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Explicit Von Neumann Stability Conditions for the c-τ Scheme—A Basic Scheme in the Development of the CE-SE Courant Number Insensitive Schemes

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April 2005

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EXPLICIT VON NEUMANN STABILITY CONDITIONS FOR THE c-τ SCHEME—A BASIC SCHEME IN THE DEVELOPMENT OF THE CE-SE COURANT NUMBER INSENSITIVE SCHEMES

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Abstract

As part of the continuous development of the space-time conservation element and solution element (CE-SE) method, recently a set of so called "Courant number insensitive schemes" has been proposed. The key advantage of these new schemes is that the numerical dissipation associated with them generally does not increase as the Courant number decreases. As such, they can be applied to problems with large Courant number disparities (such as what commonly occurs in Navier-Stokes problems) without incurring excessive numerical dissipation.

A basic scheme in the development of the Courant number insensitive schemes is the so called "c-τ scheme". It is a solver of the PDE

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

where $a \neq 0$ is a constant. At each space-time staggered mesh points (j, n), the c-τ scheme is formed by

$$u_j^n = \frac{1}{2}\left\{(1+\nu)u_{j-1/2}^{n-1/2} + (1-\nu)u_{j+1/2}^{n-1/2} + (1-\nu^2)\left[(u_{\bar{x}})_{j-1/2}^{n-1/2} - (u_{\bar{x}})_{j+1/2}^{n-1/2}\right]\right\}$$

and

$$(u_{\bar{x}})_j^n = \frac{1}{2(1+\tau)}\left[u_{j+1/2}^{n-1/2} - (1+2\nu-\tau)(u_{\bar{x}})_{j+1/2}^{n-1/2} - u_{j-1/2}^{n-1/2} - (1-2\nu-\tau)(u_{\bar{x}})_{j-1/2}^{n-1/2}\right]$$

Here: (i) u_j^n and $(u_{\bar{x}})_j^n$, respectively, denote the numerical analogues of u and $(\Delta x/4)\partial u/\partial x$ at the mesh point (j, n); (ii) $\nu \stackrel{\text{def}}{=} a\Delta t/\Delta x$ is the Courant number; and (iii) τ is an adjustable parameter $\neq -1$.

Because the c-τ scheme is formed by two rather complicated equations involving two parameters ν and τ, it were not expected that its von Neumann stability conditions could be cast into an *explicit analytical form*. Against this expectation, it will be shown rigorously in this paper that, based on the von Neumann analysis, the c-τ scheme is stable if and only if

$$\nu^2 \leq 1, \quad \tau \geq \tau_o(\nu^2), \quad \text{and} \quad (\nu^2, \tau) \neq (1, 1)$$

where

$$\tau_o(x) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if} \quad x = 0 \\ \dfrac{4 - x - 2\sqrt{2(2 - x - x^2)}}{x} & \text{if} \quad 0 < x \leq 3/11 \\ \dfrac{x - 1 + \sqrt{1 - 2x + 5x^2}}{2x} & \text{if} \quad 3/11 \leq x \leq 1 \end{cases}$$

Note that the current stability conditions are in complete agreement with those generated numerically and reported earlier.

In addition, it will be shown that: (i) $\tau_o(x)$ is continuous at $x = 0$; (ii) $\tau_o(x)$ is consistently defined at $x = 3/11$; (iii)

$$\lim_{x \to \frac{3}{11}^-} \tau_o'(x) = \lim_{x \to \frac{3}{11}^+} \tau_o'(x) = 121/90$$

where $\tau_o'(x) \stackrel{\text{def}}{=} d\tau_o(x)/dx$; (iv) $\tau_o(x)$ is strictly montonically increasing in the interval $0 < x < 1$; and (v)

$$x < \tau_o(x) < \sqrt{x}, \qquad 0 < x < 1$$

1. Introduction

As part of the continuous development of the space-time conservation element and solution element (CE-SE) method [1–11], recently a set of so called "Courant number insensitive schemes" has been reported in [9–11]. The key advantage of these new schemes is that the numerical dissipation associated with them generally does not increase as the Courant number decreases. As such, they can be applied to problems with large Courant number disparities (such as what commonly occurs in Navier-Stokes problems) without incurring excessive numerical dissipation.

A basic scheme in the development of the Courant number insensitive schemes is the so called "c-τ scheme" [11]. It is a solver of the PDE

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{1.1}$$

where $a \neq 0$ is a constant. Consider Fig. 1 and let Ω denote the set of all space-time staggered mesh points (dots in Fig. 1), where $n = 0, \pm 1/2, \pm 1, \pm 3/2, \pm 2, \ldots$, and, for each n, $j = n \pm 1/2, n \pm 3/2, n \pm 5/2, \ldots$. Then, at each $(j, n) \in \Omega$, the c-τ scheme is formed by

$$u_j^n = \frac{1}{2}\left\{(1+\nu)u_{j-1/2}^{n-1/2} + (1-\nu)u_{j+1/2}^{n-1/2} + (1-\nu^2)\left[(u_{\bar{x}})_{j-1/2}^{n-1/2} - (u_{\bar{x}})_{j+1/2}^{n-1/2}\right]\right\} \tag{1.2}$$

and

$$(u_{\bar{x}})_j^n = \frac{1}{2(1+\tau)}\left[u_{j+1/2}^{n-1/2} - (1+2\nu-\tau)(u_{\bar{x}})_{j+1/2}^{n-1/2} - u_{j-1/2}^{n-1/2} - (1-2\nu-\tau)(u_{\bar{x}})_{j-1/2}^{n-1/2}\right] \tag{1.3}$$

Here: (i) u_j^n and $(u_{\bar{x}})_j^n$, respectively, denote the numerical analogues of u and $(\Delta x/4)\partial u/\partial x$ at the mesh point (j, n); (ii)

$$\nu \stackrel{\text{def}}{=} \frac{a\Delta t}{\Delta x} \tag{1.4}$$

is the Courant number; and (iii) τ is an adjustable parameter $\neq -1$. It is shown in [12] that Eqs. (1.2) and (1.3) are consistent with a pair of PDEs with Eq. (1.1) being one of them.

Because the c-τ scheme is formed by two rather complicated equations involving two parameters ν and τ, it was not expected that its von Neumann stability conditions could be cast into an *explicit analytical form.* But to the contrary, it will be shown rigorously in this paper that, based on the von Neumann analysis, the c-τ scheme is stable if and only if

$$\nu^2 \le 1, \quad \tau \ge \tau_o(\nu^2), \quad \text{and} \quad (\nu^2, \tau) \neq (1, 1) \tag{1.5}$$

where

$$\tau_o(x) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if} \quad x = 0 \\ \dfrac{4 - x - 2\sqrt{2(2 - x - x^2)}}{x} & \text{if} \quad 0 < x \le 3/11 \\ \dfrac{x - 1 + \sqrt{1 - 2x + 5x^2}}{2x} & \text{if} \quad 3/11 \le x \le 1 \end{cases} \tag{1.6}$$

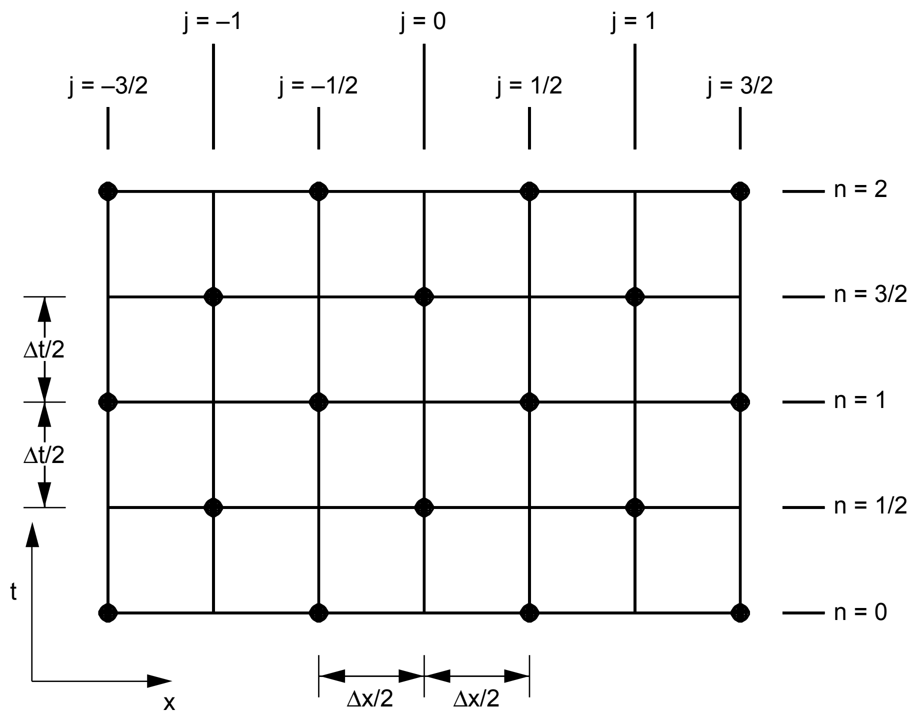

Figure 1.—A space-time mesh.

Note that the current stability conditions are in complete agreement with those generated numerically and reported earlier in [11].

In addition, it will be shown that: (i) $\tau_o(x)$ is continuous at $x = 0$; (ii) $\tau_o(x)$ is consistently defined at $x = 3/11$; (iii)

$$\lim_{x \to \frac{3}{11}^-} \tau_o'(x) = \lim_{x \to \frac{3}{11}^+} \tau_o'(x) = 121/90 \tag{1.7}$$

where $\tau_o'(x) \stackrel{\text{def}}{=} d\tau_o(x)/dx$; (iv) $\tau_o(x)$ is strictly monotonically increasing in the interval $0 < x < 1$; and (v)

$$x < \tau_o(x) < \sqrt{x}, \qquad 0 < x < 1 \tag{1.8}$$

Eqs. (1.5) and (1.8) coupled with the facts that $\tau_o(0) = 0$ and $\sqrt{\nu^2} = |\nu|$ imply that the c-τ scheme is stable if

$$\tau = |\nu| < 1 \tag{1.9}$$

On the other hand, Eqs. (1.5) and (1.8) imply that the c-τ scheme is unstable for the cases (i)

$$\nu^2 > 1 \tag{1.10}$$

and (ii)

$$\tau = \nu^2 \quad \text{and} \quad 0 < \nu^2 < 1 \tag{1.11}$$

Note that, for a reason explained in [9,11], the special c-τ scheme with Eq. (1.9) is a Courant number insensitive solver for Eq. (1.1).

The rest of the paper is outlined as follows. For any pair of ν and τ, and any phase angle θ, the amplification matrix $Q(\nu, \tau, \theta)$ that arises from the von Neumann stability analysis is presented in Sec. 2 (see Eq. (2.8)). The definition of stability (Definition 1) is then given in the same section in terms of the behaviors of $[Q(\nu, \tau, \theta)]^m$, $-\pi < \theta \le \pi$, as the integer $m \to +\infty$. In Sec. 3, Theorems 1 and 2 are introduced to link stability with the spectal radii $\rho(Q(\nu, \tau, \theta))$ of $Q(\nu, \tau, \theta)$, $-\pi < \theta \le \pi$. Based on the preliminaries given in Secs. 2 and 3, the main results are given in Sec. 4. Specifically, Sec. 4 begins with Theorem 3, in which the necessary and sufficient stability conditions are expressed implicitly in terms of a requirement on $\rho(Q(\nu, \tau, \theta))$, $-\pi < \theta \le \pi$. It is then followed by a systematic and rigorous effort to obtain the explicit solution to the above implicit conditions. Finally, conclusions and discussions are presented in Sec. 5. Moreover, to give the reader extra confidence on the main results established analytically in Theorems 34 and 35, these theorems are further validated numerically in Appendices A and B, respectively.

2. von Neumann Stability Analysis

For any $(j,n) \in \Omega$, let

$$\vec{q}(j,n) \stackrel{\text{def}}{=} \begin{pmatrix} u_j^n \\ \\ (u_{\bar{x}})_j^n \end{pmatrix} \tag{2.1}$$

$$Q_+(\nu,\tau) \stackrel{\text{def}}{=} \frac{1}{2} \begin{pmatrix} 1+\nu & 1-\nu^2 \\ \\ \dfrac{-1}{1+\tau} & -\dfrac{1-2\nu-\tau}{1+\tau} \end{pmatrix} \tag{2.2}$$

and

$$Q_-(\nu,\tau) \stackrel{\text{def}}{=} \frac{1}{2} \begin{pmatrix} 1-\nu & -(1-\nu^2) \\ \\ \dfrac{1}{1+\tau} & -\dfrac{1+2\nu-\tau}{1+\tau} \end{pmatrix} \tag{2.3}$$

where

$$1+\tau \neq 0 \tag{2.4}$$

is assumed. Then Eqs. (1.2) and (1.3) can be expreseed as

$$\vec{q}(j,n) = Q_+\vec{q}(j-1/2,\, n-1/2) + Q_-\vec{q}(j+1/2,\, n-1/2) \tag{2.5}$$

Hereafter $Q_+(\nu,\tau)$ and $Q_-(\nu,\tau)$ may be abbreviated as Q_+ and Q_-, respectively.

To study the stability of the c-τ scheme using the von Neumann analysis [1], for all $(j,n) \in \Omega$, let

$$\vec{q}(j,n) = \vec{q}^*(n,\theta)e^{i\,j\,\theta} \tag{2.6}$$

Here (i) $i \stackrel{\text{def}}{=} \sqrt{-1}$, (ii) θ, $-\infty < \theta < +\infty$, is the phase angle variation per Δx, and (iii) $\vec{q}^*(n,\theta)$ is a 2×1 column matrix. Substituting Eq. (2.6) into Eq. (2.5) and using Eq. (2.4), one has

$$\vec{q}^*(n+1/2,\theta) = Q(\nu,\tau,\theta)\vec{q}^*(n,\theta) \tag{2.7}$$

where $n = 0, \pm 1/2, \pm 1, \pm 3/2, \ldots$, and

$$\begin{aligned} Q(\nu,\tau,\theta) &\stackrel{\text{def}}{=} e^{-i\theta/2}Q_+(\nu,\tau) + e^{i\theta/2}Q_-(\nu,\tau) \\ &= \begin{pmatrix} \cos(\theta/2) - i\nu\sin(\theta/2) & -i(1-\nu^2)\sin(\theta/2) \\ \\ \dfrac{i\sin(\theta/2)}{1+\tau} & -\left[\dfrac{(1-\tau)\cos(\theta/2) + 2i\nu\sin(\theta/2)}{1+\tau}\right] \end{pmatrix} \end{aligned} \tag{2.8}$$

Because of Eq. (2.7), $Q(\nu,\tau,\theta)$ is referred to as the amplification matrix of the c-τ scheme per marching step (or per $\Delta t/2$). Also, by using Eq. (2.7), one has

$$\vec{q}^*(n+m/2,\theta) = [Q(\nu,\tau,\theta)]^m\vec{q}^*(n,\theta) \tag{2.9}$$

where $m = 1, 2, 3, \ldots$ and $n = 0, \pm 1/2, \pm 1, \pm 3/2, \ldots$.

As a result of Eq. (2.9), we have Definition 1.

Definition 1. The c-τ scheme is said to be stable with respect to a given ordered pair (ν, τ) if, for every θ, $-\infty < \theta < +\infty$, all elements of the matrix $[Q(\nu, \tau, \theta)]^m$ associated with this pair remain bounded as the positive integer $m \to +\infty$. On the other hand, the scheme is said to be unstable with respect to a given (ν, τ) if, for any θ, $-\infty < \theta < +\infty$, at least one element of the matrix $[Q(\nu, \tau, \theta)]^m$ associated with this (ν, τ) becomes unbounded as $m \to +\infty$. Hereafter, *a given (ν, τ) is said to be c-τ stable (unstable) if the c-τ scheme is stable (unstable) with respect to this (ν, τ).*

Note that: (i) Eq. (2.8) implies that, for any integer ℓ,

$$Q(\nu, \tau, \theta + 2\ell\pi) = (-1)^{\ell} Q(\nu, \tau, \theta) \tag{2.10}$$

and (ii) for any θ, $-\infty < \theta < +\infty$, there are a θ', $-\pi < \theta' \leq \pi$ and an integer ℓ such that $\theta = \theta' + 2\ell\pi$. As such, Definitions 1 is equivalent to the simplified form in which the original range of θ, i.e., $-\infty < \theta < +\infty$, is replaced by

$$-\pi < \theta \leq \pi \tag{2.11}$$

Hereafter, the simplified form of Definition 1 is assumed.

Given Definition 1, it will be shown in this paper that a given (ν, τ) is c-τ stable if and only if it satisfies Eq. (1.5). As a first step, in Sec. 3 we will answer the following question: For any given ordered set (ν, τ, θ), what are the requirements the matrix $Q(\nu, \tau, \theta)$ must meet so that all elements of the matrix $[Q(\nu, \tau, \theta)]^m$ will remain bounded as $m \to +\infty$?

3. Two Matrix Theorems

Let M be any $N \times N$ matrix with real or complex elements. By definition, the eigenspace of M is the vector space spanned by its eigenvectors. Let the dimension of this eigenspace be denoted by N'. Then $1 \leq N' \leq N$. The matrix is said to be (i) nondefective if $N' = N$ and (ii) defective if $N' < N$ [13].

Hereafter let $N = 2$. Then the eigenvalues λ_1 and λ_2 of the matrix M are the two roots of a quadratic characteristic equation. Moreover, we have Theorem 1.

Theorem 1. The matrix M is defective if and only if (i) $\lambda_1 = \lambda_2$, and (ii) $M \neq \lambda_c I$, where I is the 2×2 identity matrix and λ_c is the common value of λ_1 and λ_2.

Proof. Let $\vec{b}_1$ and $\vec{b}_2$ be two nonnull 2×1 column matrices with

$$M\vec{b}_\ell = \lambda_\ell \vec{b}_\ell, \qquad \ell = 1, 2 \tag{3.1}$$

Then, for each ℓ, $\vec{b}_\ell$ is an eigenvector of M with the eigenvalue λ_ℓ. In case that $\lambda_1 \neq \lambda_2$, it is known that $\vec{b}_1$ and $\vec{b}_2$ are linearly independent [13]. Thus $N' = 2$ and M is nondefective.

Next let $\lambda_1 = \lambda_2$ and M be nondefective. Then $N' = 2$, i.e., there exist two linearly independent 2×1 column matrices $\vec{b}_1$ and $\vec{b}_2$ that satisfy Eq. (3.1). Let

$$\vec{b}_\ell = \begin{pmatrix} b_{1\ell} \\ b_{2\ell} \end{pmatrix}, \qquad \ell = 1, 2 \tag{3.2}$$

and

$$B \stackrel{\text{def}}{=} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \tag{3.3}$$

Then, because $\lambda_1 = \lambda_2$, Eq. (3.1) can be expressed as

$$(M - \lambda_c I)B = 0 \tag{3.4}$$

where λ_c is the common value of λ_1 and λ_2. Because $\vec{b}_1$ and $\vec{b}_2$ are linearly independent, B is nonsingular [13]. Thus, B^{-1}, the inverse of B, must exist. Multiplying the expressions on the two sides of Eq. (3.4) from the right with B^{-1} leads to the conclusion that $M - \lambda_c I = 0$, i.e., $M = \lambda_c I$.

Conversely let $M = \lambda_c I$ where λ_c is any scalar. Then it can be shown easily that (i) $\lambda_1 = \lambda_2 = \lambda_c$, and (ii) any 2×1 nonnull column matrix is an eigenvector of M. The conclusion (ii) implies that $N' = 2$ and thus M is nondefective.

It has been shown that: (i) M is nondefective if $\lambda_1 \neq \lambda_2$; and (ii) in case that $\lambda_1 = \lambda_2$, M is nondefective if and only if $M = \lambda_c I$ (i.e., M is defective if and only if $M \neq \lambda_c I$) where λ_c is the common value of λ_1 and λ_2. Thus the proof is completed. **QED**.

Next let (i) m be an integer > 0; and (ii) $\rho(M)$ be the spectral radius of M, i.e.,

$$\rho(M) \stackrel{\text{def}}{=} \max\{|\lambda_1|, |\lambda_2|\} \tag{3.5}$$

Then we have Theorem 2.

Theorem 2. Every element of M^m will remain bounded as $m \to +\infty$ if and only if

$$\rho(M) \begin{cases} \leq 1 & \text{if } M \text{ is nondefective} \\ \\ < 1 & \text{if } M \text{ is defective} \end{cases} \tag{3.6}$$

Proof. According to the Jordan canonical form theorem [13], there exists a nonsingular 2×2 matrix S such that

$$M = S \Lambda S^{-1} \tag{3.7}$$

Here (i) S^{-1} is the inverse of S; (ii)

$$\Lambda \stackrel{\text{def}}{=} \begin{pmatrix} \lambda_1 & 0 \\ \\ 0 & \lambda_2 \end{pmatrix} \quad \text{if } M \text{ is nondefective} \tag{3.8}$$

and (iii)

$$\Lambda \stackrel{\text{def}}{=} \begin{pmatrix} \lambda_c & 1 \\ \\ 0 & \lambda_c \end{pmatrix} \quad \text{if } M \text{ is defective} \tag{3.9}$$

Note that λ_c in Eq. (3.9) is the common value of λ_1 and λ_2 in the defective case.

By using Eqs. (3.8) and (3.9), one has: (i)

$$\Lambda^m = \begin{pmatrix} \lambda_1^m & 0 \\ \\ 0 & \lambda_2^m \end{pmatrix} \quad \text{if } M \text{ is nondefective} \tag{3.10}$$

and (ii)

$$\Lambda^m = \begin{pmatrix} \lambda_c^m & m\lambda_c^{m-1} \\ \\ 0 & \lambda_c^m \end{pmatrix} \quad \text{if } M \text{ is defective} \tag{3.11}$$

Because (i) Eq. (3.7) implies that

$$M^m = S \Lambda^m S^{-1} \tag{3.12}$$

and (ii) Eq. (3.12) is equivalent to

$$\Lambda^m = S^{-1} M^m S \tag{3.13}$$

one can infer from Eq. (3.10) that, for the nondefective case, every element of M^m will remain bounded as $m \to +\infty$ if and only if

$$\rho(M) \leq 1 \qquad \text{(the nondefective case)} \tag{3.14}$$

On the other hand, for the defective case, by using (i) $\rho(M) = |\lambda_c|$, and (ii)

$$\lim_{m \to +\infty} |m\lambda_c^{m-1}| = \begin{cases} 0 & \text{if } |\lambda_c| < 1 \\ \\ +\infty & \text{if } |\lambda_c| \geq 1 \end{cases} \tag{3.15}$$

Eqs. (3.11)–(3.13) imply that, for the defective case, every element of M^m will remain bounded as $M \to +\infty$ if and only if

$$\rho(M) < 1 \qquad \text{(the defective case)} \tag{3.16}$$

Because Eq. (3.6) is the combined form of Eqs. (3.14) and (3.16), the proof is completed. **QED**.

At this juncture, note that the term $|m\lambda_c^{m-1}|$ grows linearly with m as $m \to +\infty$ if $|\lambda_c| = 1$. Thus, for the defective case with $|\lambda_c| = 1$, the growth rate of the magnitude of any element of M^m as $m \to +\infty$ is very low compared with the exponential growth rate associated with a nondefective or defective case with $\rho(M) > 1$. The implication of this observation will be addressed later.

4. Main Results

An immediate result of Definition 1 and Theorem 2 is Theorem 3.

Theorem 3. A given (ν, τ) is c-τ stable if and only if the condition

$$\rho(Q(\nu,\tau,\theta)) \begin{cases} \leq 1 & \text{if } Q(\nu,\tau,\theta) \text{ is nondefective} \\ \\ < 1 & \text{if } Q(\nu,\tau,\theta) \text{ is defective} \end{cases} \tag{4.1}$$

associated with the given (ν, τ) is met for all θ, $-\pi < \theta \leq \pi$.

Two immediate results of Theorem 3 are Theorems 4 and 5.

Theorem 4. A necessary condition for any given (ν, τ) to be c-τ stable is

$$\rho(Q(\nu,\tau,\theta)) \leq 1, \qquad\qquad -\pi < \theta \leq \pi \tag{4.2}$$

Theorem 5. In case that

$$\rho(Q(\nu,\tau,\theta)) \neq 1 \tag{4.3}$$

for all defective $Q(\nu,\tau,\theta)$ $(-\pi < \theta \leq \pi)$ associated with a given (ν, τ), Eq. (4.2) is also a sufficient condition for this (ν, τ) to be c-τ stable.

From Theorem 3, it becomes clear that a thorough stability study of the c-τ scheme requires a systematic investigation of the matrix $Q(\nu,\tau,\theta)$ and its eigenvalues over the entire range of ν, τ, and θ. In the following, first we shall try to narrow down the possible (ν, τ) that are c-τ stable by ruling out those that fail to satisfy Eq. (4.2).

Let $\det(M)$ denote the determinant of any square matrix M. Then any eigenvalue λ of $Q(\nu,\tau,\theta)$ satisfies the characteristic equation $\det\left(Q(\nu,\tau,\theta) - \lambda I\right) = 0$, i.e.,

$$\begin{aligned} &(1+\tau)\lambda^2 - \left[2\tau\cos(\theta/2) - i\nu(3+\tau)\sin(\theta/2)\right]\lambda \\ &- (1-\tau)\cos^2(\theta/2) - (1+\nu^2)\sin^2(\theta/2) - i\nu(1+\tau)\sin(\theta/2)\cos(\theta/2) = 0 \end{aligned} \tag{4.4}$$

Let

$$X(\nu,\tau,\theta) \stackrel{\text{def}}{=} 4\cos^2(\theta/2) + \left[4(1+\tau) - \nu^2(\tau^2+2\tau+5)\right]\sin^2(\theta/2) \tag{4.5}$$

and

$$Y(\nu,\tau,\theta) \stackrel{\text{def}}{=} 4\nu(1-\tau)\sin(\theta/2)\cos(\theta/2) \tag{4.6}$$

Then, with the aid of Eq. (2.4), Eq. (4.4) implies that $\lambda = \lambda_+(\nu,\tau,\theta)$ or $\lambda = \lambda_-(\nu,\tau,\theta)$ where

$$\lambda_\pm(\nu,\tau,\theta) \stackrel{\text{def}}{=} \frac{2\tau\cos(\theta/2) - i\nu(3+\tau)\sin(\theta/2) \pm \sqrt{X+iY}}{2(1+\tau)}, \qquad 1+\tau \neq 0 \tag{4.7}$$

Hereafter $X(\nu,\tau,\theta)$ and $Y(\nu,\tau,\theta)$ may be abbreviated as X and Y, respectively. Because the range of the phase angle ϕ in the polar form of the principal square root $\sqrt{X+iY}$ is $-\pi/2 < \phi \le \pi/2$, it can be shown that

$$\sqrt{X+iY} = \frac{1}{\sqrt{2}}\left[\sqrt{\sqrt{X^2+Y^2}+X} + i\,\mathrm{sign}(Y)\sqrt{\sqrt{X^2+Y^2}-X}\,\right] \tag{4.8}$$

where

$$\mathrm{sign}(Y) \stackrel{\mathrm{def}}{=} \begin{cases} 1 & \text{if } Y \ge 0 \\ \\ -1 & \text{if } Y < 0 \end{cases} \tag{4.9}$$

With the aid of Eq. (4.8), Eq. (4.7) implies that

$$\begin{aligned} \lambda_\pm(\nu,\tau,\theta) = \frac{1}{2(1+\tau)}\Bigg\{ & 2\tau\cos(\theta/2) \pm \frac{1}{\sqrt{2}}\sqrt{\sqrt{X^2+Y^2}+X} \\ & - i\left[\nu(3+\tau)\sin(\theta/2) \mp \frac{1}{\sqrt{2}}\,\mathrm{sign}(Y)\sqrt{\sqrt{X^2+Y^2}-X}\,\right]\Bigg\} \end{aligned} \qquad (1+\tau\neq 0) \tag{4.10}$$

Next Eq (4.10) is used to yield

$$2(1+\tau)^2(|\lambda_+|^2+|\lambda_-|^2) = 4\tau^2\cos^2(\theta/2) + \nu^2(3+\tau)^2\sin^2(\theta/2) + \sqrt{X^2+Y^2} \tag{4.11}$$

and

$$\begin{aligned} (1+\tau)^2|\lambda_+|^2|\lambda_-|^2 = {} & (1-\tau)^2\cos^4(\theta/2) + (1+\nu^2)^2\sin^4(\theta/2) \\ & + (2-2\tau+3\nu^2+\tau^2\nu^2)\sin^2(\theta/2)\cos^2(\theta/2) \end{aligned} \tag{4.12}$$

For simplicity, hereafter $\lambda_+(\nu,\tau,\theta)$ and $\lambda_-(\nu,\tau,\theta)$ may be abbreviated as λ_+ and λ_-, respectively. Next, let

$$s \stackrel{\mathrm{def}}{=} \sin^2(\theta/2), \qquad -\pi < \theta \le \pi \tag{4.13}$$

Then

$$\cos^2(\theta/2) = 1 - s \tag{4.14}$$

and, corresponding to the domain $-\pi < \theta \le \pi$, the range of s is

$$0 \le s \le 1 \tag{4.15}$$

Next, let

$$D(\nu,\tau,s) \stackrel{\mathrm{def}}{=} 2(1-\nu^2)(\tau^2-\nu^2)s^2 + \left[4\tau + (\tau^2-6\tau-3)\nu^2\right]s + 4, \qquad 0 \le s \le 1 \tag{4.16}$$

$$\begin{aligned} E(\nu,\tau,s) \stackrel{\mathrm{def}}{=} {} & \left[16\tau^2 - 8(\tau^3+4\tau^2+\tau+2)\nu^2 + (\tau^2+2\tau+5)^2\nu^4\right]s^2 \\ & + 8\left[4\tau + (\tau^2-6\tau-3)\nu^2\right]s + 16, \qquad 0 \le s \le 1 \end{aligned} \tag{4.17}$$

and

$$F(\nu,\tau,s) \stackrel{\text{def}}{=} (1-\nu^2)(\nu^2-\tau^2)s^2 - \left[2\tau(1-\tau)+(3+\tau^2)\nu^2\right]s + 4\tau, \qquad 0 \le s \le 1 \tag{4.18}$$

Then, by using Eqs. (4.5), (4.6), and (4.11)–(4.14), it can be shown that

$$E(\nu,\tau,s) = [X(\nu,\tau,\theta)]^2 + [Y(\nu,\tau,\theta)]^2 \ge 0 \tag{4.19}$$

$$D(\nu,\tau,s) - \sqrt{E(\nu,\tau,s)} = 2(1+\tau)^2\left(1-|\lambda_+|^2\right)\left(1-|\lambda_-|^2\right) \tag{4.20}$$

and

$$F(\nu,\tau,s) = (1+\tau)^2\left(1-|\lambda_+|^2|\lambda_-|^2\right) \tag{4.21}$$

As a preliminary to the future development, let

$$H(\nu,\tau,s) \stackrel{\text{def}}{=} [D(\nu,\tau,s)]^2 - E(\nu,\tau,s) \tag{4.22}$$

Then Eqs. (4.16) and (4.17) imply that

$$H(\nu,\tau,s) = 4(1-\nu^2)s^2G(\nu,\tau,s) \tag{4.23}$$

where

$$\begin{aligned} G(\nu,\tau,s) \stackrel{\text{def}}{=}& (1-\nu^2)(\tau^2-\nu^2)^2s^2 + (\tau^2-\nu^2)\left[\nu^2\tau^2+(4-6\nu^2)\tau-3\nu^2\right]s \\ &+ 4\tau\left[\nu^2\tau^2+(1-\nu^2)\tau-\nu^2\right], \quad 0 \le s \le 1 \end{aligned} \tag{4.24}$$

With the above preparations, we have Theorem 6.

Theorem 6. (A) For any (ν,τ), the condition Eq. (4.2) is equvalent to the conditions

$$D(\nu,\tau,s) \ge 0, \qquad 0 \le s \le 1 \tag{4.25}$$

$$H(\nu,\tau,s) \ge 0, \qquad 0 \le s \le 1 \tag{4.26}$$

and

$$F(\nu,\tau,s) \ge 0, \qquad 0 \le s \le 1 \tag{4.27}$$

(B) Eqs. (4.25)–(4.27) are necessary conditions for any (ν,τ) to be c-τ stable.

Proof. Part B is an immediate result of part A and Theorem 4. Thus only part A needs to be proved. To proceed, note that $|\lambda_+| \le 1$ and $|\lambda_-| \le 1$ if and only if (i)

$$\left(1-|\lambda_+|^2\right)\left(1-|\lambda_-|^2\right) \ge 0$$

and (ii)

$$\left(1-|\lambda_+|^2|\lambda_-|^2\right) \ge 0,$$

Thus, by using Eqs. (3.5), (2.4), (4.15), (4.20), and (4.21), it is easy to see that Eq. (4.2) is equivalent to Eq. (4.27) and

$$D(\nu,\tau,s) - \sqrt{E(\nu,\tau,s)} \geq 0, \qquad\qquad 0 \leq s \leq 1 \tag{4.28}$$

As a result, to complete the proof, one needs only to show that Eqs. (4.25) and (4.26) is equivalent to Eq. (4.28).

To proceed, for simplicity, in the following $D(\nu,\tau,s)$, $E(\nu,\tau,s)$, $F(\nu,\tau,s)$, $G(\nu,\tau,s)$, and $H(\nu,\tau,s)$ may be abbreviated as D, E, F, G, and H, respectively. By using the fact that $E \geq 0$ (see Eq. (4.19)), it is easy to show that the condition $D - \sqrt{E} \geq 0$ implies that (i) $D \geq 0$ and (ii)

$$D^2 - E = (D + \sqrt{E})(D - \sqrt{E}) \geq 0 \tag{4.29}$$

Thus, with the aid of Eq. (4.22), one concludes that Eq. (4.28) implies both Eqs. (4.25) and (4.26).

To show that Eqs. (4.25) and (4.26) imply Eq. (4.28), note that

$$D - \sqrt{E} = D \geq 0 \quad \text{if } D \geq 0 \text{ and } E = 0 \tag{4.30}$$

Moreover, because $D + \sqrt{E} > 0$ if $D \geq 0$ and $E > 0$, one has

$$D - \sqrt{E} = \frac{D^2 - E}{D + \sqrt{E}} \geq 0 \quad \text{if } D \geq 0,\ D^2 - E \geq 0, \text{ and } E > 0 \tag{4.31}$$

Thus, with the aid of Eqs. (4.19), (4.22), (4.30) and (4.31), one concludes that Eqs. (4.25) and (4.26) indeed imply Eq. (4.28). **QED**.

At this juncture note that, given any (ν,τ), $D(\nu,\tau,s)$, $F(\nu,\tau,s)$ and $G(\nu,\tau,s)$ are all quadratic polynomials in s and thus their minimum values in the interval $0 \leq s \leq 1$ are easy to evaluate. As will be shown, this makes the analytical study of Eqs. (4.25)–(4.27) a relatively simple one. This is very fortunate because, according to Theorem 6, these equations play key roles in the current stability study.

To proceed, note that an immediate result of Theorem 6 is Theorem 7.

Theorem 7. (i) $D(\nu,\tau,0) \geq 0$, (ii) $D(\nu,\tau,1) \geq 0$, (iii) $F(\nu,\tau,0) \geq 0$, (iv) $F(\nu,\tau,1) \geq 0$, (v) $H(\nu,\tau,0) \geq 0$, and (vi) $H(\nu,\tau,1) \geq 0$ are all necessary conditions for a given (ν,τ) to be *c*-τ stable.

To study conditions (i)–(vi) referred to above, Eqs. (4.16) (4.18), (4.23), and (4.24) are used to yield

$$D(\nu,\tau,0) = 4 \tag{4.32}$$

$$D(\nu,\tau,1) = (2-\nu^2)\tau^2 + 2(2-3\nu^2)\tau + 2\nu^4 - 5\nu^2 + 4 \tag{4.33}$$

$$F(\nu,\tau,0) = 4\tau \tag{4.34}$$

$$F(\nu, \tau, 1) = (2 + \tau + \nu^2)(\tau - \nu^2) \tag{4.35}$$

$$H(\nu, \tau, 0) = 0 \tag{4.36}$$

and

$$H(\nu, \tau, 1) = 4(1 - \nu^2)(\tau - \nu^2)^2 \left[(2 + \tau)^2 - \nu^2\right] \tag{4.37}$$

According to Eqs. (4.32) and (4.36), conditions (i) and (v) referred to in Theorem 7 are satisfied automatically. The significance of other conditions will be partially addressed in the following Theorems 8–11.

Theorem 8. $F(\nu, \tau, 0) \geq 0$ and $F(\nu, \tau, 1) \geq 0$ if and only if $\tau \geq \nu^2$.

Proof. According to Eq. (4.34), $F(\nu, \tau, 0) \geq 0$ if and only if $\tau \geq 0$. With the aid of Eq. (4.35) and the fact that $2 + \tau + \nu^2 > 0$ if $\tau \geq 0$, one concludes that $F(\nu, \tau, 0) \geq 0$ and $F(\nu, \tau, 1) \geq 0$ imply $\tau \geq \nu^2$. Conversely, it is easy to see that $F(\nu, \tau, 0) \geq 0$ and $F(\nu, \tau, 1) \geq 0$ if $\tau \geq \nu^2$. **QED**.

Theorem 9. Let $\tau \geq \nu^2$. Then $H(\nu, \tau, 1) > 0$ if and only if $\tau > \nu^2$ and $\nu^2 < 1$.

Proof. With the aid of the assumption $\tau \geq \nu^2$ and Eq. (4.37), $H(\nu, \tau, 1) > 0$ implies (i) $\tau > \nu^2$ and (ii)

$$(\nu^2 - 1)\left[\nu^2 - (2 + \tau)^2\right] > 0 \tag{4.38}$$

Because $\tau > \nu^2$ implies $\tau > 0$ and thus $\nu^2 - 1 > \nu^2 - (2 + \tau)^2$, conditions (i) and (ii) imply either (a) $\nu^2 < 1$ or (b) $\nu^2 > (2 + \tau)^2$. Case (b) can be ruled out because it along with condition (i) implies $\tau > (2+\tau)^2$, a result inconsistent with $\tau > 0$ which follows from condition (i). Thus $H(\nu, \tau, 1) > 0$ implies $\tau > \nu^2$ and $\nu^2 < 1$, if $\tau \geq \nu^2$ is assumed.

Conversely, because $(2+\tau)^2 > \tau > \nu^2$ if $\tau > \nu^2$, Eq. (4.37) implies that $H(\nu, \tau, 1) > 0$ if $\tau > \nu^2$ and $\nu^2 < 1$. Thus the proof is completed. **QED**.

Theorem 10. Let $\tau \geq \nu^2$. Then $H(\nu, \tau, 1) = 0$ if and only if at least one of the two cases: (i) $\tau = \nu^2$ and (ii) $\nu^2 = 1$, is true.

Proof. Eq. (4.37) implies that $H(\nu, \tau, 1) = 0$ if and only if at least one of the three cases: (i) $\nu^2 = 1$, (ii) $\tau = \nu^2$, and (iii) $\nu^2 = (2 + \tau)^2$, is true. Case (iii) can be ruled out because it along with the assumption $\tau \geq \nu^2$ implies $\tau \geq (2 + \tau)^2$, a result inconsistent with $\tau \geq 0$ (which follows from $\tau \geq \nu^2$). Thus the proof is completed. **QED**.

Theorem 11. Let $\tau = \nu^2$. Then $D(\nu, \tau, 1) \geq 0$ if and only if $\nu^2 \leq 1$.

Proof. Let $\tau = \nu^2$. Then Eq. (4.33) implies that

$$D(\nu, \tau, 1) = (1 - \tau)(\tau^2 + 3\tau + 4) \qquad (\tau = \nu^2) \tag{4.39}$$

With the aid of Eq. (4.39) and the fact that

$$\tau^2 + 3\tau + 4 = (\tau + 3/2)^2 + 7/4 \geq 7/4, \qquad -\infty < \tau < +\infty \tag{4.40}$$

it is easy to see that, assuming $\tau = \nu^2$, $D(\nu,\tau,1) \geq 0$ if and only if $\nu^2 \leq 1$. **QED**.

According to Theorems 8–10, the conditions (i) $F(\nu,\tau,0) \geq 0$, (ii) $F(\nu,\tau,1) \geq 0$, and (iii) $H(\nu,\tau,1) \geq 0$ require that $\tau = \nu^2$ if the conditions $\tau \geq \nu^2$ and $\nu^2 \leq 1$ are not satisfied simultaneously. On the other hand, according to Theorem 11, the condition $D(\nu,\tau,1) \geq 0$ requires that $\nu^2 \leq 1$ for the case $\tau = \nu^2$. Thus one has Theorem 12.

Theorem 12. The conditions (i) $D(\nu,\tau,1) \geq 0$, (ii) $F(\nu,\tau,0) \geq 0$, (iii) $F(\nu,\tau,1) \geq 0$, and (iv) $H(\nu,\tau,1) \geq 0$ require that $\tau \geq \nu^2$ and $\nu^2 \leq 1$. As such, Theorem 7 implies that

$$\tau \geq \nu^2 \quad \text{and} \quad \nu^2 \leq 1 \tag{4.41}$$

are necessary conditions for a given (ν,τ) to be c-τ stable.

In the following, it will be shown that only a subset of those τ and ν that satisfy the necessary conditions Eq. (4.41) will also satisfy the sufficient conditions for stability. As a prerequisite, we shall first study the conditions under which the matrix $Q(\nu,\tau,\theta)$ is defective if τ and ν satisfy Eq. (4.41). We begin with Theorem 13.

Theorem 13. Let $\tau \geq \nu^2$ and $\nu^2 \leq 1$. Then $Q(\nu,\tau,\theta)$ is defective if and only if

$$4(1+\tau) = \nu^2(\tau^2 + 2\tau + 5) \tag{4.42}$$

and

$$\cos(\theta/2) = 0 \tag{4.43}$$

Proof. Assuming $\tau \geq \nu^2$ and $\nu^2 \leq 1$, first we will show that

$$\lambda_+(\nu,\tau,\theta) = \lambda_-(\nu,\tau,\theta) \tag{4.44}$$

if and only if Eqs. (4.42) and (4.43) are satisfied. According to Eq. (4.10), Eq. (4.44) is equivalent to

$$\sqrt{X^2+Y^2} + X = 0 \quad \text{and} \quad \sqrt{X^2+Y^2} - X = 0 \tag{4.45}$$

Thus Eq. (4.44) is true if and only if

$$X = Y = 0 \tag{4.46}$$

According to Eq. (4.6), $Y = 0$ if and only if at least one of the four cases: (a) $\nu = 0$, (b) $\tau = 1$, (c) $\sin(\theta/2) = 0$, and (d) $\cos(\theta/2) = 0$, is true. For case (a) $\nu = 0$, Eqs. (4.5) and the assumption $\tau \geq \nu^2$ imply that

$$X = 4\left[1 + \tau\sin^2(\theta/2)\right] \geq 4 \qquad (\nu = 0) \tag{4.47}$$

Thus case (a) is incompatible with Eq. (4.46).

For case (b) $\tau = 1$, Eq. (4.5) implies that

$$X = 4\cos^2(\theta/2) + 8(1-\nu^2)\sin^2(\theta/2) \qquad (\tau = 1) \tag{4.48}$$

Using the assumption $\nu^2 \leq 1$, Eq. (4.48) implies that, for case (b), $X = 0$ if and only if $\nu^2 = 1$ and $\cos(\theta/2) = 0$.

Because $\cos^2(\theta/2) = 1$ if $\sin(\theta/2) = 0$, Eq. (4.5) implies that $X = 4$ if $\sin(\theta/2) = 0$. Thus case (c) is incompatible with Eq. (4.46).

Because $\sin^2(\theta/2) = 1$ if $\cos(\theta/2) = 0$, Eq. (4.5) implies that

$$X = 4(1+\tau) - \nu^2(\tau^2 + 2\tau + 5) \qquad (\cos(\theta/2) = 0) \tag{4.49}$$

if $\cos(\theta/2) = 0$. Thus, for case (d), $X = 0$ if and only if Eq. (4.42) is satisfied.

Assuming $\tau \geq \nu^2$ and $\nu^2 \leq 1$, it has been shown that $X = Y = 0$ if and only if at least one of the following two conditions: (i)

$$\tau = 1, \quad \nu^2 = 1, \quad \text{and} \quad \cos(\theta/2) = 0 \qquad \text{(i.e., case (b))}$$

and (ii)

$$\cos(\theta/2) = 0 \quad \text{and} \quad 4(1+\tau) = \nu^2(\tau^2 + 2\tau + 5) \qquad \text{(i.e., case (d))}$$

is met. Because $\tau = 1$ and $\nu^2 = 1$ form a special solution of Eq. (4.42), condition (i) is only a special case of condition (ii). Thus, assuming $\tau \geq \nu^2$ and $\nu^2 \leq 1$, Eq. (4.44) (which is equivalent to $X = Y = 0$) is true if and only if Eqs. (4.42) and (4.43) are satisfied. Moreover, with the aid of Eq. (2.8) and the fact that $\sin(\theta/2) = \pm 1$ if $\cos(\theta/2) = 0$, Eq. (4.43) also implies that one of the off-diagonal elements of $Q(\nu, \tau, \theta)$ does not vanish and thus $Q(\nu, \tau, \theta)$ is not a multiple of I. According to Theorem 1, $Q(\nu, \tau, \theta)$ is defective if and only if (i) Eq. (4.44) is true and (ii) $Q(\nu, \tau, \theta)$ is not a multiple of I. Thus the current theorem is proved. **QED**.

An immediate result of Theorem 13 is Theorem 14.

Theorem 14. The matrix $Q(\nu, \tau, \theta)$ is defective if $\tau = \nu^2 = 1$ and $\cos(\theta/2) = 0$.

To proceed, we will establish Theorem 15.

Theorem 15. Let $Q(\nu, \tau, \theta)$ be defective with $\tau \geq \nu^2$ and $\nu^2 \leq 1$. Then the special case

$$\rho(Q(\nu, \tau, \theta)) = 1 \tag{4.50}$$

occurs if and only if

$$\tau = \nu^2 = 1, \quad \text{and} \quad \cos(\theta/2) = 0 \tag{4.51}$$

Proof. As a preliminary, first we will deduce several results from the current basic assumption, i.e., $Q(\nu, \tau, \theta)$ is defective with $\tau \geq \nu^2$ and $\nu^2 \leq 1$. According to Theorem 13 and its proof, Eqs. (4.42), (4.43), and (4.46) follow immediately from the basic assumption. Also, by using Eq. (4.42) and the fact that

$$\tau^2 + 2\tau + 5 = (1+\tau)^2 + 4 \geq 4, \qquad -\infty < \tau < +\infty \tag{4.52}$$

one concludes that

$$\nu^2 = \frac{4(1+\tau)}{\tau^2 + 2\tau + 5} \tag{4.53}$$

Moreover, because $\sin(\theta/2) = \pm 1$ if $\cos(\theta/2) = 0$, with the aid of Eqs. (4.43) and (4.46), Eq. (4.10) implies that

$$\rho(Q(\nu,\tau,\theta)) = \left| \frac{\nu(3+\tau)}{2(1+\tau)} \right| \tag{4.54}$$

Next assume Eq. (4.50). Because $3 + \tau > 0$ (which follows from the assumption $\tau \geq \nu^2$), Eqs. (4.50) and (4.54) imply that

$$\nu^2 = \frac{4(1+\tau)^2}{(3+\tau)^2} \tag{4.55}$$

Eliminating ν^2 from Eqs. (4.53) and (4.55) and using the basic assumption Eq. (2.4) (which is consistent with the current assumption $\tau \geq \nu^2$), one has

$$\tau^3 + 2\tau^2 + \tau - 4 \equiv (\tau - 1)(\tau^2 + 3\tau + 4) = 0 \tag{4.56}$$

Eq. (4.56) coupled with Eq. (4.40) implies that $\tau = 1$. In turn, by using either Eq. (4.53) or Eq. (4.55), one has $\nu^2 = 1$ as a result of $\tau = 1$. Because Eq. (4.43) (i.e., $\cos(\theta/2) = 0$) is a result of the basic assumption, it has been shown that Eq. (4.51) follows from the basic assumption and Eq. (4.50).

Conversely, with the aid of (i) Theorem 1, and (ii) Eqs. (2.8) and (3.5), it can be shown by direct substitution that both the basic assumption and Eq. (4.50) are valid for the special case Eq. (4.51). Thus the proof is completed. **QED**.

Next we have Theorem 16.

Theorem 16. A given (ν,τ) satisfies Eq. (4.2) and yet is c-τ unstable if and only if $\tau = \nu^2 = 1$.

Proof. Theorems 6 and 12 imply that Eq. (4.41) is a result of Eq. (4.2). Thus, according to Theorems 5 and 15, $\tau = \nu^2 = 1$ if (ν,τ) satisfies Eq. (4.2) and is also c-τ unstable.

Conversely, Theorem 6 coupled with Eqs. (4.16), (4.18), and (4.23) implies that any (ν,τ) with $\tau = \nu^2 = 1$ satisfies Eq. (4.2). Moreover, according to Theorems 3, 14 and 15, such a (ν,τ) is also c-τ unstable. Thus the proof is completed. **QED**.

At this juncture, note that Theorems 14 and 15 state that, for the special case Eq. (4.51), $Q(\nu,\tau,\theta)$ is defective with $\rho(Q(\nu,\tau,\theta)) = 1$. Thus, according to a comment made following Eq. (3.16), for this special case, the magnitude of any element in

$[Q(\nu,\tau,\theta)]^m$ will grow not faster than linearly with m. *Because round-off errors associated with a modern computer are in the order of* 10^{-10} *or less, the instability associated with this special case generally is very mild and may not be detected even after billions of time steps have elapsed.*

Next, by combining Theorems 6, 12 and 16, one arrives at Theorem 17.

Theorem 17. A given (ν,τ) which does not satisfy Eq. (4.41) is c-τ unstable. On the other hand, a given (ν,τ) which satisfies Eq. (4.41) is c-τ stable if and only if (i) it satisfies Eqs. (4.25)–(4.27); and (ii) it does not belong to the special case $\tau=\nu^2=1$.

Compared to those given in Theorem 3, the necessary and sufficient stability conditions given in Theorem 17 are much more explicit and easier to handle. As such, this theorem will be used repeatedly in the rest of the development. In particular, it will be used to establish Theorem 18.

Theorem 18. The c-τ scheme is stable for any one of the following special cases: (a) $\nu=0$ and $\tau\geq 0$; (b) $\nu^2=1$ and $\tau>1$; and (c) $0<\nu^2<1$ and $\tau=|\nu|$.

Proof. Let $0\leq s\leq 1$ throughout this proof. Then, with the aid of Eqs. (4.16), (4.18), (4.23), and (4.24), for case (a) $\nu=0$ and $\tau\geq 0$, one has

$$D(\nu,\tau,s)=D(0,\tau,s)=2\left[(1+\tau s)^2+1\right]\geq 4 \tag{4.57}$$

$$F(\nu,\tau,s)=F(0,\tau,s)=\tau(2-s)(2+\tau s)\geq 0 \tag{4.58}$$

and

$$H(\nu,\tau,s)=H(0,\tau,s)=4s^2\tau^2(2+\tau s)^2\geq 0 \tag{4.59}$$

Because $\nu=\pm 1$ if $\nu^2=1$, for case (b) $\nu^2=1$ and $\tau>1$, one has

$$D(\nu,\tau,s)=D(\pm 1,\tau,s)=(1-\tau)^2 s+4(1-s)>0 \tag{4.60}$$

$$F(\nu,\tau,s)=F(\pm 1,\tau,s)=(1-\tau)^2 s+4(\tau-s)>0 \tag{4.61}$$

and

$$H(\nu,\tau,s)=H(\pm 1,\tau,s)=0 \tag{4.62}$$

Because $0<\nu^2<1$ and $\tau=|\nu|$ if and only if $\nu=\pm\tau$ and $0<\tau<1$, for case (c) $0<\nu^2<1$ and $\tau=|\nu|$, one has

$$D(\nu,\tau,s)=D(\pm\tau,\tau,s)=\tau(1-\tau)(8+5\tau-\tau^2)s+4(1-\tau s)>0 \tag{4.63}$$

$$F(\nu,\tau,s)=F(\pm\tau,\tau,s)=\tau(1-\tau)(\tau^2+\tau+2)s+4\tau(1-s)>0 \tag{4.64}$$

and

$$H(\nu,\tau,s)=H(\pm\tau,\tau,s)=16\tau^2(1-\tau^2)^2(1-\tau)s^2\geq 0 \tag{4.65}$$

Obviously cases (a) and (b) are special cases of the more general case defined by Eq. (4.41). Moreover, because $\nu^2 < |\nu|$ if $0 < \nu^2 < 1$, case (c) is also a special case of the more general case. In addition, none of cases (a)–(c) contains the special case $\tau = \nu^2 = 1$. With the aid of these observations and Eqs. (4.57)–(4.65), Theorem 18 follows directly from Theorem 17. **QED**.

Next let

$$\Psi \stackrel{\text{def}}{=} \{(\nu, \tau)|0 < \nu^2 < 1, \tau \geq \nu^2 \text{ and } \tau^2 \neq \nu^2\} \tag{4.66}$$

$$\Psi_- \stackrel{\text{def}}{=} \{(\nu, \tau)|0 < \nu^2 < 1, \tau \geq \nu^2 \text{ and } \tau^2 < \nu^2\} \tag{4.67}$$

and

$$\Psi_+ \stackrel{\text{def}}{=} \{(\nu, \tau)|0 < \nu^2 < 1, \tau \geq \nu^2 \text{ and } \tau^2 > \nu^2\} \tag{4.68}$$

Then Ψ_- and Ψ_+ are disjoint, and

$$\Psi = \Psi_+ \cup \Psi_- \tag{4.69}$$

Moreover, we have Theorems 19 and 20.

Theorem 19. Excluding the four special cases addressed in Theorems 16 and 18, Ψ is the set of all other (ν, τ) that satisfy the necessary stability conditions $\tau \geq \nu^2$ and $\nu^2 \leq 1$ given in Theorem 12.

Proof. Note that (i) $\tau = |\nu| > \nu^2$ if $0 < \nu^2 < 1$ and $\tau = |\nu|$; (ii) $\tau^2 = \nu^2$ if $\tau = |\nu|$, (iii) $\tau = |\nu|$ if $\tau \geq \nu^2$ and $\tau^2 = \nu^2$, and (iv) $\tau = \tau^2 = \nu^2$ implies either $\tau^2 = \nu^2 = 0$ or $\tau^2 = \nu^2 = 1$. Items (i)–(iii) imply that $0 < \nu^2 < 1$ and $\tau = |\nu|$ (which is case (c) in Theorem 18) if and only if $0 < \nu^2 < 1$, $\tau > \nu^2$, and $\tau^2 = \nu^2$. On the other hand, item (iv) implies that the case with both $0 < \nu^2 < 1$ and $\tau = \tau^2 = \nu^2$ does not exist. The proof follows from the above two observations and the facts that (i) $\tau \geq \nu^2 = 0$ if and only if $\nu = 0$ and $\tau \geq 0$, and (ii) $\tau \geq \nu^2 = 1$ if and only if either (a) $\tau = \nu^2 = 1$ or (b) $\nu^2 = 1$ and $\tau > 1$. **QED**.

Theorem 20. Eq. (4.68) is equivalent to

$$\Psi_+ = \{(\nu, \tau)|0 < \nu^2 < 1, \tau > \nu^2 \text{ and } \tau^2 > \nu^2\} \tag{4.70}$$

Proof. Note that (i) $\nu^4 > \nu^2$ if $\tau = \nu^2$ and $\tau^2 > \nu^2$, and (ii) the relations $\nu^4 > \nu^2$ and $0 < \nu^2 < 1$ are contradictory. Thus the case with $0 < \nu^2 < 1$, $\tau = \nu^2$, and $\tau^2 > \nu^2$ does not exist, i.e., Eq. (4.68) is equivalent to Eq. (4.70). **QED**.

To proceed, we will establish Theorems 21 and 22.

Theorem 21. Let $(\nu, \tau) \in \Psi$. Then

$$D(\nu, \tau, s) > 0, \qquad 0 \leq s \leq 1 \tag{4.71}$$

Proof. As a preliminary, note that Eq. (4.33) implies that

$$D(\nu,\tau,1) = (2-\nu^2)\left[\left(\tau + \frac{2-3\nu^2}{2-\nu^2}\right)^2 + \frac{2(1-\nu^2)(\nu^4+\nu^2+2)}{(2-\nu^2)^2}\right], \quad \nu^2 \neq 2 \qquad (4.72)$$

Thus

$$D(\nu,\tau,1) > 0 \quad \text{if} \quad \nu^2 < 1 \qquad (4.73)$$

Let $(\nu,\tau) \in \Psi_-$. Then Eqs. (4.16) and (4.67) imply that

$$\left[\frac{\partial^2 D(\nu,\tau,s)}{\partial s^2}\right]_{\nu,\tau} = 4(1-\nu^2)(\tau^2-\nu^2) < 0 \qquad ((\nu,\tau) \in \Psi_-) \qquad (4.74)$$

i.e., for any given $(\nu,\tau) \in \Psi_-$, the relation between the function $D(\nu,\tau,s)$ and s is represented by a curve which is concave downward on the s-D plane. Thus

$$\min_{0\le s\le 1} D(\nu,\tau,s) = \min\{D(\nu,\tau,0), D(\nu,\tau,1)\} \qquad ((\nu,\tau) \in \Psi_-) \qquad (4.75)$$

By using Eqs. (4.32) and (4.73), Eq. (4.75) implies that

$$D(\nu,\tau,s) > 0, \qquad 0 \le s \le 1 \qquad ((\nu,\tau) \in \Psi_-) \qquad (4.76)$$

Next let $(\nu,\tau) \in \Psi_+$. Then, by using Eq. (4.68) (in particular the facts that $\nu^2 < 1$ and $(1-\nu^2)(\tau^2-\nu^2) > 0$), Eq. (4.16) implies that

$$\begin{aligned} D(\nu,\tau,s) &\ge \left[4\tau + (\tau^2-6\tau-3)\nu^2\right] s + 4 \ge \left[4\tau\nu^2 + (\tau^2-6\tau-3)\nu^2\right] s + 4 \\ &= (1-\tau)^2\nu^2 s + 4(1-\nu^2 s) > 0, \qquad 0 \le s \le 1 \qquad ((\nu,\tau) \in \Psi_+) \end{aligned} \qquad (4.77)$$

It has been shown that $D(\nu,\tau,s) > 0$, $0 \le s \le 1$, for both case (a) $(\nu,\tau) \in \Psi_-$ and case (b) $(\nu,\tau) \in \Psi_+$. Because $\Psi = \Psi_- \cup \Psi_+$, the proof is completed. **QED**.

Theorem 22. Let $(\nu,\tau) \in \Psi$. Then

$$F(\nu,\tau,s) \ge 0, \qquad 0 \le s \le 1 \qquad (4.78)$$

Proof. Let $(\nu,\tau) \in \Psi_+$. Then Eqs. (4.18) and (4.68) imply that

$$\left[\frac{\partial^2 F(\nu,\tau,s)}{\partial s^2}\right]_{\nu,\tau} = 2(1-\nu^2)(\nu^2-\tau^2) < 0 \qquad ((\nu,\tau) \in \Psi_+) \qquad (4.79)$$

i.e., for any given $(\nu,\tau) \in \Psi_+$, the relation between the function $F(\nu,\tau,s)$ and s is represented by a curve which is concave downward on the s-F plane. Thus

$$\min_{0\le s\le 1} F(\nu,\tau,s) = \min\{F(\nu,\tau,0), F(\nu,\tau,1)\} \qquad ((\nu,\tau) \in \Psi_+) \qquad (4.80)$$

By using Eqs. (4.34), (4.35) and (4.70), Eq. (4.80) implies that

$$F(\nu,\tau,s) > 0, \qquad 0 \le s \le 1 \qquad\qquad ((\nu,\tau) \in \Psi_+) \tag{4.81}$$

Next let $(\nu,\tau) \in \Psi_-$. Then, by using Eq. (4.67) (in particular the facts that $(1-\nu^2)(\nu^2-\tau^2) > 0$ and $0 < \tau < |\nu| < 1$), Eq. (4.18) implies that

$$\begin{aligned}
\left[\frac{\partial F(\nu,\tau,s)}{\partial s}\right]_{\nu,\tau} &= 2(1-\nu^2)(\nu^2-\tau^2)s - \left[2\tau(1-\tau) + (3+\tau^2)\nu^2\right] \\
&\le 2(1-\nu^2)(\nu^2-\tau^2) - \left[2\tau(1-\tau) + (3+\tau^2)\nu^2\right] \\
&= -2(1-\nu^2)\tau^2 - 2\nu^4 - 2\tau(1-\tau) - (1+\tau^2)\nu^2 < 0, \\
&\quad 0 \le s \le 1 \qquad\qquad ((\nu,\tau) \in \Psi_-)
\end{aligned} \tag{4.82}$$

Thus, for any given $(\nu,\tau) \in \Psi_-$, the relation between F and s is represented by a curve on the s-F plane which has a negative slope in the interval $0 \le s \le 1$. In turn, this fact coupled with Eqs. (4.35) and (4.67) implies that

$$F(\nu,\tau,s) \ge F(\nu,\tau,1) \ge 0, \qquad 0 \le s \le 1 \qquad\qquad ((\nu,\tau) \in \Psi_-) \tag{4.83}$$

It has been shown that $F(\nu,\tau,s) \ge 0$, $0 \le s \le 1$, for both case (a) $(\nu,\tau) \in \Psi_+$ and case (b) $(\nu,\tau) \in \Psi_-$. Because $\Psi = \Psi_- \cup \Psi_+$, the proof is completed. **QED**.

According to Theorems 21 and 22, Eqs. (4.25) and (4.27) are satisfied by all $(\nu,\tau) \in \Psi$. Thus, Theorem 17 implies that a given $(\nu,\tau) \in \Psi$ is c-τ stable if and only if it satisfies Eq. (4.26). Thus, with the aid of Eqs. (4.23) and (4.66), one arrives at Theorem 23.

Theorem 23. For any given $(\nu,\tau) \in \Psi$, Eq. (4.26) is equivalent to

$$\inf_{0<s\le 1} G(\nu,\tau,s) \ge 0 \tag{4.84}$$

where the expression on the left side of the sign "$\ge$" denotes the infimum (i.e., the greatest lower bound) of $G(\nu,\tau,s)$ in the interval $0 < s \le 1$. As such, a given $(\nu,\tau) \in \Psi$ is c-τ stable if and only if it satisfies Eq. (4.84).

Because of Theorem 23, in the following we shall focus on finding those $(\nu,\tau) \in \Psi$ that satisfy Eq. (4.84).

To proceed, first we will establish Theorem 24.

Theorem 24. For any given $(\nu,\tau) \in \Psi$, let

$$s_o(\nu,\tau) \overset{\text{def}}{=} \frac{\nu^2\tau^2 + (4-6\nu^2)\tau - 3\nu^2}{2(1-\nu^2)(\nu^2-\tau^2)} \tag{4.85}$$

Let $s_o(\nu,\tau)$ be abbreviated as s_o. Then

$$\inf_{0<s\leq 1} G(\nu,\tau,s) = \begin{cases} G(\nu,\tau,s_o) & \text{if } 0 < s_o < 1 \\ G(\nu,\tau,1) & \text{if } s_o \geq 1 \\ G(\nu,\tau,0) & \text{if } s_o \leq 0 \end{cases} \tag{4.86}$$

Proof. To facilitate the proof, the domain of the function G defined in Eq. (4.24) will be extended to $-\infty < s < +\infty$. As such, for any given $(\nu,\tau) \in \Psi$ and any s with $-\infty < s < +\infty$, one has

$$\left[\frac{\partial G(\nu,\tau,s)}{\partial s}\right]_{\nu,\tau} = 2(1-\nu^2)(\tau^2-\nu^2)^2\left[s - s_o(\nu,\tau)\right] \tag{4.87}$$

and

$$\left[\frac{\partial^2 G(\nu,\tau,s)}{\partial s^2}\right]_{\nu,\tau} = 2(1-\nu^2)(\tau^2-\nu^2)^2 > 0 \tag{4.88}$$

Thus, for any given $(\nu,\tau) \in \Psi$, (i) the relation between the function $G(\nu,\tau,s)$ and s is represented by a curve which is concave upward on the s-G plane, and thus the absolute minimum of G in the interval $-\infty < s < +\infty$ occurs at where $\partial G/\partial s = 0$, i.e.,

$$s = s_o(\nu,\tau) \tag{4.89}$$

(ii) G is strictly monotonically decreasing in the interval $s < 1$ if $s_o \geq 1$; and (iii) G is strictly monotonically increasing in the interval $s > 0$ if $s_o \leq 0$. In addition, for any given (ν,τ), because G is a continuous function of s in the interval $-\infty < s < +\infty$, one also has (iv)

$$\lim_{s\to 0^+} G(\nu,\tau,s) = G(\nu,\tau,0) \tag{4.90}$$

Eq. (4.86) is a direct result of (i)–(vi). **QED**.

With the aid of Theorem 24, the bulk of the remaider of the paper will be devoted to answer a key question, i.e., given any ν with $0 < \nu^2 < 1$ (which is required by the condition $(\nu,\tau) \in \Psi$), what is the range of τ that will satisfy Eq. (4.84) and the rest of the condition $(\nu,\tau) \in \Psi$ (i.e., $\tau \geq \nu^2$ and $\tau^2 \neq \nu^2$)?

To proceed, let

$$I_\pm(x) \stackrel{\text{def}}{=} \frac{3x - 2 \pm 2\sqrt{3x^2 - 3x + 1}}{x}, \qquad 0 < x < 1 \tag{4.91}$$

and (iii)

$$J_\pm(x) \stackrel{\text{def}}{=} \frac{3x - 2 \pm \sqrt{2(x^3 - x + 2)}}{2 - x}, \qquad 0 < x < 1 \tag{4.92}$$

Hereafter, for any function $f(x)$, as usual $\sqrt{f(x)}$ denotes the principal square root of $f(x)$. As such $\sqrt{f(x)} \geq 0$ if $f(x) \geq 0$. Given Eqs. (4.91) and (4.92), one can establish Theorem 25.

Theorem 25. In the domain $0 < x < 1$, we have

$$I_+(x) > 0 \qquad (0 < x < 1) \tag{4.93}$$

$$I_-(x) < 0 \qquad (0 < x < 1) \tag{4.94}$$

$$J_+(x) > 0 \qquad (0 < x < 1) \tag{4.95}$$

and

$$J_-(x) < 0 \qquad (0 < x < 1) \tag{4.96}$$

Proof. Because

$$4(3x^2 - 3x + 1) = (3x-2)^2 + 3x^2 \tag{4.97}$$

one has

$$2\sqrt{3x^2 - 3x + 1} > |3x - 2|, \qquad x \neq 0 \tag{4.98}$$

Eqs. (4.93) and (4.94) follow directly from Eqs. (4.91) and (4.98).

Next because

$$2(x^3 - x + 2) = (3x-2)^2 + 2x(x-2)\left(x - \frac{5}{2}\right) \tag{4.99}$$

one has

$$\sqrt{2(x^3 - x + 2)} > |3x - 2|, \qquad 0 < x < 2 \tag{4.100}$$

Eqs. (4.95) and (4.96) follow directly from Eqs. (4.92) and (4.100). **QED**.

With the above preparations and the understanding that hereafter the symbol "$\Leftrightarrow$" may be used to take the place of the statement "if and only if", Theorem 26 can now be presented.

Theorem 26. (A) For any $(\nu, \tau) \in \Psi_-$, we have

$$s_o(\nu,\tau) \begin{cases} > 0 & \Leftrightarrow \quad \tau > I_+(\nu^2) \\ = 0 & \Leftrightarrow \quad \tau = I_+(\nu^2) \\ < 0 & \Leftrightarrow \quad \tau < I_+(\nu^2) \end{cases} \qquad ((\nu,\tau) \in \Psi_-) \tag{4.101}$$

and

$$s_o(\nu,\tau) \begin{cases} > 1 & \Leftrightarrow \quad \tau > J_+(\nu^2) \\ = 1 & \Leftrightarrow \quad \tau = J_+(\nu^2) \\ < 1 & \Leftrightarrow \quad \tau < J_+(\nu^2) \end{cases} \qquad ((\nu,\tau) \in \Psi_-) \tag{4.102}$$

On the other hand, (B) for any $(\nu, \tau) \in \Psi_+$, we have

$$s_o(\nu,\tau)\begin{cases} >0 & \Leftrightarrow \quad \tau < I_+(\nu^2) \\ =0 & \Leftrightarrow \quad \tau = I_+(\nu^2) \\ <0 & \Leftrightarrow \quad \tau > I_+(\nu^2) \end{cases} \qquad ((\nu,\tau)\in\Psi_+) \tag{4.103}$$

and

$$s_o(\nu,\tau)\begin{cases} >1 & \Leftrightarrow \quad \tau < J_+(\nu^2) \\ =1 & \Leftrightarrow \quad \tau = J_+(\nu^2) \\ <1 & \Leftrightarrow \quad \tau > J_+(\nu^2) \end{cases} \qquad ((\nu,\tau)\in\Psi_+) \tag{4.104}$$

Proof. As a preliminary, note that

$$\nu^2\tau^2 + (4-6\nu^2)\tau - 3\nu^2 = \nu^2\left[\tau - I_+(\nu^2)\right]\left[\tau - I_-(\nu^2)\right] \qquad (0<\nu^2<1) \tag{4.105}$$

In addition, because $\tau \geq \nu^2$ and $0 < \nu^2 < 1$ if $(\nu, \tau) \in \Psi$, Eq. (4.94) implies that

$$\tau - I_-(\nu^2) > 0, \qquad (\nu,\tau)\in\Psi \tag{4.106}$$

Because the expression on the left side of Eq. (4.105) is the numerator of the fraction on the right side of Eq. (4.85), Eq. (4.101) now follows from Eqs. (4.85), (4.105) and (4.106), and the fact that $0 < \nu^2 < 1$, and $\nu^2 - \tau^2 > 0$ if $(\nu, \tau) \in \Psi_-$.

To prove Eq. (4.102), note that Eq. (4.85) implies that, for any $(\nu, \tau) \in \Psi$,

$$s_o(\nu,\tau) - 1 = \frac{(2-\nu^2)\tau^2 + (4-6\nu^2)\tau - \nu^2(5-2\nu^2)}{2(1-\nu^2)(\nu^2-\tau^2)} \tag{4.107}$$

Also one has

$$(2-\nu^2)\tau^2 + (4-6\nu^2)\tau - \nu^2(5-2\nu^2) = (2-\nu^2)\left[\tau - J_+(\nu^2)\right]\left[\tau - J_-(\nu^2)\right] \quad (0<\nu^2<1) \tag{4.108}$$

In addition, because $\tau \geq \nu^2$ and $0 < \nu^2 < 1$ if $(\nu, \tau) \in \Psi$, Eq. (4.96) implies that

$$\tau - J_-(\nu^2) > 0, \qquad (\nu,\tau)\in\Psi \tag{4.109}$$

Because the expression on the left side of Eq. (4.108) is the numerator of the fraction on the right side of Eq. (4.107), Eq. (4.102) now follows from Eqs. (4.107)–(4.109), and the fact that $0 < \nu^2 < 1$ and $\nu^2 - \tau^2 > 0$ if $(\nu, \tau) \in \Psi_-$.

This finishes the proof of part A. Part B can be proved using a line of logic identical to that used to prove part A. The only difference that sets part B apart from part A is that $\nu^2 - \tau^2 < 0$ for the case $(\nu, \tau) \in \Psi_+$ while $\nu^2 - \tau^2 > 0$ for the case $(\nu, \tau) \in \Psi_-$. **QED**.

Next, note that Eq. (4.24) yields

$$G(\nu,\tau,1) = (\tau-\nu^2)^2\left[(2+\tau)^2-\nu^2\right] \tag{4.110}$$

and

$$G(\nu,\tau,0) = 4\tau\left[\nu^2\tau^2+(1-\nu^2)\tau-\nu^2\right] \tag{4.111}$$

In addition, for any $(\nu,\tau)\in\Psi$, Eqs. (4.24) and (4.85) also yield

$$G(\nu,\tau,s_o) = -\frac{\nu^2(1+\tau)^2\left[\nu^2\tau^2+2(\nu^2-4)\tau+9\nu^2\right]}{4(1-\nu^2)} \tag{4.112}$$

An immediate result of Eqs. (4.66) and (4.110) is Theorem 27.

Theorem 27. For any $(\nu,\tau)\in\Psi$, we have

$$G(\nu,\tau,1)\geq 0 \qquad ((\nu,\tau)\in\Psi) \tag{4.113}$$

Next let

$$K_{\pm}(x) \stackrel{\text{def}}{=} \frac{x-1\pm\sqrt{1-2x+5x^2}}{2x}, \qquad\qquad 0<x<1 \tag{4.114}$$

Then one has Theorems 28 and 29.

Theorem 28. In the domain $0<x<1$, we have

$$K_+(x)>0 \qquad\qquad (0<x<1) \tag{4.115}$$

and

$$K_-(x)<0 \qquad\qquad (0<x<1) \tag{4.116}$$

Proof. Because

$$1-2x+5x^2 = (x-1)^2+4x^2 \tag{4.117}$$

one has

$$\sqrt{1-2x+5x^2} > |x-1|, \qquad x\neq 0 \tag{4.118}$$

Eqs. (4.115) and (4.116) follow directly from Eqs. (4.114) and (4.118). **QED**.

Theorem 29. For any $(\nu,\tau)\in\Psi$, we have

$$G(\nu,\tau,0)\geq 0 \quad\Leftrightarrow\quad \tau\geq K_+(\nu^2) \qquad ((\nu,\tau)\in\Psi) \tag{4.119}$$

Proof. Note that

$$4\tau\left[\nu^2\tau^2+(1-\nu^2)\tau-\nu^2\right] = 4\tau\nu^2\left[\tau-K_+(\nu^2)\right]\left[\tau-K_-(\nu^2)\right], \quad 0<\nu^2<1 \tag{4.120}$$

In addition, because $\tau \geq \nu^2$ and $0 < \nu^2 < 1$ if $(\nu, \tau) \in \Psi$, Eq. (4.116) implies that

$$\tau - K_-(\nu^2) > 0, \qquad (\nu, \tau) \in \Psi \tag{4.121}$$

Eq. (4.119) now follows from Eqs. (4.111), (4.120) and (4.121), and the fact that $\tau \geq \nu^2$ and $0 < \nu^2 < 1$ if $(\nu, \tau) \in \Psi$. **QED**.

Next let

$$L_\pm(x) \stackrel{\text{def}}{=} \frac{4 - x \pm 2\sqrt{2(2 - x - x^2)}}{x}, \qquad 0 < x < 1 \tag{4.122}$$

Then one has Theorems 30 and 31.

Theorem 30. In the domain $0 < x < 1$, we have

$$L_+(x) > L_-(x) > 0 \qquad (0 < x < 1) \tag{4.123}$$

Proof. Note that (i)

$$2 - x - x^2 = -(x + 2)(x - 1) > 0, \qquad -2 < x < 1 \tag{4.124}$$

and (ii)

$$(4 - x)^2 - \left[2\sqrt{2(2 - x - x^2)}\right]^2 = 9x^2 > 0, \qquad x \neq 0 \tag{4.125}$$

Thus

$$4 - x = |4 - x| > 2\sqrt{2(2 - x - x^2)} > 0, \qquad 0 < x < 1 \text{ or } -2 < x < 0 \tag{4.126}$$

Eq. (4.123) is a result of Eqs. (4.122) and (4.126). **QED**.

Theorem 31. For any $(\nu, \tau) \in \Psi$, we have

$$G(\nu, \tau, s_o) \geq 0 \quad \Leftrightarrow \quad L_-(\nu^2) \leq \tau \leq L_+(\nu^2) \qquad ((\nu, \tau) \in \Psi) \tag{4.127}$$

Proof. Note that

$$\nu^2\tau^2 + 2(\nu^2 - 4)\tau + 9\nu^2 = \nu^2 \left[\tau - L_+(\nu^2)\right] \left[\tau - L_-(\nu^2)\right] \qquad (0 < \nu^2 < 1) \tag{4.128}$$

Because $1 + \tau > 0$, $\nu^2 > 0$, and $1 - \nu^2 > 0$ if $(\nu, \tau) \in \Psi$, Eqs. (4.112) and (4.128) imply that

$$G(\nu, \tau, s_o) \geq 0 \quad \Leftrightarrow \quad \left[\tau - L_+(\nu^2)\right] \left[\tau - L_-(\nu^2)\right] \leq 0 \quad ((\nu, \tau) \in \Psi) \tag{4.129}$$

if $(\nu, \tau) \in \Psi$. Because $0 < \nu^2 < 1$ if $(\nu, \tau) \in \Psi$, Eq. (4.127) now follows from Eq. (4.129) and a result of Eq. (4.123), i.e.,

$$\left[\tau - L_-(\nu^2)\right] > \left[\tau - L_+(\nu^2)\right], \qquad 0 < \nu^2 < 1 \tag{4.130}$$

QED.

With the above preliminaries, one can establish Theorem 32.

Theorem 32. (A) Let $(\nu, \tau) \in \Psi_-$. Then (ν, τ) is c-τ stable if and only if it satisfies one of the three mutually exclusive sets of conditions specified, respectively, in Eqs. (4.131)–(4.133):

$$\tau \geq J_+(\nu^2) \tag{4.131}$$

$$K_+(\nu^2) \leq \tau \leq I_+(\nu^2) \tag{4.132}$$

and

$$I_+(\nu^2) < \tau < J_+(\nu^2) \quad \text{and} \quad L_-(\nu^2) \leq \tau \leq L_+(\nu^2) \tag{4.133}$$

(B) Let $(\nu, \tau) \in \Psi_+$. Then (ν, τ) is c-τ stable if and only if it satisfies one of the three mutually exclusive sets of conditions specified, respectively, in Eqs. (4.134)–(4.136):

$$\tau \leq J_+(\nu^2) \tag{4.134}$$

$$\tau \geq I_+(\nu^2) \quad \text{and} \quad \tau \geq K_+(\nu^2) \tag{4.135}$$

and

$$J_+(\nu^2) < \tau < I_+(\nu^2) \quad \text{and} \quad L_-(\nu^2) \leq \tau \leq L_+(\nu^2) \tag{4.136}$$

Proof. Let

$$\Psi_-^{(\alpha)} \stackrel{\text{def}}{=} \{(\nu, \tau) | (\nu, \tau) \in \Psi_- \text{ and } s_o(\nu, \tau) \geq 1\} \tag{4.137}$$

$$\Psi_-^{(\beta)} \stackrel{\text{def}}{=} \{(\nu, \tau) | (\nu, \tau) \in \Psi_- \text{ and } s_o(\nu, \tau) \leq 0\} \tag{4.138}$$

$$\Psi_-^{(\gamma)} \stackrel{\text{def}}{=} \{(\nu, \tau) | (\nu, \tau) \in \Psi_- \text{ and } 0 < s_o(\nu, \tau) < 1\} \tag{4.139}$$

$$\Psi_+^{(\alpha)} \stackrel{\text{def}}{=} \{(\nu, \tau) | (\nu, \tau) \in \Psi_+ \text{ and } s_o(\nu, \tau) \geq 1\} \tag{4.140}$$

$$\Psi_+^{(\beta)} \stackrel{\text{def}}{=} \{(\nu, \tau) | (\nu, \tau) \in \Psi_+ \text{ and } s_o(\nu, \tau) \leq 0\} \tag{4.141}$$

and

$$\Psi_+^{(\gamma)} \stackrel{\text{def}}{=} \{(\nu, \tau) | (\nu, \tau) \in \Psi_+ \text{ and } 0 < s_o(\nu, \tau) < 1\} \tag{4.142}$$

Because Ψ_- and Ψ_+ are mutually exclusive, the above definitions imply that (i) $\Psi_-^{(\alpha)}$, $\Psi_-^{(\beta)}$, $\Psi_-^{(\gamma)}$, $\Psi_+^{(\alpha)}$, $\Psi_+^{(\beta)}$, and $\Psi_+^{(\gamma)}$ are mutually exclusive; (ii)

$$\Psi_- = \Psi_-^{(\alpha)} \cup \Psi_-^{(\beta)} \cup \Psi_-^{(\gamma)} \tag{4.143}$$

and (iii)

$$\Psi_+ = \Psi_+^{(\alpha)} \cup \Psi_+^{(\beta)} \cup \Psi_+^{(\gamma)} \tag{4.144}$$

Moreover, by using Theorem 26, Eqs. (4.137)–(4.142) imply

$$\Psi_-^{(\alpha)} = \{(\nu,\tau)|(\nu,\tau) \in \Psi_- \text{ and } \tau \geq J_+(\nu^2)\} \tag{4.145}$$

$$\Psi_-^{(\beta)} = \{(\nu,\tau)|(\nu,\tau) \in \Psi_- \text{ and } \tau \leq I_+(\nu^2)\} \tag{4.146}$$

$$\Psi_-^{(\gamma)} = \{(\nu,\tau)|(\nu,\tau) \in \Psi_- \text{ and } I_+(\nu^2) < \tau < J_+(\nu^2)\} \tag{4.147}$$

$$\Psi_+^{(\alpha)} = \{(\nu,\tau)|(\nu,\tau) \in \Psi_+ \text{ and } \tau \leq J_+(\nu^2)\} \tag{4.148}$$

$$\Psi_+^{(\beta)} = \{(\nu,\tau)|(\nu,\tau) \in \Psi_+ \text{ and } \tau \geq I_+(\nu^2)\} \tag{4.149}$$

and

$$\Psi_+^{(\gamma)} = \{(\nu,\tau)|(\nu,\tau) \in \Psi_+ \text{ and } J_+(\nu^2) < \tau < I_+(\nu^2)\} \tag{4.150}$$

respectively.

To proceed, note that:

(a) With the aid of (i) Eqs. (4.137) and (4.140), and (ii) Theorems 24 and 27, Theorem 23 implies that a given $(\nu,\tau) \in \Psi_-^{(\alpha)} \cup \Psi_+^{(\alpha)}$ is always *c*-τ stable.

(b) With the aid of (i) Eqs. (4.138) and (4.141), and (ii) Theorems 24 and 29, Theorem 23 implies that a given $(\nu,\tau) \in \Psi_-^{(\beta)} \cup \Psi_+^{(\beta)}$ is *c*-τ stable if and only if

$$\tau \geq K_+(\nu^2) \tag{4.151}$$

(c) With the aid of (i) Eqs. (4.139) and (4.142), and (ii) Theorems 24 and 31, Theorem 23 implies that a given $(\nu,\tau) \in \Psi_-^{(\gamma)} \cup \Psi_+^{(\gamma)}$ is *c*-τ stable if and only if

$$L_-(\nu^2) \leq \tau \leq L_+(\nu^2) \tag{4.152}$$

Theorem 32 now follows from Eqs. (4.143)–(4.150) and the facts presented in the above items (a)–(c). **QED**.

In principle, the question of whether a given (ν,τ) is *c*-τ stable can now be answered by using Theorems 12, 16, 18, 19, and 32. However, in its current complicated form, Theorem 32 is difficult to use. Fortunately, Theorem 32 can be simplified greatly and, in fact, the stability condition for the *c*-τ scheme can be cast into a rather simple explicit form. To obtain this simple form, we begin with Theorem 33.

Theorem 33. We have: (A)

$$(\nu,\tau) \in \Psi_- \quad \Leftrightarrow \quad 0 < \nu^2 < 1 \text{ and } \nu^2 \leq \tau < \sqrt{\nu^2} \tag{4.153}$$

(B) Ψ_- is not empty; and (C)

$$(\nu,\tau) \in \Psi_+ \quad \Leftrightarrow \quad 0 < \nu^2 < 1 \text{ and } \tau > \sqrt{\nu^2} \tag{4.154}$$

Proof. Because (i) $-\sqrt{\nu^2} < \tau < \sqrt{\nu^2}$ if $\tau^2 < \nu^2$, and (ii) $\tau^2 < \nu^2$ if $0 \le \tau < \sqrt{\nu^2}$, part A is an immediate result of Eq. (4.67). Part B follows from the trivial fact that $\nu^2 < \sqrt{\nu^2}$ if $0 < \nu^2 < 1$. To prove part C, note that (i) $\tau > 0$ if $\nu^2 > 0$ and $\tau \ge \nu^2$, and (ii) $\tau > \sqrt{\nu^2}$ if $\tau > 0$ and $\tau^2 > \nu^2$. Thus Eq. (4.70) implies that $0 < \nu^2 < 1$ and $\tau > \sqrt{\nu^2}$ if $(\nu,\tau) \in \Psi_+$. Conversely, because (i) $\sqrt{\nu^2} > \nu^2$ if $0 < \nu^2 < 1$; (ii) $\tau > \nu^2$ if $\tau > \sqrt{\nu^2}$ and $\sqrt{\nu^2} > \nu^2$; and (iii) $\tau^2 > \nu^2$ if $\tau > \sqrt{\nu^2}$, one concludes that $(\nu,\tau) \in \Psi_+$ if $0 < \nu^2 < 1$ and $\tau > \sqrt{\nu^2}$. **QED**.

Next let

$$c_1 \stackrel{\text{def}}{=} 3 - 2\sqrt{2} \tag{4.155}$$

$$c_2 \stackrel{\text{def}}{=} 3/11 \tag{4.156}$$

$$c_3 \stackrel{\text{def}}{=} (41 - 7\sqrt{33})/2 \tag{4.157}$$

and

$$c_4 \stackrel{\text{def}}{=} \left[\left(\sqrt{\frac{1664}{27}} + \frac{181}{27}\right)^{\frac{1}{3}} - \left(\sqrt{\frac{1664}{27}} - \frac{181}{27}\right)^{\frac{1}{3}} - \frac{2}{3}\right]^2 \tag{4.158}$$

We have (i) $c_1 \approx 0.172$, $c_2 \approx 0.273$, $c_3 \approx 0.394$ and $c_4 \approx 0.530$, and (ii)

$$0 < c_1 < c_2 < c_3 < c_4 < 1 \tag{4.159}$$

With the above preparations, we have Theorem 34.

Theorem 34. (A) In the domain $0 < x < 1$, $I_+(x)$, $J_+(x)$, $K_+(x)$, and $L_-(x)$ are strictly monotonically increasing while $L_+(x)$ is strictly monotonically decreasing; (B) we have

$$I_+(x) < x < K_+(x) < L_-(x) < J_+(x) < \sqrt{x} < L_+(x), \qquad 0 < x < c_1 \tag{4.160}$$

$$I_+(x) = x < K_+(x) < L_-(x) < J_+(x) < \sqrt{x} < L_+(x), \qquad x = c_1 \tag{4.161}$$

$$x < I_+(x) < K_+(x) < L_-(x) < J_+(x) < \sqrt{x} < L_+(x), \qquad c_1 < x < c_2 \tag{4.162}$$

$$x < I_+(x) = K_+(x) = L_-(x) < J_+(x) < \sqrt{x} < L_+(x), \qquad x = c_2 \tag{4.163}$$

$$x < K_+(x) < L_-(x) < I_+(x) < J_+(x) < \sqrt{x} < L_+(x), \qquad c_2 < x < c_3 \tag{4.164}$$

$$x < K_+(x) < L_-(x) < I_+(x) = J_+(x) = \sqrt{x} < L_+(x), \qquad x = c_3 \tag{4.165}$$

$$x < K_+(x) < L_-(x) < \sqrt{x} < J_+(x) < I_+(x) < L_+(x), \qquad c_3 < x < c_4 \tag{4.166}$$

$$x < K_+(x) < L_-(x) = \sqrt{x} < J_+(x) < I_+(x) < L_+(x), \qquad x = c_4 \tag{4.167}$$

and

$$x < K_+(x) < \sqrt{x} < L_-(x) < J_+(x) < I_+(x) < L_+(x), \qquad c_4 < x < 1 \tag{4.168}$$

(C)

$$K'_+(c_2) = L'_-(c_2) = 121/90 \tag{4.169}$$

where $K'_+(x) \stackrel{\text{def}}{=} dK_+(x)/dx$ and $L'_-(x) \stackrel{\text{def}}{=} dL_-(x)/dx$; and (D)

$$\lim_{x \to 0^+} L_-(x) = 0 \quad \text{and} \quad \lim_{x \to 1^-} K_+(x) = 1 \tag{4.170}$$

In order not to interrupt the current stream of development, the lengthy proof for Theorem 34 will be provided later in the paper. Here, with the aid of this theorem, we shall establish a simplified form of the stability condition for the c-τ scheme as given in Theorem 35.

Theorem 35. Let

$$\tau_o(x) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if} \quad x = 0 \\ L_-(x) & \text{if} \quad 0 < x \le 3/11 \\ K_+(x) & \text{if} \quad 3/11 \le x < 1 \\ 1 & \text{if} \quad x = 1 \end{cases} \tag{4.171}$$

$$\Gamma_o \stackrel{\text{def}}{=} \{(\nu, \tau) | \nu^2 \le 1, \tau \ge \tau_o(\nu^2) \text{ and } (\nu^2, \tau) \ne (1, 1)\} \tag{4.172}$$

and

$$\Gamma \stackrel{\text{def}}{=} \{(\nu, \tau) | \nu^2 \le 1 \text{ and } \tau \ge \tau_o(\nu^2)\} \tag{4.173}$$

Then: (A) $\tau_o(x)$ is continuous at $x = 0$ and $x = 1$; (B) $\tau_o(x)$ is consistently defined at $x = 3/11$; (C)

$$\lim_{x \to \frac{3}{11}^-} \tau'_o(x) = \lim_{x \to \frac{3}{11}^+} \tau'_o(x) = 121/90 \tag{4.174}$$

where $\tau'_o(x) \stackrel{\text{def}}{=} d\tau_o(x)/dx$; (D) $\tau_o(x)$ is strictly monotonically increasing in the interval $0 < x < 1$; (E)

$$x < \tau_o(x) < \sqrt{x}, \qquad 0 < x < 1 \tag{4.175}$$

(F) a given (ν, τ) is c-τ stable if and only if $(\nu, \tau) \in \Gamma_o$; and (G) a given (ν, τ) satisfies Eq. (4.2) if and only if $(\nu, \tau) \in \Gamma$.

Proof. Part A is a result of Eqs. (4.170) and (4.171). Part B follows from the fact that $L_-(3/11) = K_+(3/11) = 1/3$. Part C follows from Eqs. (4.156) and (4.169). Part D is a result of part A of Theorem 34, and parts B and C of the current theorem. Part E is a result of Eqs. (4.160)–(4.168) and (4.171).

To prove part F, one needs to show that: (i) $(\nu, \tau) \in \Gamma_o$ for any (ν, τ) that is c-τ stable; and (ii) $(\nu, \tau) \notin \Gamma_o$ for any (ν, τ) that is c-τ unstable. Here whether any particular (ν, τ) is c-τ stable is determined using Theorems 12, 16, 18, 19, and 35.

To proceed, let

$$\Phi_1 \stackrel{\text{def}}{=} \{(\nu, \tau)|\nu^2 > 1 \text{ or } \tau < \nu^2 \leq 1\} \tag{4.176}$$

$$\Phi_2 \stackrel{\text{def}}{=} \{(\nu, \tau)|\tau = \nu^2 = 1\} \tag{4.177}$$

$$\Phi_3 \stackrel{\text{def}}{=} \{(\nu, \tau)|\nu^2 = 0 \text{ and } \tau \geq 0\} \tag{4.178}$$

$$\Phi_4 \stackrel{\text{def}}{=} \{(\nu, \tau)|\nu^2 = 1 \text{ and } \tau > 1\} \tag{4.179}$$

$$\Phi_5 \stackrel{\text{def}}{=} \{(\nu, \tau)|0 < \nu^2 < 1 \text{ and } \tau = |\nu|\} \tag{4.180}$$

With the aid Theorem 19, it is seen that Ψ_-, Ψ_+, and the five sets defined above are inclusive and yet mutually exclusive, i.e., any (ν, τ) belongs to one and only one of these sets. To facilitate the proof, Ψ_- and Ψ_+, respectively, will be further divided into several disjoint subsets to be defined immediately.

Let

$$\Psi_-^{(1)} \stackrel{\text{def}}{=} \{(\nu, \tau)|0 < \nu^2 < c_2 \text{ and } \nu^2 \leq \tau < \sqrt{\nu^2}\} \tag{4.181}$$

$$\Psi_-^{(2)} \stackrel{\text{def}}{=} \{(\nu, \tau)|\nu^2 = c_2 \text{ and } \nu^2 \leq \tau < \sqrt{\nu^2}\} \tag{4.182}$$

$$\Psi_-^{(3)} \stackrel{\text{def}}{=} \{(\nu, \tau)|c_2 < \nu^2 < c_3 \text{ and } \nu^2 \leq \tau < \sqrt{\nu^2}\} \tag{4.183}$$

and

$$\Psi_-^{(4)} \stackrel{\text{def}}{=} \{(\nu, \tau)|c_3 \leq \nu^2 < 1 \text{ and } \nu^2 \leq \tau < \sqrt{\nu^2}\} \tag{4.184}$$

Because $(\nu, \tau) \in \Psi_- \Leftrightarrow 0 < \nu^2 < 1$ and $\nu^2 \leq \tau < \sqrt{\nu^2}$ (see Theorem 33), one concludes that (i) $\Psi_-^{(\ell)}$, $\ell = 1, 2, 3, 4$, are nonempty disjoint subsets of Ψ_-, and (ii)

$$\Psi_- = \cup_{\ell=1}^{4} \Psi_-^{(\ell)} \tag{4.185}$$

Next let

$$\Psi_+^{(1)} \stackrel{\text{def}}{=} \{(\nu, \tau)|0 < \nu^2 \leq c_3 \text{ and } \tau > \sqrt{\nu^2}\} \tag{4.186}$$

and

$$\Psi_+^{(2)} \stackrel{\text{def}}{=} \{(\nu, \tau)|c_3 < \nu^2 < 1 \text{ and } \tau > \sqrt{\nu^2}\} \tag{4.187}$$

Because $(\nu, \tau) \in \Psi_+ \Leftrightarrow 0 < \nu^2 < 1$ and $\tau > \sqrt{\nu^2}$ (see Theorem 33), one concludes that (i) $\Psi_+^{(1)}$ and $\Psi_+^{(2)}$, are nonempty disjoint subsets of Ψ_+, and (ii)

$$\Psi_+ = \Psi_+^{(1)} \cup \Psi_+^{(2)} \tag{4.188}$$

From the above discussion, the sets (i) Φ_ℓ, $\ell = 1,2,3,4,5$; (ii) $\Psi_-^{(\ell)}$, $\ell = 1,2,3,4$; and (iii) $\Psi_+^{(1)}$ and $\Psi_+^{(2)}$, are inclusive and yet mutually exclusive, i.e., any (ν,τ) must belong to one and only one of these sets. Part F will be proved by showing that it is valid over each of these sets in the following case-by-case discussions:

1. $(\nu,\tau) \in \Phi_1$. According to Theorem 12, any $(\nu,\tau) \in \Phi_1$ is c-τ unstable. Thus part F is true over Φ_1 if one can show that $(\nu,\tau) \notin \Gamma_o$ if $(\nu,\tau) \in \Phi_1$. Because $(\nu,\tau) \notin \Gamma_o$ if $\nu^2 > 1$ (see Eq. (4.172)), the proof for case 1 is completed if one can show that $(\nu,\tau) \notin \Gamma_o$ if $\tau < \nu^2 \le 1$.

   To proceed, note that Eq. (4.175) and the facts that $\tau_o(0) = 0$ and $\tau_o(1) = 1$ imply that
$$\nu^2 \le \tau_o(\nu^2), \qquad \nu^2 \le 1 \tag{4.189}$$
   Thus $\tau < \tau_o(\nu^2)$ if $\tau < \nu^2 \le 1$. As a result of Eq. (4.172), this in turn implies that $(\nu,\tau) \notin \Gamma_o$ if $\tau < \nu^2 \le 1$. As such part F is true over Φ_1.

2. $(\nu,\tau) \in \Phi_2$. According to Theorem 16, any $(\nu,\tau) \in \Phi_2$ is c-τ unstable. Also, according to Eq. (4.172), $(\nu,\tau) \notin \Gamma_o$ if $(\nu,\tau) \in \Phi_2$. Thus part F is true over Φ_2.

3. $(\nu,\tau) \in \Phi_3$. According to Theorem 18, any $(\nu,\tau) \in \Phi_3$ is c-τ stable. Because $\tau_o(0) = 0$, Eq. (4.172) implies that $(\nu,\tau) \in \Gamma_o$ if $(\nu,\tau) \in \Phi_3$. Thus part F is true over Φ_3.

4. $(\nu,\tau) \in \Phi_4$. According to Theorem 18, any $(\nu,\tau) \in \Phi_4$ is c-τ stable. Because $\tau_o(1) = 1$, Eq. (4.172) implies that $(\nu,\tau) \in \Gamma_o$ if $(\nu,\tau) \in \Phi_4$. Thus part F is true over Φ_4.

5. $(\nu,\tau) \in \Phi_5$. According to Theorem 18, any $(\nu,\tau) \in \Phi_5$ is c-τ stable. On the other hand, Eqs. (4.175) implies that
$$\tau_o(\nu^2) < \sqrt{\nu^2}, \qquad 0 < \nu^2 < 1 \tag{4.190}$$
   i.e., $\tau_o(\nu^2) < \sqrt{\nu^2} = |\nu|$ if $0 < \nu^2 < 1$. This coupled with Eq. (4.172) implies that $(\nu,\tau) \in \Gamma_o$ if $(\nu,\tau) \in \Phi_5$. Thus part F is true over Φ_5.

6. $(\nu,\tau) \in \Psi_-^{(1)}$. For this case, we have (i) $0 < \nu^2 < c_2$, and (ii) $\nu^2 \le \tau < \sqrt{\nu^2}$. To proceed, Note that Eqs. (4.160)–(4.162) imply that
$$I_+(\nu^2) < K_+(\nu^2), \qquad 0 < \nu^2 < c_2 \tag{4.191}$$
$$\nu^2 < L_-(\nu^2) < J_+(\nu^2) < \sqrt{\nu^2}, \qquad 0 < \nu^2 < c_2 \tag{4.192}$$
   and
$$I_+(\nu^2) < L_-(\nu^2) < J_+(\nu^2) < L_+(\nu^2), \qquad 0 < \nu^2 < c_2 \tag{4.193}$$
   Because Eq. (4.191) contradicts Eq. (4.132), Eq. (4.132) cannot be satisfied by any $(\nu,\tau) \in \Psi_-^{(1)}$. Moreover, by using Eq. (4.192), it can be shown that
$$\Psi_-^{(1)} = \Psi_-^{(1,1)} \cup \Psi_-^{(1,2)} \cup \Psi_-^{(1,3)} \tag{4.194}$$

where $\Psi_-^{(1,1)}$, $\Psi_-^{(1,2)}$, and $\Psi_-^{(1,3)}$ are nonempty disjoint sets defined by

$$\Psi_-^{(1,1)} \stackrel{\text{def}}{=} \{(\nu,\tau)|0<\nu^2<c_2 \text{ and } \nu^2 \le \tau < L_-(\nu^2)\} \tag{4.195}$$

$$\Psi_-^{(1,2)} \stackrel{\text{def}}{=} \{(\nu,\tau)|0<\nu^2<c_2 \text{ and } L_-(\nu^2) \le \tau < J_+(\nu^2)\} \tag{4.196}$$

and

$$\Psi_-^{(1,3)} \stackrel{\text{def}}{=} \{(\nu,\tau)|0<\nu^2<c_2 \text{ and } J_+(\nu^2) \le \tau < \sqrt{\nu^2}\} \tag{4.197}$$

Thus any $(\nu,\tau) \in \Psi_-^{(1)}$ must fall into one and only one of the following three sub-cases: (i) $(\nu,\tau) \in \Psi_-^{(1,1)}$, (ii) $(\nu,\tau) \in \Psi_-^{(1,2)}$, and (iii) $(\nu,\tau) \in \Psi_-^{(1,3)}$.

Let $(\nu,\tau) \in \Psi_-^{(1,1)}$. By using the relation $L_-(\nu^2) < J_+(\nu^2)$ which follows from Eq. (4.192) or Eq. (4.193), it is seen that Eq. (4.131) cannot be true for the current sub-case where $\nu^2 \le \tau < L_-(\nu^2)$. Also, the second part of Eq. (4.133), i.e., $L_-(\nu^2) \le \tau \le L_+(\nu^2)$, cannot be true for the sub-case. Moreover, for a reason given earlier, Eq. (4.132) also cannot be true for the sub-case. According to part A of Theorem 32, the above results imply that any $(\nu,\tau) \in \Psi_-^{(1,1)}$ is c-τ unstable. On the other hand, because $\tau_o(\nu^2) = L_-(\nu^2)$ if $0 < \nu^2 < c_2$ (see Eqs. (4.156) and (4.171)), one concludes that $\tau < \tau_o(\nu^2)$ and thus $(\nu,\tau) \notin \Gamma_o$ if $(\nu,\tau) \in \Psi_-^{(1,1)}$. As such it has been shown that part F is true over $\Psi_-^{(1,1)}$.

Let $(\nu,\tau) \in \Psi_-^{(1,2)}$. It follows from Eq. (4.193) that Eq. (4.133) is satisfied by any (ν,τ) with $L_-(\nu^2) \le \tau < J_+(\nu^2)$. According to part A of Theorem 32 and Eq. (4.196), this implies that any (ν,τ) in the current sub-case is c-τ stable. On the other hand, because $\tau_o(\nu^2) = L_-(\nu^2)$ if $0 < \nu^2 < c_2$, one concludes that $\tau \ge \tau_o(\nu^2)$ and thus $(\nu,\tau) \in \Gamma_o$ if $(\nu,\tau) \in \Psi_-^{(1,2)}$. As such, it has been shown that part F is true over $\Psi_-^{(1,2)}$.

Let $(\nu,\tau) \in \Psi_-^{(1,3)}$. Obviously Eq. (4.131) is true for the current sub-case where $J_+(\nu^2) \le \tau < \sqrt{\nu^2}$. According to part A of Theorem 32, this implies that any (ν,τ) in the current sub-case is c-τ stable. On the other hand, because (i) $\tau_o(\nu^2) = L_-(\nu^2)$ if $0 < \nu^2 < c_2$, and (ii) the relation $L_-(\nu^2) < J_+(\nu^2)$ is a part of Eq. (4.193), one concludes that $\tau > \tau_o(\nu^2)$ and thus $(\nu,\tau) \in \Gamma_o$ if $(\nu,\tau) \in \Psi_-^{(1,3)}$. As such, it has been shown that part F is true over $\Psi_-^{(1,3)}$.

It has been shown that part F is true over each of the three nonempty disjoint sets $\Psi_-^{(1,1)}$, $\Psi_-^{(1,2)}$, and $\Psi_-^{(1,3)}$. Eq. (4.194) now implies that part F is true over $\Psi_-^{(1)}$.

7. $(\nu,\tau) \in \Psi_-^{(2)}$. For this case, we have (i) $\nu^2 = c_2$, and (ii) $\nu^2 \le \tau < \sqrt{\nu^2}$. To proceed, Note that Eqs. (4.163) implies that

$$\nu^2 < I_+(\nu^2) = K_+(\nu^2) = L_-(\nu^2) < J_+(\nu^2) < \sqrt{\nu^2} < L_+(\nu^2), \qquad \nu^2 = c_2 \tag{4.198}$$

With the aid of Eq. (4.198), it can be shown that

$$\Psi_-^{(2)} = \Psi_-^{(2,1)} \cup \Psi_-^{(2,2)} \cup \Psi_-^{(2,3)} \cup \Psi_-^{(2,4)} \tag{4.199}$$

where $\Psi_-^{(2,1)}$, $\Psi_-^{(2,2)}$, $\Psi_-^{(2,3)}$, and $\Psi_-^{(2,4)}$ are nonempty disjoint sets defined by

$$\Psi_-^{(2,1)} \stackrel{\text{def}}{=} \{(\nu,\tau)|\nu^2 = c_2 \text{ and } \nu^2 \le \tau < L_-(\nu^2)\} \tag{4.200}$$

$$\Psi_-^{(2,2)} \stackrel{\text{def}}{=} \{(\nu,\tau)|\nu^2 = c_2 \text{ and } \tau = L_-(\nu^2)\} \tag{4.201}$$

$$\Psi_-^{(2,3)} \stackrel{\text{def}}{=} \{(\nu,\tau)|\nu^2 = c_2 \text{ and } L_-(\nu^2) < \tau < J_+(\nu^2)\} \tag{4.202}$$

and

$$\Psi_-^{(2,4)} \stackrel{\text{def}}{=} \{(\nu,\tau)|\nu^2 = c_2 \text{ and } J_+(\nu^2) \le \tau < \sqrt{\nu^2}\} \tag{4.203}$$

Thus any $(\nu,\tau) \in \Psi_-^{(2)}$ must fall into one and only one of the following four sub-cases: (i) $(\nu,\tau) \in \Psi_-^{(2,1)}$, (ii) $(\nu,\tau) \in \Psi_-^{(2,2)}$, (iii) $(\nu,\tau) \in \Psi_-^{(2,3)}$, and (iv) $(\nu,\tau) \in \Psi_-^{(2,4)}$.

Let $(\nu,\tau) \in \Psi_-^{(2,1)}$. By using the relation $L_-(\nu^2) < J_+(\nu^2)$ which follows from Eq. (4.198), it is seen that Eq. (4.131) cannot be true for the current sub-case where $\nu^2 \le \tau < L_-(\nu^2)$. Moreover, by using the relation $I_+(\nu^2) = K_+(\nu^2) = L_-(\nu^2)$ which also follows from Eq. (4.198), it is seen that Eq. (4.132) also cannot be true for the sub-case. In addition, the second part of Eq. (4.133) also cannot be true for the sub-case. According to part A of Theorem 32, this implies that any $(\nu,\tau) \in \Psi_-^{(2,1)}$ is c-τ unstable. On the other hand, because $\tau_o(\nu^2) = L_-(\nu^2)$ if $\nu^2 = c_2$, one concludes that $\tau < \tau_o(\nu^2)$ and thus $(\nu,\tau) \notin \Gamma_o$ if $(\nu,\tau) \in \Psi_-^{(2,1)}$. As such it has been shown that part F is true over $\Psi_-^{(2,1)}$.

Let $(\nu,\tau) \in \Psi_-^{(2,2)}$. By using the relation $I_+(\nu^2) = K_+(\nu^2) = L_-(\nu^2)$ which follows from Eq. (4.198), it is seen that Eq. (4.132) is true for the current sub-case where $\tau = L_-(\nu^2)$. According to part A of Theorem 32, this implies that any $(\nu,\tau) \in \Psi_-^{(2,2)}$ is c-τ stable. On the other hand, because $\tau_o(\nu^2) = L_-(\nu^2)$ if $\nu^2 = c_2$, one concludes that $\tau = \tau_o(\nu^2)$ and thus $(\nu,\tau) \in \Gamma_o$ if $(\nu,\tau) \in \Psi_-^{(2,2)}$. As such it has been shown that part F is true over $\Psi_-^{(2,2)}$.

Let $(\nu,\tau) \in \Psi_-^{(2,3)}$. By using the relation $I_+(\nu^2) = L_-(\nu^2) < J_+(\nu^2) < L_+(\nu^2)$ which follows from Eq. (4.198), it is seen that Eq. (4.133) is true for the current case where $L_-(\nu^2) < \tau < J_+(\nu^2)$. According to part A of Theorem 32, this implies that any $(\nu,\tau) \in \Psi_-^{(2,3)}$ is c-τ stable. On the other hand, because $\tau_o(\nu^2) = L_-(\nu^2)$ if $\nu^2 = c_2$, one concludes that $\tau > \tau_o(\nu^2)$ and thus $(\nu,\tau) \in \Gamma_o$ if $(\nu,\tau) \in \Psi_-^{(2,3)}$. As such it has been shown that part F is true over $\Psi_-^{(2,3)}$.

Let $(\nu,\tau) \in \Psi_-^{(2,4)}$. Obviously Eq. (4.131) is true for the current sub-case where $J_+(\nu^2) \le \tau < \sqrt{\nu^2}$. According to part A of Theorem 32, this implies that any (ν,τ) in the current sub-case is c-τ stable. On the other hand, because (i) $\tau_o(\nu^2) = L_-(\nu^2)$ if $\nu^2 = c_2$, and (ii) the relation $L_-(\nu^2) < J_+(\nu^2)$ is a part of Eq. (4.198), one concludes that $\tau > \tau_o(\nu^2)$ and thus $(\nu,\tau) \in \Gamma_o$ if $(\nu,\tau) \in \Psi_-^{(2,4)}$. As such, it has been shown that part F is true over $\Psi_-^{(2,4)}$.

It has been shown that part F is true over each of the four nonempty disjoint sets $\Psi_-^{(2,1)}$, $\Psi_-^{(2,2)}$, $\Psi_-^{(2,3)}$, and $\Psi_-^{(2,4)}$. Eq. (4.199) now implies that part F is true over $\Psi_-^{(2)}$.

8. $(\nu,\tau) \in \Psi_-^{(3)}$. For this case, we have (i) $c_2 < \nu^2 < c_3$, and (ii) $\nu^2 \leq \tau < \sqrt{\nu^2}$. To proceed, Note that Eqs. (4.164) implies that

$$\nu^2 < K_+(\nu^2) < L_-(\nu^2) < I_+(\nu^2) < J_+(\nu^2) < \sqrt{\nu^2} < L_+(\nu^2), \qquad c_2 < \nu^2 < c_3 \tag{4.204}$$

With the aid of Eq. (4.204), it can be shown that

$$\Psi_-^{(3)} = \Psi_-^{(3,1)} \cup \Psi_-^{(3,2)} \cup \Psi_-^{(3,3)} \cup \Psi_-^{(3,4)} \tag{4.205}$$

where $\Psi_-^{(3,1)}$, $\Psi_-^{(3,2)}$, $\Psi_-^{(3,3)}$, and $\Psi_-^{(3,4)}$ are nonempty disjoint sets defined by

$$\Psi_-^{(3,1)} \stackrel{\text{def}}{=} \{(\nu,\tau) | c_2 < \nu^2 < c_3 \text{ and } \nu^2 \leq \tau < K_+(\nu^2)\} \tag{4.206}$$

$$\Psi_-^{(3,2)} \stackrel{\text{def}}{=} \{(\nu,\tau) | c_2 < \nu^2 < c_3 \text{ and } K_+(\nu^2) \leq \tau \leq I_+(\nu^2)\} \tag{4.207}$$

$$\Psi_-^{(3,3)} \stackrel{\text{def}}{=} \{(\nu,\tau) | c_2 < \nu^2 < c_3 \text{ and } I_+(\nu^2) < \tau < J_+(\nu^2)\} \tag{4.208}$$

and

$$\Psi_-^{(3,4)} \stackrel{\text{def}}{=} \{(\nu,\tau) | c_2 < \nu^2 < c_3 \text{ and } J_+(\nu^2) \leq \tau < \sqrt{\nu^2}\} \tag{4.209}$$

Thus any $(\nu,\tau) \in \Psi_-^{(3)}$ must fall into one and only one of the following four sub-cases: (i) $(\nu,\tau) \in \Psi_-^{(3,1)}$, (ii) $(\nu,\tau) \in \Psi_-^{(3,2)}$, (iii) $(\nu,\tau) \in \Psi_-^{(3,3)}$, and (iv) $(\nu,\tau) \in \Psi_-^{(3,4)}$.

Let $(\nu,\tau) \in \Psi_-^{(3,1)}$. By using the relation $K_+(\nu^2) < J_+(\nu^2)$ which follows from Eq. (4.204), it is seen that Eq. (4.131) cannot be true for the current sub-case where $\nu^2 \leq \tau < K_+(\nu^2)$. Moreover, obviously Eq. (4.132) is also not true for the sub-case. In addition, by using the relation $K_+(\nu^2) < L_-(\nu^2) < I_+(\nu^2)$ which also follows from Eq. (4.204), one concludes that Eq. (4.133) also can not be true for the sub-case. According to part A of Theorem 32, the above results imply that any $(\nu,\tau) \in \Psi_-^{(3,1)}$ is c-τ unstable. On the other hand, because $\tau_o(\nu^2) = K_+(\nu^2)$ if $c_2 < \nu^2 < c_3$, one concludes that $\tau < \tau_o(\nu^2)$ and thus $(\nu,\tau) \notin \Gamma_o$ if $(\nu,\tau) \in \Psi_-^{(3,1)}$. As such it has been shown that part F is true over $\Psi_-^{(3,1)}$.

Let $(\nu,\tau) \in \Psi_-^{(3,2)}$. Obviously Eq. (4.132) is true for the current sub-case where $K_+(\nu^2) \leq \tau \leq I_+(\nu^2)$. According to part A of Theorem 32, this implies that any $(\nu,\tau) \in \Psi_-^{(3,2)}$ is c-τ stable. On the other hand, because $\tau_o(\nu^2) = K_+(\nu^2)$ if $c_2 < \nu^2 < c_3$, one concludes that $\tau \geq \tau_o(\nu^2)$ and thus $(\nu,\tau) \in \Gamma_o$ if $(\nu,\tau) \in \Psi_-^{(3,2)}$. As such it has been shown that part F is true over $\Psi_-^{(3,2)}$.

Let $(\nu,\tau) \in \Psi_-^{(3,3)}$. By using the relation $L_-(\nu^2) < I_+(\nu^2) < J_+(\nu^2) < L_+(\nu^2)$ which follows from Eq. (4.204), it is seen that Eq. (4.133) is true for the current case where $I_+(\nu^2) < \tau < J_+(\nu^2)$. According to part A of Theorem 32, this implies that any $(\nu,\tau) \in \Psi_-^{(3,3)}$ is c-τ stable. On the other hand, because (i) $\tau_o(\nu^2) = K_+(\nu^2)$ if $c_2 < \nu^2 < c_3$, and (ii) the relation $K_+(\nu^2) < I_+(\nu^2)$ is a part of Eq. (4.204), one concludes that $\tau > \tau_o(\nu^2)$ and thus $(\nu,\tau) \in \Gamma_o$ if $(\nu,\tau) \in \Psi_-^{(3,3)}$. As such it has been shown that part F is true over $\Psi_-^{(3,3)}$.

Let $(\nu,\tau) \in \Psi_-^{(3,4)}$. Obviously Eq. (4.131) is true for the current sub-case where $J_+(\nu^2) \leq \tau < \sqrt{\nu^2}$. According to part A of Theorem 32, this implies that any (ν,τ) in the current sub-case is c-τ stable. On the other hand, because (i) $\tau_o(\nu^2) = K_+(\nu^2)$ if $c_2 < \nu^2 < c_3$, and (ii) the relation $K_+(\nu^2) < J_+(\nu^2)$ is a part of Eq. (4.204), one concludes that $\tau > \tau_o(\nu^2)$ and thus $(\nu,\tau) \in \Gamma_o$ if $(\nu,\tau) \in \Psi_-^{(3,4)}$. As such, it has been shown that part F is true over $\Psi_-^{(3,4)}$.

It has been shown that part F is true over each of the four nonempty disjoint sets $\Psi_-^{(3,1)}$, $\Psi_-^{(3,2)}$, $\Psi_-^{(3,3)}$, and $\Psi_-^{(3,4)}$. Eq. (4.205) now implies that part F is true over $\Psi_-^{(3)}$.

9. $(\nu,\tau) \in \Psi_-^{(4)}$. For this case, we have (i) $c_3 \leq \nu^2 < 1$, and (ii) $\nu^2 \leq \tau < \sqrt{\nu^2}$. To proceed, Note that Eqs. (4.165)–(4.168) implies that

$$\nu^2 < K_+(\nu^2) < \sqrt{\nu^2} \leq J_+(\nu^2) \leq I_+(\nu^2) < L_+(\nu^2), \qquad c_3 \leq \nu^2 < 1 \tag{4.210}$$

With the aid of Eq. (4.210), it can be shown that

$$\Psi_-^{(4)} = \Psi_-^{(4,1)} \cup \Psi_-^{(4,2)} \tag{4.211}$$

where $\Psi_-^{(4,1)}$ and $\Psi_-^{(4,2)}$ are nonempty disjoint sets defined by

$$\Psi_-^{(4,1)} \overset{\text{def}}{=} \{(\nu,\tau) | c_3 \leq \nu^2 < 1 \text{ and } \nu^2 \leq \tau < K_+(\nu^2)\} \tag{4.212}$$

and

$$\Psi_-^{(4,2)} \overset{\text{def}}{=} \{(\nu,\tau) | c_3 \leq \nu^2 < 1 \text{ and } K_+(\nu^2) \leq \tau < \sqrt{\nu^2}\} \tag{4.213}$$

Thus any $(\nu,\tau) \in \Psi_-^{(4)}$ must fall into one and only one of the following two sub-cases: (i) $(\nu,\tau) \in \Psi_-^{(4,1)}$ and (ii) $(\nu,\tau) \in \Psi_-^{(4,2)}$.

Let $(\nu,\tau) \in \Psi_-^{(4,1)}$. By using the relation $K_+(\nu^2) < J_+(\nu^2) \leq I_+(\nu^2)$ which follows from Eq. (4.210), it is seen that none of Eqs. (4.131)–(4.133) is true for the current sub-case where $\nu^2 \leq \tau < K_+(\nu^2)$. According to part A of Theorem 32, this implies that any $(\nu,\tau) \in \Psi_-^{(4,1)}$ is c-τ unstable. On the other hand, because $\tau_o(\nu^2) = K_+(\nu^2)$ if $c_3 \leq \nu^2 < 1$, one concludes that $\tau < \tau_o(\nu^2)$ and thus $(\nu,\tau) \notin \Gamma_o$ if $(\nu,\tau) \in \Psi_-^{(4,1)}$. As such it has been shown that part F is true over $\Psi_-^{(4,1)}$.

Let $(\nu,\tau) \in \Psi_-^{(4,2)}$. By using the relation $K_+(\nu^2) < \sqrt{\nu^2} \leq I_+(\nu^2)$ which follows from Eq. (4.210), it is seen that Eq. (4.132) is true for the current sub-case where

$K_+(\nu^2) \le \tau < \sqrt{\nu^2}$. According to part A of Theorem 32, this implies that any $(\nu,\tau) \in \Psi_-^{(4,2)}$ is c-τ stable. On the other hand, because $\tau_o(\nu^2) = K_+(\nu^2)$ if $c_3 \le \nu^2 < 1$, one concludes that $\tau \ge \tau_o(\nu^2)$ and thus $(\nu,\tau) \in \Gamma_o$ if $(\nu,\tau) \in \Psi_-^{(4,2)}$. As such it has been shown that part F is true over $\Psi_-^{(4,2)}$.

It has been shown that part F is true over each of the two nonempty disjoint sets $\Psi_-^{(4,1)}$ and $\Psi_-^{(4,2)}$. Eq. (4.211) now implies that part F is true over $\Psi_-^{(4)}$.

10. $(\nu,\tau) \in \Psi_+^{(1)}$. For this case, we have (i) $0 < \nu^2 \le c_3$, and (ii) $\tau > \sqrt{\nu^2}$. To proceed, Note that Eqs. (4.160)–(4.165) imply that

$$I_+(\nu^2) \le \sqrt{\nu^2}, \qquad 0 < \nu^2 \le c_3 \tag{4.214}$$

$$K_+(\nu^2) < \sqrt{\nu^2}, \qquad 0 < \nu^2 \le c_3 \tag{4.215}$$

and

$$L_-(\nu^2) < \sqrt{\nu^2}, \qquad 0 < \nu^2 \le c_3 \tag{4.216}$$

By using Eqs. (4.214) and (4.215), one concludes that Eq. (4.135) is true for the current case where $\tau > \sqrt{\nu^2}$. According to part B of Theorem 32, this implies that any (ν,τ) in the current case is c-τ stable. On the other hand, because (i) $\tau_o(\nu^2) = L_-(\nu^2)$ if $0 < \nu^2 \le c_2$, and (ii) $\tau_o(\nu^2) = K_+(\nu^2)$ if $c_2 \le \nu^2 \le c_3$, Eqs. (4.215) and (4.216) imply that $\tau > \tau_o(\nu^2)$ and thus $(\nu,\tau) \in \Gamma_o$ if $(\nu,\tau) \in \Psi_+^{(1)}$. As such, it has been shown that part F is true over $\Psi_+^{(1)}$.

11. $(\nu,\tau) \in \Psi_+^{(2)}$. For this case, we have (i) $c_3 < \nu^2 < 1$, and (ii) $\tau > \sqrt{\nu^2}$. To proceed, Note that Eqs. (4.166)–(4.168) imply that

$$K_+(\nu^2) < \sqrt{\nu^2} < J_+(\nu^2) < I_+(\nu^2), \qquad c_3 < \nu^2 < 1 \tag{4.217}$$

and

$$L_-(\nu^2) < J_+(\nu^2) < I_+(\nu^2) < L_+(\nu^2), \qquad c_3 < \nu^2 < 1 \tag{4.218}$$

By using Eq. (4.217), one has

$$\Psi_+^{(2)} = \Psi_+^{(2,1)} \cup \Psi_+^{(2,2)} \cup \Psi_+^{(2,3)} \tag{4.219}$$

where $\Psi_+^{(2,1)}$, $\Psi_+^{(2,2)}$, and $\Psi_+^{(2,3)}$ are nonempty disjoint sets defined by

$$\Psi_+^{(2,1)} \stackrel{\text{def}}{=} \{(\nu,\tau) | c_3 < \nu^2 < 1 \text{ and } \sqrt{\nu^2} < \tau \le J_+(\nu^2)\} \tag{4.220}$$

$$\Psi_+^{(2,2)} \stackrel{\text{def}}{=} \{(\nu,\tau) | c_3 < \nu^2 < 1 \text{ and } J_+(\nu^2) < \tau < I_+(\nu^2)\} \tag{4.221}$$

and

$$\Psi_+^{(2,3)} \stackrel{\text{def}}{=} \{(\nu,\tau) | c_3 < \nu^2 < 1 \text{ and } \tau \ge I_+(\nu^2)\} \tag{4.222}$$

Thus any $(\nu, \tau) \in \Psi_+^{(2)}$ must fall into one and only one of the following three sub-cases: (i) $(\nu, \tau) \in \Psi_+^{(2,1)}$, (ii) $(\nu, \tau) \in \Psi_+^{(2,2)}$, and (iii) $(\nu, \tau) \in \Psi_+^{(2,3)}$.

Let $(\nu, \tau) \in \Psi_+^{(2,1)}$. Eq. (4.134) is true for any (ν, τ) in the current sub-case where $\sqrt{\nu^2} < \tau \leq J_+(\nu^2)$. According to part B of Theorem 32, this implies that the any $(\nu, \tau) \in \Psi_+^{(2,1)}$ is c-τ stable. On the other hand, because (i) $\tau_o(\nu^2) = K_+(\nu^2)$ if $c_3 < \tau < 1$, and (ii) the relation $K_+(\nu^2) < \sqrt{\nu^2}$ is a part of Eq. (4.217), one concludes that $\tau > \tau_o(\nu^2)$ and thus $(\nu, \tau) \in \Gamma_o$ if $(\nu, \tau) \in \Psi_+^{(2,1)}$. As such it has been shown that part F is true over $\Psi_+^{(2,1)}$.

Let $(\nu, \tau) \in \Psi_+^{(2,2)}$. By using Eq. (4.218), one concludes that Eq. (4.136) is true for the current case where $J_+(\nu^2) < \tau < I_+(\nu^2)$. According to part B of Theorem 32, this implies that any $(\nu, \tau) \in \Psi_+^{(2,2)}$ is c-τ stable. On the other hand, because (i) $\tau_o(\nu^2) = K_+(\nu^2)$ if $c_3 < \nu^2 < 1$, and (ii) the relation $K_+(\nu^2) < J_+(\nu^2)$ is a part of Eq. (4.217), one concludes that $\tau > \tau_o(\nu^2)$ and thus $(\nu, \tau) \in \Gamma_o$ if $(\nu, \tau) \in \Psi_+^{(2,2)}$. As such it has been shown that part F is true over $\Psi_+^{(2,2)}$.

Let $(\nu, \tau) \in \Psi_+^{(2,3)}$. By using the relation $K_+(\nu^2) < I_+(\nu^2)$ which follows from Eq. (4.217), one concludes that Eq. (4.135) is true for the current sub-case where $\tau \geq I_+(\nu^2)$. According to part B of Theorem 32, this implies that any $(\nu, \tau) \in \Psi_+^{(2,3)}$ is c-τ stable. On the other hand, because (i) $\tau_o(\nu^2) = K_+(\nu^2)$ if $c_3 < \nu^2 < 1$, and (ii) the relation $K_+(\nu^2) < I_+(\nu^2)$ is a part of Eq. (4.217), one concludes that $\tau > \tau_o(\nu^2)$ and thus $(\nu, \tau) \in \Gamma_o$ if $(\nu, \tau) \in \Psi_+^{(2,3)}$. As such, it has been shown that part F is true over $\Psi_+^{(2,3)}$.

It has been shown that part F is true over each of the three nonempty disjoint sets $\Psi_+^{(2,1)}$, $\Psi_+^{(2,2)}$, and $\Psi_+^{(2,3)}$. Eq. (4.219) now implies that part F is true over $\Psi_+^{(2)}$.

It has been established that part F is true over each of the sets mentioned in the paragraph immediately following Eq. (4.188). Because any (ν, τ) must belong to one and only one of these sets, the proof of part F is completed.

Finally, with the aid of Theorems 4 and 16, one can obtain part G from part F. **QED**.

As promised earlier, a proof for Theorem 34 will be provided in the remainder of the paper. As a preliminary, we have Theorem 36.

Theorem 36. In the domain $0 < x < 1$, (A) $I_+(x)$, $J_+(x)$, $K_+(x)$, and $L_-(x)$ are strictly monotonically increasing while $L_+(x)$ is strictly monotonically decreasing. Moreover, we have (B)

$$3 > I_+(x) > 0, \qquad 0 < x < 1 \tag{4.223}$$

$$3 > J_+(x) > 0, \qquad 0 < x < 1 \tag{4.224}$$

$$1 > K_+(x) > 0, \qquad 0 < x < 1 \tag{4.225}$$

$$3 > L_-(x) > 0, \qquad 0 < x < 1 \tag{4.226}$$

and

$$L_{+}(x) > 3, \qquad\qquad 0 < x < 1 \tag{4.227}$$

Proof. Let $f'(x) \stackrel{\text{def}}{=} df(x)/dx$ for any function f of x. Then (i) Eqs. (4.91) and (4.98) imply that

$$I'_{+}(x) = \frac{3x - 2 + 2\sqrt{3x^2 - 3x + 1}}{x^2\sqrt{3x^2 - 3x + 1}} > 0, \qquad\qquad 0 < x < 1 \tag{4.228}$$

(ii) Eqs. (4.92) and (4.100) imply that

$$\begin{aligned} J'_{+}(x) &= \frac{-x^3 + 6x^2 - x + 2 + 4\sqrt{2(x^3 - x + 2)}}{(2-x)^2\sqrt{2(x^3 - x + 2)}} \\ &= \frac{x^2(1-x) + 5x^2 + 1 + (1-x) + 4\sqrt{2(x^3 - x + 2)}}{(2-x)^2\sqrt{2(x^3 - x + 2)}} > 0, \quad 0 < x < 1 \end{aligned} \tag{4.229}$$

(iii) Eqs. (4.114) and (4.118) imply that

$$K'_{+}(x) = \frac{\sqrt{1 - 2x + 5x^2} - (1-x)}{2x^2\sqrt{1 - 2x + 5x^2}} > 0, \qquad\qquad 0 < x < 1 \tag{4.230}$$

(iv) Eqs. (4.122) and (4.126) imply that

$$L'_{-}(x) = \frac{2\left[4 - x - 2\sqrt{2(2 - x - x^2)}\right]}{x^2\sqrt{2(2 - x - x^2)}} > 0, \qquad\qquad 0 < x < 1 \tag{4.231}$$

and (v) Eqs. (4.122) and (4.126) imply that

$$L'_{+}(x) = -\frac{2\left[4 - x + 2\sqrt{2(2 - x - x^2)}\right]}{x^2\sqrt{2(2 - x - x^2)}} < 0, \qquad\qquad 0 < x < 1 \tag{4.232}$$

Thus part A is true.

Moreover, by using (i) Eqs. (4.91), (4.92), (4.114), and (4.122), and (ii) L'hopital's rule, one has (i)

$$\lim_{x\to 1^-} I_{+}(x) = \lim_{x\to 1^-} J_{+}(x) = \lim_{x\to 1^-} L_{-}(x) = \lim_{x\to 1^-} L_{+}(x) = 3, \text{ and } \lim_{x\to 1^-} K_{+}(x) = 1 \tag{4.233}$$

(ii)

$$\lim_{x\to 0^+} I_{+}(x) = \lim_{x\to 0^+}\left(3 + \frac{6x - 3}{\sqrt{3x^2 - 3x + 1}}\right) = 3 + (-3) = 0 \tag{4.234}$$

(iii)

$$\lim_{x \to 0^+} J_+(x) = 0 \tag{4.235}$$

(iv)

$$\lim_{x \to 0^+} K_+(x) = \lim_{x \to 0^+} \frac{1}{2} \left(1 + \frac{5x - 1}{\sqrt{1 - 2x + 5x^2}} \right) = \frac{1}{2}(1 - 1) = 0 \tag{4.236}$$

and (v)

$$\lim_{x \to 0^+} L_-(x) = \lim_{x \to 0^+} \left[-1 + \frac{2(1 + 2x)}{\sqrt{2(2 - x - x^2)}} \right] = -1 + 1 = 0 \tag{4.237}$$

part B now follows from Part A and Eqs. (4.233)–(4.237). **QED**

An immediate result of Theorem 36 and the fact that $0 < x < \sqrt{x} < 1$ if $0 < x < 1$ is given in Theorem 37.

Theorem 37. We have

$$\begin{aligned} &x < \sqrt{x} < L_+(x), \quad I_+(x) < L_+(x), \quad J_+(x) < L_+(x), \\ &K_+(x) < L_+(x) \quad \text{and} \quad L_-(x) < L_+(x), \qquad 0 < x < 1 \end{aligned} \tag{4.238}$$

Theorem 37 is but one of many algebraic relations that are needed in the proof of Theorem 34. Note that, in establishing other needed relations, several inequalities that involve the four prinicipal square roots that appear in the definitions of $I_\pm(x)$, $J_\pm(x)$, $K_\pm(x)$, and $L_\pm(x)$, i.e.,

$$\sqrt{3x^2 - 3x + 1} > 0, \qquad -\infty < x < +\infty \tag{4.239}$$

$$\sqrt{2(x^3 - x + 2)} > 0, \qquad 0 < x < 2 \tag{4.240}$$

$$\sqrt{1 - 2x + 5x^2} > 0, \qquad -\infty < x < +\infty \tag{4.241}$$

and

$$\sqrt{2(2 - x - x^2)} > 0, \qquad -2 < x < 1 \tag{4.242}$$

(which follow from Eqs. (4.97), (4.100), (4.117), and (4.124), respectively) will be used repeatedly. Also to be used often is the following algebraic property:

Property I. Let $a \geq 0$ and $b \geq 0$. Then

$$a^2 - b^2 \begin{cases} > 0 & \Leftrightarrow \quad a - b > 0 \\ = 0 & \Leftrightarrow \quad a - b = 0 \\ < 0 & \Leftrightarrow \quad a - b < 0 \end{cases} \tag{4.243}$$

With the above preparations, a set of relations will be given in Theorems 38–48.

Theorem 38. We have

$$x - I_+(x) \begin{cases} > 0 & \text{if } \ 0 < x < 3 - 2\sqrt{2} \\ = 0 & \text{if } \ x = 3 - 2\sqrt{2} \\ < 0 & \text{if } \ 3 - 2\sqrt{2} < x < 1 \end{cases} \tag{4.244}$$

Proof. Let $0 < x < 1$ throughout the proof. Then Eq. (4.91) implies that

$$x - I_+(x) = \frac{x^2 - 3x + 2 - 2\sqrt{3x^2 - 3x + 1}}{x} \tag{4.245}$$

With the aid of Property I, Eq. (4.244) is a result of Eq. (4.245) and the following relations: (i) Eq. (4.239); (ii)

$$x^2 - 3x + 2 = (x-1)(x-2) > 0 \tag{4.246}$$

(iii)

$$\begin{aligned} &(x^2 - 3x + 2)^2 - \left(2\sqrt{3x^2 - 3x + 1}\right)^2 = x^2(x^2 - 6x + 1) \\ &= x^2 \left[x - (3 + 2\sqrt{2})\right] \left[x - (3 - 2\sqrt{2})\right] \end{aligned} \tag{4.247}$$

and (iv) $0 < 3 - 2\sqrt{2} < 1 < 3 + 2\sqrt{2}$. **QED**.

Theorem 39. We have

$$x < K_+(x), \qquad\qquad 0 < x < 1 \tag{4.248}$$

Proof. Let $0 < x < 1$ throughout the proof. Then Eq. (4.114) implies that

$$K_+(x) - x = \frac{\sqrt{1 - 2x + 5x^2} - (2x^2 - x + 1)}{2x} \tag{4.249}$$

With the aid of Property I, Eq. (4.248) is a result of Eq. (4.249) and the following relations: (i) Eq. (4.241); (ii)

$$2x^2 - x + 1 = 2(x - 1/4)^2 + 7/8 \geq 7/8 \tag{4.250}$$

and (iii)

$$\left(\sqrt{1 - 2x + 5x^2}\right)^2 - (2x^2 - x + 1)^2 = 4x^3(1 - x) > 0 \tag{4.251}$$

QED.

Theorem 40. Let c_3 be the constant defined in Eq. (4.157). Then

$$\sqrt{x} - I_+(x) \begin{cases} > 0 & \text{if } \ 0 < x < c_3 \\ = 0 & \text{if } \ x = c_3 \\ < 0 & \text{if } \ c_3 < x < 1 \end{cases} \tag{4.252}$$

Proof. Let $0 < x < 1$ throughout the proof. Then Eq. (4.91) implies that

$$\sqrt{x} - I_+(x) = \frac{x\sqrt{x} - 3x + 2 - 2\sqrt{3x^2 - 3x + 1}}{x} \tag{4.253}$$

With the aid of Property I, Eq. (4.252) is a result of Eq. (4.253) and the following relations: (i) Eq. (4.239); (ii)

$$x\sqrt{x} - 3x + 2 = \left(1 - \sqrt{x}\right)\left[1 + 2\sqrt{x} + (1 - x)\right] > 0 \tag{4.254}$$

(iii)

$$\begin{aligned} &\left(x\sqrt{x} - 3x + 2\right)^2 - \left(2\sqrt{3x^2 - 3x + 1}\right)^2 = x^3 - 6x^{5/2} - 3x^2 + 4x^{3/2} \\ &= x^{3/2}\left(\sqrt{x} + 1\right)\left(\sqrt{x} - \frac{7 + \sqrt{33}}{2}\right)\left(\sqrt{x} - \frac{7 - \sqrt{33}}{2}\right) \end{aligned} \tag{4.255}$$

(iv) $0 < \left(7 - \sqrt{33}\right)/2 < 1 < \left(7 + \sqrt{33}\right)/2$; and (v) $c_3 = \left[\left(7 - \sqrt{33}\right)/2\right]^2$. **QED**.

Theorem 41. Let c_3 be the constant defined in Eq. (4.157). Then

$$\sqrt{x} - J_+(x) \begin{cases} > 0 & \text{if } \ 0 < x < c_3 \\ = 0 & \text{if } \ x = c_3 \\ < 0 & \text{if } \ c_3 < x < 1 \end{cases} \tag{4.256}$$

Proof. Let $0 < x < 1$ throughout the proof. Then Eq. (4.92) implies that

$$\sqrt{x} - J_+(x) = \frac{-x\sqrt{x} - 3x + 2\sqrt{x} + 2 - \sqrt{2(x^3 - x + 2)}}{2 - x} \tag{4.257}$$

With the aid of Property I, Eq. (4.256) is a result of Eq. (4.257) and the following relations: (i) Eq. (4.240); (ii)

$$-x\sqrt{x} - 3x + 2\sqrt{x} + 2 = \left(1 - \sqrt{x}\right)\left(x + 4\sqrt{x} + 2\right) > 0 \tag{4.258}$$

(iii)

$$
\begin{aligned}
&\left(-x\sqrt{x} - 3x + 2\sqrt{x} + 2\right)^2 - \left[\sqrt{2(x^3 - x + 2)}\right]^2 \\
&= -x^3 + 6x^{5/2} + 5x^2 - 16x^{3/2} - 6x + 8\sqrt{x} \qquad (4.259) \\
&= \sqrt{x}(2 - x)\left(\sqrt{x} + 1\right)\left(\sqrt{x} - \frac{7 + \sqrt{33}}{2}\right)\left(\sqrt{x} - \frac{7 - \sqrt{33}}{2}\right)
\end{aligned}
$$

(iv) $0 < \left(7 - \sqrt{33}\right)/2 < 1 < \left(7 + \sqrt{33}\right)/2$; and (v) $c_3 = \left[\left(7 - \sqrt{33}\right)/2\right]^2$. **QED**.

Theorem 42. We have

$$
K_+(x) < \sqrt{x}, \qquad 0 < x < 1 \tag{4.260}
$$

Proof. Let $0 < x < 1$ throughout the proof. Then Eq. (4.114) implies that

$$
\sqrt{x} - K_+(x) = \frac{2x\sqrt{x} - x + 1 - \sqrt{1 - 2x + 5x^2}}{2x} \tag{4.261}
$$

With the aid of Property I, Eq. (4.260) is a result of Eq. (4.261) and the following relations: (i) Eq. (4.241); (ii)

$$
2x\sqrt{x} - x + 1 = 2x\sqrt{x} + (1 - x) > 0 \tag{4.262}
$$

and (iii)

$$
\left(2x\sqrt{x} - x + 1\right)^2 - \left(\sqrt{1 - 2x + 5x^2}\right)^2 = 4x\sqrt{x}(1 - x)\left(1 - \sqrt{x}\right) > 0 \tag{4.263}
$$

QED.

Theorem 43. Let c_4 be the constant defined in Eq. (4.158). Then

$$
\sqrt{x} - L_-(x) \begin{cases} > 0 & \text{if } 0 < x < c_4 \\ = 0 & \text{if } x = c_4 \\ < 0 & \text{if } c_4 < x < 1 \end{cases} \tag{4.264}
$$

Proof. Unless specified otherwise. Let $0 < x < 1$ in this proof. Then Eq. (4.122) implies that

$$
\sqrt{x} - L_-(x) = \frac{2\sqrt{2(2 - x - x^2)} - (4 - x - x\sqrt{x})}{x} \tag{4.265}
$$

To proceed, note that

$$\left[2\sqrt{2(2-x-x^2)}\right]^2 - \left(4 - x - x\sqrt{x}\right)^2 = -x\sqrt{x}\, g(x) \tag{4.266}$$

where

$$g(x) \stackrel{\text{def}}{=} x\sqrt{x} + 2x + 9\sqrt{x} - 8, \qquad x \geq 0 \tag{4.267}$$

Because (i)

$$g'(x) = 3\sqrt{x}/2 + 2 + 9/(2\sqrt{x}) = 3/(2\sqrt{x})\left[(\sqrt{x}+2/3)^2 + 23/9\right] > 0, \qquad x > 0 \tag{4.268}$$

and (ii)

$$g(0) = -8 \quad \text{and} \quad g(1) = 4 \tag{4.269}$$

one concludes that $g(x)$ is strictly monotonically increasing in the interval $0 < x < 1$ and there is one and only one real root of $g(x) = 0$ in this interval. By using the standard formula for the roots of a cubic equation, it can be shown that this root is given by $x = c_4$. Moreover, Eqs. (4.268) and (4.269) imply that: (i) $g(x) < 0$ if $0 < x < c_4$; (ii) $g(x) = 0$ if $x = c_4$; and (iii) $g(x) > 0$ if $c_4 < x < 1$. As such Eq. (4.266) implies that

$$\left[2\sqrt{2(2-x-x^2)}\right]^2 - \left(4 - x - x\sqrt{x}\right)^2 \begin{cases} > 0 & \text{if } \ 0 < x < c_4 \\ = 0 & \text{if } \ x = c_4 \\ < 0 & \text{if } \ c_4 < x < 1 \end{cases} \tag{4.270}$$

With the aid of Property I, Eq. (4.264) is a result of Eqs. (4.265) and (4.270), and the the following relations: (i) Eq. (4.242); and (ii)

$$4 - x - x\sqrt{x} = 2 + (1-x) + \left(1 - x\sqrt{x}\right) > 2, \qquad 0 < x < 1 \tag{4.271}$$

QED.

Theorem 44. We have

$$L_-(x) < J_+(x), \qquad 0 < x < 1 \tag{4.272}$$

Proof. Let $0 < x < 1$ throughout this proof. Then Eqs. (4.92) and (4.122) imply that

$$J_+(x) - L_-(x) = \frac{x\sqrt{2(x^3 - x + 2)} + 2(2-x)\sqrt{2(2-x-x^2)} - (8 - 4x - 2x^2)}{x(2-x)} \tag{4.273}$$

Let

$$\beta(x) \stackrel{\text{def}}{=} x\sqrt{2(x^3 - x + 2)} + 2(2-x)\sqrt{2(2-x-x^2)} + (8 - 4x - 2x^2) \tag{4.274}$$

and

$$\beta_\pm(x) \stackrel{\text{def}}{=} 4\sqrt{(x^3 - x + 2)(2 + x)} \pm \sqrt{1-x}\,(8 + 3x - x^2) \tag{4.275}$$

Then (i)

$$\begin{aligned}
&\big[J_+(x) - L_-(x)\big]\beta(x) \\
&= \frac{8x(2-x)\sqrt{(x^3 - x + 2)(1-x)(2+x)} + 2x^5 - 12x^4 + 6x^3 + 36x^2 - 32x}{x(2-x)} \\
&= \frac{8x(2-x)\sqrt{(x^3 - x + 2)(1-x)(2+x)} - 2x(1-x)(2-x)(8 + 3x - x^2)}{x(2-x)} \\
&= 2\sqrt{1-x}\,\beta_-(x)
\end{aligned} \tag{4.276}$$

and (ii)

$$\beta_-(x)\beta_+(x) = x^5 + 9x^4 + 31x^3 + 39x^2 + 16x \tag{4.277}$$

Thus

$$\big[J_+(x) - L_-(x)\big]\beta(x)\beta_+(x) = 2\sqrt{1-x}(x^5 + 9x^4 + 31x^3 + 39x^2 + 16x) \tag{4.278}$$

By using (i) Eqs. (4.240) and (4.242); (ii) $\sqrt{(x^3 - x + 2)(2 + x)} > 0$; (iii)

$$8 - 4x - 2x^2 = 2 + 4(1-x) + 2(1-x^2) > 2 \tag{4.279}$$

and (iv)

$$8 + 3x - x^2 = 7 + 3x + (1 - x^2) > 7 \tag{4.280}$$

it follows from Eqs. (4.274) and (4.275) that

$$\beta(x) > 0 \quad \text{and} \quad \beta_+(x) > 0, \qquad\qquad 0 < x < 1 \tag{4.281}$$

Eq. (4.272) is a result of Eq. (4.281) and the fact that the expression on the right side of Eq. (4.278) is positive everywhere in the interval $0 < x < 1$. **QED**

Theorem 45. We have

$$K_+(x) - I_+(x) \begin{cases} > 0 & \text{if } \ 0 < x < 3/11 \\ = 0 & \text{if } \ x = 3/11 \\ < 0 & \text{if } \ 3/11 < x < 1 \end{cases} \tag{4.282}$$

Proof. Eqs. (4.91) and (4.114) imply that

$$K_+(x) - I_+(x) = \frac{(3-5x) - \left(4\sqrt{3x^2-3x+1} - \sqrt{1-2x+5x^2}\right)}{2x}, \qquad 0 < x < 1 \quad (4.283)$$

By using (i) Eqs. (4.239) and (4.241), and (ii)

$$\begin{aligned} &\left(4\sqrt{3x^2-3x+1}\right)^2 - \left(\sqrt{1-2x+5x^2}\right)^2 = 43x^2 - 46x + 15 \\ &= 43\left[(x-23/43)^2 + 116/(43)^2\right] \geq 116/43, \qquad -\infty < x < \infty \end{aligned} \quad (4.284)$$

an application of Property I leads to the conclusion

$$4\sqrt{3x^2-3x+1} - \sqrt{1-2x+5x^2} > 0, \qquad -\infty < x < +\infty \quad (4.285)$$

Moreover, we have

$$3 - 5x \begin{cases} \leq 0 & \text{if } \ x \geq 3/5 \\ > 0 & \text{if } \ x < 3/5 \end{cases} \quad (4.286)$$

Combining Eqs. (4.283), (4.285) and (4.286), one has

$$K_+(x) - I_+(x) < 0, \qquad 3/5 \leq x < 1 \quad (4.287)$$

To proceed, let

$$\xi(x) \stackrel{\text{def}}{=} \frac{1}{2}\left(3 - 5x + 4\sqrt{3x^2-3x+1} - \sqrt{1-2x+5x^2}\right), \qquad 0 < x < 1 \quad (4.288)$$

and

$$\xi_\pm(x) \stackrel{\text{def}}{=} 2\sqrt{(3x^2-3x+1)(1-2x+5x^2)} \pm (7x^2 - 5x + 2), \qquad 0 < x < 1 \quad (4.289)$$

Then Eqs. (4.285) and (4.286) imply that

$$\xi(x) > 0, \qquad 0 < x < 3/5 \quad (4.290)$$

In addition, by using (i) $\sqrt{(3x^2-3x+1)(1-2x+5x^2)} > 0$, $-\infty < x < +\infty$ (which follows from Eqs. (4.239) and (4.241)); and (ii)

$$7x^2 - 5x + 2 = 7\left[(x-5/14)^2 + 31/196\right] \geq 31/28, \qquad -\infty < x < +\infty \quad (4.291)$$

one has

$$\xi_+(x) > 0, \qquad 0 < x < 1 \quad (4.292)$$

Combining Eqs. (4.290) and (4.292), one arrives at the conclusion:

$$\xi(x)\,\xi_+(x) > 0, \qquad 0 < x < 3/5 \tag{4.293}$$

Next, Eqs. (4.283), (4.288), and (4.289) imply that (i)

$$[K_+(x) - I_+(x)]\,\xi(x) = \frac{\xi_-(x)}{x}, \qquad 0 < x < 1 \tag{4.294}$$

and (ii)

$$\xi_-(x)\,\xi_+(x) = 11x^4 - 14x^3 + 3x^2 = 11x^2(x-1)(x-3/11), \quad 0 < x < 1 \tag{4.295}$$

Thus

$$[K_+(x) - I_+(x)]\,\xi(x)\xi_+(x) = 11x(x-1)(x-3/11), \qquad 0 < x < 1 \tag{4.296}$$

It follows from Eqs. (4.293) and (4.296) that

$$K_+(x) - I_+(x) \begin{cases} > 0 & \text{if } \ 0 < x < 3/11 \\ = 0 & \text{if } \ x = 3/11 \\ < 0 & \text{if } \ 3/11 < x < 3/5 \end{cases} \tag{4.297}$$

Eq. (4.282) is an immediate result of Eqs. (4.287) and (4.297). **QED**.

Theorem 46. We have

$$L_-(x) - I_+(x) \begin{cases} > 0 & \text{if } \ 0 < x < 3/11 \\ = 0 & \text{if } \ x = 3/11 \\ < 0 & \text{if } \ 3/11 < x < 1 \end{cases} \tag{4.298}$$

Proof. Eqs. (4.91) and (4.122) imply that

$$L_-(x) - I_+(x) = \frac{2}{x}\left(3 - 2x - \sqrt{2(2-x-x^2)} - \sqrt{3x^2-3x+1}\right), \quad 0 < x < 1 \tag{4.299}$$

By using Eq. (4.299) and the definitions

$$\mu(x) \stackrel{\text{def}}{=} \frac{1}{2}\left(3 - 2x + \sqrt{2(2-x-x^2)} + \sqrt{3x^2-3x+1}\right), \qquad 0 < x < 1 \tag{4.300}$$

and

$$\mu_{\pm}(x) \stackrel{\text{def}}{=} 3x^2 - 7x + 4 \pm 2\sqrt{2(2-x-x^2)(3x^2-3x+1)}, \qquad 0 < x < 1 \tag{4.301}$$

one has

$$[L_-(x) - I_+(x)]\,\mu(x) = \frac{\mu_-(x)}{x}, \qquad 0 < x < 1 \tag{4.302}$$

and

$$\mu_-(x)\,\mu_+(x) = 33x^4 - 42x^3 + 9x^2 = 33x^2(x-1)(x-3/11), \quad 0 < x < 1 \tag{4.303}$$

In turn, Eqs. (4.302) and (4.303) imply that

$$[L_-(x) - I_+(x)]\,\mu(x)\,\mu_+(x) = 33x(x-1)(x-3/11), \qquad 0 < x < 1 \tag{4.304}$$

By using (i) Eqs. (4.239) and (4.242); (ii) $3 - 2x > 0$ if $x < 3/2$; and (iii)

$$3x^2 - 7x + 4 = 3(x-1)(x-4/3) > 0, \qquad x < 1 \quad \text{or} \quad x > 4/3 \tag{4.305}$$

Eqs. (4.300) and (4.301) imply that

$$\mu(x) > 0 \quad \text{and} \quad \mu_+(x) > 0, \qquad 0 < x < 1 \tag{4.306}$$

Eq. (4.298) is an immediate result of Eqs. (4.304) and (4.306). **QED**.

Theorem 47. We have

$$L_-(x) - K_+(x) \begin{cases} > 0 & \text{if } \ 0 < x < 3/11 \text{ or } 3/11 < x < 1 \\ = 0 & \text{if } \ x = 3/11 \end{cases} \tag{4.307}$$

Proof. Eqs. (4.114) and (4.122) imply that

$$L_-(x) - K_+(x) = \frac{9 - 3x - 4\sqrt{2(2-x-x^2)} - \sqrt{1-2x+5x^2}}{2x}, \qquad 0 < x < 1 \tag{4.308}$$

By using Eq. (4.308) and the definitions

$$\psi(x) \stackrel{\text{def}}{=} \frac{1}{2}\left(9 - 3x + 4\sqrt{2(2-x-x^2)} + \sqrt{1-2x+5x^2}\right), \qquad 0 < x < 1 \tag{4.309}$$

and

$$\psi_{\pm}(x) \stackrel{\text{def}}{=} 9x^2 - 5x + 4 \pm 2\sqrt{2(2-x-x^2)(1-2x+5x^2)}, \qquad 0 < x < 1 \tag{4.310}$$

one has

$$[L_-(x) - K_+(x)]\,\psi(x) = \frac{\psi_-(x)}{x}, \qquad 0 < x < 1 \tag{4.311}$$

and

$$\psi_-(x)\,\psi_+(x) = 121x^4 - 66x^3 + 9x^2 = 121x^2(x - 3/11)^2, \quad 0 < x < 1 \tag{4.312}$$

In turn, Eqs. (4.311) and (4.312) imply that

$$[L_-(x) - K_+(x)]\,\psi(x)\,\psi_+(x) = 121x(x - 3/11)^2, \qquad 0 < x < 1 \tag{4.313}$$

By using (i) Eqs. (4.241) and (4.242); (ii) $9 - 3x > 0$ if $x < 3$; and (iii)

$$9x^2 - 5x + 4 = 9\left[(x - 5/18)^2 + 119/324\right] \geq 119/36, \qquad -\infty < x < +\infty \tag{4.314}$$

Eqs. (4.309) and (4.310) imply that

$$\psi(x) > 0 \quad \text{and} \quad \psi_+(x) > 0, \qquad 0 < x < 1 \tag{4.315}$$

Eq. (4.307) is an immediate result of Eqs. (4.313) and (4.315). **QED**.

Theorem 48. Let c_3 be the constant defined in Eq. (4.157). Then we have

$$J_+(x) - I_+(x) \begin{cases} > 0 & \text{if } \ 0 < x < c_3 \\ = 0 & \text{if } \ x = c_3 \\ < 0 & \text{if } \ c_3 < x < 1 \end{cases} \tag{4.316}$$

Proof. Eqs. (4.91) and (4.92) imply that

$$J_+(x) - I_+(x) = \frac{6x^2 - 10x + 4 - \left[2(2-x)\sqrt{3x^2 - 3x + 1} - x\sqrt{2(x^3 - x + 2)}\right]}{x(2-x)}, \quad 0 < x < 1 \tag{4.317}$$

To proceed, note that Eq. (4.239) implies that

$$2(2 - x)\sqrt{3x^2 - 3x + 1} > 0, \qquad x < 2 \tag{4.318}$$

Also Eq. (4.240) implies that

$$x\sqrt{2(x^3 - x + 2)} > 0, \qquad 0 < x < 2 \tag{4.319}$$

Moreover, we have

$$\begin{aligned}
&\left[2(2-x)\sqrt{3x^2-3x+1}\right]^2 - \left[x\sqrt{2(x^3-x+2)}\right]^2 \\
&= -2x^5+12x^4-58x^3+96x^2-64x+16 \\
&= 2(1-x)(x^4-5x^3+24x^2-24x+8) \\
&= 2(1-x)\left[x^2(x^2-5x+6)+2(9x^2-12x+4)\right] \\
&= 2(1-x)\left[x^2(x-2)(x-3)+2(3x-2)^2\right] > 0, \qquad 0<x<1
\end{aligned} \tag{4.320}$$

With the aid of Eqs. (4.318)–(4.320), an application of Property I leads to the conclusion

$$2(2-x)\sqrt{3x^2-3x+1} - x\sqrt{2(x^3-x+1)} > 0, \qquad 0<x<1 \tag{4.321}$$

Next note that

$$x(2-x) > 0, \qquad 0<x<2 \tag{4.322}$$

and

$$6x^2-10x+4 = 6(x-1)(x-2/3)\begin{cases} \le 0 & \text{if } \ 2/3 \le x \le 1 \\ > 0 & \text{if } \ x<2/3 \text{ or } x>1 \end{cases} \tag{4.323}$$

By combining Eq. (4.317) with Eqs. (4.321)–(4.323), one concludes that

$$J_+(x) - I_+(x) < 0, \qquad 2/3 \le x < 1 \tag{4.324}$$

To study the case where $0<x<2/3$, let

$$\eta(x) \stackrel{\text{def}}{=} \frac{1}{2}\left[6x^2-10x+4+2(2-x)\sqrt{3x^2-3x+1} - x\sqrt{2(x^3-x+2)}\right], \ 0<x<1 \tag{4.325}$$

and

$$\eta_\pm(x) \stackrel{\text{def}}{=} 2\sqrt{2}\sqrt{(3x^2-3x+1)(x^3-x+2)} \pm (-x^3+10x^2-9x+4), \ 0<x<1 \tag{4.326}$$

By using Eqs. (4.321) and (4.323), Eq. (4.325) implies that

$$\eta(x) > 0, \qquad 0<x<2/3 \tag{4.327}$$

Moreover, because (i) $\sqrt{(3x^2-3x+1)(x^3-x+2)} > 0$, $0<x<2$ (see Eqs. (4.239) and (4.240)); and (ii)

$$-x^3+10x^2-9x+4 = x^2(1-x)+(3x-3/2)^2+7/4 > 7/4, \qquad 0<x<1 \tag{4.328}$$

Eq. (4.326) implies that

$$\eta_+(x) > 0, \qquad 0 < x < 1 \tag{4.329}$$

Next, by using Eqs. (4.157), (4.317), (4.325) and (4.326), it can be shown that (i)

$$\begin{aligned}
&\big[J_+(x) - I_+(x)\big]\eta(x) \\
&= \frac{2\sqrt{2}\,x(2-x)\sqrt{(3x^2-3x+1)(x^3-x+2)} - (x^5 - 12x^4 + 29x^3 - 22x^2 + 8x)}{x(2-x)} \\
&= \frac{2\sqrt{2}\,x(2-x)\sqrt{(3x^2-3x+1)(x^3-x+2)} - x(2-x)(-x^3 + 10x^2 - 9x + 4)}{x(2-x)} \\
&= \eta_-(x), \qquad 0 < x < 1
\end{aligned} \tag{4.330}$$

and (ii)

$$\begin{aligned}
\eta_-(x)\eta_+(x) &= -x^6 + 44x^5 - 142x^4 + 172x^3 - 89x^2 + 16x = x(1-x)^3(x^2 - 41x + 16) \\
&= x(1-x)^3(x - c_3)\left(x - \frac{41 + 7\sqrt{33}}{2}\right), \qquad 0 < x < 1
\end{aligned} \tag{4.331}$$

In turn, Eqs. (4.330) and (4.331) imply that

$$[J_+(x) - I_+(x)]\,\eta(x)\,\eta_+(x) = x(1-x)^3(x - c_3)\left(x - \frac{41 + 7\sqrt{33}}{2}\right), \quad 0 < x < 1 \tag{4.332}$$

With the aid of Eqs. (4.327), (4.329) and (4.332), and the relation

$$0 < c_3 < 2/3 < 1 < \frac{41 + 7\sqrt{33}}{2} \tag{4.333}$$

one concludes that

$$J_+(x) - I_+(x) \begin{cases} > 0 & \text{if } \ 0 < x < c_3 \\ = 0 & \text{if } \ x = c_3 \\ < 0 & \text{if } \ c_3 < x < 2/3 \end{cases} \tag{4.334}$$

Eq. (4.316) now is an immediate result of Eqs. (4.324) and (4.334). **QED**.

With the above preparations, Theorem 34 can now be proved. Part A is identical to part A of Theorem 36. Part B can be shown using Theorems 38–48 and the two relations

$$\sqrt{x} < L_+(x) \quad \text{and} \quad I_+(x) < L_+(x), \qquad 0 < x < 1 \tag{4.335}$$

which form a part of Theorem 37. Part C follows from Eqs. (4.230), (4.231), and (4.156). Part D was shown in Eqs. (4.233) and (4.237). **QED**.

Finally, note that none of the relations

$$x < J_+(x), \quad x < L_-(x), \quad \text{and} \quad K_+(x) < J_+(x), \qquad 0 < x < 1 \tag{4.336}$$

appears in Theorems 37–48. However, they can be shown using Theorem 34. As such, they can be considered as results of Theorems 38–48 and the relations Eq. (4.335).

5. Conclusions and Discussions

With the aid of many unexpected mathematical simplifications that occur along the way, it has been shown in Sec. 4 that there is an explicit analytical solution to the implicit stability conditions stated in Theorem 3. The first and perhaps the most important "break" encountered is the simple relation Eq. (4.23), i.e., $H(\nu, \tau, s)$, a quartic polynomial in s, is equal to the product of $4(1-\nu^2)s^2$ and $G(\nu, \tau, s)$, a quadratic polynomial in s. Without Eq. (4.23) and the fortunate fact that both $D(\nu, \tau, s)$ and $F(\nu, \tau, s)$ are also quadratic polynomials in s, the relatively straightforward study of the necessary stability conditions Eqs. (4.25)–(4.27) (Theorem 6) as presented in Sec. 4 would have become much more complicated.

Moreover, the fact that $F(\nu, \tau, 1)$ and $H(\nu, \tau, 1)$ can be cast into the simple factorized forms Eqs. (4.35) and (4.37), respectively, are instrumental in the successful effort to establish Eq. (4.41) as necessary conditions for stability (Theorem 12).

With the aid of Theorems 13–15, it was shown that the special case in which (ν, τ) satisfies Eq. (4.2) and yet is c-τ unstable occurs if and only if $\tau = \nu^2 = 1$ (Theorem 16). Using Theorem 16, Theorem 17 was then established to provide a set of necessary and sufficient stability conditions much more explicit and easier to handle than those given originally in Theorem 3. Based on Theorem 17, it was then shown that the c-τ scheme is stable if (a) $\nu = 0$ and $\tau \geq 0$; or (b) $\nu^2 = 1$ and $\tau > 1$; or (c) $0 < \nu^2 < 1$ and $\tau = |\nu|$ (Theorem 18).

Excluding the four special cases addressed in Theorems 16 and 18, the set Ψ defined in Eq. (4.66) is the set of all other (ν, τ) that satisfy the necessary stability conditions $\tau \geq \nu^2$ and $\nu^2 \leq 1$ (Theorem 19). To facilitate the development, Ψ is divided into two disjoint subsets Ψ_- and Ψ_+, which are defined in Eq. (4.66) and (4.67).

It turns out that Eqs. (4.25) and (4.27) are satisfied by all $(\nu, \tau) \in \Psi$ (Theorems 21 and 22). Thus, according to Theorem 17, a given $(\nu, \tau) \in \Psi$ is c-τ stable if and only if it satisfies Eq. (4.26). As such, one arrives at the conclusion that a given $(\nu, \tau) \in \Psi$ is c-τ stable if and only if it satisfies Eq. (4.84) (Theorem 23). This necessary and sufficient stability condition obviously is even simpler than those given in Theorem 17.

With the aid of Theorems 24–31, for the set Ψ, we are able to obtain the explicit solution to the necessary and sufficient stability condition Eq. (4.84) in the form given in Theorem 32. The functions $I_+(x)$, $J_+(x)$, $K_+(x)$, $L_+(x)$, and $L_-(x)$, $0 < x < 1$, that appear in Theorem 32 are defined in Eqs. (4.91), (4.92), (4.114), and (4.122).

In principle, whether a given (ν, τ) is c-τ stable can be determined by using Theorems 12, 16, 18, 19, and 32. However, by using the alternative definitions of Ψ_- and Ψ_+ given in Theorem 33, and the ordering properties Eqs. (4.160)–(4.168) given in Theorem 34, it was shown that Theorems 12, 16, 18, 19, and 32 can be combined and turned into the simple explicit form of necessary and sufficient stability conditions given in Theorem 35.

Finally note that the proof of the ordering properties Eqs. (4.160)–(4.168) is hinged on the rather incredible facts that the 4–6th order polynomials in x or $\sqrt{x}$ that appear in Eqs. (4.247), (4.251), (4.255), (4.259), (4.263), (4.295), (4.303), (4.312), and (4.331) all can be factorized and studied analytically.

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Appendix A. Numerical Validation of Theorem 34

```
      implicit real*8(a-h,o-z)
c
c     Program "ineqs.for".
c
c     This program is used to verify numerically the inequalities
c     Equations (4.160)--(4.168) (see Theorem 34).
c
c *** The functions I-plus, K-plus, L-minus, and L-plus are undefined
c *** at x=0.d0. Thus, instead of being evaluated at x=0.d0, these
c *** functions will be evaluated at x=ep where ep is a very small
c *** positive number.
c
c *** At x=1.d0, 2.d0*(2.d0-x-x**2)=0.d0. Because of round-off errors,
c *** the value of this expression may become negative when x is very
c *** close to 1.d0. As such the square root of this expression and
c *** therefore the functions L-minus and L-plus may be undefined
c *** computationally when x is too close to 1.d0. Thus, instead of
c *** being evaluated at x=1.d0, the functions will be evaluated at
c *** x=1.d0-eq where eq is a very small positive number.
c
c *** n1 = number of uniform sub-intervals in (0,c1).
c *** n2 = number of uniform sub-intervals in (c1,c2).
c *** n3 = number of uniform sub-intervals in (c2,c3).
c *** n4 = number of uniform sub-intervals in (c3,c4).
c *** n5 = number of uniform sub-intervals in (c4,1).
c
      srt(x)=dsqrt(x)
      fip(x)=(3.d0*x-2.d0+2.d0*dsqrt(3.d0*x**2-3.d0*x+1.d0))/x
      fjp(x)=(3.d0*x-2.d0+dsqrt(2.d0*(x**3-x+2.d0)))/(2.d0-x)
      fkp(x)=(x-1.d0+dsqrt(5.d0*x**2-2.d0*x+1.d0))/(2.d0*x)
      flm(x)=(4.d0-x-2.d0*dsqrt(2.d0*(2.d0-x-x**2)))/x
      flp(x)=(4.d0-x+2.d0*dsqrt(2.d0*(2.d0-x-x**2)))/x
c
      n1=17
      n2=10
      n3=12
      n4=14
      n5=47
      ep=1.d-7
      eq=1.d-12
      one=1.d0-eq
      n5m=n5-1
c
      c1=3.d0-2.d0*dsqrt(2.d0)
      c2=3.d0/11.d0
      c3=(41.d0-7.d0*dsqrt(33.d0))/2.d0
      c4=(dexp((1.d0/3.d0)*dlog(dsqrt(1664.d0/27.d0)+181.d0/27.d0))
     *   -dexp((1.d0/3.d0)*dlog(dsqrt(1664.d0/27.d0)-181.d0/27.d0))
     *   -2.d0/3.d0)**2
c
      open (unit=8,file='ineqs.txt')
      write (8,1)
      write (8,2)
      write (8,3) n1,n2,n3,n4,n5
      write (8,4) ep,eq
      write (8,2)
      write (8,10) c1,c2,c3,c4
      write (8,2)
c
      dx1=c1/dfloat(n1)
```

```
      dx2=(c2-c1)/dfloat(n2)
      dx3=(c3-c2)/dfloat(n3)
      dx4=(c4-c3)/dfloat(n4)
      dx5=(1.d0-c4)/dfloat(n5)
c
      write (8,20) ep
      write (8,30) fip(ep),ep,fkp(ep),flm(ep)
      write (8,40) fjp(ep),srt(ep),flp(ep)
      write (8,2)
      x=0.d0
      do 100 i=1,n1
      x=x+dx1
      write (8,20) x
      write (8,30) fip(x),x,fkp(x),flm(x)
      write (8,40) fjp(x),srt(x),flp(x)
100   continue
      write (8,2)
      x=c1
      do 200 i=1,n2
      x=x+dx2
      write (8,20) x
      write (8,50) x,fip(x),fkp(x),flm(x)
      write (8,40) fjp(x),srt(x),flp(x)
200   continue
      write (8,2)
      x=c2
      do 300 i=1,n3
      x=x+dx3
      write (8,20) x
      write (8,60) x,fkp(x),flm(x),fip(x)
      write (8,40) fjp(x),srt(x),flp(x)
300   continue
      write (8,2)
      x=c3
      do 400 i=1,n4
      x=x+dx4
      write (8,20) x
      write (8,70) x,fkp(x),flm(x),srt(x)
      write (8,80) fjp(x),fip(x),flp(x)
400   continue
      write (8,2)
      x=c4
      do 500 i=1,n5m
      x=x+dx5
      write (8,20) x
      write (8,90) x,fkp(x),srt(x),flm(x)
      write (8,80) fjp(x),fip(x),flp(x)
500   continue
      write (8,2)
      write (8,20) one
      write (8,90) one,fkp(one),srt(one),flm(one)
      write (8,80) fjp(one),fip(one),flp(one)
      close (unit=8)
1     format (' ***** The output for the code "ineqs.for". ***********')
2     format (' ***************************************************')
3     format (' n1 =',i3,' n2 =',i3,' n3 =',i3,' n4 =',i3,' n5 =',i3)
4     format (' ep =',g14.7,' eq =',g14.7)
10    format (' c1 =',g14.7,' c2 =',g14.7,' c3 =',g14.7,' c4 =',g14.7)
20    format (' x =',g14.7)
30    format (' fip =',g14.7,' x =',g14.7,' fkp =',g14.7,' flm ='g14.7)
```

```
40    format (' fjp =',g14.7,' srt =',g14.7,' flp =',g14.7)
50    format (' x =',g14.7,' fip =',g14.7,' fkp =',g14.7,' flm ='g14.7)
60    format (' x =',g14.7,' fkp =',g14.7,' flm =',g14.7,' fip ='g14.7)
70    format (' x =',g14.7,' fkp =',g14.7,' flm =',g14.7,' srt ='g14.7)
80    format (' fjp =',g14.7,' fip =',g14.7,' flp ='g14.7)
90    format (' x =',g14.7,' fkp =',g14.7,' srt =',g14.7,' flm ='g14.7)
      stop
      end
```

```
***** The output for the code "ineqs.for". ***********
******************************************************
n1 = 17 n2 = 10 n3 = 12 n4 = 14 n5 = 47
ep = 0.1000000E-06 eq = 0.1000000E-11
******************************************************
c1 = 0.1715729     c2 = 0.2727273     c3 = 0.3940307     c4 = 0.5302216
******************************************************
x = 0.1000000E-06
fip = 0.7549517E-07 x = 0.1000000E-06 fkp = 0.9936496E-07 flm = 0.1154632E-06
fjp = 0.1250000E-06 srt = 0.3162278E-03 flp = 0.8000000E+08
******************************************************
x = 0.1009252E-01
fip = 0.7685593E-02 x = 0.1009252E-01 fkp = 0.1019436E-01 flm = 0.1138297E-01
fjp = 0.1267669E-01 srt = 0.1004615     flp =  790.6547
x = 0.2018504E-01
fip = 0.1561019E-01 x = 0.2018504E-01 fkp = 0.2059214E-01 flm = 0.2282467E-01
fjp = 0.2547771E-01 srt = 0.1420741     flp =  394.3102
x = 0.3027757E-01
fip = 0.2378408E-01 x = 0.3027757E-01 fkp = 0.3119254E-01 flm = 0.3432655E-01
fjp = 0.3840651E-01 srt = 0.1740045     flp =  262.1877
x = 0.4037009E-01
fip = 0.3221806E-01 x = 0.4037009E-01 fkp = 0.4199420E-01 flm = 0.4589012E-01
fjp = 0.5146663E-01 srt = 0.2009231     flp =  196.1206
x = 0.5046261E-01
fip = 0.4092349E-01 x = 0.5046261E-01 fkp = 0.5299516E-01 flm = 0.5751692E-01
fjp = 0.6466170E-01 srt = 0.2246388     flp =  156.4757
x = 0.6055513E-01
fip = 0.4991230E-01 x = 0.6055513E-01 fkp = 0.6419281E-01 flm = 0.6920851E-01
fjp = 0.7799544E-01 srt = 0.2460795     flp =  130.0418
x = 0.7064765E-01
fip = 0.5919705E-01 x = 0.7064765E-01 fkp = 0.7558387E-01 flm = 0.8096653E-01
fjp = 0.9147166E-01 srt = 0.2657963     flp =  111.1570
x = 0.8074018E-01
fip = 0.6879090E-01 x = 0.8074018E-01 fkp = 0.8716441E-01 flm = 0.9279262E-01
fjp = 0.1050943     srt = 0.2841482     flp =  96.99047
x = 0.9083270E-01
fip = 0.7870770E-01 x = 0.9083270E-01 fkp = 0.9892977E-01 flm = 0.1046885
fjp = 0.1188673     srt = 0.3013846     flp =  85.96932
x = 0.1009252
fip = 0.8896199E-01 x = 0.1009252     fkp = 0.1108746     flm = 0.1166559
fjp = 0.1327948     srt = 0.3176873     flp =  77.14995
x = 0.1110177
fip = 0.9956905E-01 x = 0.1110177     fkp = 0.1229927     flm = 0.1286967
fjp = 0.1468811     srt = 0.3331933     flp =  69.93186
x = 0.1211103
fip = 0.1105449     x = 0.1211103     fkp = 0.1352774     flm = 0.1408127
fjp = 0.1611304     srt = 0.3480090     flp =  63.91469
x = 0.1312028
fip = 0.1219063     x = 0.1312028     fkp = 0.1477212     flm = 0.1530058
fjp = 0.1755472     srt = 0.3622193     flp =  58.82131
x = 0.1412953
fip = 0.1336709     x = 0.1412953     fkp = 0.1603157     flm = 0.1652779
fjp = 0.1901361     srt = 0.3758927     flp =  54.45373
x = 0.1513878
fip = 0.1458572     x = 0.1513878     fkp = 0.1730522     flm = 0.1776312
fjp = 0.2049017     srt = 0.3890859     flp =  50.66677
x = 0.1614804
fip = 0.1584845     x = 0.1614804     fkp = 0.1859211     flm = 0.1900676
fjp = 0.2198489     srt = 0.4018462     flp =  47.35156
x = 0.1715729
fip = 0.1715729     x = 0.1715729     fkp = 0.1989124     flm = 0.2025894
fjp = 0.2349824     srt = 0.4142136     flp =  44.42483
******************************************************
```

```
x = 0.1816883
x = 0.1816883     fip = 0.1851749    fkp = 0.2120452    flm = 0.2152275
fjp = 0.2503425     srt = 0.4262491    flp =  41.81622
x = 0.1918038
x = 0.1918038     fip = 0.1992835    fkp = 0.2252789    flm = 0.2279558
fjp = 0.2659001     srt = 0.4379541    flp =  39.48134
x = 0.2019192
x = 0.2019192     fip = 0.2139216    fkp = 0.2386021    flm = 0.2407767
fjp = 0.2816607     srt = 0.4493542    flp =  37.37903
x = 0.2120346
x = 0.2120346     fip = 0.2291133    fkp = 0.2520026    flm = 0.2536927
fjp = 0.2976296     srt = 0.4604722    flp =  35.47599
x = 0.2221501
x = 0.2221501     fip = 0.2448834    fkp = 0.2654682    flm = 0.2667063
fjp = 0.3138126     srt = 0.4713280    flp =  33.74499
x = 0.2322655
x = 0.2322655     fip = 0.2612574    fkp = 0.2789864    flm = 0.2798201
fjp = 0.3302153     srt = 0.4819393    flp =  32.16352
x = 0.2423810
x = 0.2423810     fip = 0.2782619    fkp = 0.2925447    flm = 0.2930369
fjp = 0.3468438     srt = 0.4923220    flp =  30.71286
x = 0.2524964
x = 0.2524964     fip = 0.2959241    fkp = 0.3061303    flm = 0.3063594
fjp = 0.3637040     srt = 0.5024902    flp =  29.37726
x = 0.2626118
x = 0.2626118     fip = 0.3142718    fkp = 0.3197307    flm = 0.3197905
fjp = 0.3808023     srt = 0.5124567    flp =  28.14342
x = 0.2727273
x = 0.2727273     fip = 0.3333333    fkp = 0.3333333    flm = 0.3333333
fjp = 0.3981449     srt = 0.5222330    flp =  27.00000
******************************************************
x = 0.2828359
x = 0.2828359     fkp = 0.3469168    flm = 0.3469816    fip = 0.3531240
fjp = 0.4157266     srt = 0.5318232    flp =  25.93797
x = 0.2929445
x = 0.2929445     fkp = 0.3604781    flm = 0.3607478    fip = 0.3736852
fjp = 0.4335657     srt = 0.5412435    flp =  24.94818
x = 0.3030531
x = 0.3030531     fkp = 0.3740056    flm = 0.3746351    fip = 0.3950460
fjp = 0.4516690     srt = 0.5505026    flp =  24.02338
x = 0.3131618
x = 0.3131618     fkp = 0.3874879    flm = 0.3886471    fip = 0.4172353
fjp = 0.4700437     srt = 0.5596086    flp =  23.15726
x = 0.3232704
x = 0.3232704     fkp = 0.4009141    flm = 0.4027873    fip = 0.4402814
fjp = 0.4886969     srt = 0.5685687    flp =  22.34430
x = 0.3333790
x = 0.3333790     fkp = 0.4142738    flm = 0.4170595    fip = 0.4642118
fjp = 0.5076361     srt = 0.5773898    flp =  21.57965
x = 0.3434876
x = 0.3434876     fkp = 0.4275569    flm = 0.4314677    fip = 0.4890530
fjp = 0.5268689     srt = 0.5860782    flp =  20.85904
x = 0.3535962
x = 0.3535962     fkp = 0.4407542    flm = 0.4460158    fip = 0.5148299
fjp = 0.5464032     srt = 0.5946396    flp =  20.17866
x = 0.3637049
x = 0.3637049     fkp = 0.4538567    flm = 0.4607081    fip = 0.5415657
fjp = 0.5662470     srt = 0.6030795    flp =  19.53515
x = 0.3738135
x = 0.3738135     fkp = 0.4668562    flm = 0.4755489    fip = 0.5692811
fjp = 0.5864085     srt = 0.6114029    flp =  18.92550
x = 0.3839221
x = 0.3839221     fkp = 0.4797451    flm = 0.4905430    fip = 0.5979939
```

```
fjp = 0.6068962     srt = 0.6196145     flp =  18.34702
x = 0.3940307
x = 0.3940307      fkp = 0.4925164     flm = 0.5056950     fip = 0.6277187
fjp = 0.6277187     srt = 0.6277187     flp =  17.79729
******************************************************
x = 0.4037587
x = 0.4037587      fkp = 0.5046894     flm = 0.5204302     srt = 0.6354201
fjp = 0.6480814     fip = 0.6572896     flp =  17.29339
x = 0.4134866
x = 0.4134866      fkp = 0.5167423     flm = 0.5353209     srt = 0.6430292
fjp = 0.6687705     fip = 0.6878137     flp =  16.81234
x = 0.4232145
x = 0.4232145      fkp = 0.5286703     flm = 0.5503721     srt = 0.6505494
fjp = 0.6897943     fip = 0.7192926     flp =  16.35257
x = 0.4329424
x = 0.4329424      fkp = 0.5404690     flm = 0.5655889     srt = 0.6579836
fjp = 0.7111613     fip = 0.7517238     flp =  15.91262
x = 0.4426703
x = 0.4426703      fkp = 0.5521347     flm = 0.5809764     srt = 0.6653348
fjp = 0.7328803     fip = 0.7851000     flp =  15.49116
x = 0.4523983
x = 0.4523983      fkp = 0.5636642     flm = 0.5965403     srt = 0.6726056
fjp = 0.7549601     fip = 0.8194089     flp =  15.08699
x = 0.4621262
x = 0.4621262      fkp = 0.5750545     flm = 0.6122865     srt = 0.6797986
fjp = 0.7774099     fip = 0.8546332     flp =  14.69900
x = 0.4718541
x = 0.4718541      fkp = 0.5863032     flm = 0.6282209     srt = 0.6869164
fjp = 0.8002391     fip = 0.8907504     flp =  14.32617
x = 0.4815820
x = 0.4815820      fkp = 0.5974081     flm = 0.6443499     srt = 0.6939611
fjp = 0.8234573     fip = 0.9277326     flp =  13.96757
x = 0.4913100
x = 0.4913100      fkp = 0.6083677     flm = 0.6606804     srt = 0.7009351
fjp = 0.8470743     fip = 0.9655470     flp =  13.62232
x = 0.5010379
x = 0.5010379      fkp = 0.6191806     flm = 0.6772194     srt = 0.7078403
fjp = 0.8711002     fip =  1.004156     flp =  13.28964
x = 0.5107658
x = 0.5107658      fkp = 0.6298458     flm = 0.6939742     srt = 0.7146788
fjp = 0.8955453     fip =  1.043517     flp =  12.96878
x = 0.5204937
x = 0.5204937      fkp = 0.6403626     flm = 0.7109527     srt = 0.7214525
fjp = 0.9204201     fip =  1.083583     flp =  12.65907
x = 0.5302216
x = 0.5302216      fkp = 0.6507306     flm = 0.7281632     srt = 0.7281632
fjp = 0.9457355     fip =  1.124304     flp =  12.35987
******************************************************
x = 0.5402169
x = 0.5402169      fkp = 0.6612284     srt = 0.7349945     flm = 0.7460974
fjp = 0.9722173     fip =  1.166769     flp =  12.06277
x = 0.5502122
x = 0.5502122      fkp = 0.6715692     srt = 0.7417629     flm = 0.7642956
fjp = 0.9991881     fip =  1.209809     flp =  11.77555
x = 0.5602075
x = 0.5602075      fkp = 0.6817533     srt = 0.7484701     flm = 0.7827683
fjp =  1.026661     fip =  1.253358     flp =  11.49765
x = 0.5702028
x = 0.5702028      fkp = 0.6917814     srt = 0.7551177     flm = 0.8015269
fjp =  1.054648     fip =  1.297353     flp =  11.22857
x = 0.5801981
x = 0.5801981      fkp = 0.7016542     srt = 0.7617073     flm = 0.8205830
fjp =  1.083164     fip =  1.341727     flp =  10.96781
```

```
x = 0.5901934
x = 0.5901934     fkp = 0.7113728     srt = 0.7682404     flm = 0.8399495
fjp =  1.112220     fip =  1.386415     flp =  10.71493
x = 0.6001886
x = 0.6001886     fkp = 0.7209383     srt = 0.7747184     flm = 0.8596397
fjp =  1.141833     fip =  1.431351     flp =  10.46950
x = 0.6101839
x = 0.6101839     fkp = 0.7303522     srt = 0.7811427     flm = 0.8796680
fjp =  1.172015     fip =  1.476471     flp =  10.23113
x = 0.6201792
x = 0.6201792     fkp = 0.7396159     srt = 0.7875146     flm = 0.9000497
fjp =  1.202782     fip =  1.521713     flp =  9.999448
x = 0.6301745
x = 0.6301745     fkp = 0.7487310     srt = 0.7938353     flm = 0.9208013
fjp =  1.234148     fip =  1.567017     flp =  9.774095
x = 0.6401698
x = 0.6401698     fkp = 0.7576992     srt = 0.8001061     flm = 0.9419404
fjp =  1.266130     fip =  1.612325     flp =  9.554745
x = 0.6501651
x = 0.6501651     fkp = 0.7665224     srt = 0.8063281     flm = 0.9634858
fjp =  1.298742     fip =  1.657582     flp =  9.341082
x = 0.6601603
x = 0.6601603     fkp = 0.7752023     srt = 0.8125025     flm = 0.9854579
fjp =  1.332001     fip =  1.702736     flp =  9.132810
x = 0.6701556
x = 0.6701556     fkp = 0.7837410     srt = 0.8186303     flm =  1.007879
fjp =  1.365924     fip =  1.747740     flp =  8.929647
x = 0.6801509
x = 0.6801509     fkp = 0.7921404     srt = 0.8247126     flm =  1.030771
fjp =  1.400528     fip =  1.792548     flp =  8.731324
x = 0.6901462
x = 0.6901462     fkp = 0.8004027     srt = 0.8307504     flm =  1.054162
fjp =  1.435830     fip =  1.837119     flp =  8.537585
x = 0.7001415
x = 0.7001415     fkp = 0.8085299     srt = 0.8367446     flm =  1.078079
fjp =  1.471848     fip =  1.881414     flp =  8.348183
x = 0.7101368
x = 0.7101368     fkp = 0.8165241     srt = 0.8426961     flm =  1.102551
fjp =  1.508601     fip =  1.925400     flp =  8.162884
x = 0.7201320
x = 0.7201320     fkp = 0.8243875     srt = 0.8486059     flm =  1.127613
fjp =  1.546108     fip =  1.969045     flp =  7.981461
x = 0.7301273
x = 0.7301273     fkp = 0.8321223     srt = 0.8544749     flm =  1.153300
fjp =  1.584388     fip =  2.012320     flp =  7.803693
x = 0.7401226
x = 0.7401226     fkp = 0.8397306     srt = 0.8603038     flm =  1.179652
fjp =  1.623462     fip =  2.055201     flp =  7.629368
x = 0.7501179
x = 0.7501179     fkp = 0.8472146     srt = 0.8660935     flm =  1.206713
fjp =  1.663349     fip =  2.097666     flp =  7.458277
x = 0.7601132
x = 0.7601132     fkp = 0.8545766     srt = 0.8718447     flm =  1.234531
fjp =  1.704072     fip =  2.139695     flp =  7.290217
x = 0.7701085
x = 0.7701085     fkp = 0.8618186     srt = 0.8775582     flm =  1.263160
fjp =  1.745652     fip =  2.181271     flp =  7.124987
x = 0.7801038
x = 0.7801038     fkp = 0.8689429     srt = 0.8832348     flm =  1.292660
fjp =  1.788111     fip =  2.222381     flp =  6.962386
x = 0.7900990
x = 0.7900990     fkp = 0.8759515     srt = 0.8888752     flm =  1.323099
fjp =  1.831472     fip =  2.263011     flp =  6.802214
```

```
x = 0.8000943
x = 0.8000943     fkp = 0.8828468     srt = 0.8944799     flm =  1.354550
fjp =  1.875760     fip =  2.303152     flp =  6.644271
x = 0.8100896
x = 0.8100896     fkp = 0.8896306     srt = 0.9000498     flm =  1.387102
fjp =  1.920998     fip =  2.342796     flp =  6.488349
x = 0.8200849
x = 0.8200849     fkp = 0.8963053     srt = 0.9055854     flm =  1.420850
fjp =  1.967213     fip =  2.381936     flp =  6.334238
x = 0.8300802
x = 0.8300802     fkp = 0.9028728     srt = 0.9110874     flm =  1.455906
fjp =  2.014428     fip =  2.420567     flp =  6.181717
x = 0.8400755
x = 0.8400755     fkp = 0.9093352     srt = 0.9165563     flm =  1.492400
fjp =  2.062673     fip =  2.458687     flp =  6.030554
x = 0.8500707
x = 0.8500707     fkp = 0.9156945     srt = 0.9219928     flm =  1.530482
fjp =  2.111973     fip =  2.496292     flp =  5.880499
x = 0.8600660
x = 0.8600660     fkp = 0.9219528     srt = 0.9273974     flm =  1.570329
fjp =  2.162358     fip =  2.533382     flp =  5.731282
x = 0.8700613
x = 0.8700613     fkp = 0.9281119     srt = 0.9327708     flm =  1.612151
fjp =  2.213857     fip =  2.569957     flp =  5.582603
x = 0.8800566
x = 0.8800566     fkp = 0.9341739     srt = 0.9381133     flm =  1.656201
fjp =  2.266500     fip =  2.606019     flp =  5.434124
x = 0.8900519
x = 0.8900519     fkp = 0.9401406     srt = 0.9434256     flm =  1.702787
fjp =  2.320318     fip =  2.641569     flp =  5.285453
x = 0.9000472
x = 0.9000472     fkp = 0.9460140     srt = 0.9487082     flm =  1.752292
fjp =  2.375344     fip =  2.676609     flp =  5.136132
x = 0.9100424
x = 0.9100424     fkp = 0.9517957     srt = 0.9539614     flm =  1.805199
fjp =  2.431612     fip =  2.711144     flp =  4.985600
x = 0.9200377
x = 0.9200377     fkp = 0.9574878     srt = 0.9591860     flm =  1.862136
fjp =  2.489155     fip =  2.745178     flp =  4.833160
x = 0.9300330
x = 0.9300330     fkp = 0.9630919     srt = 0.9643822     flm =  1.923937
fjp =  2.548009     fip =  2.778714     flp =  4.677908
x = 0.9400283
x = 0.9400283     fkp = 0.9686098     srt = 0.9695506     flm =  1.991756
fjp =  2.608213     fip =  2.811757     flp =  4.518626
x = 0.9500236
x = 0.9500236     fkp = 0.9740431     srt = 0.9746915     flm =  2.067265
fjp =  2.669803     fip =  2.844314     flp =  4.353579
x = 0.9600189
x = 0.9600189     fkp = 0.9793937     srt = 0.9798055     flm =  2.153047
fjp =  2.732819     fip =  2.876389     flp =  4.180123
x = 0.9700141
x = 0.9700141     fkp = 0.9846630     srt = 0.9848930     flm =  2.253481
fjp =  2.797304     fip =  2.907989     flp =  3.993822
x = 0.9800094
x = 0.9800094     fkp = 0.9898528     srt = 0.9899543     flm =  2.377166
fjp =  2.863299     fip =  2.939120     flp =  3.786021
x = 0.9900047
x = 0.9900047     fkp = 0.9949646     srt = 0.9949898     flm =  2.546482
fjp =  2.930849     fip =  2.969788     flp =  3.534287
*****************************************************
x =  1.000000
x =  1.000000     fkp =  1.000000     srt =  1.000000     flm =  2.999995
```

```
fjp =  3.000000     fip =  3.000000     flp =  3.000005
```

Appendix B. Numerical Validation of Theorem 35

```
      implicit real*8(a-h,o-z)
      complex*16 a,b,c,cdsqrt,x1,x2,x3,dcmplx
c
c     Program "ctausc.for".
c
c     This program is used to verify numerically the assertion made
c     in part G of Theorem 35.
c
c *** The critical value of tau, i.e., tauo(nu**2), is evaluated for
c *** each given value of nu (the Courant number).
c
c *** Given any (nu,tau), the spectral radius of the amplification
c *** matrix is a function of the phase angle theta. The least upper
c *** bound (denoted by "am") of the spectral radii over the range
c *** -pi < theta .le. pi is evaluated for each given (nu,tau).
c
c *** When nu is replaced by -nu, each of the two resulting
amplification
c *** factors (defined in Eq. (4.7)) becomes the complex conjugate of
that
c *** before sign-change. Thus the spectral radius does not change as nu
c *** is replaced by -nu. For this reason, the range of nu can be
limited
c *** to nu. ge. 0.
c
c *** When theta is replaced by -theta, each of the two resulting
c *** amplification factors also becomes the complex conjugate of
c *** that before sign-change. Thus the range of theta can be limited
c *** to 0 .le. theta .le. pi.
c
c *** Theorems 16 and 18 imply that the least upper bound am = 1 if
c *** nu = 1 and tau .ge. 1 (Note: According to Eq. (4.7), the value
c *** of the principal amplification factor = 1 when theta = 0. Thus
c *** am .ge. 1 for any (nu,tau). In turn, this implies that am = 1
c *** for any (nu,tau) which meets the condition Eq. (4.2)). Moreover,
c *** Theorems 6 and 12 imply that am > 1 if nu > 1 regardless the
c *** value assumed by tau. Thus numerical results may not be consistent
c *** with theoretical predictions at the singular case nu = 1 if
c *** round-off errors are not controlled carefully. For this reason,
c *** a statement "if (dabs(x-1.d0).lt.ep) x=1.d0" is added in the code
c *** to insure that the value of x is really "1" as intended. Here
c *** ep (>0) is an input parameter and assumes to be very small.
c
c     x = nu.
c     z = The phase angle theta of a Fourier component.
c     nx = number of the values of nu.
c
c     nt = number of the values of tau with tau>tauo (tau<tauo) for
c          each value of nu. Here tauo is the critical value of tau
c          associated with a given value of nu. Because the case with
c          tau=tauo is always considered, there are (2*nt+1) values
c          of tau associated with each value of nu, i.e.,
c          tauo*(1-dt*nt), tauo*(1-dt*(nt-1)),..., tauo*(1-dt), tauo,
c          tauo*(1+dt),..., tauo*(1+dt*(nt-1)), tauo*(1+dt*n).
c
c     nz = number of the intervals over the domain
c          0 .le. theta .le. pi.
c     xs = The initial value of nu.
c
      fkp(s) = (s-1.d0+dsqrt(5.d0*s**2-2.d0*s+1.d0))/(2.d0*s)
```

```
      flm(s) = (4.d0-s-2.d0*dsqrt(2.d0*(2.d0-s-s**2)))/s
c
      pi = 3.1415926535897932d0
      nx = 25
      nt = 5
      nz = 1000
      xs = 0.d0
      dx = 5.d-2
      dt = 1.d-4
      ep = 1.d-7
      c2 = 3.d0/11.d0
      dz = pi/dfloat(nz)
      x = xs-dx
      nzp = nz+1
      ts = 1.d0-dt*dfloat(nt+1)
      nt2p = nt*2+1
c
      open (unit=8,file='ctausc.txt')
      write (8,10)
      write (8,15)
      write (8,20) nx,nt,nz
      write (8,30) xs,dx,dt,ep
      write (8,15)
      do 200 i = 1,nx
      x = x+dx
      if (dabs(x-1.d0).lt.ep) x=1.d0
      xx = x**2
      if (xx.eq.0.d0) tauo = 0.d0
      if (xx.gt.0.d0.and.xx.le.c2) tauo = flm(xx)
      if (xx.gt.c2) tauo = fkp(xx)
      tau = tauo*ts
      dtau = tauo*dt
      do 200 j = 1,nt2p
      tau = tau+dtau
      am = 0.d0
      z = -dz
      do 100 k = 1,nzp
      z = z+dz
      z1 = dcos(z/2.d0)
      z2 = dsin(z/2.d0)
      ar = 1.d0+tau
      ai = 0.d0
      br = -2.d0*tau*z1
      bi = x*(3.d0+tau)*z2
      cr = -((1.d0 - tau)*z1**2 + (1.d0 + x**2)*z2**2)
      ci = -x*(1.d0 + tau)*z1*z2
      a = dcmplx(ar,ai)
      b = dcmplx(br,bi)
      c = dcmplx(cr,ci)
      x1 = (-b + cdsqrt(b**2 - 4.d0*a*c))/(2.d0*a)
      x2 = (-b - cdsqrt(b**2 - 4.d0*a*c))/(2.d0*a)
      a1 = cdabs(x1)
      a2 = cdabs(x2)
      am = dmax1(a1,a2,am)
100   continue
      write (8,40) x,tauo,tau,am
200   continue
      close (unit=8)
10    format (' ***** The output for the code "ctausc.for". ******')
15    format (' ****************************************************')
```

```
20    format (' nx =',i4,' nt =',i4,' nz =',i4)
30    format (' xs =',g14.7,' dx =',g14.7,' dt =',g14.7,' ep =',g14.7)
40    format (' nu =',g11.4,' tauo =',g14.7,' tau =',g14.7,
     *        ' am =',g21.14)
      stop
      end
```

```
***** The output for the code "ctausc.for". ******
*****************************************************
nx =  25 nt =   5 nz =1000
xs =  0.000000     dx = 0.5000000E-01 dt = 0.1000000E-03 ep = 0.1000000E-06
*****************************************************
nu =  0.000     tauo =  0.000000     tau =  0.000000     am =  1.0000000000000
nu =  0.000     tauo =  0.000000     tau =  0.000000     am =  1.0000000000000
nu =  0.000     tauo =  0.000000     tau =  0.000000     am =  1.0000000000000
nu =  0.000     tauo =  0.000000     tau =  0.000000     am =  1.0000000000000
nu =  0.000     tauo =  0.000000     tau =  0.000000     am =  1.0000000000000
nu =  0.000     tauo =  0.000000     tau =  0.000000     am =  1.0000000000000
nu =  0.000     tauo =  0.000000     tau =  0.000000     am =  1.0000000000000
nu =  0.000     tauo =  0.000000     tau =  0.000000     am =  1.0000000000000
nu =  0.000     tauo =  0.000000     tau =  0.000000     am =  1.0000000000000
nu =  0.000     tauo =  0.000000     tau =  0.000000     am =  1.0000000000000
nu =  0.000     tauo =  0.000000     tau =  0.000000     am =  1.0000000000000
nu = 0.5000E-01 tauo = 0.2814261E-02 tau = 0.2812854E-02 am =  1.0000001740909
nu = 0.5000E-01 tauo = 0.2814261E-02 tau = 0.2813136E-02 am =  1.0000001391629
nu = 0.5000E-01 tauo = 0.2814261E-02 tau = 0.2813417E-02 am =  1.0000001042349
nu = 0.5000E-01 tauo = 0.2814261E-02 tau = 0.2813699E-02 am =  1.0000000694331
nu = 0.5000E-01 tauo = 0.2814261E-02 tau = 0.2813980E-02 am =  1.0000000346952
nu = 0.5000E-01 tauo = 0.2814261E-02 tau = 0.2814261E-02 am =  1.0000000000000
nu = 0.5000E-01 tauo = 0.2814261E-02 tau = 0.2814543E-02 am =  1.0000000000000
nu = 0.5000E-01 tauo = 0.2814261E-02 tau = 0.2814824E-02 am =  1.0000000000000
nu = 0.5000E-01 tauo = 0.2814261E-02 tau = 0.2815106E-02 am =  1.0000000000000
nu = 0.5000E-01 tauo = 0.2814261E-02 tau = 0.2815387E-02 am =  1.0000000000000
nu = 0.5000E-01 tauo = 0.2814261E-02 tau = 0.2815669E-02 am =  1.0000000000000
nu = 0.1000     tauo = 0.1127835E-01 tau = 0.1127272E-01 am =  1.0000006672229
nu = 0.1000     tauo = 0.1127835E-01 tau = 0.1127384E-01 am =  1.0000005331916
nu = 0.1000     tauo = 0.1127835E-01 tau = 0.1127497E-01 am =  1.0000003994142
nu = 0.1000     tauo = 0.1127835E-01 tau = 0.1127610E-01 am =  1.0000002661220
nu = 0.1000     tauo = 0.1127835E-01 tau = 0.1127723E-01 am =  1.0000001328300
nu = 0.1000     tauo = 0.1127835E-01 tau = 0.1127835E-01 am =  1.0000000000000
nu = 0.1000     tauo = 0.1127835E-01 tau = 0.1127948E-01 am =  1.0000000000000
nu = 0.1000     tauo = 0.1127835E-01 tau = 0.1128061E-01 am =  1.0000000000000
nu = 0.1000     tauo = 0.1127835E-01 tau = 0.1128174E-01 am =  1.0000000000000
nu = 0.1000     tauo = 0.1127835E-01 tau = 0.1128287E-01 am =  1.0000000000000
nu = 0.1000     tauo = 0.1127835E-01 tau = 0.1128399E-01 am =  1.0000000000000
nu = 0.1500     tauo = 0.2545752E-01 tau = 0.2544479E-01 am =  1.0000013933207
nu = 0.1500     tauo = 0.2545752E-01 tau = 0.2544734E-01 am =  1.0000011137094
nu = 0.1500     tauo = 0.2545752E-01 tau = 0.2544988E-01 am =  1.0000008340988
nu = 0.1500     tauo = 0.2545752E-01 tau = 0.2545243E-01 am =  1.0000005555428
nu = 0.1500     tauo = 0.2545752E-01 tau = 0.2545498E-01 am =  1.0000002775110
nu = 0.1500     tauo = 0.2545752E-01 tau = 0.2545752E-01 am =  1.0000000000000
nu = 0.1500     tauo = 0.2545752E-01 tau = 0.2546007E-01 am =  1.0000000000000
nu = 0.1500     tauo = 0.2545752E-01 tau = 0.2546261E-01 am =  1.0000000000000
nu = 0.1500     tauo = 0.2545752E-01 tau = 0.2546516E-01 am =  1.0000000000000
nu = 0.1500     tauo = 0.2545752E-01 tau = 0.2546770E-01 am =  1.0000000000000
nu = 0.1500     tauo = 0.2545752E-01 tau = 0.2547025E-01 am =  1.0000000000000
nu = 0.2000     tauo = 0.4546499E-01 tau = 0.4544225E-01 am =  1.0000022161825
nu = 0.2000     tauo = 0.4546499E-01 tau = 0.4544680E-01 am =  1.0000017713023
nu = 0.2000     tauo = 0.4546499E-01 tau = 0.4545135E-01 am =  1.0000013266296
nu = 0.2000     tauo = 0.4546499E-01 tau = 0.4545589E-01 am =  1.0000008832441
nu = 0.2000     tauo = 0.4546499E-01 tau = 0.4546044E-01 am =  1.0000004411696
nu = 0.2000     tauo = 0.4546499E-01 tau = 0.4546499E-01 am =  1.0000000000000
nu = 0.2000     tauo = 0.4546499E-01 tau = 0.4546953E-01 am =  1.0000000000000
nu = 0.2000     tauo = 0.4546499E-01 tau = 0.4547408E-01 am =  1.0000000000000
nu = 0.2000     tauo = 0.4546499E-01 tau = 0.4547862E-01 am =  1.0000000000000
nu = 0.2000     tauo = 0.4546499E-01 tau = 0.4548317E-01 am =  1.0000000000000
nu = 0.2000     tauo = 0.4546499E-01 tau = 0.4548772E-01 am =  1.0000000000000
nu = 0.2500     tauo = 0.7146911E-01 tau = 0.7143338E-01 am =  1.0000029597671
nu = 0.2500     tauo = 0.7146911E-01 tau = 0.7144052E-01 am =  1.0000023649173
nu = 0.2500     tauo = 0.7146911E-01 tau = 0.7144767E-01 am =  1.0000017704610
```

```
nu = 0.2500     tauo = 0.7146911E-01 tau = 0.7145482E-01 am =  1.0000011789100
nu = 0.2500     tauo = 0.7146911E-01 tau = 0.7146196E-01 am =  1.0000005880919
nu = 0.2500     tauo = 0.7146911E-01 tau = 0.7146911E-01 am =  1.0000000000000
nu = 0.2500     tauo = 0.7146911E-01 tau = 0.7147626E-01 am =  1.0000000000000
nu = 0.2500     tauo = 0.7146911E-01 tau = 0.7148340E-01 am =  1.0000000000000
nu = 0.2500     tauo = 0.7146911E-01 tau = 0.7149055E-01 am =  1.0000000000000
nu = 0.2500     tauo = 0.7146911E-01 tau = 0.7149770E-01 am =  1.0000000000000
nu = 0.2500     tauo = 0.7146911E-01 tau = 0.7150484E-01 am =  1.0000000000000
nu = 0.3000     tauo = 0.1037043     tau = 0.1036525     am =  1.0000034247159
nu = 0.3000     tauo = 0.1037043     tau = 0.1036629     am =  1.0000027345694
nu = 0.3000     tauo = 0.1037043     tau = 0.1036732     am =  1.0000020464043
nu = 0.3000     tauo = 0.1037043     tau = 0.1036836     am =  1.0000013620442
nu = 0.3000     tauo = 0.1037043     tau = 0.1036940     am =  1.0000006790801
nu = 0.3000     tauo = 0.1037043     tau = 0.1037043     am =  1.0000000000000
nu = 0.3000     tauo = 0.1037043     tau = 0.1037147     am =  1.0000000000000
nu = 0.3000     tauo = 0.1037043     tau = 0.1037251     am =  1.0000000000000
nu = 0.3000     tauo = 0.1037043     tau = 0.1037354     am =  1.0000000000000
nu = 0.3000     tauo = 0.1037043     tau = 0.1037458     am =  1.0000000000000
nu = 0.3000     tauo = 0.1037043     tau = 0.1037562     am =  1.0000000000000
nu = 0.3500     tauo = 0.1424870     tau = 0.1424158     am =  1.0000034155413
nu = 0.3500     tauo = 0.1424870     tau = 0.1424301     am =  1.0000027246816
nu = 0.3500     tauo = 0.1424870     tau = 0.1424443     am =  1.0000020381007
nu = 0.3500     tauo = 0.1424870     tau = 0.1424586     am =  1.0000013549685
nu = 0.3500     tauo = 0.1424870     tau = 0.1424728     am =  1.0000006753657
nu = 0.3500     tauo = 0.1424870     tau = 0.1424870     am =  1.0000000000000
nu = 0.3500     tauo = 0.1424870     tau = 0.1425013     am =  1.0000000000000
nu = 0.3500     tauo = 0.1424870     tau = 0.1425155     am =  1.0000000000000
nu = 0.3500     tauo = 0.1424870     tau = 0.1425298     am =  1.0000000000000
nu = 0.3500     tauo = 0.1424870     tau = 0.1425440     am =  1.0000000000000
nu = 0.3500     tauo = 0.1424870     tau = 0.1425583     am =  1.0000000000000
nu = 0.4000     tauo = 0.1882382     tau = 0.1881441     am =  1.0000027984482
nu = 0.4000     tauo = 0.1882382     tau = 0.1881629     am =  1.0000022284014
nu = 0.4000     tauo = 0.1882382     tau = 0.1881817     am =  1.0000016634636
nu = 0.4000     tauo = 0.1882382     tau = 0.1882006     am =  1.0000011036658
nu = 0.4000     tauo = 0.1882382     tau = 0.1882194     am =  1.0000005490377
nu = 0.4000     tauo = 0.1882382     tau = 0.1882382     am =  1.0000000000000
nu = 0.4000     tauo = 0.1882382     tau = 0.1882570     am =  1.0000000000000
nu = 0.4000     tauo = 0.1882382     tau = 0.1882758     am =  1.0000000000000
nu = 0.4000     tauo = 0.1882382     tau = 0.1882947     am =  1.0000000000000
nu = 0.4000     tauo = 0.1882382     tau = 0.1883135     am =  1.0000000000000
nu = 0.4000     tauo = 0.1882382     tau = 0.1883323     am =  1.0000000000000
nu = 0.4500     tauo = 0.2415157     tau = 0.2413949     am =  1.0000016114364
nu = 0.4500     tauo = 0.2415157     tau = 0.2414191     am =  1.0000012754027
nu = 0.4500     tauo = 0.2415157     tau = 0.2414433     am =  1.0000009460213
nu = 0.4500     tauo = 0.2415157     tau = 0.2414674     am =  1.0000006236255
nu = 0.4500     tauo = 0.2415157     tau = 0.2414916     am =  1.0000003082242
nu = 0.4500     tauo = 0.2415157     tau = 0.2415157     am =  1.0000000000000
nu = 0.4500     tauo = 0.2415157     tau = 0.2415399     am =  1.0000000000000
nu = 0.4500     tauo = 0.2415157     tau = 0.2415640     am =  1.0000000000000
nu = 0.4500     tauo = 0.2415157     tau = 0.2415882     am =  1.0000000000000
nu = 0.4500     tauo = 0.2415157     tau = 0.2416123     am =  1.0000000000000
nu = 0.4500     tauo = 0.2415157     tau = 0.2416365     am =  1.0000000000000
nu = 0.5000     tauo = 0.3030615     tau = 0.3029100     am =  1.0000003301576
nu = 0.5000     tauo = 0.3030615     tau = 0.3029403     am =  1.0000002505240
nu = 0.5000     tauo = 0.3030615     tau = 0.3029706     am =  1.0000001771524
nu = 0.5000     tauo = 0.3030615     tau = 0.3030009     am =  1.0000001104901
nu = 0.5000     tauo = 0.3030615     tau = 0.3030312     am =  1.0000000511002
nu = 0.5000     tauo = 0.3030615     tau = 0.3030615     am =  1.0000000000000
nu = 0.5000     tauo = 0.3030615     tau = 0.3030918     am =  1.0000000000000
nu = 0.5000     tauo = 0.3030615     tau = 0.3031222     am =  1.0000000000000
nu = 0.5000     tauo = 0.3030615     tau = 0.3031525     am =  1.0000000000000
nu = 0.5000     tauo = 0.3030615     tau = 0.3031828     am =  1.0000000000000
nu = 0.5000     tauo = 0.3030615     tau = 0.3032131     am =  1.0000000000000
```

```
nu = 0.5500     tauo = 0.3732664     tau = 0.3730798     am =  1.0000000020424
nu = 0.5500     tauo = 0.3732664     tau = 0.3731171     am =  1.0000000010635
nu = 0.5500     tauo = 0.3732664     tau = 0.3731544     am =  1.0000000004564
nu = 0.5500     tauo = 0.3732664     tau = 0.3731918     am =  1.0000000001377
nu = 0.5500     tauo = 0.3732664     tau = 0.3732291     am =  1.0000000000175
nu = 0.5500     tauo = 0.3732664     tau = 0.3732664     am =  1.0000000000000
nu = 0.5500     tauo = 0.3732664     tau = 0.3733038     am =  1.0000000000000
nu = 0.5500     tauo = 0.3732664     tau = 0.3733411     am =  1.0000000000000
nu = 0.5500     tauo = 0.3732664     tau = 0.3733784     am =  1.0000000000000
nu = 0.5500     tauo = 0.3732664     tau = 0.3734157     am =  1.0000000000000
nu = 0.5500     tauo = 0.3732664     tau = 0.3734531     am =  1.0000000000000
nu = 0.6000     tauo = 0.4490661     tau = 0.4488415     am =  1.0000000004655
nu = 0.6000     tauo = 0.4490661     tau = 0.4488864     am =  1.0000000002389
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As part of the continuous development of the space-time conservation element and solution element (CE-SE) method, recently a set of so called "Courant number insensitive schemes" has been proposed. The key advantage of these new schemes is that the numerical dissipation associated with them generally does not increase as the Courant number decreases. As such, they can be applied to problems with large Courant number disparities (such as what commonly occurs in Navier-Stokes problems) without incurring excessive numerical dissipation.

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