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Mathematical Metaphors: Problem Reformulation and Analysis Strategies

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Abstract

This paper addresses the critical need for the development of intelligent or assisting software tools for the scientist who is working in the initial problem formulation and mathematical model representation stage of research. In particular, examples of that representation in fluid dynamics and instability theory are discussed. The creation of a mathematical model that is ready for application of certain solution strategies requires extensive symbolic manipulation of the original mathematical model. These manipulations can be as simple as term reordering or as complicated as discovery of various symmetry groups embodied in the equations, whereby Bäcklund-type transformations create new determining equations and integrability conditions or create differential Gröbner bases that are then solved in place of the original nonlinear PDEs. Several examples are presented of the kinds of problem formulations and transforms that can be frequently encountered in model representation for fluids problems. The capability of intelligently automating these types of transforms, available *prior* to actual mathematical solution, is advocated. Physical meaning and assumption-understanding can then be propagated through the mathematical transformations, allowing for explicit strategy development.

Introduction

Scientific computing requires access to the sophisticated numerical manipulation and computation systems. But scientists also require analytic tools that support various types of problem transformations and strategic planning for analysis. Symbolic reasoning is excellent for management of the flow of computational components and for developing alternative computational strategies as might occur in a planning system; but such tools are not yet available for computational physics driving active models and boundary conditions. Completely apart from mathematical solution or simulation, both of these features of problem manipulation and symbolic reasoning should be brought to bear at the initial problem formulation stage. One must allow altering of mathematical representations, by inverting various dependency relations, by carrying out sundry symbolic substitutions and reformulations through differential operations, by reordering partial differential equations according to their symmetry bases, and by maintaining semantics throughout these transformations -- only then can actual solution begin.

There are several stages of activity involved in computational science. The first of these begins with the scientist finding and expressing the simplest mathematical model that is believed to capture all the detail necessary and sufficient to appropriately simulate or model the physical process being addressed in the problem statement. In fact, at a basic level, this first stage is simply defining and refining the problem statement itself. Once a model has been created, then solution strategies are developed which are constrained by the limits of the mathematical description, by the assumptions made to insure computational tractability, and by the computational resource requirements as available on the particular processor architectures or distributed computing environments. Various (possibly alternative) strategies are then selected and implemented through a suitable computational language that supports the algebraic, differential and difference calculus used, and that supports the symbolic manipulation of the computational objects themselves in order to affect linkages between various canonical computational modules. Of course, this is followed by debugging, by problem or theory refinement, by general exploration or sensitivity analysis of the model itself, and eventually by publication of some results. Of these levels of computational activity, the most basic level is probably the least supported by intelligent assisting software tools. Yet this most basic level of (1) finding and expressing the appropriate mathematical model representation, (2) analyzing or ensuring its capture of essential physics, and (3) transforming it to a structure amenable for applying known solution strategies, is frequently a highly time-consuming enterprise, fraught with error and redefinition. It could be radically shortened and verified through use of reliable equation-transformation operation tools.

Most scientists carry out such initial activity off-line, and bring a more or less well-defined or reasonably-expressed problem statement to their workstations. However, it would be most useful if an interactive, sophisticated symbolic and numerical manipulation and planning system could be available to scientists at this early stage of research.

This paper considers problems in fluid dynamics and instability theory. It should be noted, however, that, it does not refer to problems in applied computational fluid dynamics. In that domain, simulations of complex flow fields are created around fixed aerodynamic structures in order to observe non-laminar flow fields and to define how to minimize these vortex sheets by optimally redesigning the structure. These problems are structurally well-defined from the outset, and many years of experience have gone into creating advanced simulation and exploration tools for these researchers. Rather, here the focus is on the scientific understanding of the conditions under which, for example, various forms of turbulent transport and energy dissipation in naturally occurring fluids such as atmospheres or planetary interiors, might grow or transform to finite amplitude secondary flow structures; or how the conditions in these fluids respond to perturbations in their flow fields so that the fluid evolves into new flow regimes. The concentration is on the fluid physics rather than on a physical structure; the concentration is on maintaining understanding and insight as mathematical assumptions and transformations propagate during analysis.

The initial representation of this type of fluid dynamics problem may be a fairly broad model description at the level of the relevant equations of motion, conservation laws, constitutive equations, and boundary conditions that embed these equations into a particular problem statement. But an actual "working" problem statement arises only from successive refinements, coordinate transformations, transformations in and out of various phase space representations, substitutions of truncated power series expansions or of new parameters, and physically or mathematically justified constraints or assumptions on this initial problem statement. In a sense, the general problem is reduced to a particular instance upon which an entire intellectual fabric may hinge. For example, later in this paper [and in the **Appendix**], a problem is outlined on isostatic recovery flow of the Earth's mantle following deglaciation of the Pleistocene ice sheets under weakly non-Newtonian rheology. One starts with equations for conservation of momentum, of mass, and with an expression for nonlinear viscosity. After much transformation, substitution, and differential operation on these equations, one *realizes* that the essential question to be addressed is to clarify the actual role of coupling of harmonic disturbances in nonlinear stress-dependent rheology and induced wavelength harmonics. Boundary conditions are then created for *that* problem, and that is the ultimate problem that is analyzed numerically.

The ***realization*** is an act of insight that cannot be anticipated at the outset of the problem definition. It arises because of the insight gained from understanding the constraints against solution methodologies embodied in the original problem statement, the assumptions, and selected problem reformulation strategies. It is a kind of insight not eagerly allocated to "assisting" systems by scientists. In particular, it is a different kind of insight than that gained from detailed numerical simulation and experimentation on the "whole" model, a process frequently allocated to high speed, vector processor architectures. It is also a different form of insight from that which arises under analysis of phase portraits of dynamical systems. Phase portraits are not currently tailored to expose the detailed physical processes responsible for transitions between bifurcated regions of the phase space. Rather, they identify variations in dynamical systems behavior as gleaned from identification of initial conditions of trajectories in the phase space, or from analysis of the evolution of eigenvalues in the complex plane as particular bifurcations parameters are varied with respect to characteristic flow parameters. However, the symmetries that arise from such analysis are the same that occur in the original differential equations, and so there should be a way of mapping between these kinds of representations to achieve insight on the physical process.

Language and Metaphor

So, why is this paper entitled "Mathematical Metaphors"? Consider a language. It has a grammar by which we understand the deeper linguistic structure. At the surface, the words are used to define and individuate concepts, and the structure allows these concepts to be interrelated. Once we have learned the grammar, and at some level the linguistic structure, we study the literature. The literature is a transformation of ideas through metaphor; understanding in the language requires metaphor because this is how the unfamiliar becomes familiar, and how new connections and relations are established. We use the language to reformulate the language.

Mathematics is also a language with a surface grammar and a deep grammatical structure. We spend many years learning the surface grammar, which is basically manipulation of symbols for identities -- everything through differential and integral calculus. The surface grammar is used to manipulate symbols that define and identify substitution possibilities wherein an equation says "the result of carrying out the calculation on the left side of the *equals* sign is the same as the result of carrying out the one on the right side." Once we get to ordinary and partial differential equations, to higher order and to nonlinear equations, or to variational calculus, we are at a new level of use in the language. There are no longer any fixed rules about what counts as a solution: a solution is any function that can be found to satisfy the equation, given the particular problem conditions. The bulk of scientific computation is this calculation by algebraic substitution of values for variables, and the substitution of calculation process for equivalent calculation process. But in science theory formation or in problem statement formulation, *prior* to these substitutions of calculations for calculations, is the metaphorical transformation of equation and model representations, operating at the level of the structure of the language, and with its attendant insight. Our theory and insight is enhanced by the metaphors we create through this analytic transformation of the problem statement. This act is not part of a 'solution methodology' but rather is transformation of problem representation while maintaining physical understanding. Just as a theory is a metaphor between model and reality, so is a problem refinement a metaphor between the original model and a solution. This level of theory formation is the level of the expressibility and subtlety of the language. It is this metaphorical level of expressibility in assisting tools that is advocated.

Discussion and Examples

In this section, several examples are presented of the types of equation transformations that occur regularly in nonlinear fluids problems and in dynamical systems generally, thereby hopefully lending clarification to the ideas of the previous section. It is difficult to rank these operations according to which are the most useful or the most necessary for near-term intelligent automation. Rather, it is hoped that some of these techniques will seem within the grasp of some of the Intelligent Systems Research Community, or that these examples will stimulate a research domain for intelligent mathematical and scientific tool development.

I. Burgers' Equation Transformations

First, consider Burgers' equation [Zwillinger, 1998, p.387-390], and look at two possible transformations of it that can make it solvable using alternative methods. The appropriate form can well depend on the physical setting in which the equation arises in a given problem, so one form may be more "intuitive" than another. Burgers' equation is generally thought of as one form of mathematical model for turbulent flow, and it is also used in the approximate theory for weak stationary shock waves in real fluids. It has the general form of a quasi-linear PDE, $L[u] = 0$, and with λ a parameter, it has the form:

$$u_t + uu_x - \lambda u_{xx} = 0 \qquad (1)$$

We can now construct what is called the Hopf transformation, in two steps (Bäcklund transformations generally). Introduce a new variable, v, and set $u = v_x$. Substitution yields the equation:

$$v_{xt} + v_x v_{xx} - \lambda v_{xxx} = 0 \qquad (2)$$

and one integration by x yields:

$$v_t + (1/2)v_x^2 - \lambda v_{xx} = 0 \qquad (3)$$

This equation looks worse than before, but if we then introduce a second transform of variables between v and a new variable w such that $v = -2\lambda \log(w)$, equation (3) reduces to a simple one-dimension diffusion equation

$$w_t - \lambda w_{xx} = 0 \qquad (4)$$

which is solvable by known methods. Of course, a solution for w yields a solution for v and hence u.
The combined transformation:

$$u = -2\lambda\ [\log(w)]_x \qquad (5)$$

is called the Hopf transformation. It is presented here in two stages, because if one stops at equation (3), there is an alternative method of solution. Furthermore, perhaps the physical problem under consideration yields this form of the transformed Burgers' equation, and the diffusion solution is not what is wanted. Equation (3) can also be solved by so-called "separation" methods. Here, a typical solution is sought in the form:

$$v = f[t + g(x)] = f(w) \qquad (6)$$

for unknown f and g . Here, $w = t + g(x)$. Upon substitution of (6) into (3), a relation arises for f and g:

$$f'(w)[1 - \lambda g''(x)] = [g'(x)]^2[\lambda f''(w) - (1/2)(f'(w))^2] \tag{7}$$

and this can be separated into:

$$\begin{aligned} [1 - \lambda g''(x)] / (g'(x))^2 &= [\lambda f''(w) - (1/2)(f'(w))^2] / f'(w) \\ &= c \quad \text{[a constant]} \end{aligned} \tag{8}$$

Hence, f and g must satisfy the system of equations:

$$\begin{aligned} \lambda f'' - (1/2)(f')^2 - cf' &= 0 \\ \lambda g'' + c(g')^2 &= 1 \end{aligned} \tag{9}$$

These equations are integrable by reduction of order. The successes from using transformations of the form (6) for Burgers' equation above has led to the generalization of such transforms for use in solutions of the Navier-Stokes equation [see **II**.], whereby the stream function is assumed to have the form

$$\psi = f[t + g(x) + h(y)] \tag{10}$$

and this then opens up another area of analysis for instability and turbulence.

This first example with Burgers' equation is fairly straight-forward, and is well known in the literature. Here, it introduces the flavor of transformation of equations tailored to specific needs. The point is that if access to suites of transformations were available in a computational system, then various experiments could be run by operating on the defining equations and discovering the symmetries that emerged. This would of course require not only the possibility of functional substitution, but also the availability of differential operation on equations as a whole so as to separate or project forms on subspaces or create, say, poloidal and toroidal component equations, and so on.

II. Navier-Stokes Equation Transformations

As a second example, consider some simple variable transformations on the Navier-Stokes equation. This equation represents conservation of momentum in fluids [Batchelor, 1967, p.147; Schlichting and Gersten, 2003, p.68 & 101ff.], and for incompressible fluids, it takes the vector-indices form:

$$\partial u_i/\partial t + u_j\, \partial u_i/\partial x_j = - 1/\rho\, \partial p/\partial x + \nu\, [\partial^2 u_i/\partial x_i \partial x_j] \tag{11}$$

where indices $i,j = 1,2,3$, and u_i is velocity, ρ is density, p is pressure, and ν is constant kinematic viscosity. Because the fluid is incompressible, the conservation of mass is expressed by a simplified form of the continuity equation, namely, $\partial u_k/\partial x_k = 0$. That means that, for the two-dimensional problem (u,v; x,y), we can define a stream function that satisfies continuity, whereby now:

$$u = \psi_y \quad \text{and} \quad v = - \psi_x \;; \;\; \text{with} \;\; \psi_{yx} - \psi_{xy} = 0 \tag{12}$$

Thus, one can see that the Navier-Stokes equations could be written in terms of the stream function ψ and the pressure gradients by substituting the above definitions, and carrying out the necessary differentiations. Normally, if one is proceeding along this path, one also then cross-differentiates the equations and subtracts to eliminate pressure, and the resulting equation is a fourth order PDE in the stream function. The partial derivatives have essentially projected the original equations onto a one family set of curves in "stream-function" space. But, instead of pursuing that path here (and because the final example carries it out in more detail), consider the transformation that arises from assuming a solution of the form:

$$u_i = - 2\nu/\phi\, [\partial\phi/\partial x_i] \tag{13}$$

If this transformation is substituted into the original Navier-Stokes equation for u_i, the resulting equation is a linear diffusion equation in ϕ :

$$\partial\phi/\partial t = \nu[\partial^2\phi/\partial x_i\partial x_j] + \{p(t,x_1,x_2,x_3)/2\rho\phi\}\ \phi \tag{14}$$

This equation can now be thought of as a model for viscous flow in a pure initial-value problem where pressure is prescribed. In this case, it turns out that the velocity field must satisfy a source distribution of

$$S(t, x_1, x_2, x_3) = - 2\nu[\partial \ln \phi/\partial x_i\partial x_j] \tag{15}$$

in order to guarantee that conservation of mass is still satisfied. If, however, $S = 0$, then the linear diffusion equation for ϕ, (14) above, along with mass conservation, transforms into a Bernoulli equation, which classically relates flow to pressure and density. On the other hand, the nonlinear form of Bernoulli equation

$$\partial\theta/\partial t + 1/2\ [\partial\theta/\partial x_i\ \partial\theta/\partial x_j] + p/\rho = 0 \tag{16}$$

goes back to a linear diffusion equation by using mass conservation and the transform $\theta = \ln \phi$.

Each of these manipulations are in the general class of reducible equations and are found in nonlinear PDE texts; they may provide guidance to scientists who are working on finding a reasonable representation for their particular problem. Such transforms and others of a similar nature would be valuable if they were available in the "working" knowledge of a computationally assisting software system.

So much for these kinds of examples: where's all the *insight* that is supposed to arise from these transformations? Where's the *metaphor*? To see that, one needs to consider an application of these types of transformations to real problems. What is outlined now in this final example is a geophysical fluid dynamics problem. Section **III** basically outlines the structure and evolution of the analysis; the actual detailed analysis, evolution, and transformation of this particular geophysics problem is presented in the **Appendix**. The problem is not solved; rather, the problem is transformed to a solvable form, and the appropriate boundary conditions are integrated into the analysis. That final analysis is then ready for programming to a numerical solution or simulation, and the exact physical assumptions inherent are carried forward into that simulation.

III. Example: Isostatic Rebound of the Earth's Mantle under Non-Newtonian Rheology

Suppose the general problem is one of understanding the nonlinear viscosity distribution in the Earth's interior. There are several sources of data that can constrain such analysis: earthquake-seismic, laboratory data on phase transitions in mantle-like rocks, gravity anomalies over large regions of the Earth, historical records on sea-level change, earthquake free-oscillation data, and changes in length of day due to tidal de-spinning of the Earth-Moon system, to name a few. It may turn out that most of this data can be supported by more simple layered linear viscosity; but for now suppose one has reason to hypothesize a nonlinear viscosity distribution. Then here is an example strategy with transformations.

The problem to be considered is the flow of the mantle of the Earth in response to the unloading (melting and removal) of the Pleistocene ice sheets about 7K to 10K years ago. We want to determine the viscosity profile in the mantle $\mu(x,y)$ based on this flow $\mathbf{v}(x,y)$ of the mantle in response to the unloading stresses. Bold terms are vector quantities. The basic equation of motion for rebound flow in this medium is the variable viscosity form of the Navier-Stokes equation, representing conservation of momentum. In vector form this equation is:

$$\rho\ D\mathbf{v}/Dt = \rho\mathbf{g} - \nabla p + [\nabla\cdot (2\mu\nabla)]\mathbf{v} + \nabla \times (\mu\nabla \times \mathbf{v}) \tag{17}$$

Conservation of mass is simply div $\mathbf{v} = 0$, as before. The inertial terms in the convective derivative $D\mathbf{v}/Dt$ can be ignored in this problem because they are small compared to the viscous terms and restoring stresses. It is assumed that the flow is confined to the (x,y) plane, with y positive vertically upward from the non-deformed surface. The next usual operation is to introduce a term for the vorticity as curl $\mathbf{v}$ into the equations of motion, which then become:

$$\begin{gathered}\nabla(p + \Phi) = - \mu\nabla \times \boldsymbol{\xi} + 2(\nabla\mu \cdot \nabla)\mathbf{v} +(\nabla\mu) \times \boldsymbol{\xi} \\ \text{with}\quad \boldsymbol{\xi} = \nabla \times \mathbf{v}\end{gathered} \tag{18}$$

Here the gravity force is written as the gradient of a potential Φ. This equation can now be operated on by taking the curl of the whole equation (ie, ∇ x :) which eliminates the potential terms and projects the vector equation onto the (x,y) plane. This operation also serves to eliminate several of the terms in the expanded form of the differential vector operator relations. The vorticity vector itself points in the transverse plane, and its components lie entirely in (x,y). After carrying out all the operations and imposing mass conservation, the two-dimensional form of the single equation of motion can be written explicitly in terms of u, v, and μ, with partial differentiation in x, y:

$$\mu(v_{xxx} - u_{xxy} + v_{xyy} - u_{yyy}) + 2[\mu_x(v_{xx} - u_{xy}) + \mu_y(v_{xy} - u_{yy})] + 2\mu_{xy}(v_y - u_x) + (\mu_{xx} - \mu_{yy})(v_x + u_y) = 0 \tag{19}$$

The horizontal and vertical velocity components can again be represented in terms of a Stokes stream function $\psi(x,y)$

$$u = \psi_y \quad \text{and} \quad v = -\psi_x \tag{20}$$

which becomes the single dependent variable describing the flow. Substitution yields:

$$\mu(\psi_{xxxx} + 2\psi_{xxyy} + \psi_{yyyy}) + 2[\mu_x(\psi_{xxx} + \psi_{xyy}) + \mu_y(\psi_{xxy} + \psi_{yyy})] + 4\mu_{xy}\psi_{xy} + (\mu_{xx} - \mu_{yy})(\psi_{xx} - \psi_{yy}) = 0 \tag{21}$$

Solution of this fourth-order, viscous-dominant equation of motion is sought in terms of $\psi(x,y)$. This equation is equivalent to Brennen's equation for linear viscosity [Brennen, 1974, p. 3995, his eqn. (9)], but (21) includes both horizontal and vertical variations in the viscosity. Integrations of ψ would then give u and v, and if an explicit relation is made for the viscosity in terms of the flow strain-rates, then this viscosity profile can be found as well. Conversely, a viscosity form could be assumed in order to start the analysis, and iterations would then converge to a self-consistent system.

The difficulty is that the viscosity $\mu(x,y)$ is now a complicated function of the stress field or strain-rate field in the medium; hence it is also a function of $\psi(x,y)$. The equation of motion is thus highly nonlinear in $\psi(x,y)$, and exact admissible solutions are not known. One cannot assume a simple modal solution for ψ, such as a superposition of Fourier components, because in the nonlinear problem, as the deformed surface rebounds, a new stress field is induced by which given disturbance harmonics will couple and induce other harmonics. In addition, the change in strain rates during recovery will alter the viscosity profile sensed by the rebounding nonlinear fluid. That is, μ must be considered as the functional $\mu[\psi(x,y)]$. Hence, the form of a general solution to this equation of motion must be derived which is appropriate for describing the rebounding flow in such a non-Newtonian medium. Transformations (math metaphors) become necessary,because one hopes to preserve physical understanding at the same time. Once a general solution $\psi(x,y)$ is found, it can be used in conjunction with various boundary conditions to develop an equation which describes the time-change (relaxation) of the deformed surface. This rate can be compared with projections from data for verification of the model.

In order to derive the general solution $\psi(x,y)$, it is necessary to impose physically reasonable assumptions on the nature of the viscosity variation, and to indicate the extent to which flow in the medium is coupled to this viscosity variation. Ideally, these assumptions should also yield mathematical tractability! One relation to select is a viscosity variation that can be described as weakly spatially coupled (*i.e.*, the viscosity at one position is dependent on the viscosity of material nearby). One also thus has to assume that the stream function behaves smoothly over a narrow bandwidth about some prescribed wavelength, even though the overall variation across the spectrum may be significant. These assumptions are converted to mathematical constraints on the form of the solution $\psi(x,y)$ and on the form of the transformation analysis during reduction of the equations. It turns out that this approach provides both significant physical insight into the nonlinear rheology problem as well as mathematically tractable analysis.

An acceptable method to advance beyond the statement of this fourth-order equation is to Fourier Transform it, solve the resulting equation in the (k,h) transform domain for Ψ(k,h), then inverse transform this solution back to yield an analytic form for $\psi(x,y)$. This analytic form of $\psi(x,y)$ can then be differentiated appropriately, now with some understanding of the role of coupling of harmonics, and an actual relaxation equation can be developed for the rate of change of the deformed surface under specified initial displacement or stress conditions. By using our $\psi(x,y)$, and combining the expression for vanishing normal stress at the deformed free surface with a kinematic surface condition that ties the vertical velocity of that surface to the rebounding motion there, a partial differential equation is derived which describes the time evolution and rebound of the initially specified depressed land. Solution of this relaxation equation is the ultimate goal; it can be carried out numerically using finite-difference or multi-grid techniques.

All work prior to articulating this relaxation equation can be considered symbolic manipulation and exploration in theory formation and problem statement reformulation. In this summary, the presentation has been simplified, in that the analysis in the Fourier Transform domain turns out to be rather involved. The convolutions integrals that are solved in (k,h) are of course the various products in (x,y), such as $\mu_x\psi_{xyy}$, and solution of these integrals is what requires the stringent assumption on the form of nearest neighbor "awareness" of the viscosity relations. In several instances, truncated series expansions are needed, and these are substituted (more metaphor). At one point, an interesting first-order PDE arises to be solved for the transformed stream function $\Psi(k,h)$:

$$A(k,h)\ \partial\Psi(k,h)/\partial k\ +\ B(k,h)\ \partial\Psi(k,h)/\partial h\ +\ C(k,h)\ \Psi(k,h)\ =\ 0 \tag{22}$$

This equation is then solved along characteristic paths in (k,h) space by a transformation of variables. The subsidiary equation

$$dh/dk\ = B(k,h)/A(k,h) = \text{function}(k,h,k',h') \tag{23}$$

defines the characteristic paths in (k,h) along which the solution must exist, and this yields a constraint on the relation between the two wavenumbers k and h (representing the transform from x and y). The primed k and h are not derivatives, but rather here represent the width of the pulse for μ in the convolution integrals in the k and h directions. This constraint dictates the form of the final solution, but it also yields insight into the nature of the coupling of the harmonics from the relaxing deformation. For a suitable transformation of variables, the first-order equation above becomes:

$$A(\zeta,\theta)\ \partial\Psi(\zeta,\theta)/\partial\zeta\ +\ C(\zeta,\theta)\ \Psi(\zeta,\theta)\ =\ 0 \tag{24}$$

In this problem, the appropriate transformation of variables turns out to be:

$$\theta(k,h) = hk^{-b/a} \quad \text{and} \quad \zeta(k,h) = k \tag{25}$$

where b and a are also functions of the pulse width. This transformation leads to a solution of the first-order equation in (ζ,θ) space, and inverse transforming this yields a solution for $\Psi(k,h)$. Finally, the $\Psi(k,h)$ solution is inverse Fourier Transformed to yield the desired solution for our original stream function $\psi(x,y)$. Plus, the nonlinearity of the viscosity is completely incorporated in this solution due to the convolution integrals.

This $\psi(x,y)$ can now be analytically differentiated, and finally an equation can be developed for the relaxation of the surface (see equation (A72) in the **Appendix**). It is this final equation that becomes the particular problem to be solved in this physical scenario. All the rest is preparation for that.

Summary

A wide variety of symbolic manipulation of equations can occur during the initial problem statement formulation stage in a theoretical applied math problem. Frequently, scientists bring fully formed problem statements to their workstation to seek numerical solutions or symbolically guided variation of parameters of a well-formed problem. However, in many of those cases, a tremendous amount of off-line analysis has been accomplished ahead of runtime, and this work can be referred to as problem statement formulation: starting from general equations of motion and transforming and reducing the equations to a solvable representation. This enterprise is in urgent need of intelligent mathematical manipulation tools by which the scientist can explore various transformation possibilities and see what physical constraints are intimately tied to each kind of transformation or assumption. This theory formation stage of theoretical work requires the capability to perform substitution of variables, substitution of truncated series representations for variables, changes in independent variables so as to reformulate equations in alternative spaces, carrying out integrations such as Fourier Transforms so that analytic results can be viewed and manipulated, inverse transforming back while retaining original meaning, and re-writing equations according to various symmetry operations or from differential vector operations. In an ideal case, a user might be able to start with conservation laws for mass, momentum, and energy, and write them as balance equations in "english" or in pseudo-code, and later replace words with mathematical terms representing the processes. That would truly be metaphorical! In all of these operation, it is necessary to maintain a background knowledge whereby vector symbols retain meaning or body forces really can be considered as gradients of potentials, and so on. In addition to these intelligent data structures, planning systems could be exploited to manage the progress of the calculations, while retaining the experience of alternative strategies. Understanding would then propagate through the various transformations. Such systems would be very valuable and broadly used were they to come under development.

References

Batchelor, G.K. (1967) An Introduction to Fluid Dynamics. Cambridge University Press, London. 615p.

Brennen, C. (1974) *Isostatic recovery and the strain rate dependent viscosity of the Earth's mantle*. Jour. Geophys. Res., vol.79, p.3993-4001.

Dennemeyer, R. (1968) Introduction to Partial Differential Equations and Boundary Value Problems. McGraw Hill Publishing Company; New York. 376p.

Gröbner, W., and N. Hofreiter (1965) Integral Tables: Vol. 1, Definite Integrals [in German]. Springer-Verlag, Vienna. 166p.

Gröbner, W., and N. Hofreiter (1966) Integral Tables: Vol. 2, Indefinite Integrals [in German]. Springer-Verlag, Vienna. 204p.

Schlichting, H., and K. Gersten (2003) Boundary Layer Theory, 8th ed. Springer-Verlag, Berlin, 801p.

Zwillinger, D. (1998) Handbook of Differential Equations, 3rd ed. Academic Press, Boston, MA. 801p.

APPENDIX: The Nonlinear Mantle Viscosity Problem and its Transformations

I. Derivation of the Equations of Motion and the Stream Function Solution

This appendix addresses one particular strategy for formulating the problem of the gravitational relaxation and rebound of a nonlinear viscous Earth's mantle after unloading. Insofar as one is trying to understand the physics of the rebound behavior, the following analytic work helps to isolate the effects of those processes that drive the global behavior. The fundamental exploratory work here is not to solve a rebound problem, but rather to use this rebound problem formulation as a domain within which to explore the impacts of various assumptions in the analysis. The particular strategy of the method presented here is as follows: start with expressions of conservations equations for mass and momentum. Conservation of mass for incompressible fluids allows the velocity field to be represented in terms of one variable, the stream function $\psi(x,y)$. This stream function can also explicitly incorporate the variable viscosity $\mu(x,y)$, because $\mu[\psi(x,y)]$, and this can be inverted. The products of various derivatives of ψ and μ become convolution products in the Fourier Transformed momentum equations. The function $\boldsymbol{\Psi}(k,h)$ that satisfies these equations is derived through a series of algebraic manipulations and integral transforms. Inverse transforming $\boldsymbol{\Psi}(k,h)$, and using that derived $\psi(x,y)$ in the boundary conditions allows derivation of the relaxation equation of the free-surface $\eta(x,y,t)$. This equation describes the time rate of change of the rebounding surface under the constraints and assumptions explicitly made in the particular strategy outlined here. Other strategies are possible with different outcomes, and these are discussed as options in the text.

The problem to be considered is the flow of the mantle of the Earth in response to the unloading (melting) of the Pleistocene ice sheets about 7K to 10K years ago. The goal is to determine the viscosity profile in the mantle $\mu(x,y)$ based on this flow $\mathbf{v}(x,y)$ of the mantle in response to the unloading stresses. Bold terms are vector quantities. The basic equation of motion for rebound flow in this medium is the variable viscosity form of the Navier-Stokes equation, representing conservation of momentum. In vector form this equation is:

$$\rho[\partial \mathbf{v}/\partial t + \mathbf{v} \cdot \nabla \mathbf{v}] = \rho\, D\mathbf{v}/Dt = \rho \mathbf{g} - \nabla p + [\nabla \cdot (2\mu\nabla)]\mathbf{v} + \nabla \times (\mu\nabla \times \mathbf{v}) \qquad \text{(A1)}$$

Conservation of mass requires div $\mathbf{v} = 0$. The inertial terms in the convective derivative $D\mathbf{v}/Dt$ can be ignored in this problem because they are small compared to the viscous terms and restoring stresses. It is assumed that the flow is confined to the (x,y) plane, with y positive vertically upward from the non-deformed surface. The basic equation of motion thus reduces to one of Stokes flow for variable viscosity:

$$-\nabla(p + \rho g y) - \mu\nabla \times \nabla \times \mathbf{v} + 2(\nabla\mu \cdot \nabla)\mathbf{v} + (\nabla\mu) \times (\nabla \times \mathbf{v}) = 0 \qquad \text{(A2)}$$

Equation (A2) can be rewritten in terms of the vorticity, curl $\mathbf{v}$, by introducing the vector

$$\xi = \nabla \times \mathbf{v} \qquad \text{(A3)}$$

into the equation of motion, which then becomes:

$$\nabla(p + \Phi) = -\mu\nabla \times \xi + 2(\nabla\mu \cdot \nabla)\mathbf{v} + (\nabla\mu) \times \xi \qquad \text{(A4)}$$

Here the gravity force is written as the gradient of a potential, Φ. This equation can now be operated on by taking the curl of the whole equation (ie, $\nabla \times :$) which eliminates the potential terms and projects the vector equation onto the (x,y) plane. This operation also serves to eliminate several of the terms in the expanded form of the differential vector operator relations. Taking the curl of equation (A4) yields:

$$0 = \mu\nabla^2\xi - \nabla(\nabla\mu \cdot \xi) + 2(\xi \cdot \nabla)\nabla\mu + 2(\nabla\mu \cdot \nabla)\xi$$
$$+ 2[(\nabla_j\nabla_m\mu)\nabla_m] \times \mathbf{v} - \xi\nabla^2\mu \qquad \text{(A5)}$$

wherein the repeated subscript m in the operator $(\nabla_j \nabla_m \mu)\nabla_m$ is summed over spatial coordinates for each operator ∇_j. For the two-dimensional problem, all flow is confined to the (x,y) plane so that all transverse partials $\partial/\partial z$ and the Eulerian velocity component $w = \mathbf{v}\mathbf{e_3}$ vanish. Furthermore, the vorticity as defined in (A3) becomes:

$$\xi = (\partial v/\partial x - \partial u/\partial y)\,\mathbf{e_3} = \xi_z\,\mathbf{e_3} \tag{A6}$$

and points in the transverse plane so that its components lie entirely in (x,y). Here u and v are the x- and y-components of the velocity vector. With this restriction, the terms $-\nabla(\nabla\mu \cdot \xi)$ and $2(\xi \cdot \nabla)\nabla\mu$ in equation (A5) vanish as well.

After carrying out all the operations and imposing mass conservation, the two-dimensional form of the single equation of motion can be written explicitly in terms of u, v, and μ, with partial differentiation in x, y:

$$\begin{aligned}&\mu(v_{xxx} - u_{xxy} + v_{xyy} - u_{yyy})\\ &\quad + 2[\mu_x(v_{xx} - u_{xy}) + \mu_y(v_{xy} - u_{yy})]\\ &\qquad + 2\mu_{xy}(v_y - u_x) + (\mu_{xx} - \mu_{yy})(v_x + u_y) = 0\end{aligned} \tag{A7}$$

The horizontal and vertical velocity components can be represented in terms of a Stokes stream function $\psi(x,y)$, by which

$$u = \psi_y \quad \text{and} \quad v = -\psi_x \qquad \text{(satisfying div } \mathbf{v} = 0\text{)}. \tag{A8}$$

$\psi(x,y)$ then becomes the single dependent variable describing the incompressible flow. Substitution yields:

$$\begin{aligned}&\mu(\psi_{xxxx} + 2\psi_{xxyy} + \psi_{yyyy})\\ &\quad + 2[\mu_x(\psi_{xxx} + \psi_{xyy}) + \mu_y(\psi_{xxy} + \psi_{yyy})]\\ &\qquad + 4\mu_{xy}\psi_{xy} + (\mu_{xx} - \mu_{yy})(\psi_{xx} - \psi_{yy}) = 0\end{aligned} \tag{A9}$$

This equation is equivalent to Brennen's linear viscous equation [Brennen, 1974, p. 3995, his eqn. (9)], but (A9) includes both horizontal and vertical variations in the viscosity. Solution of this fourth-order, viscous-dominant equation of motion is sought in terms of $\psi(x,y)$. Integrations of ψ would then give the velocity components u and v, and if an explicit relation is made for the viscosity in terms of the flow strain-rates, then this viscosity profile can be found as well. Conversely, a viscosity form can be assumed in order to start the analysis, and iterations can converge to a self-consistent system.

The problem is that the viscosity $\mu(x,y)$ is now a complicated function of the stress field or strain-rate field in the medium; hence it is also a function of $\psi(x,y)$. The equation of motion (A9) is thus highly nonlinear in $\psi(x,y)$, and exact admissible solutions are not known. One cannot assume a simple modal solution for ψ, such as a superposition of Fourier components, because in the nonlinear problem, as the deformed surface rebounds, a new stress field is induced by which given disturbance harmonics will couple and induce other harmonics. In addition, the change in strain rates during recovery will alter the viscosity profile sensed by the rebounding nonlinear fluid. That is, μ must be considered as the functional $\mu[\psi(x,y)]$. Hence, the form of a general solution to this equation of motion must be derived which is appropriate for describing the rebounding flow in such a non-Newtonian medium. Transformations become necessary, but we hope to preserve physical understanding at the same time. Once a general solution $\psi(x,y)$ is found, it can be used in conjunction with various boundary conditions to develop an equation which describes the time-change (relaxation) of the deformed surface. This rate can be compared with projections from data for verification of the model.

An acceptable method to operate on equation (A9) is to Fourier Transform it, solve the resulting equation in the (k,h) transform domain for the transformed stream function $\boldsymbol{\Psi}$(k,h), then inverse transform this solution back to yield an analytic form for $\psi(x,y)$. This analytic form of $\psi(x,y)$ can then be differentiated appropriately, now with some understanding of the role of coupling of harmonics in the nonlinear problem, and an actual relaxation equation can be developed for the rate of change of the deformed surface under specified initial displacement or stress conditions. By using our $\psi(x,y)$, and combining the expression for vanishing normal stress at the deformed free surface with a kinematic surface condition that ties the vertical velocity of that surface to the rebounding motion there, a partial differential equation is derived which describes the time evolution and rebound of the initially specified depressed land. Solution of this equation is the ultimate goal; it can be carried out numerically using finite-difference or multigrid techniques.

By this analysis method, convolution integrals arise, in the transform domain, that represent the products that occur in equation (A9), such as $\mu\psi_{xxxx}$, $\mu_x\psi_{xyy}$, and so on. These integrals, however, contain the unknown variable $\boldsymbol{\Psi}(k,h)$ for which a solution is sought. Hence, in order to evaluate the convolution integrals, either the behavior of $\boldsymbol{\Psi}(k,h)$ must be known or hypothesized in advance, or some physical assumptions must be made. Appropriate assumptions are discussed below.

Let (k,h) represent the Fourier Transform variables for (x,y), and let $\boldsymbol{\mu}(k,h)$ and $\boldsymbol{\Psi}(k,h)$ represent the transformed variables for $\mu(x,y)$ and $\psi(x,y)$. Here we have initially assumed that $\mu = \mu[\psi(x,y)]$ can be represented explicity in terms of $\mu(x,y)$. Then the Fourier transform of equation (A9) yields the following equation:

$$\begin{aligned}
&\boldsymbol{\mu}(k,h) * [k^4\boldsymbol{\Psi}(k,h) + 2k^2h^2\boldsymbol{\Psi}(k,h) + h^4\boldsymbol{\Psi}(k,h)] \\
&+ 2k\boldsymbol{\mu}(k,h) * [k^3\boldsymbol{\Psi}(k,h) + kh^2\boldsymbol{\Psi}(k,h)] \\
&+ 2h\boldsymbol{\mu}(k,h) * [k^2h\boldsymbol{\Psi}(k,h) + h^3\boldsymbol{\Psi}(k,h)] \\
&+ 4kh\boldsymbol{\mu}(k,h) * kh\boldsymbol{\Psi}(k,h) \\
&+ [k^2\boldsymbol{\mu}(k,h) - h^2\boldsymbol{\mu}(k,h)] * [k^2\boldsymbol{\Psi}(k,h) - h^2\boldsymbol{\Psi}(k,h)] = 0 \qquad \text{(A10)}
\end{aligned}$$

Equation (A10) defines an integral equation in which the integrands are the convolved products of appropriate $\boldsymbol{\mu}(k,h)$ and $\boldsymbol{\Psi}(k,h)$ terms. This equation can be simplified to a solvable equation under reasonable physical assumptions on the nature of the viscosity variation and on the extent to which flow in the medium is coupled to this viscosity variation. Ideally, these assumptions should also yield mathematical tractability! I have selected a viscosity variation that can be described as weakly spatially coupled (the viscosity at one position is dependent on the viscosity of material nearby), and I have also had to assume that the stream function behaves smoothly over a narrow bandwidth about some prescribed wavelength even though the overall variation across the spectrum may be significant. These assumptions are converted to mathematical constraints on the form of the solution $\psi(x,y)$ and on the form of the transformation analysis during reduction of the equations. It turns out that this approach provides both significant physical insight into the nonlinear rheology problem as well as mathematically tractable analysis.

If the viscosity μ were constant (Newtonian rheology), its Fourier Transform would be a delta function so as to recover the $\boldsymbol{\Psi}(k,h)$-terms appropriately throughout the convolution integrals. If, instead, the viscosity μ is assumed to be only slightly different from the Newtonian constant value, but is also weakly dependent at each point (x,y) on the viscosity at a small finite distance away, then its transform can be represented by a *delta-like* function: instead of a pure delta function at each point (k,h), $\boldsymbol{\mu}(k,h)$ would have some finite bandwidth represented by $2k'$ and $2h'$ centered at each k and h. For convenience, we define the $\boldsymbol{\mu}(k,h)$ to be unity (normalized viscosity) in the range $\pm k'$ and $\pm h'$ about k and h, and to be zero outside this range. This square-pulse definition approximates the envisioned delta-like function and acts as a square filter which when convolved with the $\boldsymbol{\Psi}(k,h)$-terms, greatly simplifies the convolution integrals. The viscosity variation is absorbed in the evaluation of the convolution integrals which are now non-vanishing only in a limited range. Thus, for example, we can identify $2\mu_x\psi_{xyy}$ in the transform domain as:

$$2k\boldsymbol{\mu}(k,h) * kh^2\boldsymbol{\Psi}(k,h) = 2\int_{k-k'}^{k+k'}\int_{h-h'}^{h+h'} (k-\xi)\xi\eta^2\boldsymbol{\Psi}(\xi,\eta)\, d\xi d\eta \qquad \text{(A11)}$$

By restricting the half-widths k' and h' to be small enough, the transformed stream function $\boldsymbol{\Psi}(k,h)$ can be assumed to vary only slightly from uniform over 2k' and 2h', whereby it may be represented as a Taylor series approximation in the integrand, truncated to linear terms only:

$$\boldsymbol{\Psi}(\xi,\eta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} (1/n!m!)\left\{\partial^{(n)}/\partial\xi^{(n)}\ \partial^{(m)}/\partial\eta^{(m)}\left[\boldsymbol{\Psi}(k,h)\right]\right\}(\xi-k)^n\,(\eta-h)^m$$

$$= \boldsymbol{\Psi}(k,h) + (\xi-k)\,\partial\boldsymbol{\Psi}(k,h)/\partial\xi + (\eta-h)\,\partial\boldsymbol{\Psi}(k,h)/\partial\eta + O(n,m > 1) \qquad \text{(A12)}$$

The example convolution integral above then becomes:

$$2k\boldsymbol{\mu}(k,h) * kh^2\boldsymbol{\Psi}(k,h) =$$

$$2\int_{k-k'}^{k+k'}\int_{h-h'}^{h+h'} (k-\xi)\xi\eta^2\left\{\boldsymbol{\Psi}(k,h) + (\xi-k)\,\partial\boldsymbol{\Psi}(k,h)/\partial\xi + (\eta-h)\,\partial\boldsymbol{\Psi}(k,h)/\partial\eta\right\}d\xi d\eta \qquad \text{(A13)}$$

By substituting the expansion (A12) into all convolution integrals, as in the above example, equation (A10) becomes an integral equation to be solved for $\boldsymbol{\Psi}(k,h)$. Note that the derivatives

$$\partial\boldsymbol{\Psi}(k,h)/\partial\xi \qquad \text{and} \qquad \partial\boldsymbol{\Psi}(k,h)/\partial\eta$$

are constant and integrate to

$$\partial\boldsymbol{\Psi}(k,h)/\partial k \qquad \text{and} \qquad \partial\boldsymbol{\Psi}(k,h)/\partial h$$

All other functions of ξ and η integrate to functions of k and h. The integrations are straightforward but tedious. After much algebra, the following partial differential equation results for $\boldsymbol{\Psi}(k,h)$:

$$\left[(8/3)(k'^3h')(k^3 + kh^2)\right]\partial\boldsymbol{\Psi}(k,h)/\partial k + \left[(8/3)(k'h'^3)(h^3 + hk^2)\right]\partial\boldsymbol{\Psi}(k,h)/\partial h$$

$$+ \left[(4k'h')(k^2 + h^2)^2 + (4/3)(k'^3h' - k'h'^3)(k^2 - h^2)\right]\boldsymbol{\Psi}(k,h) = 0 \qquad \text{(A14)}$$

This equation can be conveniently written as:

$$A(k,h)\,\partial\boldsymbol{\Psi}(k,h)/\partial k + B(k,h)\,\partial\boldsymbol{\Psi}(k,h)/\partial h + C(k,h)\,\boldsymbol{\Psi}(k,h) = 0 \qquad \text{(A15)}$$

where:

$$A(k,h) = [(8/3)(k'^3h')(k^3 + kh^2)] = ak(k^2 + h^2)$$

$$B(k,h) = [(8/3)(k'h'^3)(h^3 + hk^2)] = bh(k^2 + h^2) \tag{A16}$$

$$C(k,h) = [(4k'h')(k^2 + h^2)^2 + (4/3)(k'^3h' - k'h'^3)(k^2 - h^2)]$$

$$= c(k^2 + h^2)^2 + [(a-b)/2](k^2 - h^2)$$

where a, b, and c represent the derived constant functions of k' and h'.

Equation (A15) is then solved along characteristic paths in (k,h) space by a transformation of variables. The subsidiary equation

$$dh/dk = B(k,h)/A(k,h) = bh/ak \tag{A17}$$

defines the characteristic paths in (k,h) along which the solution of (A15) must exist. That is, equation (A17) defines the manner in which the solution $\boldsymbol{Y}$ (k,h) must change in k for a given change in h. Solution of (A17) yields the relation:

$$hk^{-b/a} = d = \text{constant} \qquad \text{with} \qquad b/a = (h'/k')^2 \tag{A18}$$

Thus, solution of the convolution integrals, such as (A12), with the above constraints from the resulting partial differential equation (A15), requires a stringent assumption on the form of nearest neighbor "awareness" of the viscosity functions and the flow stream function. That is, relation (A18) is a constraint on the relationship between the two wavenumbers k and h (representing the spatial transforms of x and y), and the terms k' and h'. Recall, the primed k and h are not derivatives, but rather represent the width of the pulse for μ in the convolution integrals in the k and h directions. This constraint dictates the form of the final solution, but it also yields insight into the nature of the coupling of the harmonics from the relaxing deformation. If k' and h' were equal, then the viscosity nonlinearity would be isotropic in that the width of the nearest neighbor coupling would be the same in both the horizontal and vertical directions. Furthermore, the b/a exponent would be unity, and the wavenumbers h and k representing the variation of y in x and y would be within a constant of each other. Thus, we need to somehow determine the constant ‘d’ in (A18).

Consider the constant viscosity problem to be the Newtonian limit problem of equation (A9). In that case, with $\mu = \mu^{(0)} + \varepsilon\mu^{(1)} + O(\varepsilon^2)$ for some small parameter ε, and $\mu^{(0)}$ = constant, equation (9) reduces to:

$$\mu^{(0)} (\psi_{xxxx} + 2\psi_{xxyy} + \psi_{yyyy}) = 0 \tag{A19}$$

for the Newtonian problem. Whatever the solution is to the full problem (A9), we want to be able to recover the solution to (A19) in the limit of constant viscosity. This ties the relation between h and k to that which must result in the Newtonian limit as well. Equation (A19) admits a modal solution of the form:

$$\psi(x,y) = A \exp[i(kx + hy)] \tag{A20}$$

for some arbitrary amplitude A. Substitution of this solution into (A19) yields a dispersion relation that must hold between h and k

$$(k^2 + h^2)^2 = 0 \qquad \text{or} \qquad \pm(k^2 + h^2) = 0 \qquad \text{or} \qquad h = \pm ik \tag{A21}$$

The physically reasonable solution requires that the response flow be attenuated at depth away from the deformed surface as e^{-hy} , so to achieve this we must select h = +ik. Hence the particular solution of (A19) must be of the form

$$\psi(x,y) = A\, e^{ikx}\, e^{ihy} = A\, e^{ikx}\, e^{-ky} \tag{A22}$$

But in the Newtonian limit, h' and k' also vanish uniformly, so b/a approaches unity so that the constant d in (A18) must equal +i. For the non-Newtonian problem, (A18) must hold for varying h and k and for varying h' and k' (with h'/k' not necessarily unity), but the product $hk^{-b/a} = i$ must always hold along characteristic paths in (k,h) space. This constraint is carried forward into the form of the general solution we are deriving here for $\psi(x,y)$ as a solution to equation (A9).

For a certain transformation of variables, the first-order equation (A15) with (A16), that arose from solutions of the convolution integrals, can be transformed to a first-order equation of the form:

$$A(\zeta,\theta)\ \partial\boldsymbol{\Psi}(\zeta,\theta)/\partial\zeta \ + \ C(\zeta,\theta)\ \boldsymbol{\Psi}(\zeta,\theta) \ = \ 0 \tag{A23}$$

In this problem, the appropriate change of variables needed to achieve this transformation from (A15), with characteristics as defined in (A18), turns out to be:

$$\zeta(k,h) = k \qquad \text{and} \qquad \theta(k,h) = hk^{-b/a}$$

or

$$k = \zeta \qquad \text{and} \qquad h \ = \ \theta k^{+b/a} \ = \ \theta\zeta^{b/a} \tag{A24}$$

(See, for example, discussion in Dennemeyer, 1968, p.12-15). The coefficients A(k,h) and C(k,h) in (A15) or (A16) become, under this transformation:

$$A(\zeta,\theta) \ = \ a\zeta[\zeta^2 + \theta^2\zeta^{2b/a}]$$

$$C(\zeta,\theta) \ = \ c[\zeta^2 + \theta^2\zeta^{2b/a}] \ + \ [(a\text{-}b)/2][\zeta^2 - \theta^2\zeta^{2b/a}] \tag{A25}$$

If equation (A23) is integrated with respect to ζ while holding θ fixed, then $\boldsymbol{\Psi}(\zeta,\theta)$ must satisfy the general form:

$$\boldsymbol{\Psi}(\zeta,\theta) \ = \ f[\theta]\ \exp\left[\ -\!\int C(\zeta,\theta)\ /\ A(\zeta,\theta)\ d\zeta\ \right] \ = \ f[\theta]\ \chi(\zeta,\theta) \tag{A26}$$

where $f[\theta]$ is an arbitrary function. The exponential integral part $\chi(\zeta,\theta)$ satisfies (A23) alone, and inverse transforming its variables using (A24) yields a *particular* solution, s(k,h), of equation (A15), whereby

$$s(k,h) \ = \ \chi[\zeta(k,h),\ \theta(k,h)] \ = \ \chi[k,\ \theta(k,h)] \tag{A27}$$

The *general* solution of equation (A15) is then:

$$\boldsymbol{\Psi}(k,h) \ = \ f[\theta(k,h)]\ s(k,h) \tag{A28}$$

where $f[\theta]$ is arbitrary.

Accordingly, solution of equation (A15) is reduced to solving equation (A26) for the exponential integral part

$$\chi(\zeta,\theta) \ = \ \exp\left[\ -\!\int C(\zeta,\theta)\ /\ A(\zeta,\theta)\ d\zeta\ \right]\ , \tag{A29}$$

substituting the result into equation (A27), and re-transforming the variables back from (ζ,θ) to (k,h) using the relations (A24). A form of $f[\theta]$ is then selected, and that function times s(k,h) yields the general solution desired. Continuing the strategy, when the $\boldsymbol{\Psi}(k,h)$ solution (A28) is inverse Fourier Transformed, it yields the desired solution for our original stream function $\psi(x,y)$, solving equation (A9). Plus, the nonlinearity of the viscosity is completely incorporated in this solution due to the form of the convolution integrals selected. This $\psi(x,y)$ can then be analytically differentiated, and finally an equation can be developed for the relaxation of the deformed land surface. It is this final equation that becomes the particular problem to be solved in this physical scenario.

To solve (A26), equation (A29) must be solved. From (A25), the ratio of C/A becomes the needed integrand in equation (A29) as follows:

$$C(\zeta,\theta) / A(\zeta,\theta) = [c/a][\zeta^2 + \theta^2\zeta^{2b/a}] /\zeta$$
$$+ [(a-b)/2a][\zeta^2 - \theta^2\zeta^{2b/a}] /\zeta[\zeta^2 + \theta^2\zeta^{2b/a}] \qquad (A30)$$

or

$$-\int C(\zeta,\theta) / A(\zeta,\theta)\, d\zeta = - [c/a]\int \zeta\, [1 + \theta^2\zeta^r]\, d\zeta$$
$$- [(a-b)/2a]\int \{[1 - \theta^2\zeta^r] /\zeta[1 + \theta^2\zeta^r]\}\, d\zeta \qquad (A31)$$

where $r = (2b/a) - 2$ for notational simplification. Note that if $b = a$ (*i.e.*, $h' = \pm k'$) as in the Newtonian problem, then r vanishes. Evaluation of the integrals in (A31) requires evaluation of four integrals:

(i) $\quad - [c/a]\int \zeta\, d\zeta$

(ii) $\quad - [c/a]\, \theta^2\int \zeta^{r+1}\, d\zeta$

(iii) $\quad - [(a-b)/2a]\int \zeta^{-1} /[1 + \theta^2\zeta^r]\, d\zeta$ $\qquad$ (A32)

(iv) $\quad - [(a-b)/2a]\, \theta^2\int \zeta^{r-1} /[1 + \theta^2\zeta^r]\, d\zeta$ $\qquad$ with $r = (2b/a) - 2$

Integrals (i), (ii), and (iv) are trivial and yield:

(i) $\quad - [c/a]\int \zeta\, d\zeta = - [c/2a]\, \zeta^2$

(ii) $\quad - [c/a]\, \theta^2\int \zeta^{r+1}\, d\zeta = - [c/2b]\, \theta^2\, \zeta^{r+2}$ $\qquad$ (A33)

(iv) $\quad - [(a-b)/2a]\, \theta^2\int \zeta^{r-1} /[1 + \theta^2\zeta^r]\, d\zeta = - (1/4) \ln [1 + \theta^2\zeta^r]$

Integral (iii) is not quite so straightforward. In the case that $h' \approx k'$, $b/a = (h'/k')^2$ is close to but not equal to unity, and $r = (2b/a) - 2$ is very small but not zero. Then for small r, the function ζ^r in integral (iii) can be expanded as

$$\zeta^r = 1 + r \ln\zeta + (r \ln\zeta)^2 /2! + ... \qquad (A34)$$

which for small r can be truncated to

$$\zeta^r \approx 1 + r \ln\zeta \tag{A35}$$

Under this restriction of small r and of $h' \approx k'$, integral (iii) in (A32) becomes approximately:

$$\approx - [(a-b)/2a] \int \zeta^{-1} / [(1 + \theta^2) + (\theta^2 r) \ln \zeta] \, d\zeta = - [(a-b)/2a] \int \zeta^{-1} / [p + q \ln \zeta] \, d\zeta \tag{A36}$$

By letting $U = \ln z$ so that $dU = dz/z$, and with p and q defined implicitly as in (A36), integral (iii) becomes:

$$- [(a-b)/2a] \int [p + qU]^{-1} \, dU = - [(a-b)/2a] \, (1/q) \ln (p + qU)$$

or

$$- [(a-b)/2a] \int \zeta^{-1} / [1 + \theta^2 \zeta^r] \, d\zeta = - [(a-b)/2a] \, (1/\theta^2 r) \ln \left(1 + \theta^2 + \theta^2 r \ln \zeta\right) \tag{A37}$$

The sum of equations (A33) and (A37) are the required solutions to the integral (A31). But the solution (A37) holds for small r only, and it is singular if r vanishes. Thus $h' \approx k'$ but $h' \neq k'$. Hence, under the analytic formulation here, the "windows" of nearest-neighbor interaction for the viscosity, as explicitly represented in the convolution integrals, **must** be slightly different in the horizontal and vertical dimensions. The viscosity is not isotropic, and coupling interactions are different vertically from horizontally.

Combining equations (A33) and (A37) yields:

$$\begin{aligned} -\int C(\zeta,\theta) / A(\zeta,\theta) \, d\zeta \approx & - [c/2a] \, \zeta^2 - [c/2b] \, \theta^2 \zeta^{r+2} - (1/4) \ln [1 + \theta^2 \zeta^r] \\ & - [(a-b)/2a] \, (1/\theta^2 r) \ln \left(1 + \theta^2 + \theta^2 r \ln \zeta\right) \end{aligned}$$

then using again $\zeta^r \approx 1 + r \ln\zeta$ and the definition for r, the integral (A31) simplifies to:

$$-\int C(\zeta,\theta) / A(\zeta,\theta) \, d\zeta \approx - [c/2a]\zeta^2 - [c/2b]\theta^2 \zeta^{2b/a} + [(1 - \theta^2)/4\theta^2] \ln [1 + \theta^2 \zeta^r] \tag{A38}$$

Working backwards, this solution can now be substituted into equation (A29) for $\chi(\zeta,\theta)$ yielding:

$$\chi(\zeta,\theta) = \exp \left[- [c/2a]\zeta^2 - [c/2b]\theta^2 \zeta^{2b/a} \right] \left(1 + \theta^2 \zeta^r\right)^{[(1 - \theta^2)/4\theta^2]} \tag{A39}$$

Recalling the variable changes given in equations (A24), equation (A39) can be rewritten in terms of the original variables (k,h), and then it can be substituted into equation (A27) for the particular solution s(k,h):

$$s(k,h) \;=\; \exp\Big[-[c/2a]k^2 \;-[c/2b]h^2\Big] \Big(1 + (h^2/k^2)\Big)^{[(1 - (hk^{-b/a})^2)/(2hk^{-b/a})^2]} \qquad \text{(A40)}$$

However, because solution of the subsidiary equation (A17) dictates that relation (A18) must hold, and that the product $hk^{-b/a}$ in this relation must be $+i$, the exponent in equation (A40) reduces to -1/2, yielding the particular solution to be:

$$s(k,h) \;=\; \exp\Big[-[c/2a]k^2 \;-[c/2b]h^2\Big] \Big(1 + (h^2/k^2)\Big)^{-1/2} \qquad \text{(A41)}$$

from which is derived the general solution to equation (A15):

$$\boldsymbol{\Psi}(k,h) \;=\; f[\theta(k,h)]\; s(k,h) \;=\; f\,[hk^{-b/a}]\,\exp\Big[-[c/2a]k^2 \;-[c/2b]h^2\Big]\Big(1 + (h^2/k^2)\Big)^{-1/2} \qquad \text{(A42)}$$

Recall that $f\,[hk^{-b/a}]$ is arbitrary; one may select it as follows. Because the characteristics requires $hk^{-b/a}$ be constant along paths in (k,h), and because the boundary conditions in the Newtonian case require $h = ik$, so that the constant must be $+i$, *select* f to be the particular case of unity by choosing f as:

$$f\,[hk^{-b/a}] \;=\; [hk^{-b/a}]^4 \;=\; (+i)^4 \;=\; 1$$

Then the general solution reduces to:

$$\boldsymbol{\Psi}(k,h) \;=\; \exp\Big[-[c/2a]k^2 \;-[c/2b]h^2\Big]\Big(1 + (h^2/k^2)\Big)^{-1/2} \qquad \text{(A43)}$$

This is the solution to equation (A15), derived from equation (A10), which must now be inverse Fourier Transformed to yield a general solution to equation (A9) for $\psi(x,y)$. This is accomplished by integrating $\boldsymbol{\Psi}$ from (A43) as follows:

$$\psi(x,y) \;=\; (1/2\pi)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \boldsymbol{\Psi}(k,h)\, e^{ihy}\, e^{ikx}\, dh\, dk \qquad \text{(A44)}$$

Now, instead of carrying out this integration immediately, consider some physical constraints in order to **experiment** with the behavior of this solution in terms of the coupling of response harmonics during rebound. Assume first that $\psi(x,y)$ could actually be well represented in the horizontal x-direction by superposition of independent Fourier components. In particular, consider the assumption in which a modal solution of the form e^{ikx} in the horizontal direction is sufficient to describe the flow. Then, even though the stress field is changing as the rebound occurs, that change in the horizontal direction is slight compared to that changing in the vertical y-direction so that there is very little coupling of changing horizontal viscosity distribution to changing horizontal flow velocity u. This assumption leads to a simplification of integral (A44) in that we can now assume no integration over k, and that (perhaps within a factor of $1/2\pi$) (A44) reduces to

$$\psi(x,y) \;=\; G(k,y)e^{ikx} \qquad \text{(A45)}$$

Then the half-inverse transform G(k,y) of $\boldsymbol{\Psi}$(k,h) , now for each fixed wavenumber k, is defined by:

$$G(k,y) = (1/2\pi)\int_{-\infty}^{\infty} \boldsymbol{\Psi}(k,h)\, e^{ihy}\, dh$$

or

$$G(k,y) = (1/2\pi) \exp\left[-[c/2a]k^2\right]\int_{-\infty}^{\infty} [1 + (h^2/k^2)]^{-1/2} \exp\left[-[c/2b]h^2\right]\, e^{ihy}\, dh \qquad (A46)$$

To evaluate this integral, first expand the $1 + (h^2/k^2)$ term as the series

$$[1 + (h^2/k^2)]^{-1/2} = 1 - (1/2)\,(h/k)^2 + (1{\cdot}3/2{\cdot}4)\,(h/k)^4 - (1{\cdot}3{\cdot}5/2{\cdot}4{\cdot}6)\,(h/k)^6 + \cdots$$

or

$$[1 + (h^2/k^2)]^{-1/2} = 1 + \sum_{\nu=1}^{\infty} (-1)^{\nu} \left\{\prod_{m=0}^{\nu-1} [(1+2m)/(2+2m)]\right\} (h/k)^{2\nu} \qquad (A47)$$

Let A(k) be defined as the function of k outside the integral over dh in (A46):

$$A(k) = (1/2\pi) \exp\left[-[c/2a]k^2\right] \qquad (A48)$$

Then substituting this definition and the series expansion (A47) into the integral (A46) yields the following integral:

$$G(k,y)/A(k) = \int_{-\infty}^{\infty} \exp\left[-[c/2b]h^2 + ihy\right]\, dh \quad +$$

$$+ \sum_{\nu=1}^{\infty} (-1)^{\nu} \left\{\prod_{m=0}^{\nu-1} [(1+2m)/(2+2m)]\right\}(1/k)^{2\nu}\int_{-\infty}^{\infty} h^{2\nu} \exp\left[-[c/2b]h^2 + ihy\right] dh \qquad (A49)$$

The first integral of (A49) is evaluated using a Gaussian identity, whereby, let:

$$\int_{-\infty}^{\infty} \exp\left[-[c/2b]h^2 + ihy\right]\, dh = \int_{-\infty}^{\infty} \exp\left[-\alpha h^2 + ihy\right] dh \quad ; \ \alpha = c/2b$$

Then recognizing that

$$\int_{-\infty}^{\infty} e^{ihy}\, dh = \int_{-\infty}^{\infty} \cos(hy)\, dh$$

and that

$$\int_{-\infty}^{\infty} \exp\,[-\alpha h^2] \cos\,(hy)\, dh \;=\; (\pi/\alpha)^{1/2}\;\exp\,[-y^2/4\alpha]\;;\;\; 1/\alpha = 2b/c\;;\;\; 1/4\alpha = b/2c$$

By direct substitution, the first integral of (A49) becomes:

$$\int_{-\infty}^{\infty} \exp\,[-[c/2b]h^2 + ihy]\;\; dh \;=\; [2\pi b/c]^{1/2}\;\exp\,[-\,(b/2c)y^2] \qquad \text{(A50)}$$

The second integral of (A49) is evaluated as follows. Let $p = c/2b$ and $2q = iy$. Then it takes the form:

$$\int_{-\infty}^{\infty} h^{2\nu} \exp\,[-[c/2b]h^2 + ihy]\; dh \;=\; \int_{-\infty}^{\infty} h^{2\nu} \exp\,[-ph^2 + 2qh]\; dh$$

which admits the series solution (see, for example, Gröbner and Hofreiter, 1966, vol 2, form (6), p. 65):

$$=\;\; (2\nu)!\,\exp(q^2/p)\;[\pi/p]^{1/2}\,(q/p)^{2\nu}\;\sum_{r=0}^{\nu}\,[1/(2\nu-2r)!\;r!]\;(p/4q^2)^r \quad ;\quad p>0$$

Substituting back for $p = c/2b$ and $q = iy/2$ gives the desired function of y as:

$$\int_{-\infty}^{\infty} h^{2\nu} \exp\,[-[c/2b]h^2 + ihy]\; dh \;=$$

$$=\; [2\pi b/c]^{1/2}\;\exp\,[-\,(b/2c)y^2]\;[2biy/2c]^{2\nu}\;\sum_{r=0}^{\nu}\,[(2\nu)!/(2\nu-2r)!\;r!]\;(-c4/2b4y^2)^r$$

$$=\; [2\pi b/c]^{1/2}\exp\,[-\,(b/2c)y^2]\;(i)^{2\nu}\;(b/c)^{2\nu}\;\sum_{r=0}^{\nu}\,[(2\nu)!/(2\nu-2r)!\;r!]\;\;(-1/2)^r\,(b/c)^{-r}\,y^{2\nu-2r}$$

$$=\; [2\pi b/c]^{1/2}\exp\,[-\,(b/2c)y^2]\;(i)^{2\nu}\sum_{r=0}^{\nu}\,[(2\nu)!/(2\nu-2r)!\;r!]\;\;(-1/2)^r\,(b/c)^{2\nu-r}\,y^{2\nu-2r} \qquad \text{(A51)}$$

Then substituting solutions (A51) and (A50) appropriately into the integral solution (A49), the inverse transform of G(k,y), for fixed k, becomes:

$$G(k,y) \;=\; A(k)\;[2\pi b/c]^{1/2}\exp\,[-\,(b/2c)y^2]\;\Big\{1 + I(k,y)\Big\} \qquad \text{(A52)}$$

where, noting that $(-1)^{\nu}(i)^{2\nu} = 1$, the series $I(k,y)$ is:

$$I(k,y) = \sum_{\nu=1}^{\infty}\left\{ \prod_{m=0}^{\nu-1} [(1+2m)/(2+2m)]\right\} k^{-2\nu} \sum_{r=0}^{\nu} [(2\nu)!/(2\nu-2r)!\, r!]\, (-1/2)^r\, (b/c)^{2\nu-r}\, y^{2\nu-2r} \qquad \text{(A53)}$$

and where $A(k)$ is defined in equation (A48). Note that this representation for $G(k,y)$ has **not** been transformed from k-space to x-space. Rather a modal dependence in the x-direction has been assumed, yielding equation (A45). The assumption that $\psi(x,y) = G(k,y)e^{ikx}$ rather than the integral of $G(k,y)e^{ikx}$ implies that a modal dependence of ψ in the x-direction is acceptable, and the complete Fourier inverse transform is not necessary. Physically, this means one assumes either that no new harmonics are induced in the x-direction; or that any given change in the vertical stress field **does** induce changes in the x-direction, but that the change is small enough that once these harmonics are defined, they can be superposed to yield the total stream function $\psi(x,y)$ through iteration over the k wavenumbers.

Note that because the original motivation for this work was to understand the actual role of coupling of the nonlinear, stress-dependent viscosity field to changes in the rebound response flow, and to understand the changes of the viscosity field due to these changes in flow stresses, it is rather artificial to now make simplifying assumptions such as (A45) that partially eliminates the physics that was argued as fundamental to this problem! However, in an effort to keep track of the effect of various approximations and their relation to the resulting physical solution, it is worthwhile imposing and relaxing these assumptions sequentially so that the role each plays in the response flow can be isolated. This procedure then becomes a form of “sensitivity analysis” on the structure of the family of solutions to the response flow problem. So, for example, another reasonable action might now be to return to the original inverse Fourier Transform, as outlined in equation (A44), and carry out the full transform rather than making the simplifying assumption (A45) of modal behavior in the horizontal direction. But we have not yet seen the effect of this assumption on a resulting relaxation equation. Hence, before assumption (A45) is replaced, it is best to continue in this mode so that there are final results to compare when the more complicated problem is solved. In addition, it is not yet clear what we wish to intend as the essence of nonlinear rheology in this problem. At a fundamental level, the problem involves recognizing that the viscosity in the planetary interior is assumed to be a function of the changing stress field; but that functional relation is yet to be specified. If the viscosity were an explicit function of the stress, then in order to develop the viscosity profiles, one would need to calculate the stress field for the deformed medium either by alternative methods, or by assuming that the flow field derived here could be written in terms of flow strain-rates, and that the viscosity is a function of these derived strain-rates. The latter is the preferable strategy, but even given that, there are still options. For example, we need to determine the specific functional relation for $\mu(x,y)$ as a function of derivatives of $\psi(x,y)$. Ideally, the form selected could be justified on the basis of either experimental rock-deformation data or on convective instability models or residual gravity anomalies. Alternatively, a proxy representation that mirrors the structure of the viscosity profile could be assumed, such as an exponential approximation. In this analysis one had to make assumptions on the nature of the coupling between the k and h wavenumbers, and on the square-wave approximation to nearest-neighbor coupling over the k' and h' intervals just so that an analytic solution of the convolution integrals could proceed. This constraint must be maintained in order to maintain applicability of the solutions already derived. Hence, the square wave approximation in the transform space implies that the viscosity must look like a product of sinc functions in (x,y)-space, using wavenumbers k' and h'. But even that restriction does not yet yield a closed problem. It is still open to selection, specifically, how one wishes to indicate the coupling between the viscosity changes in the vertical flow and those in the horizontal flow. For example, should the nonlinearity in changes in wavenumber h induce additional vertical response modes or should it induce both vertical and horizontal modes, seen as changes in k as well. Similarly, should changes in k harmonics induce changes in k, changes in k and h, or induce no new modes at all, as is assumed in equation (A45). Because of all of these various complexities, the problem progresses sequentially and by explicit tracking of the effect of the assumptions. All that one is committed to is the belief that as a certain surface displacement begins to rebound, the stress field in the mantle material, to depths on the order of the scale of the horizontal scale of the displacement, will change. And, thus the viscosity field, which is a function of this changing stress field will also change, so that the form and rate of the rebound will change. This stress or strain-rate dependence of viscosity is what is meant by the nonlinear rheology in this problem. The evaluation of specifically how that dependence induces other response characteristics is the core investigation of this analysis.

Given all these caveats, consider again the equation derived from making assumption (A45), namely equation (A52) above. Use this solution for $\psi(x,y)$ to derive the time-change (relaxation) equation for the rebounding surface. That equation must then be solved in conjunction with boundary conditions, which are now derived.

II. Derivation of the Boundary Conditions

The boundary conditions to be imposed at the rebounding surface, as represented by the solution (A52) and (A53) for (A45) above, are: a dynamic free surface condition that determines the driving normal stress, and two kinematic or matching conditions. The kinematic conditions at the free surface require that the horizontal velocity vanish at the displace surface, and that the vertical velocity exactly correspond to the time rate of change of that displaced free surface, say $y = \eta(x,t)$. Because horizontal flows are negligible compared with the vertical rebound, and because the initial vertical displacement η is small compared to the horizontal scale of the rebounding feature, the kinematic conditions can be linearized to order η and be applied at y = 0. Hence to order η, the kinematic conditions become:

$$u = \partial\psi/\partial y = 0 \qquad \text{at } y = 0 \tag{A54}$$

$$v = -\partial\psi/\partial x = \partial\eta/\partial t \qquad \text{at } y = 0 \tag{A55}$$

Evaluation of the derivatives of ψ in both of these conditions, as well as in the normal stress condition presented below, requires evaluation of the derivatives of G(k,y) and hence of I(k,y) from equations (A52) and (A53). For convenience, the various derivatives that will be needed are presented together here. From (A52):

$$\partial G(k,y)/\partial y = - A(k)\, [2\pi b/c]^{1/2} \exp\left[- (b/2c)y^2\right] \left\{(by/c)[1 + I(k,y)] - \partial I(k,y)/\partial y\right\} \tag{A56a}$$

and

$$\partial^2 G(k,y)/\partial y^2 = - A(k)\, [2\pi b/c]^{1/2} \exp\left[- (b/2c)y^2\right] \cdot\left\{(b/c)(1 - by^2/c)[1 + I(k,y)] + (2by/c)\, \partial I(k,y)/\partial y - \partial^2 I(k,y)/\partial y^2\right\} \tag{A56b}$$

with A(k) defined in (A48); and from (A53):

$$\partial I(k,y)/\partial y = \sum_{\nu=1}^{\infty}\left\{\prod_{m=0}^{\nu-1} [(1+2m)/(2+2m)]\right\}k^{-2\nu} \cdot \sum_{r=0}^{\nu} [(2\nu)!/(2\nu-2r)!\, r!]\, (-1/2)^r\, (b/c)^{2\nu-r}\, (2\nu-2r)y^{2\nu-2r-1} \tag{A57a}$$

and

$$\partial^2 I(k,y)/\partial y^2 = \sum_{\nu=1}^{\infty}\left\{\prod_{m=0}^{\nu-1} [(1+2m)/(2+2m)]\right\}k^{-2\nu} \cdot \sum_{r=0}^{\nu} [(2\nu)!/(2\nu-2r)!\, r!]\, (-1/2)^r\, (b/c)^{2\nu-r}\, (2\nu-2r)(2\nu-2r-1)y^{2\nu-2r-2} \tag{A57b}$$

Using relations (A52), (A53) for our general solution (A45), and substituting in the required derivative (A56a), kinematic condition (A54) requires that:

$$u|_{y=0} = [\partial\psi/\partial y]|_{y=0} = e^{ikx}\,[\partial G(k,y)/\partial y]|_{y=0} = 0$$

or that

$$- A(k)\, e^{ikx}\, [2\pi b/c]^{1/2} \exp[-(b/2c)y^2]|_{y=0} \Big\{(by/c)[1 + I(k,y)] - \partial I(k,y)/\partial y\Big\}\Big|_{y=0} = 0 \qquad \text{(A58)}$$

But this relation is automatically satisfied because both

$$(by/c)[1 + I(k,y)] \qquad \text{and} \qquad \partial I(k,y)/\partial y$$

vanish at $y = 0$, for all values of ν and r in the appropriate series summations for $\partial I(k,y)/\partial y$.

Condition (A55) requires that at $y = 0$:

$$v = \partial\eta/\partial t = -\partial\psi/\partial x = -ike^{ikx}\, G(k,y)|_{y=0} \qquad \text{(A59)}$$

Here, $G(k,y)$ evaluated at $y = 0$ becomes simply a function of k, but it is not just the amplitude $A(k)$ times $[2\pi b/c]^{1/2}$ as might be expected from first glance at equation (A52) and (A53). The series $I(k,y)$ in (A53) does not vanish at $y = 0$ because before $I(k,y)$ is evaluated at $y = 0$, the series must be expanded. When the expansion is carried out, there are terms in which y^0 appear, specifically whenever $2\nu = 2r$. These terms are of course equal to unity and are no longer a function of y. They hence do not vanish when the series is evaluated for $y = 0$. Thus, from (A52):

$$G(k,y)|_{y=0} = A(k)\, [2\pi b/c]^{1/2} \exp[-(b/2c)y^2]|_{y=0} \Big\{1 + I(k,y)|_{y=0}\Big\}$$

$$= A(k)\, [2\pi b/c]^{1/2} \Big\{1 + I(k,y)|_{y=0}\Big\} \qquad \text{(A60)}$$

and

$$I(k,y)|_{y=0} = -(1/2k^2)(b/c) + (9/8k^4)(b/c)^2 - (225/48k^6)(b/c)^3 + \cdots$$

$$= \sum_{\nu=1}^{\infty}(-1)^\nu\Big\{\prod_{m=0}^{\nu-1}[(1+2m)^2/(2+2m)]\Big\}(1/k)^{2\nu}\,(b/c)^\nu \qquad \text{(A61)}$$

Thus, with $A(k)$ defined in (A48), the second kinematic condition (A55) reduces to:

$$\partial\eta/\partial t = -ik\, e^{ikx}\, A(k)\, [2\pi b/c]^{1/2}$$

$$\cdot \Big\{1 + \sum_{\nu=1}^{\infty}(-1)^\nu\Big\{\prod_{m=0}^{\nu-1}[(1+2m)^2/(2+2m)]\Big\}(1/k)^{2\nu}\,(b/c)^\nu\Big\} \qquad \text{(A62)}$$

In addition to these two kinematic conditions, (A58) and (A62), the normal stress must vanish at the free, displaced surface $y = \eta(x,t)$. In this problem, the normal stress σ_{yy} at the surface is just the hydrostatic pressure there plus stresses due to the rebounding viscous flow. Hence, this condition says that:

$$\sigma_{yy} = p(y) = 0 \qquad \text{on } y = \eta \tag{A63}$$

An equation for p(y) must be determined from the vertical component of the original Navier-Stokes equation, (A2), using the stream function definition (A8) for the velocity field. This equation is:

$$p_y(y) = -\rho g - \mu(\psi_{xxx} + \psi_{xyy}) - \mu_x(\psi_{yy} - \psi_{xx}) - 2\mu_y\psi_{xy} \tag{A64}$$

Integration over y of equation (A64) provides the required relation for p(y) which must then vanish on $y = \eta$. In order to integrate (A64), we must have an explicit relation for the viscosity $\mu(x,y)$ and its derivatives. Because the viscosity has been defined in the transform domain as a square-wave filter of half-widths k' and h', centered on k and h, so that the convolution integrals could be evaluated, the viscosity in (x,y) space must be represented by a product of sinc functions that would yield this square filter in (k,h) space. Hence, for consistency in this analysis, the viscosity must be represented as:

$$\mu(x,y) = \frac{\sin 2\pi k'x}{2\pi k'x}\,\frac{\sin 2\pi h'y}{2\pi h'y} = \mathrm{sinc}(\alpha x)\,\mathrm{sinc}(\beta y) \tag{A65}$$

where the wavenumbers α and β are $\alpha = 2\pi k'$ and $\beta = 2\pi h'$. With this definition, the required derivatives of viscosity in equation (A64) become:

$$\begin{aligned} \mu_x(x,y) &= \mathrm{sinc}(\beta y)\left\{(\cos \alpha x)/x - (\sin \alpha x)/\alpha x^2\right\} \\ \mu_y(x,y) &= \mathrm{sinc}(\alpha x)\left\{(\cos \beta y)/y - (\sin \beta y)/\beta y^2\right\} \end{aligned} \tag{A66}$$

The stream function terms in equation (A64) are defined in terms of G(k,y), whereby:

$$\begin{aligned} \psi_{xxx} &= -ik^3 e^{ikx}\, G(k,y) \\ \psi_{xyy} &= ik\, e^{ikx}\, \partial^2 G(k,y)/\partial y^2 \\ \psi_{yy} &= e^{ikx}\, \partial^2 G(k,y)/\partial y^2 \\ \psi_{xx} &= -k^2 e^{ikx}\, G(k,y) \\ \psi_{xy} &= ik\, e^{ikx}\, \partial G(k,y)/\partial y \end{aligned} \tag{A67}$$

and G(k,y) is defined in (A52) and (A53), A(k) defined in (A48), and with the derivatives being presented in (A56a,b) and (A57a,b). Using all the relations (A67), (A66), and (A65), the integration of equation (A64) yields p(y) needed for the normal stress condition (A63). This integration can be written as follows:

$$p(y) = -\rho g y + e^{ikx} A(k)\, [2\pi b/c]^{1/2} \left\{ D_1\int_0^y F_1 dy + D_2\int_0^y F_2 dy + D_3\int_0^y F_3 dy \right\} \tag{A68}$$

where the factors D and integrands F are:

$$D_1 = D_1(x; k, k', h') = ik^3(\sin \alpha x)/ \alpha x + k^2(\cos \alpha x)/x - k^2(\sin \alpha x)/ \alpha x^2$$
$$= ik^3 \, \mathrm{sinc}(\alpha x) + k^2/x \, [(\cos \alpha x) - \mathrm{sinc}(\alpha x)]$$

$$D_2 = D_2(x; k, k', h') = ik(\sin \alpha x)/ \alpha x - (\cos \alpha x)/x + (\sin \alpha x)/ \alpha x^2$$
$$= ik \, \mathrm{sinc}(\alpha x) - (1/x)[\cos \alpha x - \mathrm{sinc}(\alpha x)]$$

$$D_3 = D_3(x; k, k') = 2ik(\sin \alpha x)/ \alpha x = 2ik \, \mathrm{sinc}(\alpha x)$$

(A69)

$$F_1 = F_1(y; k, h') = \mathrm{sinc}(\beta y) \, \exp[-(b/2c)y^2] \left\{1 + I(k,y)\right\}$$

$$F_2 = F_2(y; k, h') = \mathrm{sinc}(\beta y) \, \exp[-(b/2c)y^2] \cdot \left\{(b/c)(1 - by^2/c)[1 + I(k,y)] + (2by/c)\, \partial I(k,y)/\partial y - \partial^2 I(k,y)/\partial y^2\right\}$$

$$F_3 = F_3(y; k, h') = (1/y)[\cos \beta y - \mathrm{sinc}(\beta y)] \, \exp[-(b/2c)y^2] \cdot \left\{(by/c)[1 + I(k,y)] - \partial I(k,y)/\partial y\right\}$$

Using these definitions in (A68), condition (A63) yields an equation for the displaced surface η:

$$\rho g \eta = e^{ikx} A(k) \, [2\pi b/c]^{1/2} \left\{ D_1(x; k, k', h') \int_0^y F_1(y; k, h')dy + D_2(x; k, k', h') \int_0^y F_2(y; k, h')dy + D_3(x; k, k') \int_0^y F_3(y; k, h')dy \right\}\Big|_{y=\eta} \tag{A70}$$

or, by representing the integrals in (A70) by a general function $\mathbf{T}(x, y; k, k', h')$, evaluated at $y = \eta$:

$$\rho g \eta = \mathbf{T}(x, y; k, k', h')\big|_{y=\eta} \, e^{ikx} A(k) \, [2\pi b/c]^{1/2} \tag{A71}$$

III. The Relaxation-Equation for Rebound under Prescribed Nonlinear Viscosity

By inverting (A71) for the normal stress boundary condition and solving for $\rho g \eta / \mathbf{T}$, equation (A71) can be combined with (A62) by eliminating the common factor $e^{ikx} A(k) [2\pi b/c]^{1/2}$. This then yields an equation for the time rate of change of the displaced surface:

$$\partial\eta/\partial t \;=\; -ik\rho g \left[\eta \,/\, \mathbf{T}(x, y; k, k', h')\big|_{y=\eta}\right] \cdot$$

$$\cdot \left\{ 1 + \sum_{\nu=1}^{\infty} (-1)^{\nu} \left\{ \prod_{m=0}^{\nu-1} \left[(1+2m)^2/(2+2m)\right] \right\} (1/k)^{2\nu} (b/c)^{\nu} \right\} \qquad \text{(A72)}$$

Equation (A72) is the relaxation equation which must be solved for $\eta(x,t)$, given an initial $\eta(x,0)$, at each lateral point x for a given wavenumber k. At the same time, k must be iterated before composition for a total wavenumber distribution, and the integrals contained in the function $\mathbf{T}(x, y; k, k', h')$ must be solved during each iteration on k and change of parameters k' and h'.

(A72) is the equation that has been sought. It describes the time rate of change of the rebounding surface under the constraints and assumptions explicitly made in the particular strategy outlined here. Were the problem to have been one of linear viscosity, this same relaxation equation would have had the simpler form of:

$$\partial\eta/\partial t \;=\; \left[\rho g/2\mu k\right]\eta \qquad \text{(A73)}$$

This relaxation equation can be solved analytically to yield:

$$\eta \;=\; \eta_o \exp[t/\tau_R] \,,$$

where the relaxation time $\tau_R \;=\; 2\mu k/\rho g$ and $\tau_R^{-1} = (\eta)^{-1}[\partial\eta/\partial t]$

Ultimately, numerical, iterative solutions to equation (A72) could be carried out. These would yield relaxation times that would describe the rebound behavior and its variation over time due to the nonlinear coupling of the stresses with the viscosity.

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14. ABSTRACT
This paper addresses the critical need for the development of intelligent or assisting software tools for the scientist who is working in the initial problem formulation and mathematical model representation stage of research. In particular, examples of that representation in fluid dynamics and instability theory are discussed. The creation of a mathematical model that is ready for application of certain solution strategies requires extensive symbolic manipulation of the original mathematical model. These manipulations can be as simple as term reordering or as complicated as discovery of various symmetry groups embodied in the equations, whereby Bäcklund-type transformations create new determining equations and integrability conditions or create differential Gröbner bases that are then solved in place of the original nonlinear PDEs. Several examples are presented of the kinds of problem formulations and transforms that can be frequently encountered in model representation for fluids problems. The capability of intelligently automating these types of transforms, available prior to actual mathematical solution, is advocated. Physical meaning and assumption-understanding can then be propagated through the mathematical transformations, allowing for explicit strategy development.

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mathematical methods, analytic transformations, geophysics, fluid dynamics, nonlinear viscosity

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