

To derive the complete solution we replicate the technic used in section 4. We take a solution and find the frontiers(the conditions on quantities and the sign of μ_{ij}s). We then look for a solution if one of these conditions fails. We then find the binding constraints and then we check the conditions of existence of a solution. We always obtain a system of P equations with P unknowns. We have N equations $(\frac{\partial U}{\partial q})$ and the missing equations are equality conditions on quantities, obtained by examining the attractions and the different possible paths from one type to another. And so on...(see figure 4). This has been done with Mathematica 3.0 (a mathematical software).

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