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SPECIFIC IMPULSE AND MASS FLOW RATE ERROR

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INTRODUCTION

Specific impulse is defined in words in many ways. Very early in any text on rocket propulsion a phrase similar to "specific impulse is the thrust force per unit propellant weight flow per second" will be found.**(2)** It is only after seeing the mathematics written down does the definition mean something physically to scientists and engineers responsible for either measuring it or using someone's value for it. A very typical mathematical definition is: **(3)**

$$I_{sp} = \frac{\int_0^T F_t dt}{g\int_0^T \dot{m}_e dt} \tag{1}$$

where F_t is the thrust and $\dot{m}_e$ is the mass flow rate. The dot indicates a time derivative as usual and g is the acceleration due to gravity. If the numerator and denominator of (1) are multiplied by the inverse of the time period, the two integrals take on a new meaning:

$$I_{sp} = \frac{\frac{1}{T}\int_0^T F_t dt}{g\frac{1}{T}\int_0^T \dot{m}_e dt} \tag{2}$$

Now the integrals are averages and the specific impulse can be more correctly defined as the average thrust produced by the rocket engine divided by (g times) the average mass flow rate. The units of specific impulse are seconds. This definition is much more amenable to both scientists and engineers. The procedure followed in most textbooks will be reviewed here for completeness and to act as an introduction to the modifications to come later.

The usual freshman physics development of the rocket flight equations begins with the conservation of momentum equation: **(4)**

$$\sum p = 0 \tag{3}$$

$$m_R v_R - m_e v_e = 0 \tag{4}$$

where m_R is the mass of the rocket, m_e is the mass of the exhausted propellant, and v_R and v_e are the velocities of the rocket and the exhausted propellant respectively. Taking the time derivative of momentum gives the force:

$$\frac{dp}{dt} = F \tag{5}$$

and applying this to the momentum equation gives:

$$\frac{d}{dt}\left(m_R v_R - m_e v_e\right) = 0 \tag{6}$$

$$\frac{d}{dt}\left(m_R v_R\right) - \frac{d}{dt}\left(m_e v_e\right) = 0 \tag{7}$$

The last term is the rocket thrust:

$$F_t = \frac{d}{dt}\left(m_e v_e\right) \tag{8}$$

Performing the derivative correctly, knowing that the mass of the rocket is time dependent gives:

$$F_t = m_e \dot{v}_e + v_e \dot{m}_e \tag{9}$$

Now if v_e is a constant, then of course $\dot{v}_e$ = 0 so that:

$$F_t = v_e \dot{m}_e \tag{10}$$

Substituting this into the I_{sp} equation (1) gives:

$$I_{sp} = \frac{\int_0^T v_e \dot{m}_e}{g\int_0^T \dot{m}_e dt} \tag{11}$$

and again, assuming v_e is a constant and if it is further assumed that $\dot{m}_e$ is also a constant, then:

$$I_{sp} = \frac{v_e \dot{m}_e \int_0^T dt}{g \dot{m}_e \int_0^T dt} \tag{12}$$

then:

$$I_{sp} = \frac{v_e}{g} \tag{13}$$

which is the normally encountered result, which is perfectly acceptable if both the mass flow rate and the exhaust velocity are constants.**(5)** This is indeed often the case, or is at least assumed to be the case, for most rocket propulsion measurements. In this investigation however the interest is in determining how an error in the mass flow rate measurement affects the specific impulse, so these simplifications cannot be used.

THE INSTANTANEOUS SPECIFIC IMPULSE

The very use of the word impulse implies an integration over time so perhaps it is not appropriate to use the term as it will be used in the following argument but for historical reasons it will be used anyway. An instantaneous specific impulse (which of course literally doesn't make much sense) is defined to be:

$$I_{sp}^* \equiv \frac{1}{2} \frac{F_t}{g \dot{m}_e} \tag{14}$$

where the asterisk indicates that the term is related to the classical definition but not identical to it. The particular interest in this investigation is in how the specific impulse varies with mass flow rate so it is logical to try to determine the derivative of the specific impulse with respect to the mass flow rate:

$$\frac{dI_{sp}^*}{d\dot{m}_e} = \frac{1}{2g} \frac{d}{d\dot{m}_e} \left(\frac{F_t}{\dot{m}_e} \right) \tag{15}$$

$$= \frac{1}{2g} \left[\frac{-F_t}{(\dot{m}_e)^2} + \frac{1}{\dot{m}_e} \frac{dF_t}{d\dot{m}_e} \right] \tag{16}$$

Now equation (9) gives the correct form for F_t, so substituting this into (16) gives:

$$\frac{dF_t}{d\dot{m}_e} = \frac{d}{d\dot{m}_e}\left[m_e\dot{v}_e + v_e\dot{m}_e\right] \tag{17}$$

$$= m_e\frac{d\dot{v}_e}{d\dot{m}_e} + \dot{v}_e\frac{dm_e}{d\dot{m}_e} + v_e + \dot{m}_e\frac{dv_e}{d\dot{m}_e} \tag{18}$$

where all the terms have units of velocity as they should. Substituting (18) into (16) and rearranging gives:

$$\frac{dI_{sp}^*}{d\dot{m}_e} = \frac{1}{2g}\left\{\frac{m_e}{\dot{m}_e}\frac{d\dot{v}_e}{d\dot{m}_e} + \frac{\dot{v}_e}{\dot{m}_e}\frac{dm_e}{d\dot{m}_e} + \frac{dv_e}{d\dot{m}_e} - \frac{m_e\dot{v}_e}{(\dot{m}_e)^2}\right\} \tag{19}$$

At this point it is tempting to write an equation relating the mass of the exhausted propellant to the mass flow rate and the elapsed time. Such an equation would look like:

$$m_e = \dot{m}_e t \tag{20}$$

and this would greatly simplify (19) thereby leading to a quick solution but there is a problem with this if thought goes beyond freshman physics. Assume that (20) is true for a moment, then:

$$\dot{m}_e = \frac{d}{dt}\left(\dot{m}_e t\right) \tag{21}$$

$$= \dot{m}_e + t\ddot{m}_e \tag{22}$$

which implies

$$\ddot{m}_e = 0 \tag{23}$$

and this certainly is not physical for the problem at hand since the principle interest here is in determining how a change in $\dot{m}_e$ affects I_{sp}^*, so $\ddot{m}_e$ must exist. It is then tempting to assume that perhaps $\ddot{m}_e$ is a constant, which would also help in simplifying (19). This implies that:

$$\dot{m}_e = At \tag{24}$$

where A is a constant. This requires $\dot{m}_e$ to be a linear function of time which might be correct for some applications but is too restrictive for the general problem at hand, so it is important to stay with equation (19) as the fundamental result. This is somewhat

analogous to a classical mechanics problem with non-constant acceleration. Such a problem does not come up often but it is nevertheless quite physical.

Examine one term of equation (19):

$$\frac{\dot{v}_e}{\dot{m}_e}\frac{dm_e}{d\dot{m}_e} \tag{25}$$

This can be rewritten as:

$$\frac{dv_e}{dt}\frac{dt}{dm_e}\frac{dm_e}{d\dot{m}_e} = \frac{dv_e}{d\dot{m}_e} \tag{26}$$

which does simplify equation (19):

$$\frac{dI_{sp}^*}{d\dot{m}_e} = \frac{1}{2g}\left\{\frac{m_e}{\dot{m}_e}\frac{d\dot{v}_e}{d\dot{m}_e} + 2\frac{dv_e}{d\dot{m}_e} - \frac{m_e\dot{v}_e}{(\dot{m}_e)^2}\right\} \tag{27}$$

The last term in (19) can also be simplified:

$$\dot{v}_e\frac{m_e}{(\dot{m}_e)^2} = \frac{m_e}{\dot{m}_e}\left(\frac{\dot{v}_e}{\dot{m}_e}\right) = \frac{m_e}{\dot{m}_e}\frac{dv_e}{dm_e} \tag{28}$$

so that equation (19) now becomes:

$$\frac{dI_{sp}^*}{d\dot{m}_e} = \frac{1}{2g}\left\{\frac{m_e}{\dot{m}_e}\frac{d\dot{v}_e}{d\dot{m}_e} + 2\frac{dv_e}{d\dot{m}_e} - \frac{m_e}{\dot{m}_e}\frac{dv_e}{dm_e}\right\} \tag{29}$$

which is the fundamental result of this investigation.

Rewriting equation (29) slightly:

$$\frac{dI_{sp}^*}{d\dot{m}_e} = \frac{1}{g}\frac{dv_e}{d\dot{m}_e} + \frac{1}{2g}\frac{m_e}{\dot{m}_e}\left(\frac{d\dot{v}_e}{d\dot{m}_e} - \frac{dv_e}{dm_e}\right) \tag{30}$$

Now if the last term was not present, then:

$$\frac{dI_{sp}^*}{d\dot{m}_e} = \frac{1}{g}\frac{dv_e}{d\dot{m}_e} \tag{31}$$

Integrating:

$$\int dI_{sp}^{*} = \frac{1}{g}\int dv_e \tag{32}$$

gives:

$$I_{sp}^{*} = \frac{v_e}{g} \tag{33}$$

which is the classical textbook result (equation 13) obtained if the exhaust velocity and mass flow rate are assumed to be constants.

The last term in equation (30):

$$\frac{1}{2g}\frac{m_e}{\dot{m}_e}\left(\frac{d\dot{v}_e}{d\dot{m}_e} - \frac{dv_e}{dm_e}\right) \tag{34}$$

can be thought of as a "correction" to $\frac{dI_{sp}^{*}}{d\dot{m}_e}$ for the case of v_e and $\dot{m}_e$ being non-constant. So the classical specific impulse can be written in terms of the instantaneous specific impulse as:

$$I_{sp}^{*} = I_{sp} + \frac{1}{2g}\int \frac{m_e}{\dot{m}_e}\left(\frac{d\dot{v}_e}{d\dot{m}_e} - \frac{dv_e}{dm_e}\right)d\dot{m}_e \tag{35}$$

This is the final result.

DISCUSSION

As an exercise and example, it is possible to construct plausible functions for the derivatives in the last term of equation (30) to get an idea of how the specific impulse error depends on the error in the mass flow rate. It seems reasonable to assume that the mass of the expelled propellant increases in time, but that dependence is usually not known, so as a guess, assume that:

$$m_e(t) = At^n \tag{36}$$

where A is a constant and n represents a collection of random numbers whose average is 1. For n = 1 the dependence is linear which might make sense for some applications. If equation (36) is assumed to be true then:

$$\dot{m}_e(t) = nAt^{n-1} \tag{37}$$

Now the dependence of the exhaust velocity on the mass flow rate is required. It seems that there shouldn't be much of a dependence here so it is assumed that:

$$v_e = Bm_e^q \tag{38}$$

where B is a constant and q is defined similar to n except the average is 0. This says that the exhaust velocity does not vary much with the mass flow rate. This leads to:

$$\frac{dv_e}{dm_e} = qBm_e^{q-1} \tag{39}$$

For the dependence of the time derivative of the exhaust velocity on the mass flow rate, a dependence similar to equation (36) is assumed:

$$\dot{v}_e = C\dot{m}_e^j \tag{40}$$

where C is a constant and j is defined like *n* and *q* and it has an average of 1. This then leads to:

$$\frac{d\dot{v}_e}{d\dot{m}_e} = jC\dot{m}_e^{j-1} \tag{41}$$

As a reality check, notice that the time derivative of the mass flow rate is:

$$\ddot{m}_e(t) = n(n-1)At^{n-2} \tag{42}$$

which is not a constant in time, and that was required from the outset. However, if *n* is allowed to take its average value then the derivative is zero, which might be reasonable as a steady state condition. The final functional form needed is the dependence of the exhaust velocity on the mass flow rate. This is assumed to be simply linear:

$$v_e = D\dot{m}_e \tag{43}$$

so that the derivative is a constant:

$$\frac{dv_e}{d\dot{m}_e} = D \tag{44}$$

The functional assumptions made thus far allow the basic result (equation 30) to be written in terms of a time variable only:

$$\frac{dI_{sp}^{*}}{d\dot{m}_e} = \frac{D}{g} + \frac{1}{2gn}\left[jC(nA)^{j-1}t^{(n-1)(j-1)+1} - qBA^{q-1}t^{n(q-1)+1}\right] \quad (45)$$

RESULTS

The result of fundamental interest is that of equations (30) and (35). The traditional specific impulse can be written in terms of a new instantaneous specific impulse. This new term is useful in that the parameters are more readily measurable and applicable in experiments where high speed data can be obtained. The assumptions made in the discussion may not be very physical but do allow a visualization of how the error in the specific impulse due to an error in the mass flow rate might change in time. To plot the results of equation (45) a range of values was chosen for n, j, and q which met the basic criteria for the averages:

$$n = 0, 1, 2 \qquad q = -1, 0, 1 \qquad j = 0, 1, 2$$

A curve was generated for each of the 27 possible combinations of the indices and then the curves were averaged. The result is shown in Figure 1. A general increase in the derivative of the instantaneous specific impulse with respect to the mass flow rate is an indication of an increase in the error expected in the instantaneous specific impulse when an error in the mass flow rate is present. If the derivative was zero it would mean that the instantaneous specific impulse does not depend on the mass flow rate, which is non-physical. If the derivative was a constant then that would mean that the instantaneous specific impulse changes linearly with mass flow rate, which is the textbook result. The results in Figure 1 indicate that the derivative depends on the chosen variable (time) in a nonlinear way, which is a more physical result given the initial assumptions taken at the beginning of this research.

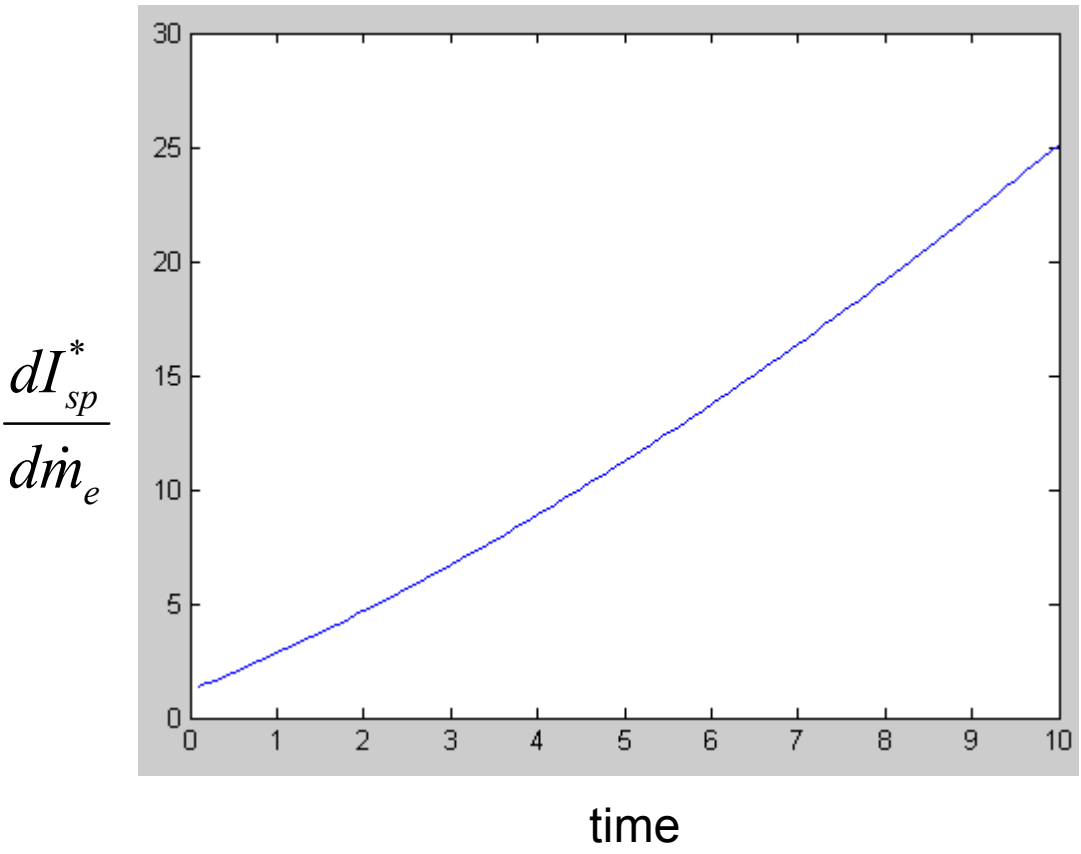

Figure 1: A plot of equation (45) showing the dependence of the derivative of the instantaneous specific impulse with respect to the mass flow rate versus time.

CONCLUSIONS

The initial goal of this research was to determine how an error in the measure of the mass flow rate in a rocket engine would affect the specific impulse. This well defined problem developed into an examination of the specific impulse itself and why it is used and what the limitations are when it is measured. In order to describe the physics of the problem, a new term had to be defined that was related to the specific impulse and it was called the instantaneous specific impulse for lack of a better term. The definition proved to be useful in that the parameters could be measured more readily than the integral required in the usual definition and the new term could be written in terms of the old one with the addition of a correction which took care of the non-constant behavior of the exhaust velocity and the mass flow rate, which was assumed from the beginning. A simple, perhaps nonrealistic set of dependencies on time were later assumed for the various parameters so that a visual representation of the results could be derived.

REFERENCES

1. Tsiolkovsky. *Investigating Space with Rocket Devices*, Nauchnoye Obozreniye (Science Review), Russia, 1898.
2. Sutton, George P., *Rocket Propulsion Elements*, John Wiley and Sons, 5th Edition, 1986, p. 3
3. ibid p.21
4. *University Physics*, F. Sears and M. Zemanski, Addison-Wesley, 4th Edition, 1970, p. 201
5. Benson,Tom, NASA Glenn Learning Tech., http://www.grc.nasa.gov/WWW/K-12