Probabilistic Parameter Uncertainty Analysis of Single Input Single Output Control Systems

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Abstract

The current standards for handling uncertainty in control systems use interval bounds for definition of the uncertain parameters. This type of approach gives no information about the likelihood of system performance but simply gives the response bounds. When used in design, current methods of μ-analysis and H^∞ can lead to overly conservative controller designs. With these methods worst case conditions are weighted equally with the most likely conditions. This research explores a unique approach for probabilistic analysis of control systems. Current reliability methods are examined, First Order Reliability Methods and Monte Carlo using sampling procedures such as Hammersley Sequence Sampling, showing the strong areas of each in handling probability. A hybrid method is developed using these reliability tools for efficiently propagating probabilistic uncertainty through classical control analyses problems. The method developed is applied to classical Bode and Step response analysis as well as analysis methods that explore the effects of the uncertain parameters on stability and performance metrics. The benefits of using this hybrid approach for calculating the mean and variance of response cumulative distribution functions are shown. Results of the probabilistic analysis of a missile pitch control system show the added information provided by this hybrid analysis. Finally, a probability of stability analysis is performed on both the missile pitch control problem and a benchmark non collocated mass spring system.

Contents

List of Figures

Nomenclature

α	= Angle of Attack
β	= Distance to most probable point (reliability index)
Δ	= Delta Uncertainty Block
δ_{ε}	= Elevon Fin Deflection
Φ	= Standard Normal Cumulative Distribution
ϕ_R	= Inverse Radix
ω	= Natural Frequency
ζ	= Damping Ratio
A_z	= Normal Body Acceleration
f_X	= Joint Probability Density Function of X
F_X	= Cumulative Distribution Function of X
$G(u)$	= Limit State Function in Standard Normal Space
$g(x)$	= Limit State Function in Physical Space
J	= Performance Function
k	= Spring Constant
$K(s)$	= System Controller Function
m	= Mass
p	= Integer Value
Pf	= Probability of Failure
p_i	= Digits of Integer p
Pr	= Probability
q	= Pitch Rate
R	= Radix
$\mathbf{U}$	= Uniform Distribution
V	= Velocity
x	= Specific Instance of Parameters in Physical Space
$\mathbf{X}$	= Uncertain Parameters in Physical Space

Acronyms

CDF	= Cumulative Distribution Function
fmincon	= Matlab Gradient Based Optimization Function
FORM	= First Order Reliability Method
HSS	= Hammersley Sequence Sampling
LHS	= Latin Hypercube Sampling
MATPA	= Matlab based FORM Analysis Code
MCS	= Monte Carlo Sampling
MPP	= Most Probable Point
MV	= Mean Value
PDF	= Probability Density Function
rmodel	= MATLAB Random Stable Transfer Function Generator
SISO	= Single Input Single Output
SORM	= Second Order Reliability Method

Chapter 1

Introduction

The demand to improve performance of modern and future aerospace vehicles is going to continue to grow as we push new limits. Retaining the level of safety seen in aerospace vehicles will be just as demanding in the future. Increasing system performance while maintaining reliability or safety requirements can be provided by uncertainty based design methods. Using probabilistic information about the uncertainty can help to develop systems that are not overly conservative on performance simply to ensure an acceptable response to extreme conditions. The goal of this research is the development of a method for incorporating probabilistic uncertainty into classical control systems analysis tools.

1.1 Probabilistic Uncertainty

Using the definition in [1] uncertainty based design can be split into two categories based on desired results, robust design and reliability based design. Robust design seeks insensitivity to small uncertainties, while reliability based design seeks a probability of failure less than some limit. A large amount of work has been done on robust design with respect to control systems, however less work has been done incorporating probability or reliability based design in controls. In both controls and aerospace arenas, the traditional design process has been done using norm-bounded descriptions of uncertainties, essentially safety factors and knockdown factors. While these safety factors give limits to the problem, information about the likelihood

of particular events is ignored. Such methods can lead to overly conservative designs that sacrifice performance to accommodate the worst case conditions. A probabilistic approach to uncertainty uses information about the likelihood of parameters in determining the likelihood of the response. Figure 1-1 shows a comparison between norm-bounded uncertainty (all values have equal likelihood) and probabilistic uncertainty, where information about the likelihood of parameter values in included. This comparison provides the focus for this research, to develop classical control analysis that includes probabilistic uncertainties to aid in finding the best controller that meets both performance and safety requirements.
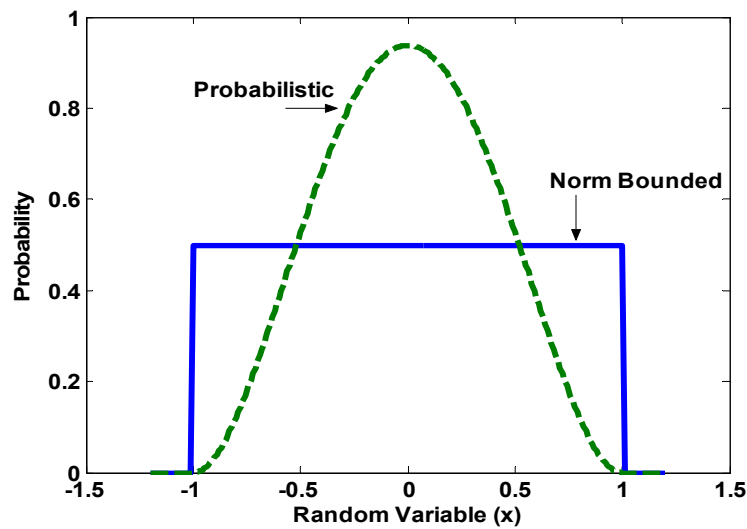
Figure 1-1: Norm bounded Uncertainty vs. Probabilistic Uncertainty

To further expand on the concept of probabilistic analysis of Single Input Single Output (SISO) control systems, the topic can be explained with a bit more clarity. While this research focuses on uncertainty analysis, the type of uncertainty design that the analysis will support must be considered. When probabilistic uncertainty is included, robust design looks at conditions near the mean reducing sensitivity to small variations about that mean. Reliability based design is concerned with conditions near the tails of the probability density function (PDF), ensuring that the probability of the system response being outside a safe range is below

a given limit. Along with two types of uncertainty based design are two types of uncertainty; model uncertainty where the physics defining the problem are only approximately correct, and parameter uncertainty where basic coefficients in the governing equations of the system are uncertain. Throughout this paper uncertainty will be pertaining to parameter uncertainty, model uncertainty will be excluded. Given probabilistic parameter uncertainty a probabilistic definition of the system response be used to make decisions about the reliability and robustness of the system. Figure 1-2 diagrams the propagation of parameter uncertainty through a process that produces a response distribution, which is the goal of the tools contained in this paper. An example of the process in Figure 1-2 with respect to classical control analysis methods would be a Bode or step response.
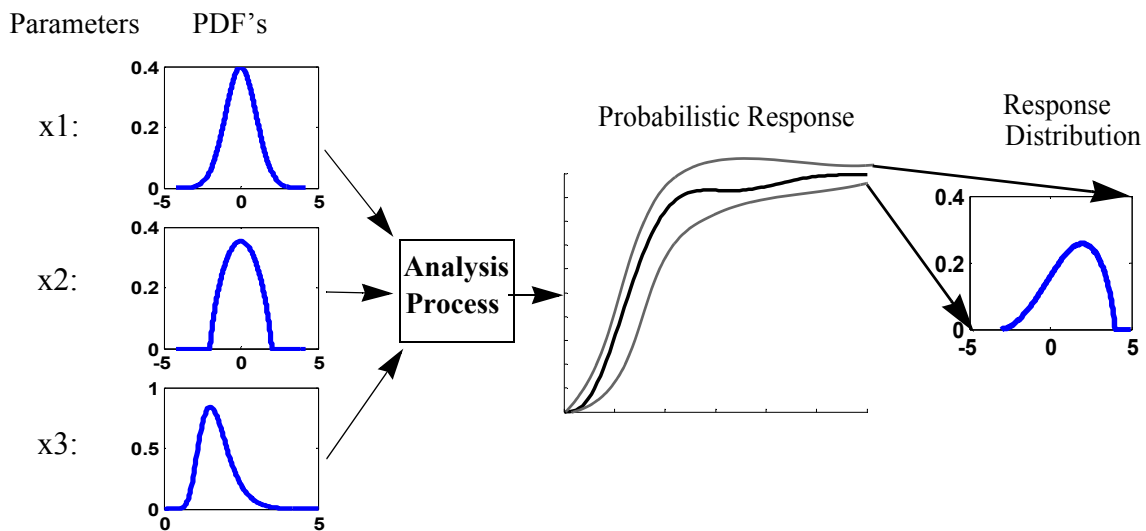
Figure 1-2: Parameter Uncertainty Propagation

1.2 Current State of Probabilistic Control Analysis

A review of the state of current research pertaining to control design and analysis of systems with probabilistic parameter uncertainties indicate that there exists only a small amount of literature pertaining directly to this topic. A large amount of work has been done in

the field of robust design, however; this work predominately uses norm-bounded uncertainty containing no information about the probability distribution of parameters. A few papers like those directed by Stengel [2],[3] take into account probability when working with parameter uncertainty in control systems and will be discussed in section 1.3. In this research, a different approach is described for using probabilistic parameter information and reliability methods to analyze systems. There has also been a large amount of research in the past two decades on reliability analysis, mostly coming from the civil/structures engineering field and is gaining more use in other engineering disciplines. The dominant amounts of information pertaining directly to either classical control analysis or reliability analysis led to splitting the survey of current work into a section on controls and a section on probabilistic design methods. The methods that include probability in control analysis are incorporated into section 1.3 on controls.

1.3 Controls

Classical control design techniques are those frequency domain and graphical techniques pioneered by Bode, Nyquist, Nichols, others [4] [5]. The developmental efforts of these researchers laid the groundwork for analysis of control systems and methods for describing stability robustness. Gain and phase margins are the most widely used metric to express robustness. There has been extensive work over the years on robust control design facing parameter uncertainty, however; these methods have been based on norm bounded uncertainty. Methods for handling robust control design grew as complexity of systems increased, leading to current techniques of μ-analysis and H^{∞} design. The structured singular value, μ, is a mathematical object used to analyze the effects of uncertainty in linear algebra problems, particularly helpful in analysis of effects due to parameter uncertainty on stability[6]. The μ

framework is based on linear fractional transformations used to separate the uncertainties from matrices representing the system. The desired separation is seen in Figure 1-3, where Δ is a
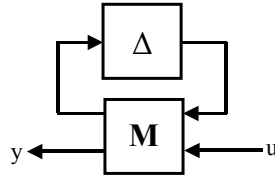
Figure 1-3: Uncertainty Separated Into Delta Block

diagonal matrix of individual parameter uncertainties and **M** is the transformed system with interconnections to the uncertainty. µ-analysis then uses a set of tools to connect the system with controllers or other system matrices and analyze the effects of different values for individual uncertainties on the overall system performance.

H^∞ is a controller optimization technique that best meets certain performance criteria. It can also be used with µ-analysis to produce an optimal controller that is still robustly stable given the system uncertainties [7]. More on these methods are included in references [6], [8], [1]. It can be seen in references[8] and [9] that current µ-analysis approach still considers uncertainty in the system as a norm bounded set. The drawback of this approach is that all uncertain values are given an equal likelihood of occurrence. Realistically most physical random variables have some sort of probabilistic distribution. Thus µ-analysis and H^∞ methods of robust control are designing for the worst case scenario by giving extreme conditions the same importance as the most probable conditions[10]. Both of these methods attempt to reduce the conservatism in the design imposed by interval bounds on the uncertain parameters. There is still the downfall of designing for extreme cases with this description of the uncertain variables. When designs are developed using norm-bounded uncertainties, systems often lack the performance characteristics that could be achieved for the most likely cases.

There has also been some work done on robust pole placement for system design with uncertainty. Again this approach uses interval bounds to define uncertainties in the system[11],[12]. This approach has the same problem of not being able to design for better performance at the most likely cases. Probabilistic information, not just interval bounds, about the uncertainty is required to design for performance at the most likely cases and still meet requirements at extreme values,

A few research investigations have looked at incorporating the probabilistic parameter information into the design process. Stochastic robustness of linear time invariant systems has been analyzed by looking at probability distributions of the closed loop eigenvalues[13]. Probabilistic robustness is then measured by the probability that all eigenvalues lie in the left half plane. In this work, Monte Carlo simulation (MCS) is used to find the distribution of eigenvalues in the complex plane; the stochastic robustness is then the probability of stability for the system. Continued work in the stochastic robustness analysis has included other performance metrics and has been applied to designing an optimal controller that reduces the probabilities of unacceptable performance[13],[2],[3]. All of these works use only MCS for probability calculations and focus on designing a controller for a system with specified parameter uncertainties. This research focuses on analysis of system responses with defined parameter distributions and how varying uncertainty affects probability of stability.

1.4 Probabilistic and Reliability Analysis

A lot of research also exists in the areas of reliability analysis and reliability methods. Reliability methods are based on the concept of a limit state function that separates a failure region from a safe region[14]. The definition of failure can be defined as any undesirable behavior in the system. This limit state function, $g(x)$, separates the failure region $g(x) \leq 0$ from

the safe region g(x)>0, so the probability of failure Pf =P[g(x)≤0]. Pf is calculated with the

following integral, where f_X is the joint PDF of the random variables, X, that is,

$$Pf = P[g(x) \le 0] = \int_{g(x) \le 0} f_X(x)dx \tag{1.1}$$

This integral can become unmanageable for high dimensional systems or when the

algorithm for g(x) is complicated. Numerical error is also a problem for very low

probabilities[14]. Reliability methods are a way of approximating a solution to the given

integral with reduced computational effort. A few of the reliability based design tools are

Monte Carlo analysis, First Order Reliability Method (FORM), and Second Order Reliability

Methods (SORM).

1.4.1 Sampling and Monte Carlo

Monte Carlo is a direct numerical simulation tool that is simple but can be computationally

intense. Random samples are generated with the desired distributions for uncertain parameters,

and then the system is simulated with each set of generated samples. The number of results

produced in the failure region is divided by the total number of results giving an approximate

Pf. If an indicator function is defined such that it has the value 1 in the failure region and 0 in

elsewhere, i.e. equation (1.2),

$$I_{g(x)} = \begin{cases} 0 & g(x) > 0 \\ 1 & g(x) \le 0 \end{cases} \tag{1.2}$$

then the integral in equation (1.1) can be rewritten as seen in equation (1.3).

$$Pf = \int\limits_{g(x) \leq 0} f_X(x)dx = \int\limits_{-\infty}^{\infty} I_{g(x)}f_X(x)dx \tag{1.3}$$

The indicator function forces integration of only the failure region while allowing the integral to be evaluated for all (x). Forming the Pf integral as in equation (1.3) shows that probability of failure is also the expected value of $I_{g(x)}$. Monte Carlo uses a discrete evaluation of the expected value integral to approximate the probability of failure integral. The expected value of the indicator function can be approximated by the summation of $I_{g(x)}$ divided by the number of evaluations, as shown in the following,

$$Pf = \int\limits_{g(x) \leq 0} f_X(x)dx \approx \frac{1}{N} \sum_{i=1}^{N} I_{g(x_i)} \tag{1.4}$$

in other words, the number of samples in the failure region divided by the total number of samples.

Although the Monte Carlo method is a simple concept, computational time can significantly increase as greater accuracy is needed or as the simulated system becomes more complicated. The absolute error in approximating the Pf integral from regular Monte Carlo is of $O(N^{-1/2})$, this slow rate of convergence leads to a very high number of samples to approximate low probability levels. Computation time increases because the function has to be evaluated for each sample. With complex systems, a large numbers of functions evaluations can be cost prohibitive. With Pf approximated by the expected value of the indicator function a representation of accuracy can be developed based on sample size, since the indicator function is a boolean variable (either 0 or 1). Stengel[13] discuses the number of samples required for the approximate Pf to be within an interval around the true Pf, given some confidence level. For

example to be 95% confident that the approximated Pf will be within 10% of a true Pf of 0.99 requires 10^5 random samples. The difficulty with pure Monte Carlo sampling arises because you need to increase the number of samples by a factor of 10 for each added decimal place in the Pf that is being approximated. 10^7 random samples are required to be 95% confident that the approximated Pf falls with 10% of 0.9999. Computationally this can become very demanding, a function that requires 0.05 seconds to be evaluated will spend 1.4 hours evaluating 10^5 samples and 140 hours evaluating 10^7 samples. Using MCS when approximating small probabilities has serious drawbacks.

There has been research on sampling techniques to reduce the number of simulations required to produce the same level of accuracy. One of the more popular methods is Latin Hypercube Sampling (LHS). This method of sampling is approached by taking the distribution and dividing it into *n* intervals of equal probability, then selecting a value from each of the intervals. The *n* samples produced with LHS cover the range of the distribution in much fewer samples than would be required to cover the range with purely random samples[15]. LHS's advantage over MCS is it's more uniform spread of points across the sample-space, with LHS this benefit reduces as the dimensions of the parameter space increases. A newer method that has become more widely used is Hammersley Sequence Sampling (HSS)[16]. HSS is considered a quasi-MC sampling method because deterministic points are used instead of random points. Hammersley points are used to divide a unit hypercube, providing uniform sample points across the sample space. Since the points are chosen on a unit hypercube, they are transformed to the given parameter distributions providing sample points for simulation. This method produces good coverage of the distribution with a greatly reduced set of sample points.[16]

1.4.2 Reliability Methods

Some of the early methods of analytical probabilistic analysis are the Mean Value (MV) method, response surface method, and differential analysis[17]. All three methods are very similar, the MV method and differential analysis methods are based on generating a taylor series expansion of the response surface about the nominal values of the uncertain parameters. With the MV method the moments of the approximate function are used to determine and approximate Pf. The differential analysis method produces the taylor series expansion and then partial derivatives of the response surface are calculated, helping to define the shape of the response surface used to approximate Pf. The response surface methods are very similar to MV methods, however; where the MV method finds a taylor series expansion of the true performance function, the response surface approximates the performance function with a simpler function, often a second order polynomial. After defining the simpler approximate performance function, the response surface method proceeds the same as the MV method. An in depth survey of reliability methods can be found in [14], [17].

First Order Reliability Method (FORM) and Second Order Reliability Method (SORM) are methods that have come into much wider use in the past decade [18]. These methods are related to the response surface method since response surface is approximated with a simpler function. The goal of FORM is to compute failure probabilities efficiently, while avoiding the particular errors due to problem formulation seen in other methods. The FORM method takes specific advantage of transforming the problem into a standard normal space (u-space), where uncertain parameters are independent with standard normal distributions. The uniformity and exponential decay properties of u-space can be used to reduce error from response surface approximation as well as simplifying the Pf calculation. The transformation and limit state approximation are the

basis of the FORM and SORM techniques. FORM approximates the limit state function with a tangent hyper-plane, a linear or "First-Order" approximation, while SORM approximates the limit state function with a Hyper-parabola, a "Second-Order" approximation. SORM can have a dramatic effect of reducing error from limit state approximation, but it comes at the computational cost of having to calculate derivatives of the limit state surface. Both FORM and SORM are strong in regions of low probability, however the approximation error increases as the limit state function nears the origin in standard normal space.

1.5 Probabilistic Analysis of SISO systems

A probabilistic view of Classical Control analysis for SISO systems will be a beneficial step in providing information on performance of systems with parameter uncertainty. The goal of this research is to show a probabilistic-based method for control systems analysis of SISO systems with parameter uncertainty. The extreme conditions do not dominate the design process by incorporating probability in the uncertainty. The most probable cases can be used to achieve a desired performance while extreme cases can still be considered. The nominal system, or system with mean values of all uncertain parameters, give one response but, a third dimension to the traditional response plots is added when you add probability because each response is represented with a distribution. Representing the probabilistic information on classical response plots must be incorporated to provide clear understandable plots. One method is the use of probabilistic confidence bounds.

1.5.1 Classical Response Analysis

Probabilistic response plot analysis looks at probabilistic analysis of the bode magnitude and phase plots as well as the step response. With probabilistic response plots, the added

dimension of probability is used to give real confidence intervals that represent the range of probable system responses. The idea pursued is to take these existing controls tools (bode, step response) add to them the probabilistic analysis tools (HSS, FORM) providing a new capability to evaluate system performance.

The hybrid approach developed mixes sampling and FORM to solve the problem. Using both methods allows for the strengths of each tool to be used. Sampling computation is quick in midrange probabilities and FORM approximation error is small in low probability regions. The appropriate combination of the two methods can produce a cumulative distribution function (CDF) of the system response with accurate representation through the middle and the tails of the CDF. One advantage of mixing these methods in a hybrid approach is reduced computational effort. As discussed in section 1.4.1 sampling alone can be extremely expensive to reach low levels of probability. Using Form allows for specific computations of these much lower levels of probability. A few FORM analyses can reach the levels of probability that would require a number of samples many orders of magnitude larger. Once the full CDF has been generated, it is easy to represent the desired confidence intervals for the system response.

A representation of probabilistic bode response is shown in Figure 1-4. If the Bode response is
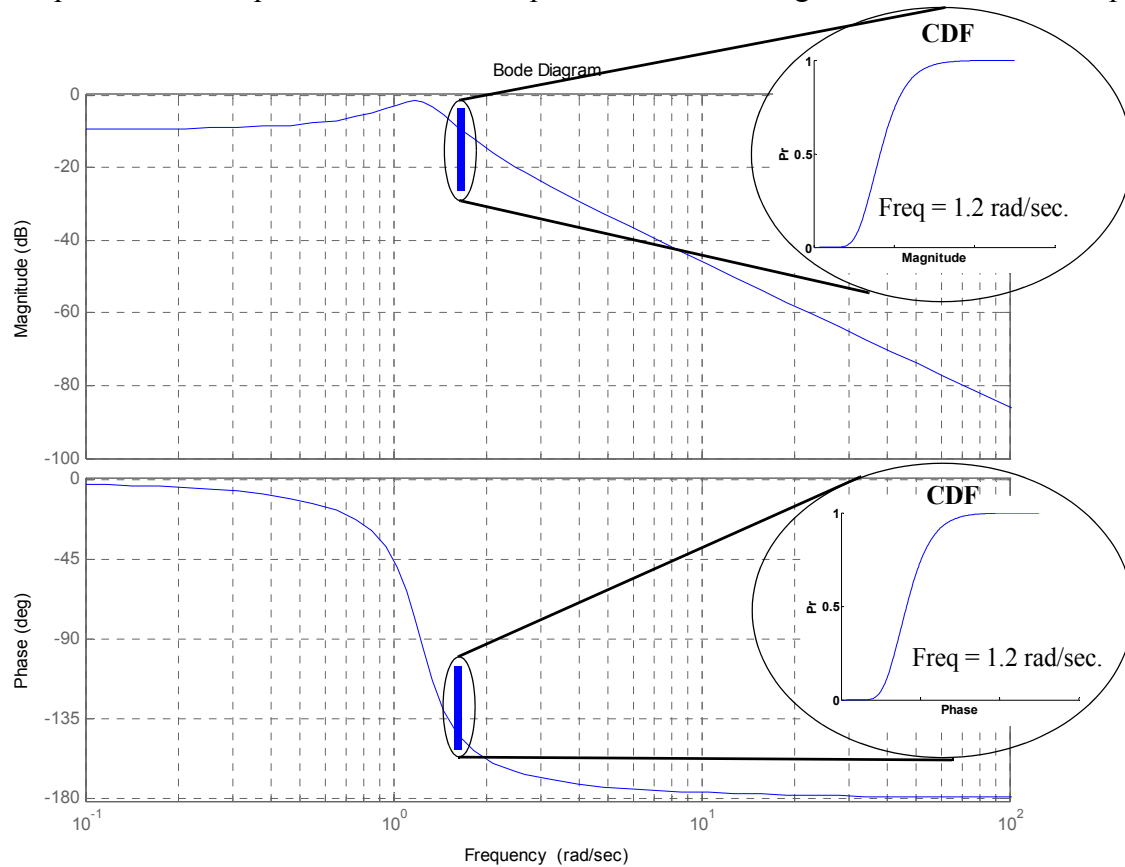
Figure 1-4: Probabilistic Bode Response

considered to be probabilistic, a cross section of the response at any frequency would produce a probabilistic distribution. Figure 1-4 shows the cross section of a specific frequency produces a CDF representing the probability that the Bode response will be less than given magnitude or phase values. A probabilistic step response would have the same structure, where at each instant in time the response values of all possible systems could be represented as a distribution.

1.5.2 Parameter Space Analysis

Parameter analysis examines how the parameter space affects the performance of the system. The concept of the largest stable hypercube has been explored with norm bounded uncertainty to determine how much certain parameters can change before becoming detrimental to the system. The norm bounded set of parameters can be scaled until instability is reached. This largest stable hypercube is simply the largest parameter space. By adding a probabilistic distribution to the parameters not only can the largest stable hypercube be found, but also the rate at which the probability of instability increases. For example, consider two systems, system **A** parameters can be scaled by a factor of 2 with guaranteed stability. However, continuing to increase the parameter scaling factor to 2.5 may lead to 30% probability of instability. Now, system **B** is only stable when the parameter space is scaled by a factor of 1.5, but a continued increase shows the parameters can be scaled to 2.5 with only 1% probability of instability. System **A** may be considered more robust; however, if a small probability of instability is acceptable, system **B** may be a much more desirable system. Clearly, this information about the rate at which instability increases can only be obtained with a probabilistic approach. Requirements would provide the method for choosing the most desirable system. Parameter space analysis looks at this probability of instability problem as well as how the size of the parameter space affects specific performance metrics.

The largest variation allowed in the parameters to still ensure stability is very useful. A probabilistic approach to the analysis allows for the added depth of understanding how the system will continue to perform if this largest variation is exceeded. Similar analysis can provide added information to other performance metrics to determine how the changes in the parameter distributions affect the performance characteristics. Items such as rise time, peak

value, and settling time of a step response can be analyzed to see how the mean of the performance metric compares to the nominal value as the parameters space varies. Analyses like these could aid in reducing costs of systems while maintaining a level of performance characteristics and meeting a required level of risk.

This chapter has introduced the ideas of probabilistic controls and has presented a method for approaching these types of problems. A hybrid approach to the problem was chosen to take advantage of the strengths of both sampling and FORM methods. A full CDF of the system response is desired, so with the hybrid approach FORM is used to resolve the tails, areas of low probability, while Monte Carlo excels at filling in the mid regions of the CDF. The development of the hybrid method and its benefits is presented. A review of sample problems and a comparison of this analysis technique to current methods are presented next.

Chapter 2

Reliability Methods

2.1 Sampling and Monte Carlo

The Monte Carlo method is based on simulating a system with a set of sample points. A sample is a vector or ordered set of the form $\mathbf{x}=(x1,x2,...xN)$, where N in the number of uncertain parameters. This vector is a specific instance selected at random from the set of random variables $\mathbf{X}$. The most important part of the Monte Carlo method is generating the sample points. Pseudo-Monte Carlo sampling, also known simply as Monte Carlo sampling (MCS), is the most well known method. MCS consists of the pseudo-random number generation of n samples on a k-dimensional hypercube. The 'pseudo-' implies that the random numbers are produced with an algorithm intended to imitate a truly random natural process. Random numbers may be repeated exactly given the seed used in the random number algorithm. The "pseudo-" prefix may be dropped though it is still implied throughout this paper. With MCS and many sampling methods, samples are generated over a uniform distribution U(0,1), then inversely mapped the CDF of the desired distribution to produce the desired samples. The approximation error (see section 1.4.1) when approximating an integral when using Monte Carlo sampling is dependent on the even distribution of the sample points not on the randomness [16]. With a limited sample size, purely random sampling can lead to clumping of sample points or areas of the sample space not adequately represented. Uniformity is key to efficient sampling techniques so alternate methods of generating sample points can

considerably improve the MC simulation results, two such methods are stratified sampling and low discrepancy sampling.

2.1.1 Stratified Sampling Methods (Latin Hypercube)

The goal of stratified sampling techniques is to produce a more uniform distribution of sample points throughout the sample space.[19] The basic concept is to divide the sample space into bins of equal probability, and then generate a random sample inside of each unique bin. By dividing the sample space into bins before selecting the random samples, better overall coverage is achieved compared to MCS. Stratified methods also give the user the ability to control the number of bins, ensuring a desired number of samples in given probability ranges. One popular variant of the basic stratified sampling technique is Latin Hypercube sampling (LHS). As a stratified technique, the sample space is again divided into unique bins of equal probability, then a reduced set of samples are randomly selected in the sample space. With LHS the randomly selected samples have two major constraints:

- each sample is randomly placed inside a bin
- all one dimensional projections of samples shall have one and only one sample in each bin.

A visual representation of LHS is illustrated in Figure 2-1. In Figure 2-1 points are selected
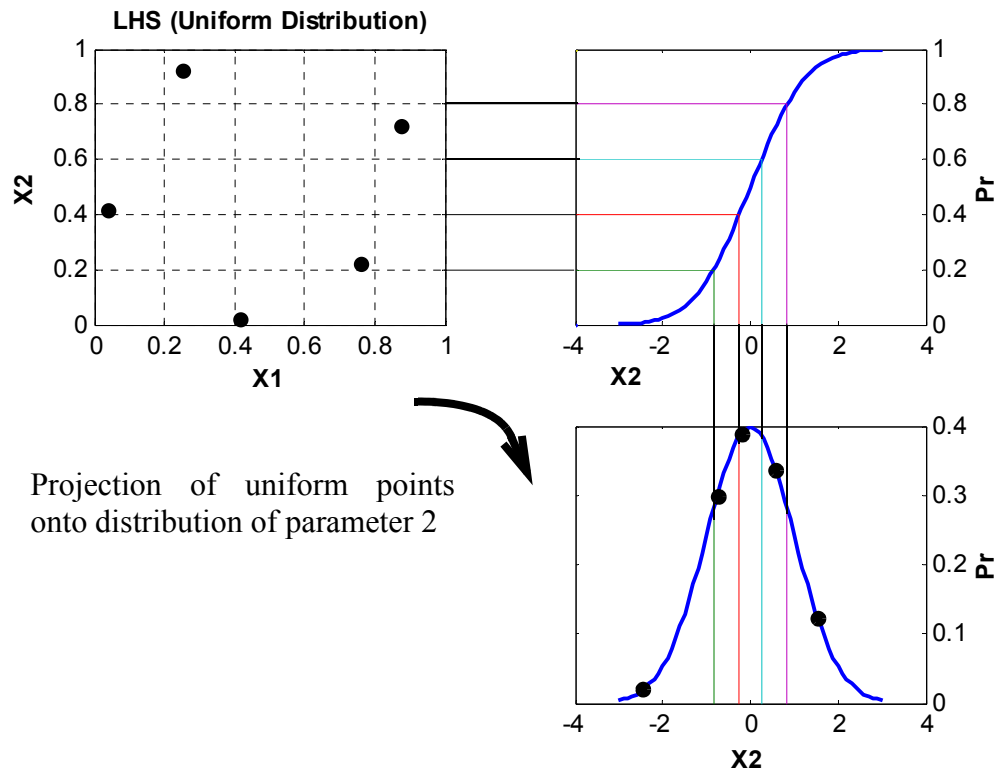
Figure 2-1: Stratified Sampling

from a two dimensional space with a uniform distribution. The points are inversely mapped

through normal CDF to produce samples with good coverage of the true parameter distribution.

The same process can be done to map the points to samples for parameter x_1. It can be seen in

the lower right portion of Figure 2-1, that if the points are projected to either axis that only one

point falls in each bin. LHS can provide a more accurate estimate of the mean with the same

number of samples as MCS or basic stratified sampling. There are however, a few drawbacks

to the LHS method. Since there are multiple arrangements of bins containing samples that meet

the two previously mentioned constraints, care must be taken to reduce spatial correlation of

the sample points. It is easily seen that sampling along the diagonals would meet the two

constraints. However, this would be an undesirable choice since it counters the goal of

uniformly covering the sample space. Highly correlated sample points can also lead to other

less desirable results[19]. A third 'soft' criterion is included in LHS algorithms to minimize correlation of sample points. One drawback of the LHS method is that the uniform quality of the sample points decreases as the dimension k of the sample space increases, however; with LHS the error of the estimate is still reduced compared to MCS with the same number of samples, or similar results can be achieved with a fewer number of sample points.

2.1.2 Low Discrepancy Methods and Hammersley Sequence Sampling

Another class of sampling methods is quasi-Monte Carlo Methods[20], with the explicit goal of producing an evenly distributed set of sample points over the sample space. The word quasi- is used because the sample points contain no randomness, instead they are chosen by a strictly deterministic algorithm. The goal again is to produce evenly distributed sample points throughout the sample space, while not having a high correlation between the points, i.e. not forming a regular grid. Another term for this type of method is low discrepancy sampling, where discrepancy is a measure of how close the sample points are from an ideal uniform distribution. This ideal uniform distribution can be thought of as a set of points that are all equidistant from each other and unstructured, or have no regular pattern.

One variant of these quasi-Monte Carlo methods is Hammersley Sequence sampling (HSS) described by Kalagnanam and Diwekar [16] which uses the Hammersley sequence to generate n uniformly distributed samples on a k-dimensional hypercube. This low discrepancy method has an advantage over techniques like LHS in that is selects points for uniformity over all dimensions of the hypercube, where LHS primarily focuses on uniformity across one dimension. HSS sample points keep their uniformity as the number of dimensions increases. The differences in sampling techniques can be seen in Figure 2-2 showing the uniformity of the

HSS points. The benefits of the HSS method and the ability to get similar MC results with a greatly reduced set of sample points led to the use of HSS points in all the sampling used in this research.
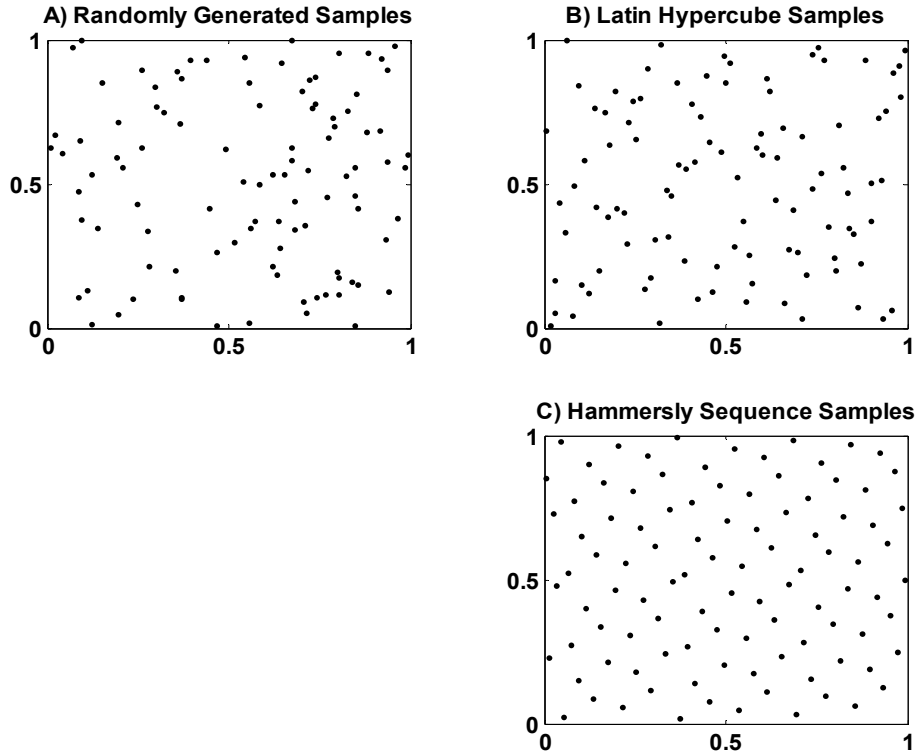
Figure 2-2: Monte Carlo Sampling Methods (100 points) A) Random Sample generation, B) Latin Hypercube Samples, C) Hammersley Sequence Samples.

Described in Kalagnanam [16] and Giunta [19], the Hammersley sequence is based on the inverse radix notation using prime numbers as the radix- R. Radix notation of an integer p is a sum of the digits of p multiplied by powers of the base, or radix.

$$p = p_m p_{m-1} \cdots p_1 p_o$$
$$p = p_o R^o + p_1 R^1 + \ldots + p_m R^m$$

(2.1)

for example in base 10 the number 516 in radix notation looks like, $516 = 6 \cdot 10^0 + 1 \cdot 10^1 + 5 \cdot 10^2$. Reversing the digits of p about the decimal point generates a

unique fraction between 0 and 1, known as the inverse radix number, 0.615 in the case of the example.

$$\phi_R(p) = 0.p_o p_1 p_2 \ldots p_m$$
$$\phi_R(p) = p_o R^{-1} + p_1 R^{-2} + \ldots + p_m R^{-m-1} \tag{2.2}$$

The Hammersley points of a k-dimensional hypercube are generated using

$$x_k(p) = \left[\frac{p}{N} \ \phi_{R_1}(p) \ \phi_{R_2}(p) \ \ldots \ \phi_{R_{k-1}}(p) \right] \tag{2.3}$$

where R_i are the first k-1 prime numbers, and p=1,2,3...,N. These N Hammersley points are distributed on the unit hypercube $[0,1]^k$ (see Figure 2-2c for a two dimensional representation). Given the CDF of each parameter distribution the Hammersley points can be inversely transformed to give a low discrepancy sequence of sample points in the parameter space.

2.1.3 Justification For Using HSS

A simple demonstration is given to show the benefits of low discrepancy sampling techniques. Given a distribution with a known mean and variance apply each sampling method and evaluate the mean and variance of the sample points. The level of Pr achievable with each sampling technique is still 1/N. The benefit of LHS and HSS methods is the reduced error bounds. The narrower error bounds result in a more accurate computation of the mean with the same number of samples, or an equivalent mean calculation with far fewer sample points.

Using a standard normal distribution, Figure 2-3 shows the results of 200 samples for different
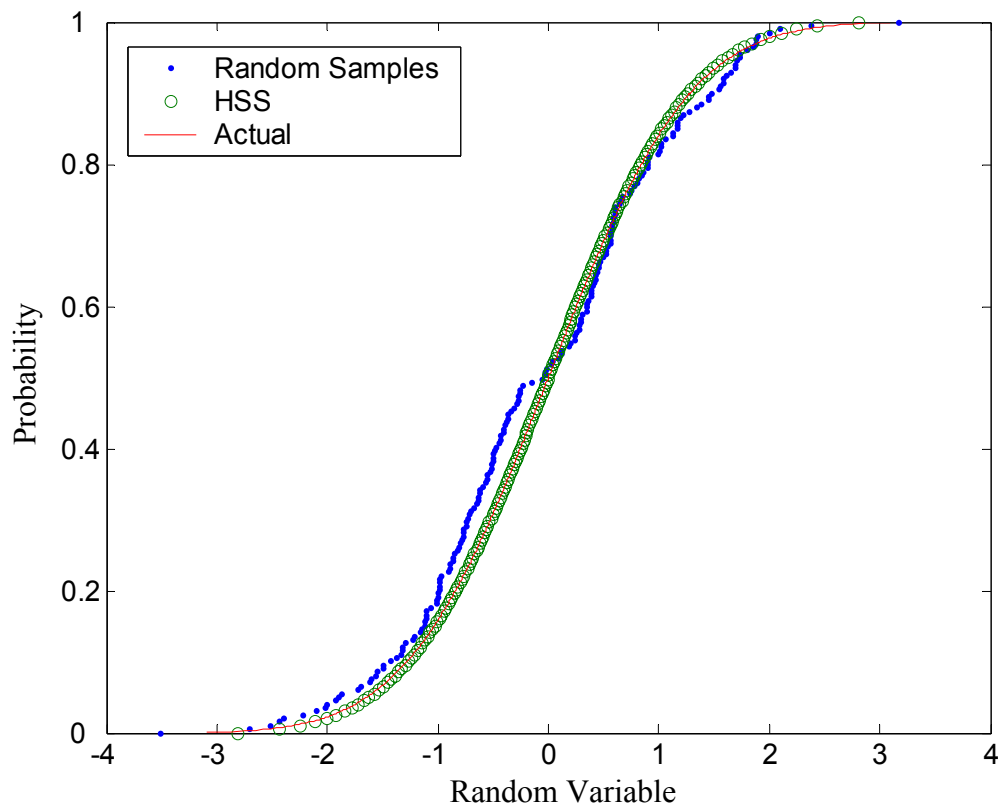
Figure 2-3: Comparison of Sampling Methods for a Standard Normal Distribution

sampling schemes. The more accurate representation of the HSS samples is evident. A

comparison of the mean and variance calculations can be seen in Table 2-1 and Table 2-2

Table 2-1: Comparison of Mean Calculations for Different Sampling Methods

Sampling Method	100	1000	10000	100000
MCS	0.00057607	0.0012104	2.1091e-005	6.9766e-005
LHS	8.5421e-005	1.0392e-005	4.175e-007	9.8505e-008
HSS	1.7347e-017	3.9248e-016	1.2248e-015	1.2098e-016
Actual mean value = 0				

respectively.These benefits of HSS drove the decision for the use of HSS in the methods

Table 2-2: Comparison of Variance Calculations for Different Sampling Methods

Sampling Method	100	1000	10000	100000
MCS	0.93405	0.98866	0.99869	0.99968
Error	*0.06595*	*0.01134*	*0.00131*	*0.00032*
LHS	0.93734	0.98960	0.99853	0.99981
Error	*0.06266*	*0.01040*	*0.00147*	*0.00019*
HSS	0.93026	0.98886	0.99845	0.99981
Error	*0.06973*	*0.01113*	*0.00154*	*0.00019*
Actual variance = 1				

developed. LHS showed a slight reduction in variance error however, the significant improvement of HSS in the mean computation was the deciding factor for choosing HSS. Being a deterministic set also allowed for easy repeatability of simulations.

2.2 First Order Reliability Methods (FORM)

Reliability methods are based on finding regions of failure and regions of safety of a given system with uncertain parameters. Each random variable **X** is represented by a probabilistic distribution. A scalar state function g(x) is defined which produces a metric of interest given a specified set of the random parameters. This state function is used to separate the safe region from the failure region, and is formulated so that g(x)>0 defines the safe region and $g(x) \leq 0$ defines the failure region. The condition that separates failure and safety, g(x)=0, is know as the limit state function. Probability of failure can then be defined by the integral shown in equation (1.1) As mentioned before, with high dimensions this integral can be very difficult and unmanageable. The ability to find the Pf without directly integrating the integral is highly desirable.

The goal of FORM is to simplify integration by calculating Pf based on an analytical approximation of the limit state function. These methods are especially effective when looking at very low levels of probability of failure, where traditional sampling methods become

excessively expensive. Reliability methods simplify the problem of performing multidimensional integration with a method of transforming the problem into a standard normal space and approximating the limit state surface with a simpler lower order hyper surface.

2.2.1 Transformation to Standard Normal Space

Standard normal space (u-space) is defined so that all random variables are statistically independent, with normal distributions having zero mean and unit variance. In u-space all random variables are defined by the standard normal density function.

$$f_U(u) = \left(\frac{1}{(2\pi)^{1/2}}\right) \cdot e^{(-(1/2) \cdot u^T u)} \text{[21]} \tag{2.4}$$

U-space has several desirable advantages for approximating the limit state surface. Most notable are the exponential decay of the probability density, and the symmetry about the origin. The exponential decay in the u-space allows for good approximations of Pf with a hyperplane since the probability attributed to the area between the actual and approximate g(x) reduces exponentially with the distance from the point where g(x) is approximated. Symmetry of u-space simplifies the approximation because the direction to the hyperplane does not affect the

approximation only the distance. The transformation of the random variables to u-space, shown in Figure 2-4, is the first step of the FORM process.
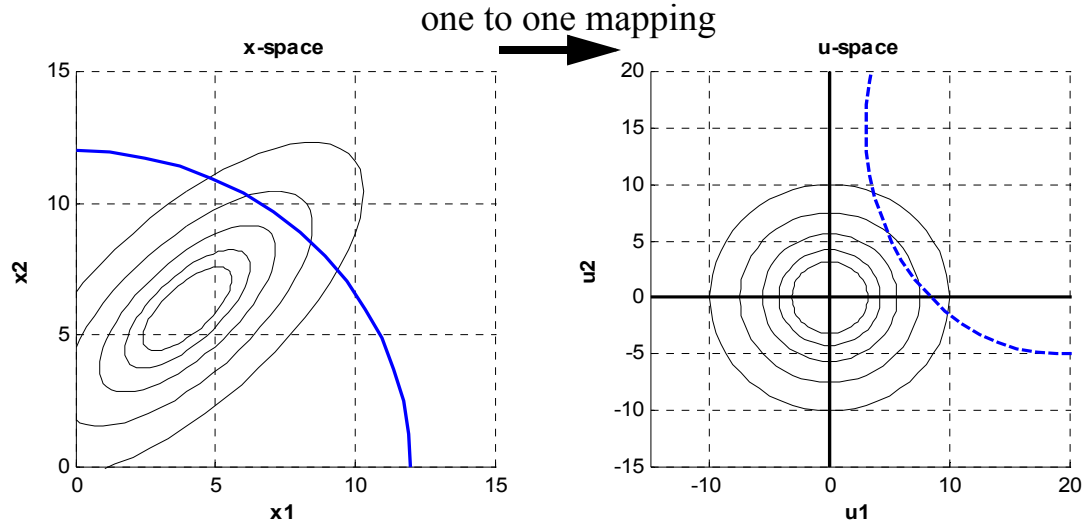
Figure 2-4: Nonlinear Transformation from x-space to u-space

The transformation process takes the variables from their native distributions in the physical space, x-space, through a one-to-one nonlinear mapping into u-space. The simplest form, the Hasofer-Lind Transformation, can be used when the random variables **X** have normal distributions and are uncorrelated. The normal distributions then must simply be shifted to a standard normal distribution, by

$$U_i = \frac{x_i - \mu_{X_i}}{\sigma_{X_i}} \tag{2.5}$$

For variables that are independent but not normally distributed the following diagonal transformation may be used:

$$u_i = \Phi^{-1}[F_{X_i}(x_i)] \tag{2.6}$$

Where Φ is the standard normal cumulative distribution function (CDF) and F_{X_i} is the CDF of the random variable **X**. Transformations that are more complex exist to handle correlated random variables, such as the Nataf and Roseblatt transformations. See reference [14] for a review of these and other transformations. For the present research, only uncorrelated random variables were used.

2.2.2 Most Probable Point Determination

The most probable point (MPP) is the point in u-space closest to the origin on the limit state function. The symmetric exponential decay of u-space means the point closest to the origin is going to have the highest probability, relating to the mostly likely point of failure. As the most probable point of failure, the MPP is the desired location for the limit state approximation. A nonlinear constrained optimization is used to find the MPP.

$$
\begin{aligned}
&\min |u| \\
&\text{subject to} \quad G(u) = 0
\end{aligned}
\tag{2.7}
$$

The MPP can be inversely transformed back to x-space for a better physical representation of the most likely point of failure. For much of the investigation performed the *fmincon* function, a gradient based optimization tool in MATLAB, is used for finding the MPP.

2.2.3 Limit State Approximation and Probability of Failure Calculation

After the transformation to u-space and finding the MPP, the limit state function can be approximated with a tangent hyper-surface. With FORM, the approximation is a tangent hyperplane (with SORM the approximation is a paraboloid). The largest contributing area to the probability of failure is the region near the MPP, therefore the Pf can be well approximated as the area beyond the tangent hyper-surface. This is where the uniformity and exponential

decay of the normal distribution is helpful in reducing the significance of error in approximation of the limit state hyper surface. SORM has the advantage of being able to reduce error resulting from highly curved limit state function, however SORM comes with added an complexity in calculating the Pf.

The Pf is approximated as the area on the failure side of the tangent hyper-surface. Since FORM uses a tangent hyper-plane, the value of working in u-space is apparent at this point. As seen in Figure 2-5, the Pf can be approximated with FORM simply using the distance from the origin to the MPP. This distance β is also known as the reliability index.

$$P_f \approx \Phi(-\beta) \tag{2.8}$$

Finding the Pf for SORM is not quite as simple because you are approximating with a second order hyper surface, however finding the area on the failure side of the surface is significantly easier than finding the area of the failure region of the original g(x).
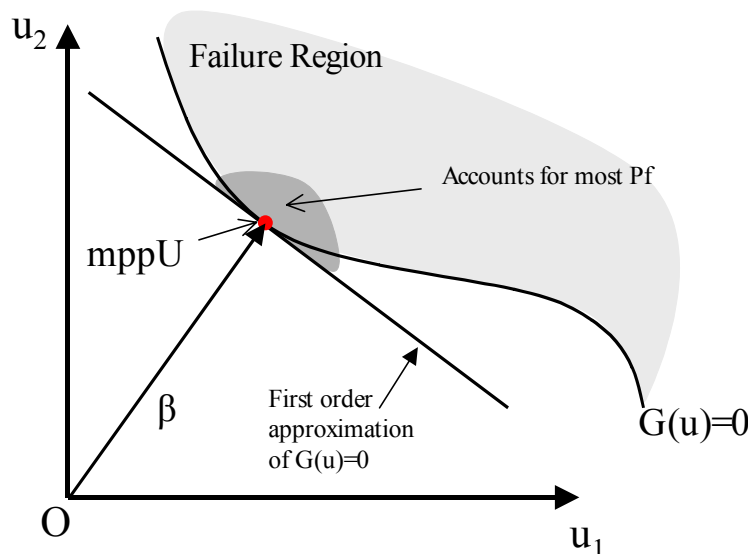

Figure 2-5: Schematic of Limit State Approximation

All FORM calculations were based on the MATPA tool developed at NASA Langley.

27

Chapter 3

Hybrid Approach

The idea for using a hybrid method to approach probabilistic SISO analysis is to take advantage of the strengths of two different techniques of reliability analysis. Sampling excels in the central region of the probability distribution scale and FORM excels in end regions of the probability scale, each technique is then used in its strong area. The data from this third dimension of information is then used to provide information about the probable performance of the system. The following sections describe a way to put these tools together to analyze the effects of probabilistic parameter uncertainty on SISO systems.

3.1 General Hybrid Method

Given a control system with deterministic parameters, it will produce a single response curve. For example, a Bode magnitude plot shows the magnitude of the system steady state response due to a sinusoidal input over a range of frequencies. If slightly different parameters for the system are applied, the Bode magnitude will obviously change. When the system parameters are defined in a probabilistic manner, the system response will be probabilistic in nature. In the example of Bode magnitude, at each frequency the magnitude can be represented by a probabilistic distribution of the magnitude response of that frequency. A specific set of parameters 'x' will produce a response $C(x)$ at one frequency or time. The CDF of this response can be thought of as giving the probability that the system response will be greater than some

reference level, C(x)>Ref. The approach to finding this distribution with reliability methods is to write the limit state function as g(x)=Ref-C(x). Thus the failure region g(x)<0 is defined by C(x)>Ref. Using reliability tools the full CDF can be found by sweeping across the full range of reference levels between Pr[Ref-C(x)]=0 to Pr[Ref-C(x)]=1.

The shape of this response distribution is unknown so sampling is used as the first step of the hybrid approach. HSS points are generated, then applied to the function C(x), producing a first approximation of the response distribution. Using a low number of HSS points (e.g. 200) allows for a good definition of the midrange of the CDF, including the general shape of the distribution. Using sampling data to determine a starting point, FORM is used to resolve the probability in the tails of the CDF down to a predetermined level. The hybrid method then takes both sets of data and combines them to produce a full CDF of the system response at that instance. The entire process must be repeated at each desired frequency or time to generate a full probabilistic representation of the system response. Computational time obviously grows with each additional instance for which the response CDF must be calculated, however; taking advantage of matrix based operations in MATLAB can help to improve computational effort. The sampling data can be computed over the full frequency range with one function call. The FORM process involves a scalar optimization, which must be performed independently at each frequency or time interval.

3.2 Tail Refinement Process

Sampling was used to find the midrange of the CDF, FORM is then used to refine the tail regions of the CDF. The true CDF is unknown a priori, so the sampling data can be used to determine a starting point for the FORM calculations. Given Pf=Pr[Ref-C(x)≤0] the C(x) values from sampling can be used as the first initial Ref values for FORM. The first FORM

computation is performed at the fourth sample from each end of the CDF generated from sampling to allow for an overlap between FORM and sampling data. Section 3.4 describes the logic for choosing the 4th point, and the process for combining the sampling and FORM data. Performing the FORM calculations at the location of the Ref value generated from a sample evaluation guarantees a feasible problem with a known approximate solution. The FORM problem becomes infeasible if the PF is zero, therefore feasibility is ensured because there exists some level of probability of failure at this Ref value. When Pf=0 the limit state function is mapped to infinity during the transformation to u-space. The optimization problem of minimizing |u| subject to G(u)=0 is ill posed if no finite u can produce G(u)=0. Aside from ensuring a feasible problem, performing the first FORM computation at a sample point allows for a smart choice of initial conditions to be used for the optimization. The specific sample point produced a value for the response function C(x), this reference value is then used in FORM to solve $Pr[g(x) \leq Ref - C(x)]$. If the sample point produces C(x) and is then used as the Ref value, the sample point x should be a good initial condition for finding an accurate first form solution. Using the initial conditions that produced the reference condition aids in reducing convergence time of the optimization.

Determining a step or ΔRef value to the next FORM analysis is done using the slope of the last three sample points. This technique places the second FORM point within the region of sample points, again ensuring a feasible problem. The reference value is only moved a small amount so the FORM problem is very similar to the first. The results of the first FORM computation can be used as initial conditions for the second computation. Providing these smarter initial conditions reduces the number of optimization iterations for the new FORM calculation.

At Each location of the SISO response analyses the FORM calculations are performed many times, FORM computations are slow enough that only the desired number computations to define the tail of the CDF should be calculated. With the shape and limit of the tail unknown, it is difficult to evenly space the desired number of FORM calculations. It is known that the CDF ranges from 0 to 1 on the y-axis so a vertical spacing can be defined and used to determine the reference step size. For example, a FORM solution is desired at probability levels decreasing by a factor of 10 (Pf=1e-2, 1e-3, e-4...). With the shape of the tail still unknown a method for determining the reference step value for each new FORM computation must be developed in an attempt to achieve the FORM results at the desired levels of probability.

The step determination method is slightly different for the first, second, and all remaining steps. With no prior FORM data, the first step was chosen based on the average spacing of the last three sample points. This averaging gives a rough estimate of the slope of the CDF tail, and again ensures that a solution to the FORM problem exists since there is a known probability of failure. A least squares fitting of data with extrapolation has been applied for determining the remaining steps (2 - N). A review of many resultant CDFs showed a second order exponential decay function best represented the tail of most CDF's. For determining the second step only two FORM calculations exist so a first order model, $Pf = a \cdot e^{b \cdot x}$ is used, where Pf is known and the new x is desired. The step is then the difference between x at the desired Pf and the previous x. For remaining steps, third and higher, calculations are done with the same least squares method, however; using a second order exponential decay model, given as,

$$Pf = a \cdot e^{b \cdot x} \cdot e^{c \cdot x^2} \qquad (3.1)$$

where a, b, and c are coefficients of the fitted curve. Figure 3-1 shows the results of the
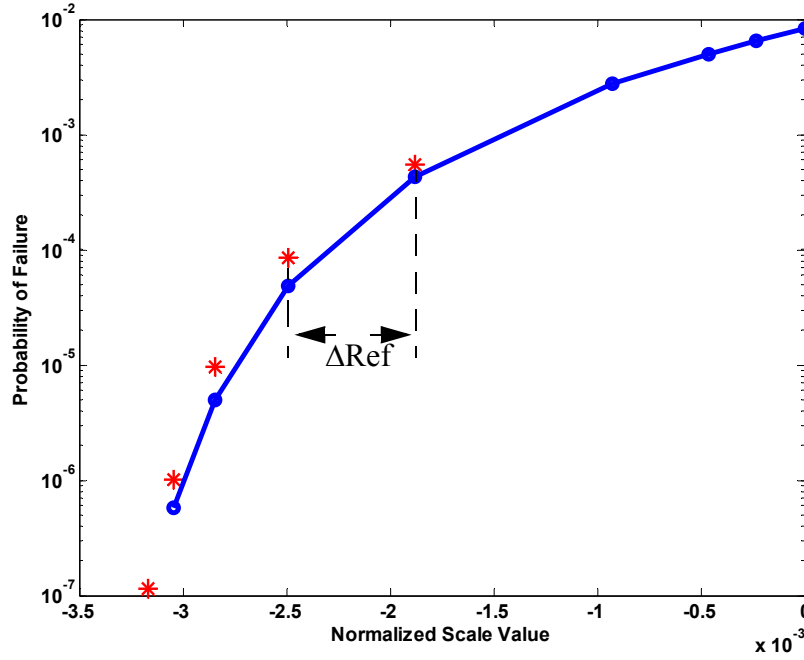
Figure 3-1: FORM Step Prediction

exponential decay extrapolation with asterisks representing predicted Pf at given locations while the circles are the calculated Pf at that value. One modification was required for use of the least squares technique, the x values must be normalized so that the least squares matrix in equation (3.2) remains invertible as the x values become large.

$$\log(Pf) = \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \cdot \begin{bmatrix} \log(a) \\ b \\ c \end{bmatrix} \tag{3.2}$$

Equation (3.2) represents the least squares equation used to find the coefficients of the exponential decay function of equation (3.1). The coefficients are then used for the extrapolation to find the new x value that will produce the desired probability.

One of the primary reasons that this more elaborate extrapolation scheme was developed was to prevent FORM from attempting zero probability computations. The FORM process uses a transformation to standard normal space, when Pf is zero the distance to the MPP is infinite since the limit state function is transformed to infinity. This event leads to an infeasible problem. It is desired to avoid this scenario because in general FORM takes a significantly longer time to not converge to a solution than it takes to converge. When FORM does not converge it produces no beneficial information other than it did not work, the added computational time makes this an undesirable scenario. A series of safeguards were developed to avoid or limit the occurrence of failed FORM computations.

3.3 Capturing Abnormal Occurrences

FORM computations use a gradient based optimization, and are not guaranteed to produce a solution. A few issues exist that can cause the optimization to not converge are as follows:

- Infeasibility of FORM (Pf =0 or 1)

- Limit state function discontinuities

- Nonsmooth limit state functions

- Complicated limit state functions requiring extensive function evaluations with given initial conditions.

For these reasons, safeguards have been implemented into the algorithm to improve the performance of the hybrid method. When the FORM computation fails before the desired level of probability is reached, an attempt to alter specific conditions to find a converged solution is desirable. With the goal of keeping computational time low, two safeguards were put in place to attempt recovery from a failed FORM calculation. If the FORM computation fails, determining if the problem is actually feasible is the primary step in finding a solution. If the Pf

33

is truly zero or one, the limit state function is transformed to infinity in u-space making the FORM problem infeasible. A feasibility test uses a non gradient based optimizer to find the closest point to the limit state function contained within the parameter space. A vector is defined in u-space, from the origin to this point, then a set of samples along an extended portion of this vector are transformed back to x-space. When evaluating these transformed samples, if a sign change is found the problem is feasible, and if no sign change is found the problem is considered infeasible. With feasibility of the problem known, there are two options. First, if the problem is infeasible the initial conditions of the problem may not have been well suited for the problem. New initial conditions are selected, half way between the infeasible and previous feasible locations, and the FORM problem is computed again. If the second attempt also fails, the hybrid analysis is not completed for that specific frequency, or time. The second option when FORM is found to be not feasible then the reference value is outside the possible response range and must be stepped back. A new reference value is chosen half way between the failed and previous successful computations. As before this is only attempted once to facilitate the quick computation for the entire response. The individual issues that cause the FORM failures can be scrutinized separately if the information at that specific frequency or time is needed.

3.4 Hybrid Data Processing and Representation

After sampling and FORM computations, probability of failure data exist for each respective method. These two sets of data must be combined to form one continuous monotonically increasing CDF. Both Methods are approximations and may not exactly match, therefore, FORM and Sampling data overlap in the hybrid method helping facilitate a smoother combination of the data. For FORM, approximation error increases in u-space as the limit state

function approaches the origin. A probability of failure level of 2% was assumed as an upper limit for trusting FORM solutions. Above this level of probability the approximation error is likely to be significant. However, sampling is more accurate when there are a significant number of points in the failure region compared to the total number of samples evaluated. Sampling results with less than 2% of the samples in the failure region were assumed less accurate than a FORM solution at that probability level. For this research 200 sample points were typically used, so less than 5 was considered not as accurate as FORM. The reasoning behind the transition between FORM and sampling is based on an assumption of when to trust FORM and when to trust sampling. The data combination logic was defined to achieve a transition between FORM and sampling, using a few safeguards to ensure a smooth and monotonically increasing final CDF. Logic must be specified for the combination when the points don't exactly line up. The logic used is as follows:

- The end 4 points of the sampling are discarded due to lack of accuracy.

- If all FORM points have a Pf less than the 5th sample point from end of the CDF, they are appended to the sampling Pf vector.

- If any FORM points result in PF greater than the 5th sample point from end of the CDF, they are discarded and the remaining points are appended to the Pf vector.

The 5th sample point from the end is assumed as the limit between when FORM is trusted and where sampling is trusted. The assumed limit is the justification for discarding FORM points greater than this 5th sample point from the end of the CDF. The combination logic is used for both tails of the CDF, and is necessary to insure a proper CDF.

One of the desires of generating the data to produce a full CDF of the system response is the ability to calculate the mean and variance of the system response. Both pieces of information

are very useful in the analysis of SISO systems. Depending on the distributions of the parameters and the characteristics of the system the mean response may or may not follow the response of the system with nominal parameter values. Representing the spread of the CDF, the variance can also be useful if comparing multiple systems to determine which system will have the narrowest range of responses. Given the relation between the CDF and the PDF

$$f(x) = \frac{dF(x)}{dx} \tag{3.3}$$

where F(x) is the CDF and f(x) is the PDF. The expected value is calculated using the CDF data as follows.

$$[E] = \int_{-\infty}^{\infty} x \cdot f_X(x) dx = \int_{0}^{1} x \cdot dF_X(x) \tag{3.4}$$

Similarly the variance is calculated in the following equation.

$$[V] = \int_{-\infty}^{\infty} (x - [E])^2 \cdot f_X(x) dx = \int_{0}^{1} (x - [E])^2 \cdot dF_X(x) \tag{3.5}$$

Representing the entire distribution along with the system response is unwieldy and difficult to interpret, leading to a method of representing the response by its mean, upper, and lower confidence bounds. The confidence bounds represent some percentage of system responses will be within these bounds.

The data from both methods (e.g. sampling and FORM) representing the CDF are discrete, and will not likely have a datum point exactly coinciding with the desired confidence interval. Thus, the data must be curve fitted to interpolate where the probability limit lies. A spline interpolate works poorly because of the generated CDF data lacks smoothness, which produces overshoot in the interpolate. By definition monotonicity must be maintained since the CDF is

the integral of an integrand that is always positive. For these reasons a piecewise cubic Hermite interpolating polynomial was chosen for fitting the output CDF data. This type of interpolating polynomial is produced in MATLAB with the *pchip* command. Given data x and y defining the CDF, this Hermite interpolating polynomial produces P(x) which is the cubic interpolate on the interval $x_i < x < x_{i+1}$, for every interval of the data. This method was chosen because overshoot is not encountered with non-smooth data and the piecewise cube Hermite polynomial uses slopes at x_i and x_{i+1} to preserve the shape of the given data. The *pchip* command eliminates problems found with splines fit, with respect to the monotonicity of the CDF data. Clearly illustrated in Figure 3-2 that the spline interpolate does not provide the necessary
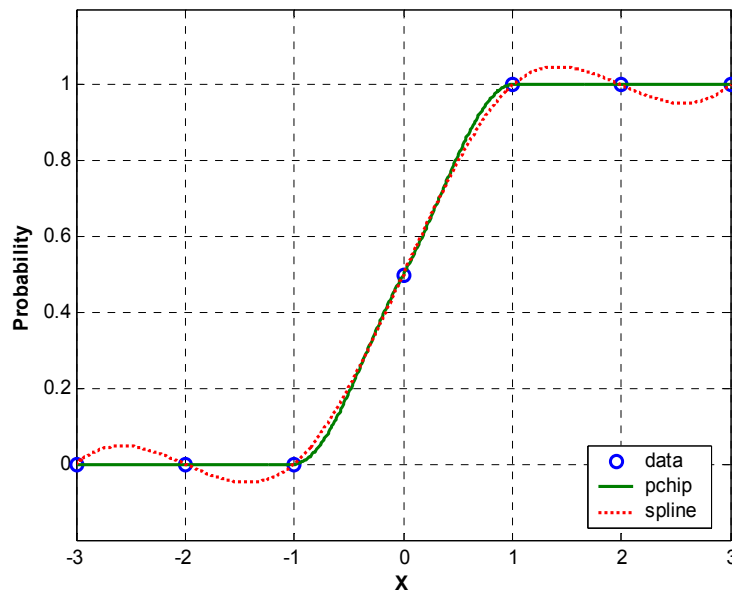

Figure 3-2: Pchip vs. Spline Curve Fitting

monotonically increasing function, where the pchip interpolation provides a feasible CDF.

Once the CDF data is smoothly represented, the value of the CDF is found for the upper and lower bounds. This representative data is then used to produce the resultant response plot,

whether it is a Bode response or a step response. Having the fully defined CDF also allows quick representation of the response with any other desired confidence intervals without having to regenerate the response.

3.5 Response Analysis Issues

The hybrid method of response analysis works for a wide range of systems. In general, good performance is achieved but the automated method of performing FORM calculations does not guarantee finding probability levels to the desired limits. FORM does provide a benefit to defining the CDF but also possesses it's own difficulties. One issue arises from using an exponential decay model when extrapolating the tails of the CDF and determining a ΔRef. If the tail of this CDF does not fit the model, the extrapolation technique may perform poorly preventing a converged FORM solution. An example of this occurs when using uniform distributions for the uncertain parameters. The often sharp drop off in probability of the response CDF makes the exponential decay model less efficient and may miss the point where the probability drops suddenly. An unknown shape of the CDF tails and a large possibility of response distributions means any extrapolation method is unlikely to perform well in all cases. One solution to this problem may be the inverse MPP problem, instead of choosing a *Ref* value and finding the Pf with FORM, the Pf could be given and perform an inverse problem to find the *Ref* value that produces the chosen Pf. This would eliminate any extrapolation technique because desired levels of Pf would be able to be exactly chosen. No research has been performed on this problem, so the inverse MPP approach remains as future work.

A second issue causing difficulty for the hybrid method originates with the FORM process. There are cases where the Pf reduces, although the MPP does not move significantly farther away from the origin in u-space. With multiple FORM calculations performed to develop the

end of the CDF, the limit state function normally moves farther from the origin with each new calculation. The FORM process uses a first order approximation of the limit state function so the probability of failure is found directly from the distance the MPP is from the origin. Equation (1.1) showed that Pf is found by integrating over the failure region, therefore the Pf is reduced when this area is smaller even if the MPP does not move farther from the origin, in u-space. If the MPP does not move significantly farther from the origin, the reduction of probability is not captured by FORM and the approximation error increases. An example of this is illustrated in Figure 3-3, where the approximated Pf stays constant as the limit state function
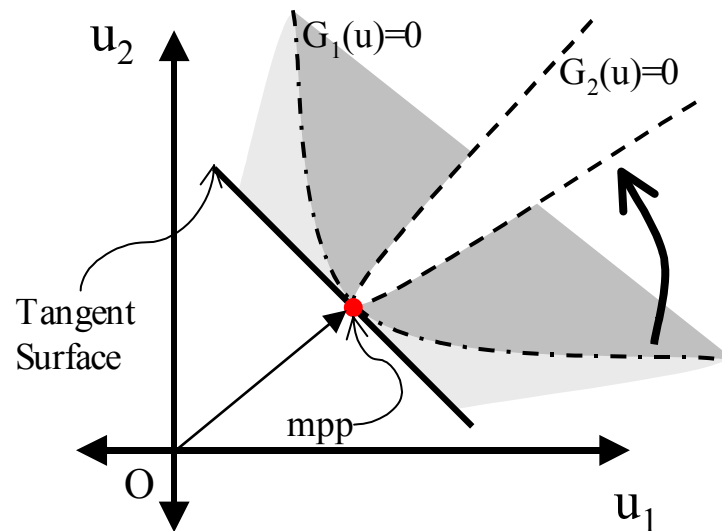

Figure 3-3: First Order Approximation Problem

shifts from $G_1(u)$ to $G_2(u)$. As FORM is calculated at each different location, the limit state function increases its curvature instead of moving away from the origin. Occasionally seen in Bode magnitude plots around the system poles, FORM calculations may never reach the desired limit if the MPP does not shift away from the origin in u-space. The implementation of SORM has the potential to improve accuracy in probability of failure calculation by reducing the error in approximation of the highly curved limit state function.

An issue that can lead to limited low probability computations of a CDF is found when the limit state function becomes non-smooth or highly erratic. This issue is particular to the use of a gradient based optimizer in the FORM process, not the overall FORM concept. Having many minimums or sharp edges in the limit state function can prevent the gradient-based FORM optimization from converging to a solution, or converging to the correct solution. Designed as a general tool for system analysis, the hybrid method cannot accommodate or work around all of these issues. Most difficulties can be determined from the resulting information of the overall analysis. Adjustments to the developed method for specific problems can often solve these issues, so that full CDF's to the desired level of probability may be achieved.

The last difficulty is specific to phase representation and is not just a problem with the hybrid method. If the variation in the uncertain parameters causes the phase response to range greater than 2π, producing a representation of the CDF becomes difficult. The current process developed does not have the ability to accommodate a CDF that spans a range greater than 2π.

3.6 Extending to Parameter Space Analysis

While probabilistic response plots analyzed a specific set of distributions, parameter space analysis is the concept of looking at how large the uncertain parameters can be allowed to expand, before undesirable system metrics occur. The parameter space is defined for this work as, the hypercube containing all possible combinations of the uncertain parameters. The standard approach finds the amount parameters can expand before undesirable metrics are found. Approaching this analysis probabilistically allows the parameter space to be expanded beyond the first undesirable metric, finding the probability that undesirable performance metrics will occur. For the stability example, if this hypercube is allowed to expand beyond the onset of instability the growth rate of probability of instability can be measured. This can give

an insight to the allowable range of parameters for a predefined acceptable probability of instability and an indication of the robustness of the system. The basic concept of the hybrid method is the same however the approach or application is different between the probabilistic response plots and the parameter space analysis. Two types of parameter space analysis are explored, performance metric analysis and probability of instability analysis. The performance metric technique most closely follows the process used in the probabilistic response plots, while probability of instability analysis shares the same basic tools, but the approach is different.

3.6.1 Performance Metric Analysis

Few modifications were necessary to adapt the system response hybrid techniques to analyze parameter space with respect to performance characteristics. Given the full set of uncertain parameters (parameter space) and a performance metric, a CDF of all possible performance metric results can be found. Sampling and FORM are used in the same way as described in sections 3.1 - 3.4. HSS samples are used to give a quick approximation of the performance metric CDF midrange. In this case $C(x)$ is the function that generates the desired performance metric; rise time, peak value, or settling time. Using the sampling as a reference, FORM is used to finish generating the CDF with the limit state function again defined as $g(x)=Ref-C(x)$. 'Ref' is a value used to step away until the low level of probability is reached. This full CDF now describes the range and likelihood of the performance metric results.

Instead of performing the computations of another time or frequency as in the probabilistic response plots, the size of the parameter space is increased and another performance CDF is generated. The mean and variance of these CDFs are computed and then compared with the

performance of the nominal system. A new specific performance metric function J, has been defined in this research. The expected value of J or the variance of J is compared to the performance metric of the nominal system, J_o, in a simple ratio, $E(J)/J_o$ or $V[J]/J_o$. Plots of these two ratios can provide information about how expanding the parameter space, increasing the amount of uncertainty, affects the performance metric defined by J.

3.6.2 Probability of Instability Analysis

As a parameter space analysis tool, stability analysis is very different from the performance metric analysis. The main difference when looking at probability of stability is that a full CDF is never desired. The performance metric analysis finds a full CDF of the metric at each increasing amounts of uncertainty. In probability of instability analysis, if the failure region is a closed space, the probability of instability may never reach a value of one. This difference is the reason that the approach is so different between the two parameter space analyses.

The basic concept is; given a system with some nominal parameter values, how much uncertainty can be allowed before instability is possible in the system. Allowing for probabilistic definitions of the uncertain parameters lets the analysis be taken a step further than conventional approaches to determining stability bounds, where the rate at which probability of instability increases can be determined. It is desired to explore how the parameter space can be enlarged before instability onset changes depending on the shape, or scaling factor, of the increasing parameter space. Largely dependent on how much is known about the

parameter uncertainty, there are many ways to scale this hypercube of the parameter space.

Four such methods are considered;

- Uniform percentage of mean values,

- Ratio based on the most probable point of failure in x-space (mppX),

- Ratio based on the closest point on g(x)=0 to the nominal values,

- Ratio based on the gradient of g(x).

These are discussed next.

Each method is trying to find the probability of instability based on the size of the parameter space, but each is different on how they select the relative scaling between each side of the hypercube. The last three scaling methods are illustrated in Figure 3-4.
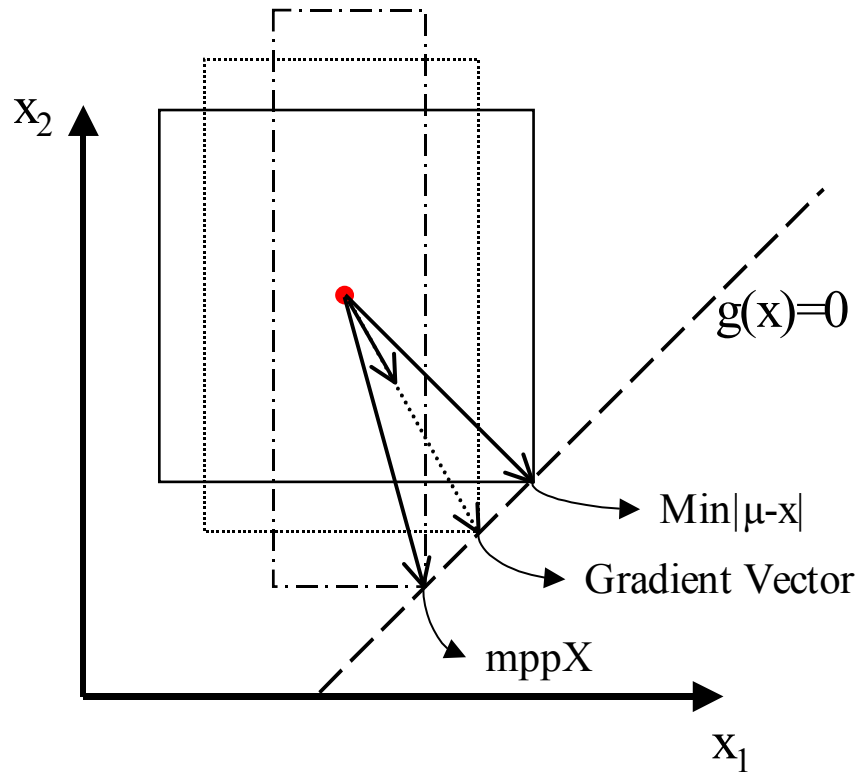

Figure 3-4: Hypercube Scaling Methods

The first method explored was the uniform percent scaling of the mean values. This was chosen as it appears the most common amongst non-probabilistic parametric uncertainty

analysis. Each parameter is defined as the range $\eta_i \cdot (1 \pm \eta_i \Delta)$ where η_i is the mean, or nominal value. This generates a norm bounded set or uniform distribution with all values having equal likelihood. Using a simple uniform distribution does not show the probabilistic benefits, however; it allows for comparison with previous works. The analysis can easily incorporate distribution information when known. The process is the same when using distributions with bounded support with the shape only affecting the calculated probability levels. The largest stable hypercube is initially unknown so sampling is used to find a rough guess of when the parameter space produces unstable systems. A very small amount of uncertainty is allowed then 200 HSS samples are evaluated to check stability. The size of the hypercube is increased until probability of stability is no longer 100%. Having bounded the transition between zero probability of instability and non-zero probability of instability, a bisection technique is used with HSS sampling to find a hypercube producing a low probability of instability. After narrowing down the probability, FORM computations are performed at steps down to a low level of probability of instability. These FORM calculations provide the data for representing how the probability of instability increases as the parameter space increases.

A second method for scaling the hypercube is to find the most probable point of failure in x-space, MPP in x-space, and let the vector from the mean values to the MPP in x-space be the vector to one corner of the hypercube, see Figure 3-4. This method requires some general knowledge about the parameter distributions, not just the mean value. The parameter space is first set very large, though still with the given distribution shapes. A FORM analysis provides a MPP in x-space used to define the new scaling ratio. This method has the benefit of knowing exactly what parameter space range will be the largest stable hypercube. Using the MPP in x-

space as the scale for the hypercube ensures that the hypercube will touch the limit state function first at this point. The hypercube be can slightly increased with this scaling, performing FORM calculations as it grows to find probability of instability.

The previous two methods assumed some previous knowledge of the uncertain parameter distributions. Transformations to u-space require knowledge of the uncertain parameter distributions, however, basic information of the parameter distributions may be unknown. One way to determine a hypercube scaling factor without this knowledge is to work in x-space. The closest point to the nominal parameter values on the limit state function, g(x)=0, is found and used to set the scaling ratio of the hypercube, see Figure 3-4. This closest point is found from the following constraint equations, where η is again the nominal parameter value.

$$
\begin{aligned}
&\min \ |\eta - x| \\
&\text{subject to} \quad g(x) \ = \ 0
\end{aligned}
\tag{3.6}
$$

This minimum distance point in x-space, similar to MPP in x-space method, gives the largest set of parameters with this scaling that will maintain stability. The parameter space hypercube is then increased from this starting size using FORM to see how the probability of instability increases as the hypercube grows.

The final technique used for developing a hypercube scaling was to use gradient information of the state function, where the ith hypercube element can be written as,

$$
\nabla_i(g) \ = \ \frac{\partial g}{\partial x_i} \approx \frac{\Delta g}{\Delta xi}
\tag{3.7}
$$

A finite differencing approach was used to find the gradient information of the limit state function seen in equation (3.7). This finite differencing was done using Δx_i as the difference between 95% and 105% of the nominal value of x_i. Unlike some methods, it is not immediately

known what range of parameters will first introduce instability using this method of determining hypercube scaling. The gradient information gives a vector that points in the direction the greatest rate of change in g(x). Points are selected along the direction of this vector until a sign change is g(x) is found, indicating the transition into the failure region. A bisection technique can be used to find a more precise value for the point on the limit state surface. From this point FORM can be used to find the probability of instability in a similar way as the previous methods.

Each of these scaling methods will provide the largest parameter values allowable, given that scaling factor, that ensure 100% stability. The additional probabilistic information can give insight to how quickly the probability of instability in the system grows when these limiting values are exceeded.

Chapter 4

Analysis of Hybrid Method

4.1 Definition of Example Problem #1

Two different example problems on uncertainty analysis were chosen from the literature to compare with the newly developed hybrid method. The first example, developed by Wise [22][7][23], is a missile pitch autopilot system with four uncertain parameters. The following aerodynamic equations and nominal aerodynamic stability derivatives represent a trim angle-of-attack of 16 degrees, Mach 0.8, and altitude of 4000 ft. With a linearized set of equations, the pitch dynamics decouple from the roll-yaw dynamics of the missile system. The state space representation of the pitch dynamics is,

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} Z_\alpha & 1 \\ M_\alpha & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} Z_\delta \\ M_\delta \end{bmatrix} \delta_e \tag{4.1}$$

where α is angle-of-attack, q is pitch rate, δ_e is elevon fin deflection. The uncertain parameters are the dimensional aerodynamic stability derivatives with the following nominal values used: Z_α=-1.3046 (1/s), Z_δ=-0.2142 (1/s), M_α=47.7109 (1/s^2), and M_δ=-104.8346 (1/s^2). The corresponding accelerometer and gyro output equations are,

$$\begin{bmatrix} A_z \\ q \end{bmatrix} = \begin{bmatrix} VZ_\alpha & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} VZ_\delta \\ 0 \end{bmatrix} \delta_e \tag{4.2}$$

where A^z is normal body acceleration and V is velocity. Having a damping ratio of $\zeta=0.6$ and natural frequency of $\omega=113.0$ (rad/s) the dynamics of the elevon fin actuator are governed by equation (4.3), where δ_{ec} is the commanded elevon deflection.

$$\ddot{\delta}_e = -2\zeta\omega\dot{\delta}_e - \omega^2\delta_e + \omega^2\delta_{ec} \tag{4.3}$$

Incorporating the actuator dynamics the linearized missile dynamics can be represented in the following transfer function.

$$G(s) = \begin{bmatrix} \dfrac{\omega^2 V(Z_\delta s^2 + Z_\alpha M_\delta - Z_\delta M_\alpha)}{(s^2 - Z_\alpha s - M_\alpha)(s^2 + 2\zeta\omega s + \omega^2)} \\[4mm] \dfrac{\omega^2(M_\delta s + M_\alpha Z_\delta - M_\delta Z_\alpha)}{(s^2 - Z_\alpha s - M_\alpha)(s^2 + 2\zeta\omega s + \omega^2)} \end{bmatrix} = \begin{bmatrix} \dfrac{A_z}{\delta_{ec}} \\[4mm] \dfrac{q}{\delta_{ec}} \end{bmatrix} \tag{4.4}$$

See reference [23] for a full development of the system.

A classical autopilot structure is given to control the commanded elevon fin deflection, δ_{ec}, based on the normal body acceleration and pitch rate outputs. A block diagram of the system with the two controller blocks is given in Figure 4-1. The two controllers have the following
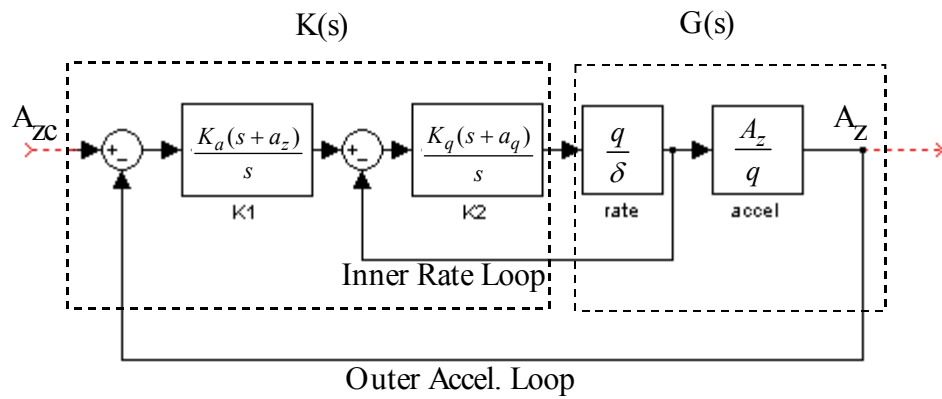

Figure 4-1: Classical Pitch Autopilot

values: K_a=-0.0015, K_q=-0.32, a_z=2.0, and a_q=6.0. The overall system provides the normal body acceleration, A_z, in response to an acceleration command. This missile pitch system allowed the hybrid method to be compared to the analysis done by Wise[23]. The original papers [22][7][23] did not use a probabilistic representations of these parameters, however assumed distributions were given to each of the parameters. All four uncertain parameter was assumed to have a beta distribution having shape coefficients of 2 and 2 with limits of plus and minus 50% of the nominal value.

4.2 Probabilistic Response Plots

The Bode and step response analysis were explored first, with the missile pitch problem. Some results of the hybrid method producing probabilistic response plots are shown in this section. The performance of the hybrid method was also analyzed. One hybrid method analysis was looking at the benefits of the hybrid approach when calculating the mean and variance of the distributions. The hybrid approach was also compared with a standard μ-analysis technique.

4.2.1 Bode Analysis

The Bode response provides information about how physical system responds to sinusoidal inputs over a range of frequencies after all transients have died out[24]. Introducing probabilistic information into this classical analysis tool can expand the benefits of this analysis, showing what frequency ranges are most affected by the parametric uncertainty. Frequency response techniques such as Bode analysis must represent both parts of the complex data, magnitude and phase. Because the FORM process is optimizing a scalar output the two portions of the complex data must be analyzed separately, forcing Bode magnitude and phase

plots to be generated separately with the hybrid method. Using the missile pitch autopilot system described in section 4.1, probabilistic Bode magnitude and phase responses can be seen in Figure 4-2. The hybrid method propagated the parametric uncertainty through the system
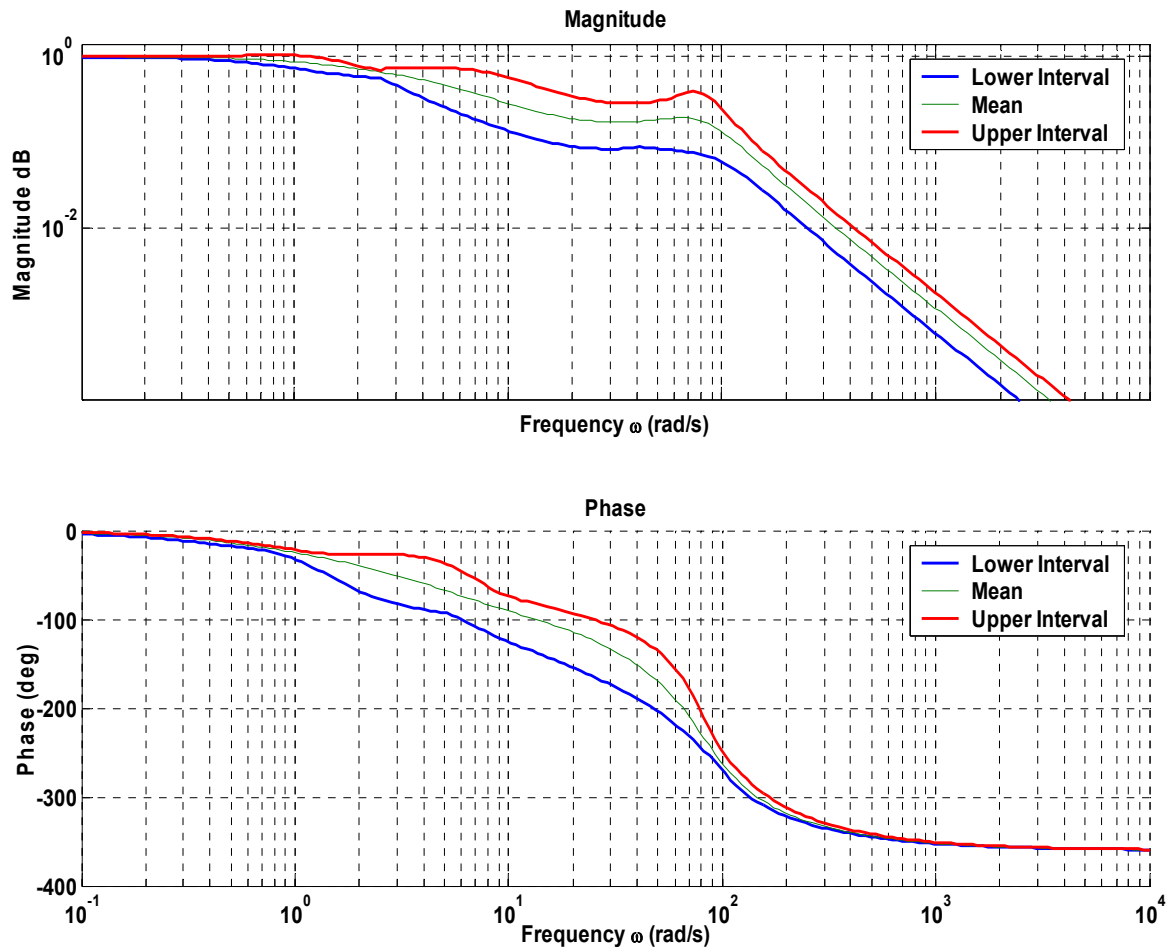

Figure 4-2: Probabilistic Bode Analysis

providing a full distribution of the magnitude and phase at each frequency. Representing this probabilistic information in the traditional Bode plot without adding a third axis produces cluttered and difficult to read graphs, as mentioned in section 3.4. The probabilistic information in Figure 4-2 is represented by the mean, upper, and lower confidence intervals. This technique simplifies representation of the data to produce a clean readable plot. For this analysis,

confidence bounds of 99.999% are represented, which means 99.999% of all responses will be

below the upper bound and 99.999% of the responses will be above the lower bound. Although,

only one set of bounds is displayed, the entire CDF has been calculated, therefore confidence

bounds of any level less than the FORM computation limit can be displayed without

reanalyzing the system. Viewing the results seen in Figure 4-2 shows the system is always

stable with the assumed distributions on the parameters.

4.2.2 Step Response Analysis

Similar to the Bode Response Analysis, a probabilistic step response provides information

not found with norm-bounded uncertainty techniques. The hybrid method does not change

when applied to the step response; only the limit state function is different. The limit state

function that represents the step response of the system is evaluated at individual time intervals

in the same style as the probabilistic bode response. This probabilistic step response of the

missile pitch problem can be seen in Figure 4-3. The variance is calculated, along with the
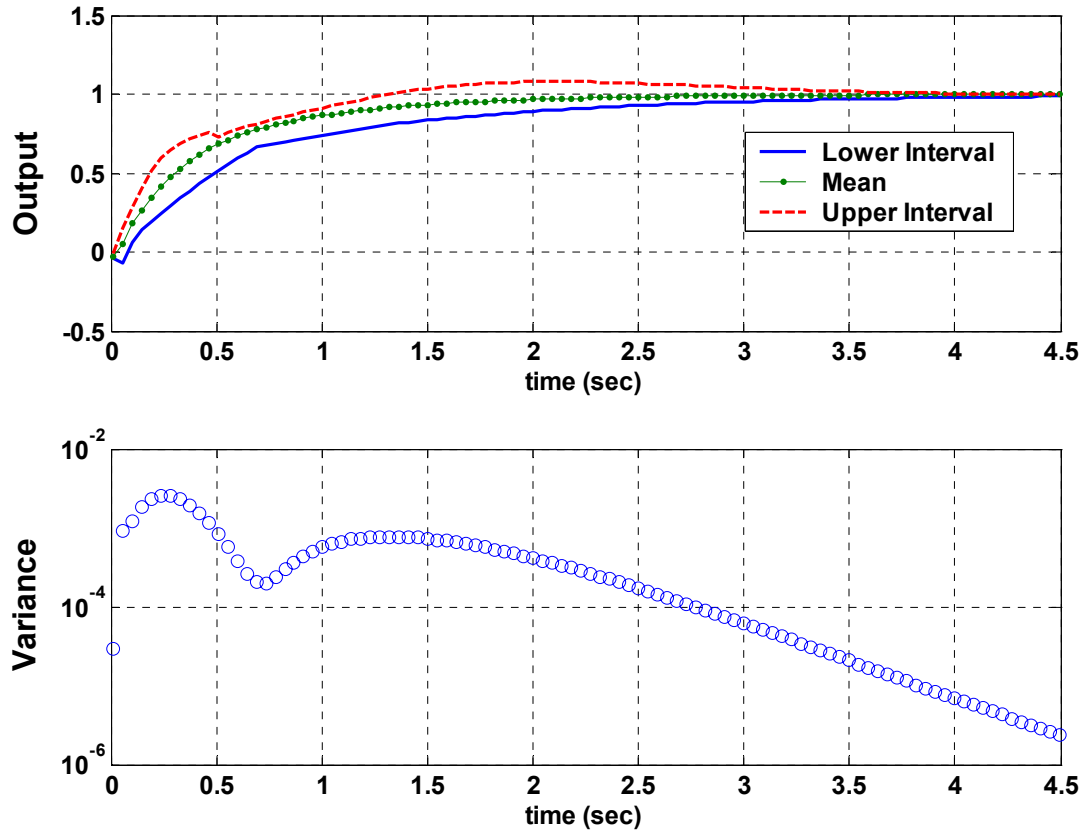

Figure 4-3: Probabilistic Step Response

mean and confidence intervals, for all the points of the response. The mean and variance of the

response are only available with a probabilistic representation of the response plots. This

probabilistic step response depicts areas that are more affected by the uncertainty. The first 0.5

seconds for example have a large variation in response while the variance drops as the system

reaches its steady state value.

4.2.3 Comparing the Hybrid Method with Standard Uncertainty Analysis

Verification that the hybrid method is indeed producing accurate results requires that it be

compared with an existing method for uncertainty analysis. It was stated earlier that the

standard uncertainty analysis and design methods in controls, μ-analysis and H^∞, are dominated by 'worst-case' scenarios, essentially the vertices of the parameter hyperspace. While this technique neglects considerations of likelihood of the response, it is useful to compare the hybrid method to this standard procedure. Giving all parameters uniform distributions to define the uncertainty allows the hybrid method to be compared directly to the current standard of a delta block representation of the uncertainty (see Figure 1-3). The delta block representation is the current basis for most uncertainty analysis which does not produce probabilistic information, however the two different methods should produce the same bounds for the system response. Figure 4-4 shows the resulting comparison of the Bode magnitude analysis of a simple mass-spring-damper problem. It can be seen that the hybrid method
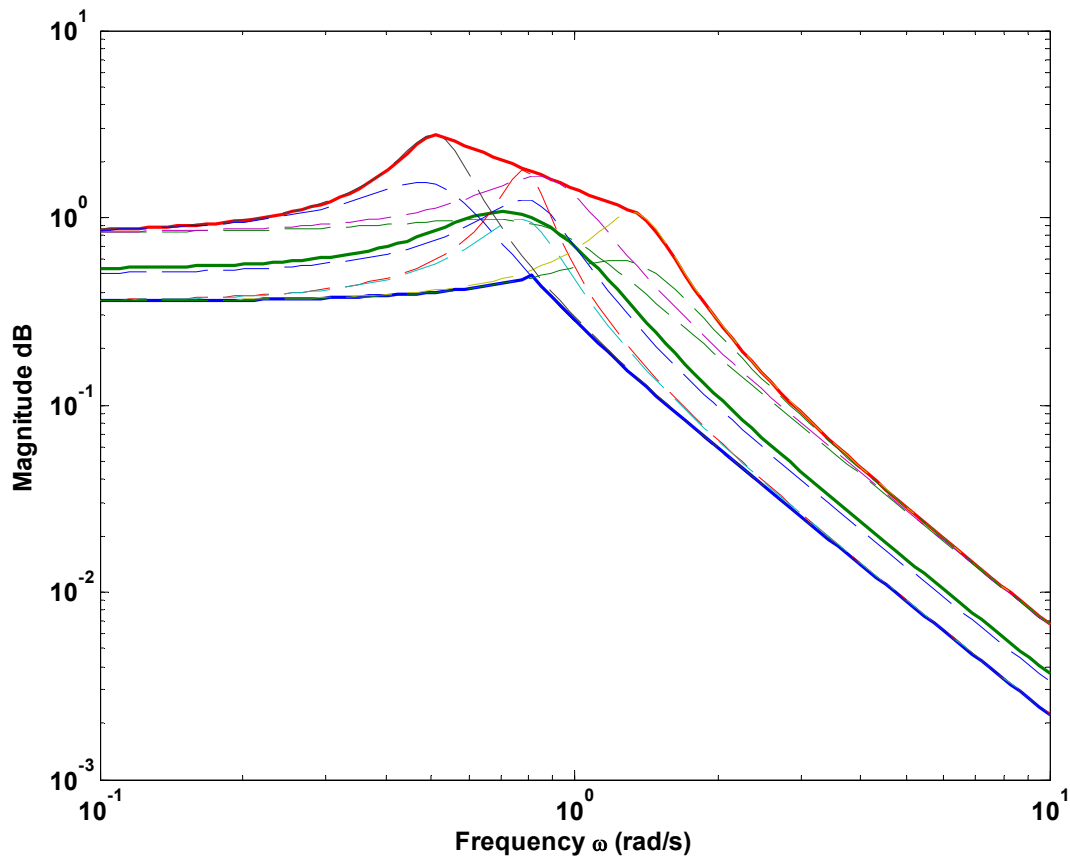

Figure 4-4: Hybrid Method Compared to Δ Block Representation of Uncertainty

produces very similar bounds compared to an analysis of the system with the uncertainty defined in a delta block. In Figure 4-4 eight dashed lines represent responses from the eight vertices of the parameter space for this system. These eight lines are not easily seen in Figure 4-4, because they lie on top of each other for some frequencies. Small differences of the bounds can be related to the hybrid method producing bounds of 99.999%, while the verticies of the parameter space represent bounds of 100%. Nevertheless, this comparison gives confidence that the hybrid method is producing accurate results. A follow-up analysis of 10,000 HSS points ensured that all responses were contained within the response envelope.

4.2.4 Mean and Variance Benefits of Hybrid Approach

There are two reasons why FORM analysis has been used in the Hybrid method. Because of Safety considerations a probabilistic analysis of control systems in the aerospace field requires handling very low levels of probability, requiring knowledge of distributions extending into the tails. Secondly, when computing mean and variance of the response distributions, significant error can result from distributions inadequately defined in the low probability regions. Approached with sampling only the number of sample points must be increased by orders of magnitude to lower the achievable probability value as seen in section 1.4.1. An analysis of the mean computations of a system shows that the addition of FORM calculations can reduce the error in the calculated mean. This benefit in the mean calculation with the hybrid method

(shown in Figure 4-5) is similar to increasing the number of sample points by a factor of 100.
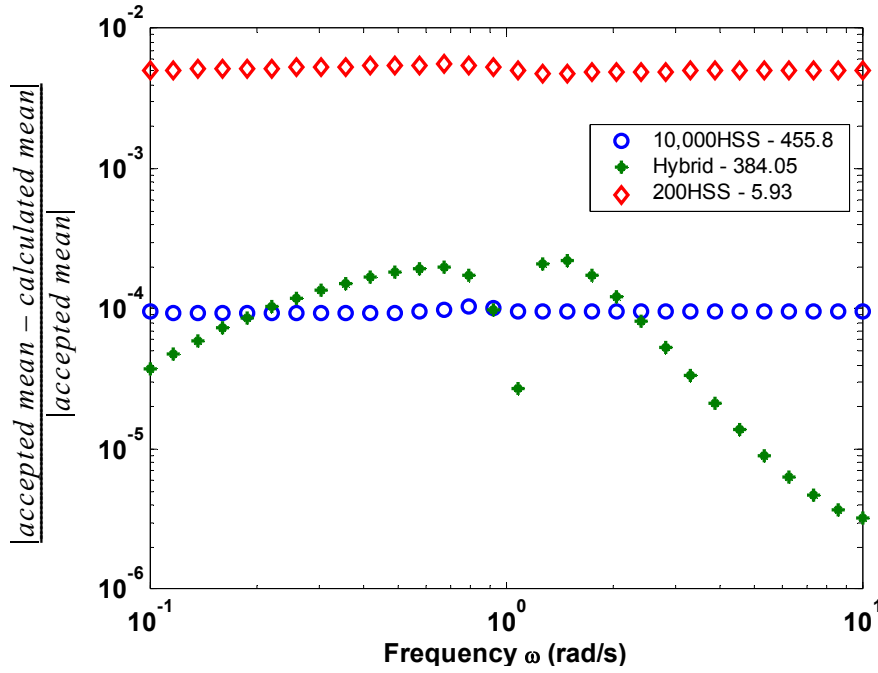

Figure 4-5: Mean Computation Error for Hybrid and HSS Methods

The comparison analysis in Figure 4-5 shows the error in the mean computations of a Bode phase analysis in the spring mass system (section 4.2.3) with three uncertain parameters. An analytic result for the exact mean is not available, an assumed 'true' answer was found from a Monte Carlo simulation with 300,000 samples. The mean response at each frequency was then calculated using a trapezoidal integration technique for solving equation (3.4). Three cases; 200 HSS, 10,000 HSS, and the hybrid approach, were performed and the mean values of the response were calculated. Figure 4-5 shows the difference between these cases and the accepted value $\frac{|accepted\ mean - calculated\ mean|}{|accepted\ mean|}$. The comparison shows how adding FORM results on the tails of the distributions generated by 200 HSS points improves the mean

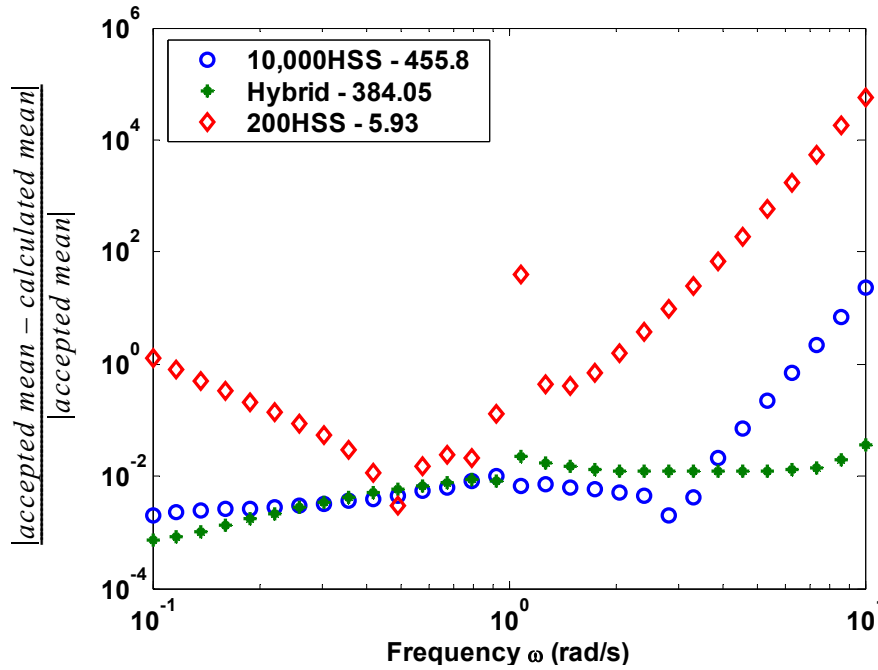result similar to the level achieved by using 10,000 HSS samples. Figure 4-6 is a similar

Figure 4-6: Variance Computation Error for Hybrid and HSS Methods

analysis of the variance calculation, equation (3.5), also showing the benefit of added FORM

calculations in the Hybrid method. The legends in both Figure 4-5 and Figure 4-6 show the

computation time in seconds for evaluating the probabilistic response for each method. The

hybrid method, at 384 seconds, is a quicker than computation of the 10,000 HSS points, at 455

seconds. Although the time increase is only modest for this example the hybrid method

generally still provides information at lower probabilities than 10,000 HSS evaluations. For

comparison the 300,000 MCS computation took approximately 37 hours.

4.3 System Response Code Testing

The hybrid method of probabilistic response plots was developed to be generic and to be

applicable to a wide range of system sizes. To test the technique, a simple way to generate a

large number of different uncertain test systems was needed. This scalable test system generates stable transfer functions with an arbitrary number of random variables. The scalable testing model was used both for systematic testing of the hybrid method and for computational effort analysis.

4.3.1 Scalable Testing Techniques

The scalable model is built using the real portion of the system poles as random variables. Using the poles as the basis for the random variables was done both for simplicity and for producing systems of similar style while they were scaled. The technique developed starts by using the *rmodel* function in Matlab to define a random stable transfer function. Considering complex conjugate pairs as one variable the real component of each pole is given a probabilistic distribution. All distributions were lognormal, with the nominal pole value used as the mean, and a standard deviation defined by 10% of the pole value. Although all poles are defined by lognormal distributions, using the pole value in determining the standard deviation gave a wide range of distribution shapes. The order of the system transfer function could now be arbitrarily set, thereby quickly producing systems of order *n.* Since there are an unknown number of conjugate pairs, the number of random variables is less than *n*. This drawback of basing the scalable model off the *rmodel* function is that the number of random variables cannot be directly specified. While the number of poles is specified, an unknown number of complex poles will cause some variation in the number of random variables. Nevertheless, this scalable model allowed for extended testing of the hybrid software to ensure it wasn't overly specialized for the few specific example problems being analyzed.

4.3.2 Computational Effort Analysis

One of the main benefits of the scalable testing model was the ability to generate an extensive computational effort analysis relating the CPU time with the number of uncertain variables. Figures 4-7 through 4-9 show the computational time in minutes for an increasing number of uncertain parameters. The time represented is the time necessary to complete a full Bode magnitude or phase response plot using the hybrid method evaluated at 20 evenly space frequencies. Some variation is expected even in systems with the same number of random variables since the exact number of FORM calculations cannot be specified, therefore the number of FORM computations at each frequency is not the same. This computation analysis is meant to look at the trend of the hybrid method computing a full system response. The computation time as a function of the number of random variables is depicted in Figure 4-7.
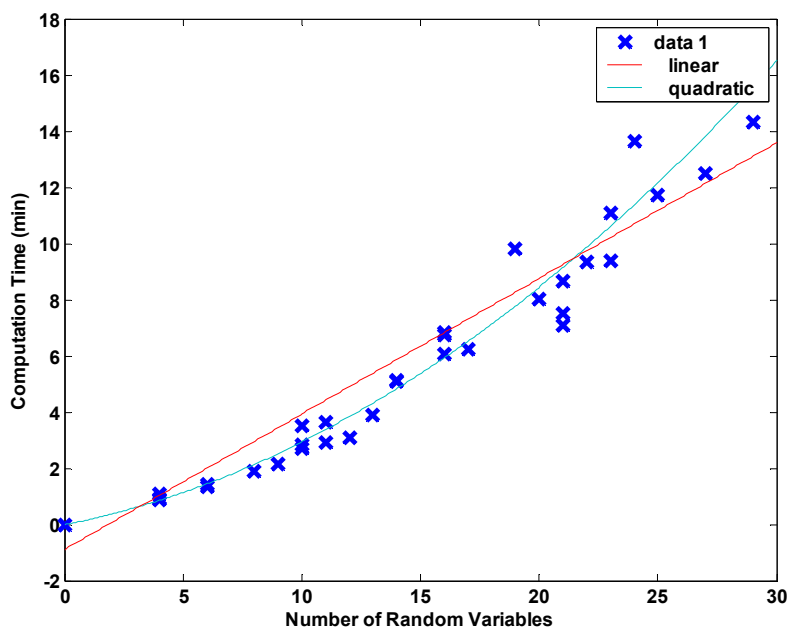

Figure 4-7: Bode Magnitude Computational Time Analysis

Fitting a second order polynomial through the data points shows the apparent second order growth trend in CPU time. For the Bode magnitude data seen in Figure 4-7 the norm of the

58

residuals for the fitted data was 8.8 for the linear fit and 6.3 for the quadratic curve fit. It becomes evident that as the number of random variables increases the computational effort will eventually become prohibitively costly. The complexity of the limit state function is also a major contributor to the computational time. While a more complicated limit state functions increase computational time, the second order growth remains evident. This can be seen in the difference between time analysis of the Bode magnitude and Bode phase plots. The computational effort for the Bode phase plot is depicted in Figure 4-8, it can be seen that the
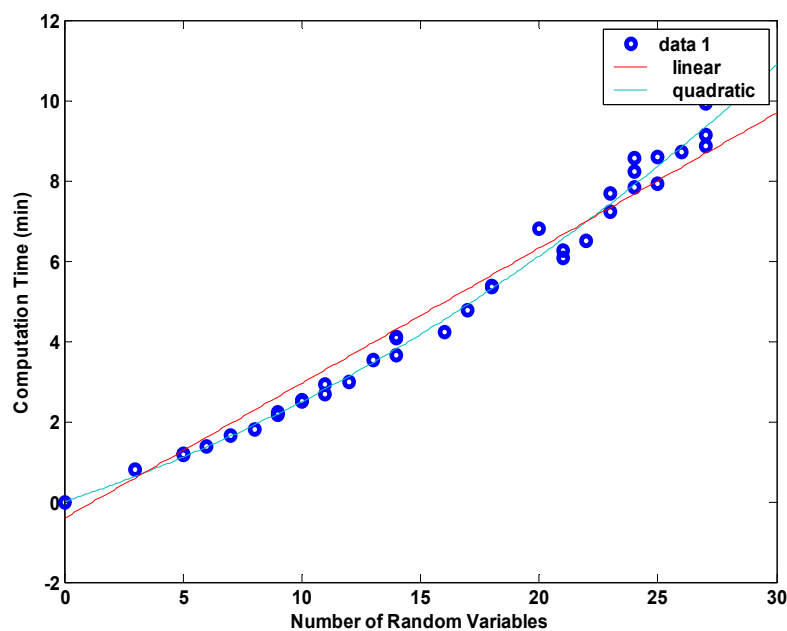

Figure 4-8: Bode Phase Computational Effort Analysis

data has a similar quadratic curve as Figure 4-7, however; the phase plot computations are faster. For the Bode phase data, the norm of the residuals was 3.2 for the linear fit and 1.7 for the quadratic fit. The step response is a more complicated function to evaluate than either the Bode magnitude or phase, thus it is expected to require more computation time. Illustrated in

Figure 4-9, the time to compute the step response is significantly greater than for either Bode
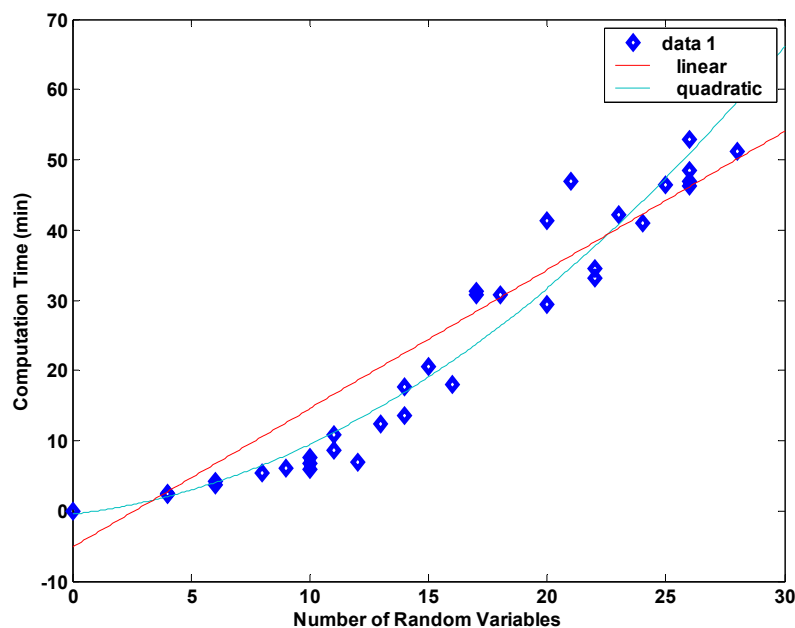

Figure 4-9: Step Response Computational Effort Analysis

responses. The quadratic curve fit again fits the data better than a linear fit, with the norm of the

residuals being 40.57 for the linear and 25.9 for a quadratic fit.

4.4 Definition of Example Problem #2

The second example problem chosen from the literature is a two-mass-spring system

depicted in Figure 4-10, with nominal parameters $m_1=m_2=1$ and $k=1$ [26]. A position sensor is
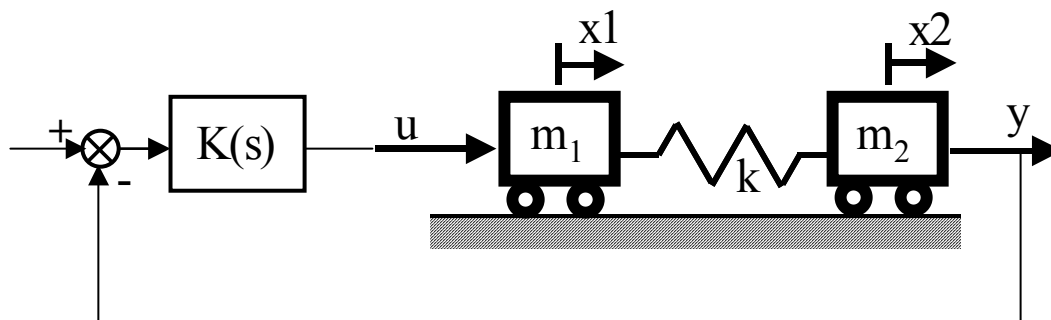

Figure 4-10: Non collocated two-mass-spring system

located on m_2 and the controller input acts on m_1 for this non-collocated problem. The transfer function representing the systems is given as

$$TF(s) = \frac{y}{u} = \frac{\dfrac{k}{(m_1 m_2)}}{s^2 \left[s^2 + k \left(\dfrac{m_1 + m_2}{m_1 m_2} \right) \right]} \tag{4.5}$$

A number of papers were written using different techniques to produce a controller for the system in Figure 4-10, given uncertainty bounds on all three parameters of plus and minus 50%. This example again uses beta distributions with shaping coefficients 2 and 2, to represent the parameter uncertainties. However, parameter space analysis adjusts the limits of the distribution to analyze affects on system characteristics. Each controller submitted was supposed to be stable for the entire range of uncertain parameters and meet a number of different performance criteria. Stengel and Marrison [25] performed a robustness comparison of the submitted controllers. The transfer functions for seven of the controllers from reference [25] follow:

$$A \Rightarrow K(s) = \frac{40.42(s + 2.388)(s + 0.350)}{(s + 163.77)[s^2 + 2(0.501)(0.924)s + (0.924)^2]} \tag{4.6}$$

$$B \Rightarrow K(s) = -\frac{42.78(s - 1.306)(s + 0.1988)}{(s + 73.073)[s^2 + 2(0.502)(1.182)s + (1.182)^2]} \tag{4.7}$$

$$C \Rightarrow K(s) = -\frac{0.599(s - 1.253)(s + 1.988)}{[s^2 + 2(0.502)(1.182)s + (1.182)^2]} \tag{4.8}$$

$$D \Rightarrow K(s) = \frac{19881(s + 100)(s + 0.212)[s^2 + 2(0.173)(0.733)s + (0.733)^2]}{[s^2 + 2(0.997)(51.16)s + (51.16)^2][s^2 + 2(0.838)(16.44)s + (16.44)^2]} \tag{4.9}$$

$$E \Rightarrow K(s) = -\frac{5.369(s - 0.348)(s + 0.0929)}{[s^2 + 2(0.832)(2.21)s + (2.21)^2]} \tag{4.10}$$

$$F \Rightarrow K(s) = \frac{2246.3s + (0.237)[s^2 - 2(0.32)(1.064)s + (1.064)^2]}{(s + 33.19)(s + 11.79)[s^2 + 2(0.90)(2.75)s + (2.75)^2]} \tag{4.11}$$

61

$$H \Rightarrow K(s) = -\frac{2.13(s + 0.145)(s - 0.98)(s + 3.43)}{[s^2 + 2(0.82)(1.59)s + (1.59)^2][s^2 + 2(0.46)(2.24)s + (2.24)^2]} \qquad (4.12)$$

However, the original equations are from the following references [28] A-C, [29] D, [30] E, and [31] F. The remaining three controllers were unable to be reproduced and give a stable system. About half of controllers in equations (4.6) through (4.12) have a leading negative sign to account for inconsistency in negative feedback representation of the original paper.

4.5 Parameter Space Analysis

The probabilistic response plots were developed first, then the hybrid method was reapplied to explore parameter space analysis. Both performance metric analysis and probability of instability were found to produce good results, however; the performance metric analysis exhibited the need for a more problem dependent approach. Most of the parameter space analysis was performed using beta distributions, a bounded support distribution that has two shape parameters a and b that prescribe the curvature within the support of the distribution.

4.5.1 Performance Metric Analysis

While many control system performance metrics exist, a few specific performance metrics were used for the development of the probabilistic performance metric analysis discussed here. The specific metrics were rise time, peak value, and settling time. There were a few early hurdles in adapting the hybrid method to performance metric analysis. While the response plot analysis developed limit state functions directly from response equations, the performance metrics analyzed in this research required limit state function to be produced based on a discretely sampled step response. A coarse spacing of time values produced a very nonsmooth limit state function making the gradient based optimization in MATPA (see section 2.2.3)

perform poorly. Interpolation of the step response around the rise time helped to alleviate this problem.

Another necessary adaptation was that the deterministic definitions of some system characteristics do not work in a probabilistic context. This was first noticed in the definition of rise time; rise time is the time it takes the system to go from 10% to 90% of the steady state value. Parameter variations can alter not only the response speed, but also the steady state value. Evaluating the rise time with a varying steady state value can cause an erratic limit state function, inhibiting the use of FORM. For this research, the definition of rise time was modified to the time for the system to go from 10% to 90% of the steady state value of the nominal system. With these two modifications, the hybrid method is able to provide full CDFs of the given metric, allowing accurate calculations of the mean performance. With uncertain parameters defined having beta distributions, Figure 4-11 shows how the mean rise time of
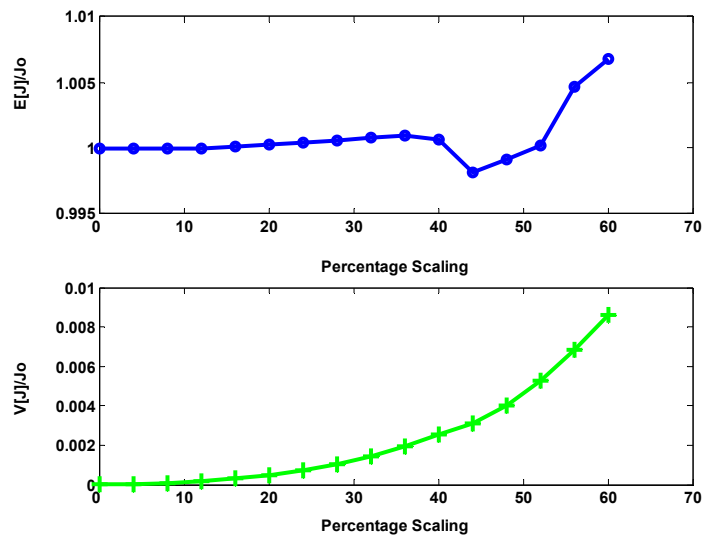

Figure 4-11: Performance Metrics as a Function of Scaled Parameter Space

missile pitch control system (see section 4.1) changes as the bounded supports of the parameter space are increased. Calculated at the same time and also included in Figure 4-11 is the plot of

V[J]/J$_o$, representing how the variance of the performance metric values increases as the supports of the parameter space are increased. For the example shown in Figure 4-11 it can be seen that with uncertain parameters having a range of up to plus and minus 40% of the nominal values, the expected value of the rise time stays very close to the rise time value of the nominal system. As the uncertainty is increased the distribution of rise time spreads out, however stays centralized about the nominal system.

By only changing the limit state function to represent a different performance metric such as peak value, the exact same analysis method can be used. Each performance metric requires a different limit state function, and each brought its own unique difficulties. Settling time provided a few more difficulties than those found working with rise time analysis. With settling time defined as the time the response last exceeds 2% of the steady state value, uncertainty can cause large jumps in the settling time value as different oscillations of the response are the last to exceed 2% deviation from the steady state value. A slight modification to the definition of settling time is harder to define than it was for rise time. This issue inhibits the use of FORM reducing the accuracy of the mean calculations. For the settling time analysis the true hybrid approach only works well when there is enough knowledge of the system step response to know these jumps in settling time value do not exist. For systems where these issues do not arise the settling time analysis can be performed in the same manner as the rise time and peak

value analysis. Figure 4-12 illustrates the Settling time analysis on the missile pitch control
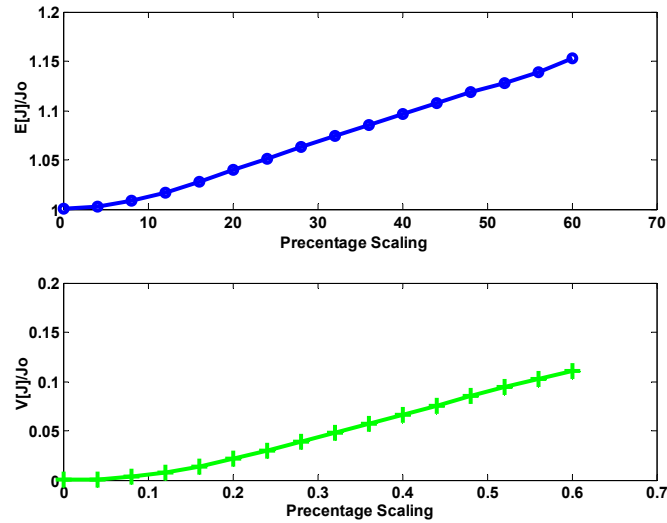

Figure 4-12: Performance Metrics as a Function of Scaled Parameter Space

problem. The expected value analysis in Figure 4-12 shows that with increasing amounts of uncertainty in this system the expected settling time is slower than the nominal system

This performance metric analysis of the parameter space provides insight into system performance as the uncertainty of the parameters is allowed to expand. The knowledge of how desired performance metrics are affected by growing uncertainty bounds can be one additional tool to help find the most desirable control system, however, cost must also be considered. As an uncertainty analysis method, the performance metric analysis is less robust than the probabilistic response plots and requires more knowledge of the system response to ensure an analysis of the performance metric is achievable. There are cases when the analysis is not valid such as peak value analysis of an overdamped system, or cases where FORM becomes non-beneficial such as when gaps in settling time are produced. These added complexities make the performance metric analysis a much more system specific analysis tool.

4.5.2 Probability of Instability

The first step in the probability of instability analysis is to compare results with existing hypercube analyses in literature. The missile pitch control problem, see section 4.1, defined by Wise[23] was analyzed using various methods to find the largest percentage scaling of the parameters before instability in the system is found. The baseline in his work was a Monte Carlo analysis resulting in bounds on the parameters of 60-61% of the mean values before instability is allowed in the system. The uniform percentage method of scaling produces a similar result of 60.4%, as well as the percentages of instability beyond this limit. Figure 4-13 shows the percent probability of instability versus the scaling factor. The scaling factor is the
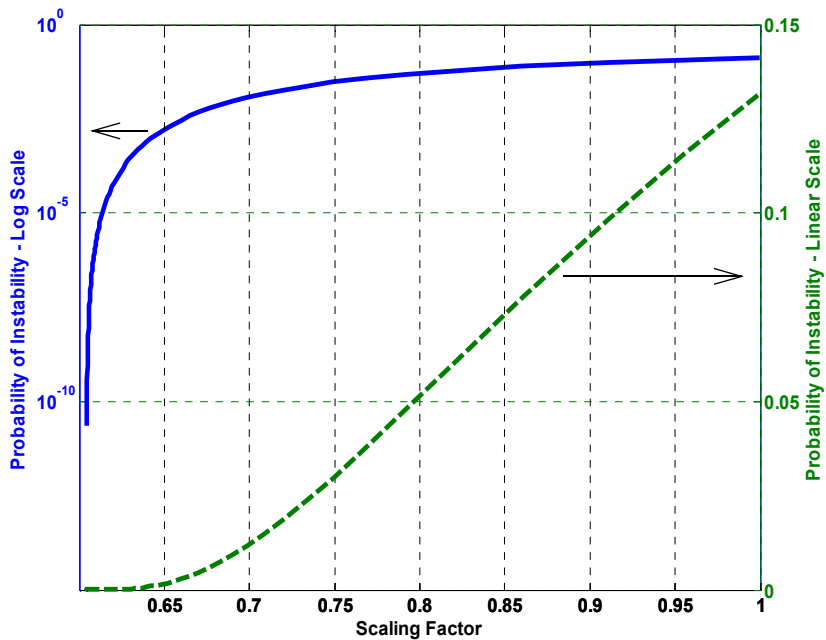

Figure 4-13: Uniform Percentage Scaling of Missile Pitch

percentage of the mean parameter values that defines the bounds of uncertain parameters. In this figure the same set of data is plotted twice, once with a logarithmic scale seen on the left and the other with a linear scale seen on the right. The data was plotted on a logarithmic scale to show the very low probability levels not noticeable on the linear scale. Finding results that

correlate well with previous research gives confidence that the hybrid method is producing accurate results.

Analyzing the system with a norm bounded set, or uniform distribution, is beneficial for comparison to non-probabilistic analysis. The real advantage to the hybrid method is being able to incorporate distributions such as beta distributions. Most physical parameters do not have a uniform distribution but one where each value has a different likelihood. The hybrid method allows the system with different distributions defining the uncertain parameters to be analyzed and compared. A system analyzed with beta distributions having the same support but different shape parameters produces different results. Figure 4-14, shows that the point at which stability
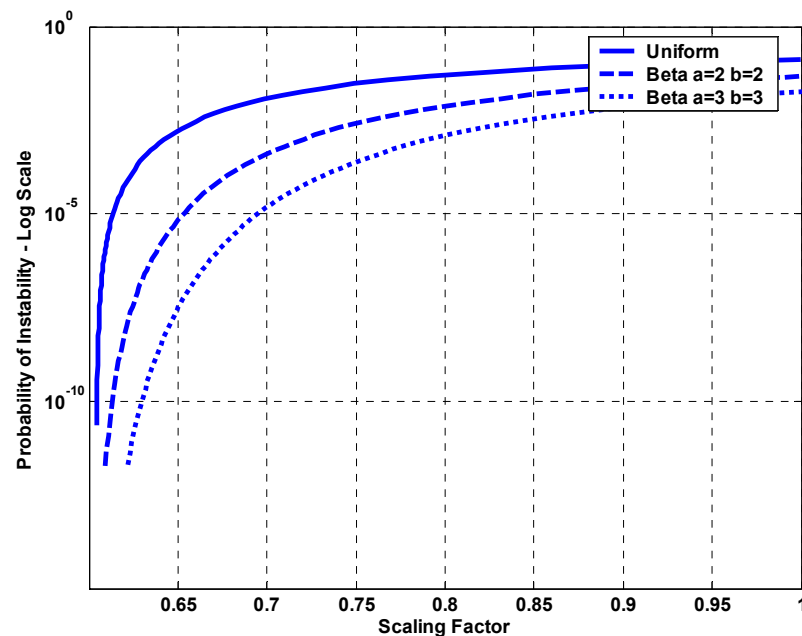

Figure 4-14: Effects of Parameter Distributions on Probability of Instability

is first violated stays similar, however; the rate that probability of instability increases does change with different parameter distributions. This provides better information about the limits on the parameter space if a specified probability of instability is acceptable. If a specified

probability of instability is acceptable, the shape of the distributions representing the uncertain parameters is important. Assuming a distribution for parameters has consequences if the uncertain parameters do not closely represent the assumed distribution. If the uncertain parameters are not closely represented by a uniform distribution, conservative results may be produced.

Another type of probability analysis compares multiple controllers for a given system. Using the Benchmark example describe in section 4.4 all controllers are compared in Figure 4-

15 and Figure 4-16. This analysis used the uniform percentage scaling method, and the x-axis
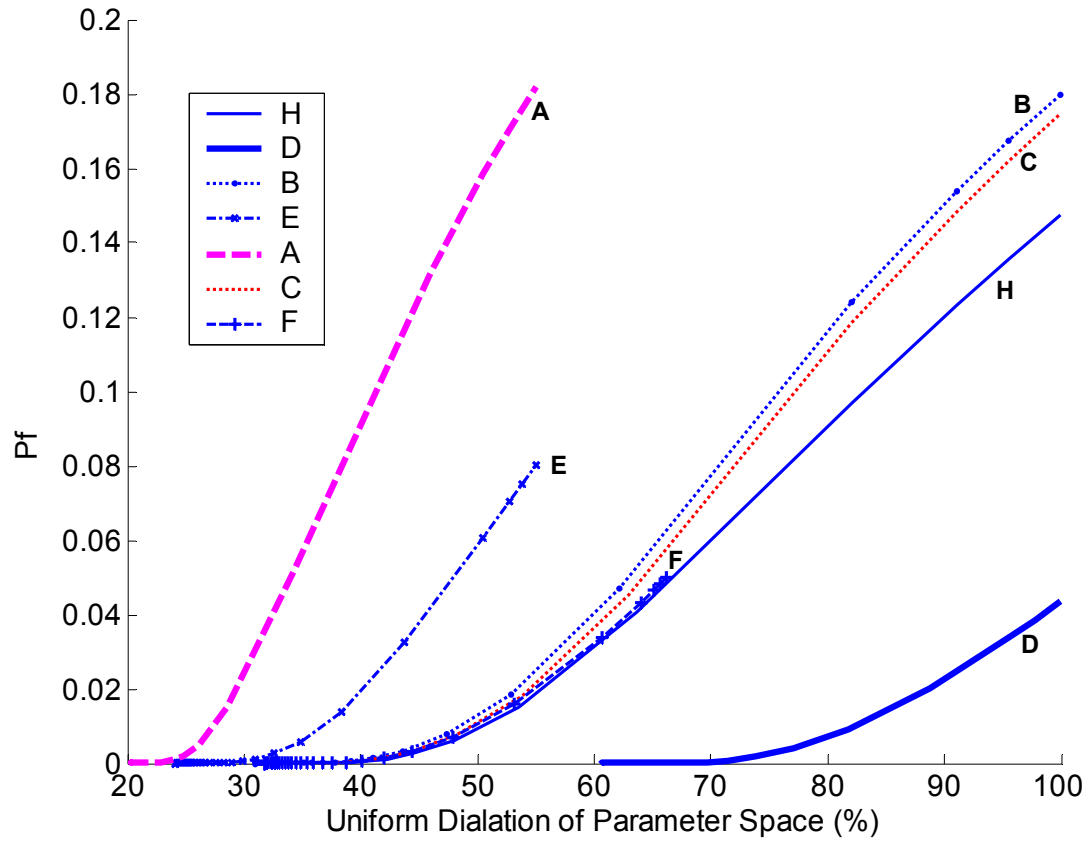

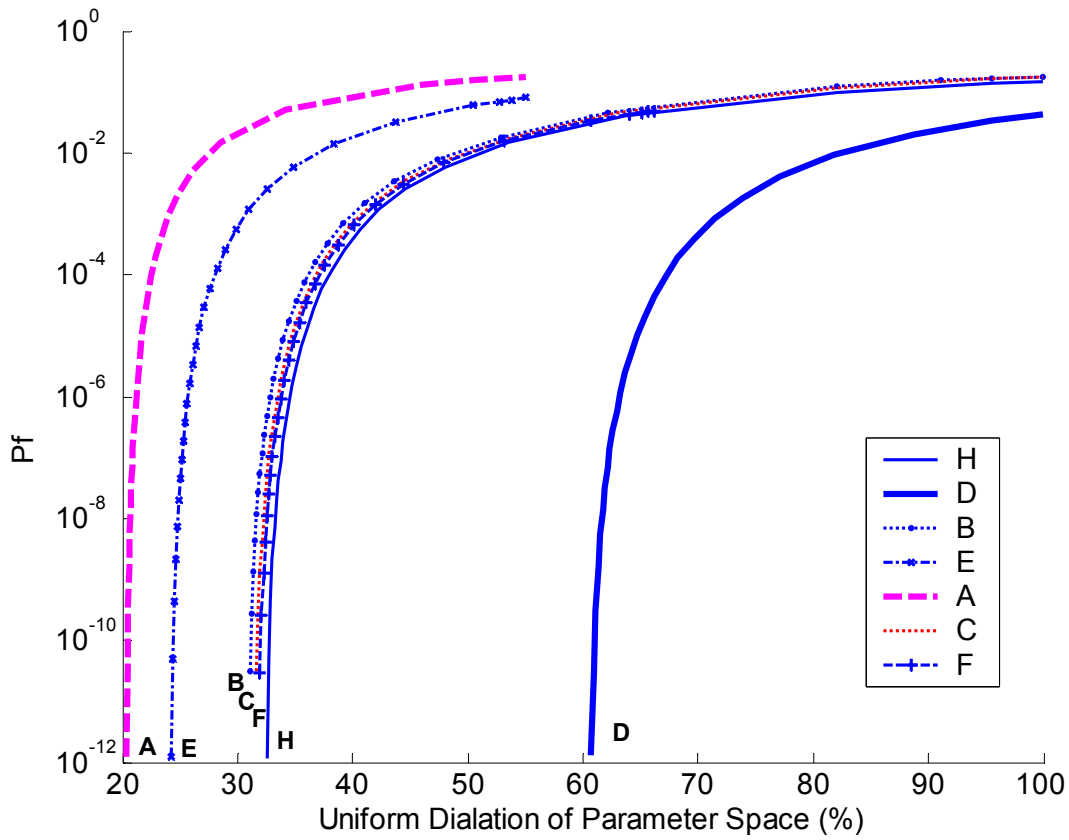**Figure 4-15:** Comparison of Benchmark Problem Controllers, Linear Scale

Figure 4-16: Comparison of Benchmark Problem Controllers, Log Scale

values represent the percentage variation of the uncertain variables. That is the x-axis shows the

percentage change in each of the parameters in the system. Controller D is significantly more

robust than the others, while controller A is the least robust. Controller D allows the uncertain

parameters to vary by 60% of the nominal value before any probability of instability, while

controller A will only tolerate a range of parameters within 20% of the nominal values before

any probability of instability. While each controller will accept a different amount of

uncertainty, all have similar rate of growth in probability of instability. Ranked for robustness,

controller D would be the best choice for the system. The results for all the controllers compare

well with the analysis of the controllers in reference [25], which also shows controller D as the

most stable.

As described in section 3.6 there are many ways the parameter space hypercube can be scaled to fit the needs of each problem. Each hypercube scaling technique was used to analyze the stable parameter space for the missile pitch problem. Table 4-1 shows the nominal values

Table 4-1: Comparison of Hypercube Scaling Techniques on Missile Pitch Problem

Nominal Values	Uniform Percentage	mppX	Closest point in x-space	Gradient of g(x)
Z_α=-1.3046	0.788	0.1528	1.3046	1.2383
Z_δ=-0.2142	0.1293	0.0040	9.3E-07	0.6113
M_α=47.711	28.817	9.5976	2.4E-06	0.0197
M_δ=-104.83	63.317	76.599	9.7E-07	0.00877

for the different variables and the delta values each scaling technique provides for the extent of a stable parameter space. The hypercubes represented by Table 4-1 are the nominal value plus and minus the delta given in the column of each scaling technique. The uniform percentage column represents a hypercube with each parameter having 60.4% variation about the nominal values. Table 4-1 shows that each of the different methods produces a different range of parameters that leads to instability. The requirements of the problem would determine which technique provides the best results.

No one scaling method seems to provide the best overall answer, however; each has its own unique degenerate cases where the method produces poor results. Although unlikely to occur at the same time, both the method using mppX and the method using the closes point in x-space have the same type of problem. The problem arises if the closest point on the limit state

function falls on or near an axis of a random variable, see Figure 4-17. When this condition
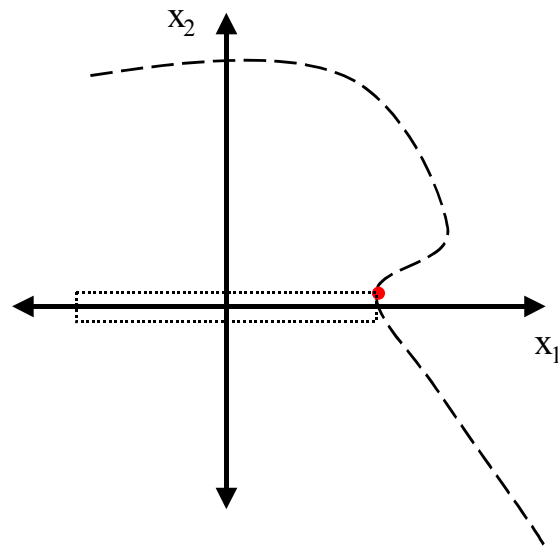

Figure 4-17: Closest Point Hypercube Scaling Problem

exists, the hypercube may be disproportionately sensitive in the direction of that parameter. Though accurate, the parameter space bounds that result may misrepresent how much parameters may vary before inducing instability. The finite differencing method also has a drawback. The finite differencing used to find the gradient of g(x) is done at the mean values of the system, the resultant vector may not point in the direction of the first point of contact

between the hypercube and the limit state surface. See Figure 4-18 for a representation of this
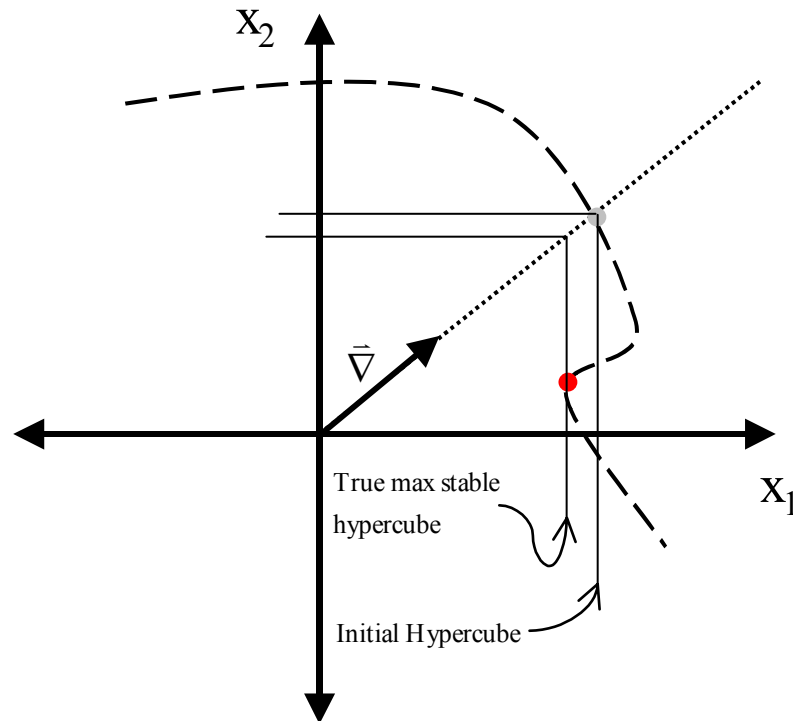

Figure 4-18: Gradient Based Hypercube Scaling Difficulty

problem. With this method points were taken along the gradient vector to find when the hypercube transitions across the limit state function. If the gradient vector doesn't point to the first point of contact, techniques must be developed to find the largest hypercube still within the stable parameter space. This problem can easily be detected if a significant probability of instability is computed at the starting hypercube. When detected, one method for solving the issue is to use the MPP found with the FORM calculation and shrink the hypercube until this point is on the surface of the hypercube. This process may need to be repeated if the scaling of the hypercube allows for a portion of the limit state function to stay within the hypercube. However, once that starting hypercube is found with zero probability of instability the analysis can proceed as usual.

Chapter 5

Conclusions

The increasing demand on aerospace control systems requires high performance characteristics as well as being robust to uncertainties. Individually the two requirements often oppose each other. A probabilistic approach can produce control systems that both improve performance as well as improve robustness. A hybrid method for approaching the analysis of SISO systems with parameter uncertainty in a probabilistic manner has been investigated. A missile pitch example and spring mass example were used to explore results of the hybrid method. Incorporating the FORM tools helped provide definition in regions of low probability without the hundreds of thousands of sample evaluations required with Monte Carlo techniques.

The developed hybrid method adapted quite well to probabilistic response plots, in both the frequency and time domain. Applied to a range of the system response, a probabilistic definition of the specific response plot was easily found. Confidence bounds provide response limits and indicate the likelihood of the system response exceeding these bounds. These plots also provide information about areas of the response that are more affected by the parameter uncertainty. The main difficulty of the hybrid method was the developed extrapolation method used for selecting FORM locations, although it worked well for most response distributions.

Expanding from the probabilistic response plot application, the hybrid method was applied to parameter space analysis. Due to the nature of many system characteristics the performance metric analysis required more restructuring of the problem to fit the hybrid method tools. Also for this type of analysis some prior knowledge is necessary to ensure the performance metric being analyzed is legitimate over the range of responses seen with increasing uncertainty. However the performance metric analysis is able to produce useful information on changes in the uncertainty affected performance metrics. The missile pitch example showed how the distribution of rise time stays centralized evenly distributed about the nominal parameters. Results of a similar analysis showed a skewed distribution for settling time where as uncertainty increased more systems had a settling time later than the nominal system.

The probability of instability analysis performed quite well across a wide range of systems. While the scaling of the parameter space is arbitrary, four techniques were given and discussed. Any one of these four techniques, or some other scaling, can be used for most generic systems when determining the largest parameter uncertainties before a probability of instability exists. The hybrid method provided information beyond pervious research by mapping the growth in probability of instability as the amount of uncertainty in the system increased.

The developed hybrid method was found to perform well both for producing probabilistic response plots and analyzing the effects of varying uncertainty with parameter space analysis. As a preliminary study this paper has shown many benefits and possibilities of probabilistic control analysis. This research has shown that the hybrid method for probabilistic analysis provides previously unavailable information about system responses due to parameter uncertainty

Chapter 6

Future Work

One of the most noticeable areas of future work is moving from the analysis phase to the design phase. It is mentioned in the introduction, there has been some research performed using sampling to incorporate probability into control design, however including low probability analysis tools such as FORM could improve results. Related directly to the hybrid method of analysis, the addition of SORM could improve results over the use of FORM in some cases.

This research looked at Bode and step responses as two classical analysis tools, however there are also root locus, Nyquist, and Nichols plots. The hybrid approach to including probability into the system analysis has not been applied to these analysis methods. The main challenge with both of these tools is that they display a combined representation of the complex data. Nyquist plots represent phase versus magnitude while root locus represents the real vs. imaginary portions of the data. This makes the problem much more difficult because you are looking at a joint probability distribution that may have high correlation. The difficulty that prevented the hybrid approach from being applied to these methods was the inability to separate the joint distribution into independent distributions. The use of FORM inhibited the ability to accommodate these two analysis tools.

There are many ways for efficiency improvements to be made in the analysis methods, particularly in the response analysis. Since FORM must be performed separately at each

frequency, this analysis could be easily segmented and performed on a distributed computing system. This would allow for greatly reduced analysis time for a large number of uncertain parameters, or systems that are more complex. Another option would be to arrange the software for the hybrid method to perform a slightly more extensive sampling analysis of the overall system, and then only performing the more detailed analysis at desired locations.

Briefly discussed in section 3.5 was the issue of the inverse MPP problem. In this research, the standard FORM problem was used to find the probability of failure and extrapolation techniques were developed attempting to find the desired probability of failure results. The inverse problem would allow for the probability of failure to be prescribed and the conditions that produce this probability would be found. Instead of finding the MPP, the distance from the MPP to the origin is prescribed and the conditions that cause the closest point of the limit state function to be this prescribed distance from the origin. The inverse problem could allow for a specified number of FORM computations at prescribed levels.

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14. ABSTRACT

The current standards for handling uncertainty in control systems use interval bounds for definition of the uncertain parameters. This approach gives no information about the likelihood of system performance, but simply gives the response bounds. When used in design, current methods of m-analysis and can lead to overly conservative controller design. With these methods, worst case conditions are weighted equally with the most likely conditions. This research explores a unique approach for probabilistic analysis of control systems. Current reliability methods are examined showing the strong areas of each in handling probability. A hybrid method is developed using these reliability tools for efficiently propagating probabilistic uncertainty through classical control analysis problems. The method developed is applied to classical response analysis as well as analysis methods that explore the effects of the uncertain parameters on stability and performance metrics. The benefits of using this hybrid approach for calculating the mean and variance of responses cumulative distribution functions are shown. Results of the probabilistic analysis of a missile pitch control system, and a non-collocated mass spring system, show the added information provided by this hybrid analysis.

15. SUBJECT TERMS

Probability; Control Analysis; Probabilistic Control; Hammersley; First Order Reliability Methods

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