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A Universal Formula for Extracting the Euler Angles

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Introduction

Recently, the authors completed a study[1] of the Davenport angles, which are a generalization of the Euler angles for which the initial and final Euler axes need not be either mutually parallel or mutually perpendicular or even along the coordinate axes. During the conduct of that study, those authors discovered a relationship which can be used to compute straightforwardly the Euler angles characterizing a proper-orthogonal direction-cosine matrix for an arbitrary Euler-axis set satisfying

$$\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2 = 0 \quad \text{and} \quad \hat{\mathbf{n}}_3 \cdot \hat{\mathbf{n}}_1 = 0 , \tag{1}$$

which is also satisfied by the more usual Euler angles we encounter commonly in the practice of Astronautics. Rather than leave that relationship hidden in an article with very different focus from the present Engineering note, we present it and the universal algorithm derived from it for extracting the Euler angles from the direction-cosine matrix here. We also offer literal "code" for performing the operations, numerical examples, and general considerations about the extraction of Euler angles which are not universally known, particularly, the treatment of statistical error.

Development of the Formula

Let $\hat{\mathbf{n}}_1$, $\hat{\mathbf{n}}_2$, $\hat{\mathbf{n}}_3$ be a sequence of three (unit) Euler axes, and φ, ϑ, ψ be the associated Euler angles. Then the direction-cosine matrix D corresponding to these

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Euler axes and Euler angles is given by

$$D = R(\hat{\mathbf{n}}_3, \psi)\, R(\hat{\mathbf{n}}_2, \vartheta)\, R(\hat{\mathbf{n}}_1, \varphi) \equiv R(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_3\,;\, \varphi, \vartheta, \psi) \tag{2}$$

where $R(\hat{\mathbf{n}}, \theta)$ denotes the direction-cosine matrix[2] of a rotation about an axis $\hat{\mathbf{n}}$ through an angle θ. Davenport[3] and Ref. 1 showed that such Euler angles exist for any proper-orthogonal D and any set of Euler axes satisfying Eq. (1). Thus, in this work, our interest is not limited to Euler axes chosen from the set $\mathcal{E} \equiv \{\pm\hat{\mathbf{1}}, \pm\hat{\mathbf{2}}, \pm\hat{\mathbf{3}}\}$, where

$$\hat{\mathbf{1}} \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{2}} \equiv \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \hat{\mathbf{3}} \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{3}$$

although in practice this is the case which occurs with greatest frequency.

The relationship discovered in Ref. 1 is

$$\begin{aligned} D &= C^T R(\hat{\mathbf{1}}, \lambda)\, R(\hat{\mathbf{3}}, \hat{\mathbf{1}}, \hat{\mathbf{3}}\,;\, \varphi, \vartheta - \lambda, \varphi)\, C \\ &\equiv C^T R(\hat{\mathbf{1}}, \lambda)\, R_{313}(\varphi, \vartheta - \lambda, \varphi)\, C \end{aligned} \tag{4}$$

with

$$\lambda = \arctan_2[(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2) \cdot \hat{\mathbf{n}}_3,\ \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_3] \tag{5}$$

$$C = [\,\hat{\mathbf{n}}_2 \quad (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2) \quad \hat{\mathbf{n}}_1\,]^T \tag{6}$$

where the matrix in Eq. (6) has been indicated by its column vectors, and $\arctan_2(y, x)$ returns the value of $\tan^{-1}(y/x)$ in the correct quadrant.

Writing

$$O \equiv R_{313}(\varphi, \vartheta - \lambda, \psi) \tag{7}$$

we can solve Eq. (4) as

$$O = R^T(\hat{\mathbf{1}}, \lambda)\, C\, D\, C^T \tag{8}$$

It is considerably easier to extract the values of $(\varphi, \vartheta, \psi)$ from O than from D directly.

From the familiar formula

$$R_{313}(\varphi, \vartheta', \psi) = \begin{bmatrix} c\psi\, c\varphi - s\psi\, c\vartheta'\, s\varphi & c\psi\, s\varphi + s\psi\, c\vartheta'\, c\varphi & s\psi\, s\vartheta' \\ -s\psi\, c\varphi - c\psi\, c\vartheta'\, s\varphi & -s\psi\, s\varphi + c\psi\, c\vartheta'\, c\varphi & c\psi\, s\vartheta' \\ s\vartheta'\, s\varphi & -s\vartheta'\, c\varphi & c\vartheta' \end{bmatrix}. \tag{9}$$

with $c\varphi \equiv \cos\varphi$, $s\varphi \equiv \sin\varphi$, etc., and $\vartheta' = \vartheta - \lambda$, we have immediately

$$\vartheta = \lambda + \cos^{-1} O_{33} \tag{10a}$$

and, for $\lambda < \vartheta < \lambda + \pi$,

$$\varphi = \arctan_2(O_{31}, -O_{32}) \quad \text{and} \quad \psi = \arctan_2(O_{13}, O_{23}) \tag{10bc}$$

For $\vartheta = \lambda$ or $\vartheta = \lambda + \pi$ the arguments of Eqs. (10b) and (10c) all vanish, and the two equations have no unique solution for φ and ψ. In those two special cases,

O depends only on $\varphi - \psi$ or $\varphi + \psi$, respectively. Thus, for $\vartheta = \lambda$ one can write at best

$$\varphi - \psi = \arctan_2(O_{12} - O_{21}, O_{11} + O_{22}) \tag{11a}$$

and for $\vartheta = \lambda + \pi$

$$\varphi + \psi = \arctan_2(O_{12} + O_{21}, O_{11} - O_{22}) \tag{11b}$$

Typically, in these cases, one sets $\psi = 0$. Equations (11) are much better behaved numerically than the usual formulas[2] with slightly simpler arguments in the $\arctan_2$ functions. One or the other of these equations are also better behaved numerically than Eqs. (10bc) *near* $\theta = \lambda$ or $\theta = \lambda + \pi$, respectively. Unfortunately, there are no available numerically well-behaved equations for both φ and ψ in these regions.

If the Euler axes are chosen from $\mathcal{E}$, or from any orthonormal set of 3×1 matrices, then λ may take on the value $-\pi/2$, 0, $\pi/2$, or π (mod 2π). When this makes the range of ϑ inconvenient the angles may be replaced by their equivalents according to[1]

$$(\varphi, \vartheta, \psi) \longleftrightarrow (\varphi + \pi, 2\lambda - \vartheta, \psi - \pi) \mod 2\pi \tag{12}$$

Frequently one desires that ϑ be in the range $0 \le \vartheta \le \pi$.

Thus, given D, $\hat{\mathbf{n}}_1$, $\hat{\mathbf{n}}_2$, $\hat{\mathbf{n}}_3$, the algorithm for extracting the Euler angles from the (proper-orthogonal) direction-cosine matrix is:

Given D, $\hat{\mathbf{n}}_1$, $\hat{\mathbf{n}}_2$, $\hat{\mathbf{n}}_3$:

- Set observability flag to "poor."
- Compute λ and C from Eqs. (5) and (6)
- Compute O from Eq. (8)
- Compute ϑ from Eq. (10a)
- If $|\vartheta - \lambda| \ge \epsilon$ and $|\vartheta - \lambda - \pi| \ge \epsilon$, ($\epsilon$ is machine- and problem-dependent)
  - Set observability flag to "good."
  - Compute φ and ψ from Eqs. (10b) and (10c)
- Else
  - Set $\psi = 0$
  - If $|\vartheta - \lambda| < \epsilon$, compute φ from Eq. (11a)
  - If $|\vartheta - \lambda - \pi| < \epsilon$, compute φ from Eq. (11b)
- Adjust angles according to Eq. (12) if necessary
- The outputs are φ, ϑ, ψ, and the observability flag.

Note that the tests above refer to the value of the argument (mod 2π) which is smallest.

Our result should be compared to that of Kolve,[5] who, instead of performing analytical operations on the direction-cosine matrix, does a special accounting of the indices. Kolve's method is applicable only to Euler axes which are parallel to the coordinate axes. Thus, Kolve's method cannot be applied to the second of the numerical examples below.

We note that φ and ψ cannot be calculated unambiguously using only Eqs. (11a) and (11b). The solution of each of the two equations yield numerical results for

$$\varphi - \psi + 2m\pi \qquad \text{and} \qquad \varphi + \psi + 2n\pi$$

respectively, where m and n are integers. By taking linear combinations of these two quantities, we can obtain numerical results for

$$\varphi + (m+n)\pi \qquad \text{and} \qquad \psi - (m-n)\pi$$

and neither $m+n$ nor $m-n$ need be even integers. Thus, both φ and $\varphi+\pi$ and both ψ and $\psi+\pi$ are possible solutions, which is unacceptable. It follows that we cannot use Eq. (11a) and Eq. (11b) alone to calculate φ and ψ.

Instead of the program we have given following Eq. (12) we can use Eq. (11a) when $O_{33} \geq 0$ to solve for $\varphi-\psi$ or Eq. (11b) when $O_{33} < 0$ to solve for $\varphi+\psi$ and to supplement either of these solutions with that for φ or ψ from either Eq. (10a) or Eq. (10b), respectively. The resulting φ and ψ will not suffer from the ambiguity of multiples of π, but only from the usual ambiguity of multiples of 2π, which causes us no distress.

When $|\vartheta-\lambda|$ is close to 0 or π, the alternate method will yield an accurate value for $\varphi \pm \psi$ for one choice of the sign, but the value for the solution from Eq. (10a) or Eq. (10b) (and its accuracy) will be the same as for the program above. When this is combined with the $\varphi \pm \psi$ to obtain the remaining angle, that angle will suffer then from the same lack of significance as that calculated from this work's proposed program. Thus, whether one uses the program proposed above for the individual angles or the alternate method is purely a matter of esthetic taste.

Numerical Examples

Example 1

As a simple example, consider the computation of φ, ϑ, and ψ for a 3-1-2 set of Euler axes with true values $\varphi = 45$ deg, $\vartheta = 30$ deg and $\psi = 20$ deg. The resulting direction-cosine matrix is

$$D = R(\hat{\mathbf{2}}, 20\text{ deg})\, R(\hat{\mathbf{1}}, 30\text{ deg})\, R(\hat{\mathbf{3}}, 45\text{ deg})$$

$$= \begin{bmatrix} 0.5435 & 0.7854 & -0.2962 \\ -0.6124 & 0.6124 & 0.5000 \\ 0.5741 & -0.0904 & 0.8138 \end{bmatrix} \tag{13}$$

One finds straightforwardly that

$$C = I_{3\times 3} \qquad \text{and} \qquad \lambda = \arctan_2(1, 0) = \pi/2 \tag{14}$$

Performing the multiplications of Eq. (8) yields

$$O = \begin{bmatrix} 0.5435 & 0.7854 & -0.2962 \\ -0.5741 & 0.0904 & -0.8138 \\ -0.6124 & 0.6124 & 0.5000 \end{bmatrix} \tag{15}$$

and applying Eqs. (10) yields

$$\varphi = 45.0000, \qquad \vartheta = 30.0000, \qquad \psi = 20.0000. \tag{16}$$

as expected. Deviations occur only in the 14th decimal place.

Example 2

To appreciate the power of this algorithm consider the following more complex example example:

$$\hat{\mathbf{n}}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \qquad \hat{\mathbf{n}}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \qquad \hat{\mathbf{n}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{17}$$

The Euler-axis set is orthonormal but not proper orthonormal, and two of the axes are certainly not along body coordinate axes. Let the direction-cosine matrix be

$$D = R(\hat{\mathbf{n}}_3, 20 \text{ deg})\, R(\hat{\mathbf{n}}_2, 30 \text{ deg})\, R(\hat{\mathbf{n}}_1, 45 \text{ deg}) = \begin{bmatrix} 0.9929 & 0.1171 & 0.0216 \\ -0.0887 & 0.6063 & 0.7903 \\ 0.0795 & -0.7866 & 0.6124 \end{bmatrix} \tag{18}$$

Then

$$C = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & -1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \tag{19a}$$

and

$$\lambda = \arctan_2(-1, 0) = -\pi/2 \tag{19b}$$

Performing the multiplications of Eq. (8) yields

$$O = \begin{bmatrix} 0.7854 & 0.5435 & 0.2962 \\ 0.0904 & -0.5741 & 0.8138 \\ 0.6124 & -0.6124 & -0.5000 \end{bmatrix} \tag{20}$$

and applying Eqs. (10) yields

$$\varphi = 45.0000, \qquad \vartheta = 30.0000, \qquad \psi = 20.0000. \tag{21}$$

as expected. Again, deviations from the input values occur only in the 14th decimal place.

Note that although the Euler angles have the same values in the two examples, the direction-cosine matrices O are not identical. This is because the middle angle for O is not ϑ but $\vartheta - \lambda$ and λ has different values in the two examples. Also, ϑ from the first example had a value outside the interval $0 \leq \vartheta \leq \pi$ and required adjustment according to Eq. (12). Note the similarities (if not equality) of the matrix entries, although they may differ by a sign and not always be in the same place. These similarities are due to the fact that In our examples above λ has the value $\pi/2$ or $-\pi/2$ so that the transformation of Eq. (12) is of a rather trivial sort. Had the two λ had very different values from multiples of $\pi/2$, then the similarity of the matrix elements might not be present. That situation will occur, however, only when the three rotation axes are chosen from a non-orthonormal set (but satisfying Eq. (1)), in which case we are dealing not with Euler angles but with the Davenport angles.[1,3]

Statistical Considerations

Since radiation-hardened spacecraft computers are routinely available now which implement IEEE Standard 754 for double precision in its numerical computations, questions of numerical precision are not the deciding factor in choosing the value of ϵ but rather estimation accuracy. However, since the direction-cosine matrix usually arises from an estimation process, it is of interest to study how the estimation errors affect the extraction of Euler angles.

The covariance matrix of the attitude is best represented in terms of the attitude increment vector $\Delta\xi$, which we now define.[2] Let D^* be the estimated direction-cosine matrix and D^{true} the true direction-cosine matrix. Then the two can be related by a very small rotation according to

$$D^* = \left(I_{3\times3} + [[\Delta\xi]]\right) D^{\text{true}} + O(|\Delta\xi|^2) \tag{22}$$

with

$$[[\Delta\xi]] \equiv \begin{bmatrix} 0 & \Delta\xi_3 & -\Delta\xi_2 \\ -\Delta\xi_3 & 0 & \Delta\xi_1 \\ \Delta\xi_2 & -\Delta\xi_1 & 0 \end{bmatrix} \tag{23}$$

The attitude increment vector $\Delta\xi$ is thus the rotation vector of a very small rotation. The attitude covariance matrix is best defined as the covariance matrix of $\Delta\xi$. Thus

$$P_{\xi\xi} \equiv E\{\Delta\xi\Delta\xi^T\} \tag{24}$$

where $E\{\cdot\}$ denotes the expectation. This definition of the covariance matrix has the advantage of being independent of the choice of primary reference axes and transforms in the usual way under a change of the body axes. It is immune to the diseases which effect the covariance matrix expressed in terms of the Euler angles, which we will encounter below.

The statistical errors in the Euler angles are given by the 3×1 array

$$\Delta\phi \equiv \begin{bmatrix} \Delta\varphi \\ \Delta\vartheta' \\ \Delta\psi \end{bmatrix} = \begin{bmatrix} \varphi^* - \varphi^{\text{true}} \\ \vartheta'^* - \vartheta'^{\text{true}} \\ \psi^* - \psi^{\text{true}} \end{bmatrix} \tag{25}$$

where again $\vartheta' = \vartheta - \lambda$ appears in the formulas, because this, effectively, is the quantity which we are extracting ultimately from the direction-cosine matrix. This error vector in the Euler angles can be related to the attitude increment vector by

$$\Delta\phi = M^{-1}(\varphi, \vartheta', \psi)\,\Delta\xi \tag{26}$$

which is Eq. (412) of Ref. 2. For a 3-1-3 set of Euler angles (Eq. (300) of Ref. 2)

$$M^{-1}(\varphi, \vartheta', \psi) = \frac{1}{\sin\vartheta'} \begin{bmatrix} \sin\psi & \cos\psi & 0 \\ \sin\vartheta'\cos\psi & -\sin\vartheta'\cos\psi & 0 \\ -\cos\vartheta'\sin\psi & -\cos\vartheta'\cos\psi & \sin\vartheta' \end{bmatrix} \tag{27}$$

which may be factored as

$$M^{-1}(\varphi, \vartheta', \psi) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -\cot\vartheta' & 1 \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{28}$$

which clearly becomes infinite as ϑ' approaches 0 or π. Note that the second matrix in Eq. (28) is proper orthogonal.

For the sake of example, let us assume that

$$P_{\xi\xi} = \sigma^2 I_{3\times3} \tag{29}$$

then

$$P_{\phi\phi} = M^{-1} P_{\xi\xi} \left(M^{-1}\right)^T = \sigma^2 \begin{bmatrix} 1 & 0 & -\cot\vartheta' \\ 0 & 1 & 0 \\ -\cot\vartheta' & 0 & \cot^2\vartheta' \end{bmatrix} \tag{30}$$

so that the variance of ψ is

$$\sigma_\psi^2 = \sigma^2 \cot^2\vartheta' \tag{31}$$

Had we not chosen the attitude covariance matrix in our example to be a multiple of the 3×3 identity matrix, we would have found that both σ_φ and σ_ψ become infinite at $\vartheta' = 0$ or π.

The estimate for ψ becomes meaningless, obviously, when σ_ψ becomes equal to 180 deg or 648,000 arcsec. If σ is equal to 1.0 arcsec, this occurs when ϑ' is equal to 0.3 arcsec or $180\,\text{deg} - 0.3\,\text{arcsec}$. For σ equal to 1.0 deg, it occurs when ϑ' is equal to 0.2 deg or 179.8 deg. Thus, as we approach the extrema of ϑ', statistical significance becomes lost before numerical significance is lost, except for computers whose wordlength is much smaller than that needed to accommodate the IEEE standard for double precision floating point numbers.

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