NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM

No. 1224

LAME'S WAVE FUNCTIONS OF THE ELLIPSOID OF REVOLUTION

By J. Meixner

Translation of ZWB Forschungsbericht Nr. 1952, June 1944

Washington
April 1949

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LAMÉ'S WAVE FUNCTIONS OF THE ELLIPSOID OF REVOLUTION*

By J. Meixner

1. INTRODUCTION

Lamé's wave functions result by separation of the wave equation in elliptic coordinates and by integration of the ordinary differential equations thus originating. They are a generalization of Lamé's potential functions which originate in the same manner from the potential equation. Lamé's wave functions are applied for boundary value problems of the wave equation for regions of space bounded by surfaces of a system of confocal ellipsoids and hyperboloids.

For general elliptic coordinates Lamé's wave functions have not been fully calculated so far. Except for a few general properties, not much is known about them. More consideration was given to Lamé's wave functions for the case of rotationally symmetrical elliptic coordinates (called for short, Lamé's wave functions of the ellipsoid of revolution). However, even for these functions few results are in existence compared with those for the better known special functions of mathematical physics, such as cylindrical and spherical functions.

The first more detailed investigation of Lamé's wave functions of the ellipsoid of revolution was made by Niven (reference 1) who with their aid treated a heat-conduction problem in the ellipsoid of revolution. However, the numerical values of the coefficients of his series developments in terms of spherical and cylindrical functions as they are given for the lowest indices contain several errors which were taken over into the report by Strutt (reference 2). A more extensive investigation with a greater number of applications was made by Maclaurin (reference 3). Möglich (reference 4), whose mathematical investigation of Lamé's wave equation is based on certain linear homogeneous integral equations, obtained results of a

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*"Die Laméschen Wellenfunktionen des Drehellipsoids." Zentrale für wissenschaftliches Berichtswesen der Luftfahrtforschung des Generalluftzeugmeisters (ZWB) Berlin-Adlershof, Forschungsbericht Nr. 1952, June 1944.

more general character. Strutt (reference 2) gives a survey of the state of the theory of Lamé's wave functions in 1932; he also demonstrates on a large number of examples from acoustics, electrodynamics, optics, wave mechanics, and theory of wave filters, the manifold possibilities of application for these functions.

Of the treatises published in the meantime, an investigation by Hanson (reference 5), which contains several new details, should be mentioned, as well as a treatise by Morse (reference 6) on addition theorems, that is, on the development of the plane wave and the spherical wave in terms of Lamé's wave functions, furthermore, a number of treatises on the wave-mechanical treatment of the ion of the hydrogen molecule (reference 7). Kotani (reference 8) deals with integral equations for Lamé's wave functions. In particular, a treatise by Chu and Stratton (reference 9) should be pointed out which settles exhaustively the problem (treated so far only incompletely) of the continuation of the solutions of equation (2.4g) for large and small argument and shows in detail how the entire theory of Mathieu's functions results as a special and boundary case from the general theory of Lamé's wave functions. Finally, a treatise by Bouwkamp (reference 10) on the theoretical and numerical treatment of diffraction on a circular aperture is to be mentioned which, for the first time, contains more detailed numerical material concerning Lamé's wave functions of the ellipsoid of revolution.

The main task of the present report on Lamé's wave functions of the ellipsoid of revolution will be to compile their most important properties in such a manner that these functions take on a form which facilitates their application. In this connection an investigation of the solutions of the ordinary homogeneous linear differential equations of the second order, which originate with separation of the wave equation in rotationally symmetrical elliptic coordinates, is of importance; further, it has to be determined what is to be understood in these solutions by functions of the first and second kind, their normalization as well as the description of the behavior of these solutions in different domains of the independent variables, in particular, their asymptotic behavior. Here belongs also the indication of a method of numerical calculation of these functions and the presentation of numerical tables.

For the purpose of clarity it was necessary to generalize and supplement the existing material in some respects and to simplify some of the calculations and proofs. Therewith the theory of Lamé's wave functions of the ellipsoid of revolution as a whole would seem to have reached a development equivalent to the theory of Mathieu's functions, and it probably even is somewhat simpler; Mathieu's functions, namely, represent not a regular but a singular special case of Lamé's wave functions.

2. THE BASIC EQUATION

2.1 Rotationally Symmetrical Elliptic Coordinates

In dealing with the rotationally symmetrical elliptic coordinates one must distinguish between those with oblate and those with prolate ellipsoids of revolution. Accordingly, one adds in the numbers denoting the formulas, the letter a or g, respectively, where both cases appear. x, y, z are the Cartesian, ξ, η, φ the elliptic coordinates. c is a positive constant. Then it follows that:

$$\left.\begin{aligned} &x = c\sqrt{(1 - \eta^2)(\xi^2 + 1)}\cos\varphi, \quad y = c\sqrt{(1 - \eta^2)(\xi^2 + 1)}\sin\varphi, \quad z = c\xi\eta \\ &0 \leqq \xi < \infty, \quad -1 \leqq \eta \leqq 1, \quad 0 \leqq \varphi \leqq 2\pi \end{aligned}\right\} \tag{2.1a}$$

$$\left.\begin{aligned} &x = c\sqrt{(1 - \eta^2)(\xi^2 - 1)}\cos\varphi, \quad y = c\sqrt{(1 - \eta^2)(\xi^2 - 1)}\sin\varphi, \quad z = c\xi\eta \\ &1 \leqq \xi < \infty, \quad -1 \leqq \eta \leqq 1, \quad 0 \leqq \varphi \leqq 2\pi \end{aligned}\right\} \tag{2.1g}$$

The symbols are the same as in the collection of formulas by Magnus and Oberhettinger (reference 11) to which reference is made also with respect to the transformation of the wave equation to elliptic coordinates and with respect to the separation of the wave equation.

2.2 Separation of the Wave Equation

Solutions of the wave equation in three dimensions are to be determined. (k = wave number.)

$$\Delta u + k^2 u = 0 \tag{2.2}$$

of the form

$$u = f_1(\xi)f_2(\eta)f_3(\varphi) \tag{2.3}$$

Then the ordinary differential equations

$$\frac{d}{d\xi}\left[\left(1 + \xi^2\right)\frac{df_1}{d\xi}\right] + \left(\frac{\mu^2}{1 + \xi^2} + k^2c^2\xi^2 - \lambda\right)f_1 = 0 \tag{2.4a}$$

$$\frac{d}{d\eta}\left[\left(1 - \eta^2\right)\frac{df_2}{d\eta}\right] + \left(-\frac{\mu^2}{1 - \eta^2} + k^2c^2\eta^2 + \lambda\right)f_2 = 0 \tag{2.5a}$$

$$\frac{d}{d\xi}\left[\left(1 - \xi^2\right)\frac{df_1}{d\xi}\right] + \left(-\frac{\mu^2}{1 - \xi^2} - k^2c^2\xi^2 + \lambda\right)f_1 = 0 \tag{2.4g}$$

$$\frac{d}{d\eta}\left[\left(1 - \eta^2\right)\frac{df_2}{d\eta}\right] + \left(-\frac{\mu^2}{1 - \eta^2} - k^2c^2\eta^2 + \lambda\right)f_2 = 0 \tag{2.5g}$$

$$\frac{d^2f_3}{d\varphi^2} = -\mu^2 f_3 \tag{2.6}$$

are valid for f_1, f_2, and f_3.

λ and μ^2 are the separation parameters. First, they are assumed to be any complex numbers. They can only be determined for a given boundary value problem. In particular, μ need not be an integer; this can be recognized, for instance, in the treatment of an inside space problem in a sector $0 \leqq \varphi \leqq \varphi_0$ of an ellipsoid of revolution.

2.3 Reduction to a Differential Equation

The differential equation (2.4g) is designated as the basic equation. (2.5g) is identical with it; (2.4a) is transformed into it when ξ is replaced by $\pm i\xi$ and k^2c^2 by $-k^2c^2$. Therewith the investigation of the differential equations (2.4a), (2.5a), and (2.5g) is reduced to that of the differential equation (2.4g). The basic domain, however, is not the same for all cases; it extends from -1 to 1 in the cases (2.5a) and (2.5g), from 1 to ∞ in the case (2.4g), whereas the basic domain of the differential equation (2.4a) in the transformation to (2.4g) will be changed to the domain from 0 to $i\infty$ (or else $-i\infty$). It proves, therefore, to be necessary to investigate the differential equation (2.4g) in the entire complex ξ-plane.

2.4 Transformations of the Basic Equation

The basic equation represents a special case of the linear homogeneous differential equation of the second order with four extra essential singularities, two of which are made to join to one essential singularity. The latter is at infinity, the two remaining extra essential singularities are at 1 and -1. The present investigation of the basic equation will start with connecting its solutions with the solutions of limiting cases of the basic equation. For $k^2c^2 = 0$, the basic equation is transformed into the differential equation of the spherical functions and their associated functions or, as they will be called here, of the general spherical functions. If one lets the two singularities at 1 and -1 combine into a single singularity at $\xi = 0$, there originates, aside from an elementary transformation, the differential equation of the cylindrical functions. This is brought about by the substitution

$$\zeta = \gamma \xi_1 \tag{2.7}$$

and the abbreviation

$$\gamma = kc \tag{2.8}$$

if one then performs the limiting process $\gamma \to 0$. From (2.4g) there originates with

$$f_1 = \left(\zeta^2 - \gamma^2\right)^{\mu/2} \zeta^{-\mu} v(\zeta) \tag{2.9}$$

the differential equation

$$\left(\zeta^2 - \gamma^2\right)\frac{d^2v}{d\zeta^2} + 2\zeta\left(1 + \frac{\mu}{\zeta^2}\gamma^2\right)\frac{dv}{d\zeta} + \left[-\lambda + \zeta^2 - \frac{\mu(\mu + 1)}{\zeta^2}\gamma^2\right]v = 0 \tag{2.10}$$

In the transition from (2.4g) to (2.4a) ζ is transformed into itself and γ^2 need only be replaced by $-\gamma^2$. For large distances, that is, $r^2 = x^2 + y^2 + z^2 >> c^2$

$$\zeta^2 = k^2r^2\left[1 - \frac{(x^2 + y^2)c^2}{r^4} + O\left(\frac{c^4}{r^4}\right)\right] \tag{2.11a}$$

$$\zeta^2 = k^2r^2\left[1 + \frac{(x^2 + y^2)c^2}{r^4} + O\left(\frac{c^4}{r^4}\right)\right] \tag{2.11g}$$

are valid.

Another important limiting case of the basic equation occurs if, of the two singularities of the basic equation located at finite distance, one or both move to infinity. Then the differential equation of Laguerre's and Hermite's orthogonal functions, respectively, is formed. This limiting case will yield the asymptotics of the eigenvalues and eigenfunctions for large absolute value of γ.

2.5 Connection with Mathieu's Functions

Mathieu's functions are, in connection with Lamé's wave functions, obtained in two ways. They appear, as is well known, in the separation of the wave equation in the coordinates of the elliptic cylinder and must, therefore, also appear in the limiting case of Lamé's wave functions for the ellipsoid with three axes when one axis becomes infinitely long. However, Mathieu's differential equation is also obtained, except for an elementary transformation, if μ in (2.4g) is set equal to $\pm 1/2$. This also indicates that it is useful to consider the basic equation not only for μ that are integers, but rather for arbitrary coefficients ν and μ. The theory of Mathieu's functions is, therefore, a special case of the theory of Lamé's wave functions of the ellipsoid of revolution. Although the present report does not yield new results of Mathieu's functions, it demonstrates how they fit into a more general picture.

3. SPHERICAL AND CYLINDRICAL FUNCTIONS

3.1 A Few Formulas for Spherical Functions

The most important formulas and theorems for spherical and cylindric functions needed below are compiled and a few estimates for these functions are given, which will be necessary for considerations on uniform convergence of certain series in terms of such functions. Magnus and Oberhettinger (reference 11) is again referred to concerning the notation and additional formulas. The general spherical functions $\underline{P}_\nu^\mu(\xi)$ and $\underline{Q}_\nu^\mu(\xi)$ both satisfy the differential equation

$$\frac{d}{d\xi}\left[(1 - \xi^2)\frac{d}{d\xi}\underline{P}_\nu^\mu(\xi)\right] + \left[\nu(\nu + 1) - \frac{\mu^2}{1 - \xi^2}\right]\underline{P}_\nu^\mu(\xi) = 0 \qquad (3.1)$$

and both satisfy the recursion formula

$$(2\nu + 1)\xi\underline{P}_\nu^\mu(\xi) = (\nu - \mu + 1)\underline{P}_{\nu+1}(\xi) + (\nu + \mu)\underline{P}_{\nu-1}^\mu(\xi) \qquad (3.2)$$

from which by three times repeated application

$$\xi^2\underline{P}_\nu^\mu(\xi) = \frac{(\nu - \mu + 2)(\nu - \mu + 1)}{(2\nu + 1)(2\nu + 3)}\underline{P}_{\nu+2}^\mu(\xi) + \frac{2\nu^2 + 2\nu - 2\mu^2 - 1}{(2\nu - 1)(2\nu + 3)}\underline{P}_\nu^\mu(\xi)$$

$$+\frac{(\nu + \mu)(\nu + \mu - 1)}{(2\nu - 1)(2\nu + 1)}\underline{P}_{\nu-2}^\mu(\xi) \qquad (3.3)$$

is obtained. Further, the series presentation which converges for $|\xi| > 1$ is given:

$$Q_\nu^\mu(\xi) = \pi^{1/2} e^{\mu\pi i} \left(\xi^2 - 1\right)^{\mu/2} \xi^{-\mu} (2\xi)^{-\nu-1} \sum_{p=0}^{\infty} \frac{\Gamma(\nu + \mu + 2p + 1)}{\Gamma\left(\nu + p + \frac{3}{2}\right)\Gamma(p + 1)} (2\xi)^{-2p} \tag{3.4}$$

where $\arg(\xi^2 - 1) = 0$ when ξ is real and >1, $\arg \xi = 0$ when ξ is real and >0. In order to obtain the uniqueness of the general spherical functions a branch cut is put from $\xi = -\infty$ over -1 to 1.

3.2 A Few Estimates for Spherical Functions

For the general spherical functions of the second kind the integral presentation

$$Q_\nu^\mu(\xi) = ie^{(\mu-\nu)\pi i} \frac{2^\mu \Gamma\left(\mu + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{4\pi \sin(\nu + \mu)\pi} \left(\xi^2 - 1\right)^{\mu/2} z^{-\nu-\mu-1} \int_0^{(1+,0+,1-,0-)} u^{\nu+\mu} (1 - \lambda)^{-1/2-\mu} \times \left(1 - \frac{u}{z^2}\right)^{-1/2-\mu} du \tag{3.5}$$

is valid.

Therein $z = \xi + \sqrt{\xi^2 - 1}$; the sign of the root must be selected so that $|z| \geqq 1$ (one need hardly be afraid that z could be mistaken for the Cartesian coordinate z). In order to estimate $\underline{Q}^{\mu}_{\nu+r}(\xi)$ for $r \rightarrow \pm\infty$ one forms the absolute values of the individual factors ahead of and in the integral. $|u|^r$ in the integral is replaced by its maximum value. This latter can, by suitable selection of the path of integration, be made smaller than $(1 + \delta)^r$ for $r > 0$, smaller than δ^r for $r < 0$, where δ is an arbitrarily small but fixed positive number independent of r, ξ. Then one obtains

$$\left|\underline{Q}^{\mu}_{\nu+r}(\xi)\right| \leq \left\{\begin{matrix} (1 + \delta)^r \\ \delta^r \end{matrix}\right\} |z|^{-r} \, \Phi^{\mu}_{\nu}(\xi) \qquad \begin{matrix} \text{for } r = 0, 1, 2, \ldots \\ \text{for } r = -1, -2, \ldots \end{matrix} \tag{3.6}$$

$\Phi^{\mu}_{\nu}(\xi)$ is a restricted positive function of ξ independent of r in each closed domain excluding the points $\xi = \pm 1, \infty$. Only the case of $\nu + \mu$ being an integer requires special consideration. For $\nu + \mu = 0, 1, 2, \ldots$ it can be demonstrated that the estimate (3.6) is retained; for $\nu + \mu = -1, -2, \ldots$ the estimate above is valid for $\underline{Q}^{\mu}_{\nu+r}(\xi) / \Gamma(\nu + r + \mu + 1)$ instead of $\underline{Q}^{\mu}_{\nu+r}(\xi)$. The case $\mu = -\frac{1}{2}, -\frac{3}{2}, \ldots$ is not to be excluded, as can be shown easily.

The general spherical functions of the first kind are given by the integral representation

$$\underline{P}^{\mu}_{\nu}(\xi) = \frac{i}{2\pi} 2^{\mu} \frac{\Gamma\left(\mu + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \left(\xi^2 - 1\right)^{\mu/2} z^{-\nu-\mu-1} \oint^{(1+, z^2-)} u^{\nu+\mu} (1 - u)^{-1/2-\mu} \left(1 - \frac{u}{z^2}\right)^{-1/2-\mu} du \tag{3.7}$$

It is valid for $-2\pi < \arg z^2 < 2\pi$ with the provision that the path of integration for $\pi < \arg z^2 < 2\pi$ and $-2\pi < \arg z^2 < -\pi$, respectively, leads past the left of the point $u = 0$ and is also to be returned there. The estimate has to be made as above. The maximum of $|u|^r$ is, if the path of integration is suitably selected, smaller than $(|z|^2 + \delta)^r$ for positive r's and smaller than δ^r for negative r's, where δ is a number of the conditions indicated above, so that

$$\begin{aligned} P^{\mu}_{\nu+r}(\xi) &\leqq \left(|z| + \frac{\delta}{|z|}\right)^r \psi^{\mu}_{\nu}(\xi) \qquad \text{for } r = 0, 1, 2, \ldots \\ &\leqq |z|^{-r} \bar{\psi}^{\mu}_{\nu}(\xi) \qquad \text{for } r = -1, -2, -3, \ldots \end{aligned} \tag{3.8}$$

respectively, where $\psi^{\mu}_{\nu}(\xi)$ and $\bar{\psi}^{\mu}_{\nu}(\xi)$ are positive, restricted functions independent of r in each closed domain excluding the points $\xi = \pm 1, \infty$.

3.3 A Few Formulas for Cylindrical Functions

For the following it is more convenient to introduce not the cylindrical functions themselves but rather the functions

$$\psi_{\nu}(\zeta) = \sqrt{\frac{\pi}{2\zeta}}\, F_{\nu+1/2}(\zeta), \qquad n_{\nu}(\zeta) = \sqrt{\frac{\pi}{2\zeta}}\, N_{\nu+1/2}(\zeta) \tag{3.9}$$

They both satisfy the differential equation

$$\frac{d^2\psi_{\nu}}{d\zeta^2} + \frac{2}{\zeta}\frac{d\psi_{\nu}}{d\zeta} + \left[1 - \frac{\nu(\nu+1)}{\zeta^2}\right]\psi_{\nu} = 0 \tag{3.10}$$

and the recursion formulas

$$\frac{2\nu + 1}{\zeta} \frac{d\psi_\nu}{d\zeta} = \frac{\nu}{2\nu - 1} \psi_{\nu-2} + \frac{2\nu + 1}{(2\nu - 1)(2\nu + 3)} \psi_\nu - \frac{\nu + 1}{2\nu + 3} \psi_{\nu+2} \quad (3.11)$$

$$\frac{2\nu + 1}{\zeta^2} \psi_\nu = \frac{1}{2\nu - 1} \psi_{\nu-2} + \frac{2(2\nu + 1)}{(2\nu - 1)(2\nu + 3)} \psi_\nu + \frac{1}{2\nu + 3} \psi_{\nu+2} \quad (3.12)$$

Besides, the simpler recursion formula

$$\frac{2\nu}{\zeta} Z_\nu(\zeta) = Z_{\nu+1}(\zeta) + Z_{\nu-1}(\zeta) \quad (3.13)$$

is to be noted for the cylindrical functions from which (3.12) is obtained by repeated application. Finally, the consistently convergent series development

$$\psi_\nu(\zeta) = \frac{1}{2}\pi^{1/2}\left(\frac{\zeta}{2}\right)^\nu \sum_{p=0}^{\infty} \frac{\left(\frac{i\zeta}{2}\right)^{2p}}{\Gamma(p + 1)\Gamma\left(\nu + p + \frac{3}{2}\right)} \quad (3.14)$$

is given with $\arg \zeta = 0$ if ζ is real and >0.

3.4 An Estimate for Cylindrical Functions

For the cylindrical functions $Z_{\nu+r}(\zeta)$ one obtains by repeated application of (3.13)

$$Z_{\nu+r}(\zeta) = \left(\frac{2}{\zeta}\right)^r \frac{\Gamma(\nu + r)}{\Gamma(\nu)}\left[1 - \frac{r - 1}{(\nu + r - 1).\nu.1!}\left(\frac{\zeta}{2}\right)^2 + \frac{(r - 2)(r - 3)}{(\nu + r - 1)(\nu + r - 2)\nu(\nu + 1)2!}\left(\frac{\zeta}{2}\right)^4 \cdot \cdot \cdot\right] Z_\nu(\zeta)$$

$$-\left(\frac{2}{\zeta}\right)^{r-1} \frac{\Gamma(\nu + r)}{\Gamma(\nu + 1)}\left[1 - \frac{r - 2}{(\nu + r - 1)(\nu + 1)1!}\left(\frac{\zeta}{2}\right)^2\right.$$

$$\left. + \frac{(r - 3)(r - 4)}{(\nu + r - 1)(\nu + r - 2)(\nu + 1)(\nu + 2)2!}\left(\frac{\zeta}{2}\right)^4 \cdot \cdot \cdot\right] Z_{\nu-1}(\zeta) \qquad (3.15)$$

The number of the sum terms in brackets to be included is $\frac{r}{2} + 1$ for even and $\frac{r + 1}{2}$ for odd positive r's. If $\operatorname{Re} \nu \geqq 1$, one obtains from (3.15) the following estimate

$$\left|Z_{\nu+r}(\zeta)\right| \leqq \left|\frac{2}{\zeta}\right|^r \frac{\Gamma(\nu + r)}{\Gamma(\nu)}\left[\left|Z_\nu(\zeta)\right| F_0(i\,|\zeta|) + \left|Z_{\nu-1}(\zeta)\right| F_1(i\,|\zeta|)\right] \qquad (3.16)$$

Therein $r = 1, 2, 3, \ldots$. A corresponding estimate may be obtained for $r = -1, -2, -3, \ldots$. Since the singular points of the cylindrical functions lie at 0 and ∞, the function

$$\left|Z_{\nu+r}(\zeta)\right| \left|\frac{\zeta}{2}\right|^r \frac{1}{\left|\Gamma(\nu + r)\right|} \qquad (3.17)$$

is in every closed domain, excluding the points 0 and ∞, restricted (considered as a function of ζ); the upper limit does not depend on r. If $\operatorname{Re} \nu < 1$, this relation is valid at least for such r's for which $\operatorname{Re} (\nu + r) \geqq 2$. A corresponding estimate is valid for negative r's.

4. THE X-FUNCTIONS OF THE FIRST AND SECOND KIND

4.1 Definition of the X-Functions of the First and Second Kind

Since for $\gamma = 0$ the basic equation (2.4g) is joined to the differential equation (3.1) of the general spherical functions, it suggests itself to develop the solutions of the basic equation in terms of spherical functions. One formulates the two at first formal series

$$X_\nu^{\mu(1)}(\xi;\gamma) = \sum_{r=-\infty}^{\infty} i^r a_{\nu,r}^\mu(\gamma) \underline{P}_{\nu+r}^\mu(\xi) \tag{4.1}$$

$$X_\nu^{\mu(2)}(\xi;\gamma) = \sum_{r=-\infty}^{\infty} i^r a_{\nu,r}^\mu(\gamma) \underline{Q}_{\nu+r}^\mu(\xi) \tag{4.2}$$

and attempts to determine the coefficients $a_{\nu,r}^\mu(\gamma)$ and the index ν in such a manner that these two series formally satisfy the basic equation and converge. The further problem will be to investigate the convergence properties of the two series (4.1) and (4.2) in order to determine that, for the two series, one has to deal with analytic functions which, in general, are linearly independent solutions of the basic equation.

For the coefficients $a_{\nu,r}^\mu(\gamma)$ the indices ν,μ and the argument γ will be omitted where there is no danger of confusion; the same applies to the coefficients to be introduced later for series developments of a similar kind. The summation index r assumes only even values. The term with $r = 0$ in the two series (4.1) and (4.2) is designated as the principal term of the series. In the solutions (4.1) and (4.2) of the basic equation an arbitrary constant factor remains undetermined. It may be determined in some way. Then the series (4.1) is denoted as X-function of the first kind and the series (4.2) as X-function of the second kind with the argument ξ and with the indices ν,μ with the parameter γ. It will be found that the index ν is determined by the separation parameter λ; more accurately, there exists a functional relation between λ, ν, μ, and γ which is expressed by

$$\lambda = \lambda_{\nu}^{\mu}(\gamma) \tag{4.3}$$

The series (4.1) and (4.2) are now inserted in the basic equation, the differential quotients of the spherical functions are eliminated by means of the differential equation of the spherical functions (3.1), and the factor ξ^2 of the spherical functions is eliminated by application of (3.3). Then there appears an infinite sum of spherical functions with coefficients independent of ξ which is equal to zero. The disappearance of the individual coefficient is sufficient to this end. This leads to the conditional equations

$$\frac{1}{\gamma^2}\Phi_r a_r = q_r a_{r-2} + p_r a_{r+2} \qquad (r = 0, \pm 2, \pm 4, \ldots) \tag{4.4}$$

with the abbreviation

$$\left.\begin{aligned}
\Phi_r &= -\lambda + (\nu + r + 1)(\nu + r) + \gamma^2 \frac{2(\nu + r + 1)(\nu + r) - 2\mu^2 - 1}{(2\nu + 2r + 3)(2\nu + 2r - 1)} \\
p_r &= \frac{(\nu + r + \mu + 2)(\nu + r + \mu + 1)}{(2\nu + 2r + 5)(2\nu + 2r + 3)} \\
q_r &= \frac{(\nu + r - \mu)(\nu + r - \mu - 1)}{(2\nu + 2r - 1)(2\nu + 2r - 3)}
\end{aligned}\right\} \tag{4.5}$$

4.2 General Qualities of the Coefficients a_r

The recursion formula for the coefficients a_r is interpreted as a difference equation. In order to avoid complications, the case of real fractional values of one-half for ν is completely excluded and the case of real integers for $\nu + \mu$ and $\nu - \mu$, respectively, is postponed. Concerning the behavior of the coefficients a_r at infinity, a simple formulation can be obtained according to Kreuser (reference 12). The equations

$$\lim_{r \to \infty} \sup \sqrt[r]{\frac{|a \pm r|}{r!}} = \frac{2}{|\gamma|} \tag{4.6}$$

or

$$\lim_{r \to \infty} \sup \sqrt[r]{|a \pm r| r!} = \frac{|\gamma|}{2} \tag{4.7}$$

are valid.

If the behavior of the coefficients a_r at infinity is given by (4.6) at least for negative or positive r's, they increase too strongly to make a convergence of series (4.1) and (4.2) possible. Therefore, a solution of the difference equation (4.4) is to be found which shows the behavior (4.7) for $r \to \infty$ as well as for $r \to -\infty$. Although there always exists an exact solution which behaves for $r \to -\infty$ as indicated in (4.7), this solution will in general exhibit for $r \to -\infty$ the behavior (4.6). Only for certain distinct values of the parameter ν (free so far), the behavior (4.7) prevails for both $r \to \infty$ and $r \to -\infty$; inversely, in this manner distinct values of λ are coordinated to each value of ν. For $\gamma = 0$ the conditions are particularly simple. There becomes for all r's

$$\left[\lambda - (\nu + r + 1)(\nu + r)\right] a_r = 0 \qquad (r = 0, \pm 2, \pm 4, \ldots)$$

Thus λ can, for a given ν, assume any of the values $(\nu + r + 1) \times (\nu + r)$. It is determined by the requirement that the series (4.1) and (4.2) should be reduced to the principal term for this case, which leads to $\lambda = \nu(\nu + 1)$. Now it is further required that under $\lambda_\nu^\mu(\gamma)$ always the value should be understood which goes over to $\nu(\nu + 1)$ for $\gamma \rightarrow 0$. The existence of such a distinct λ-value to each given ν, μ, and ϵ and its uniqueness will not be proved here; it follows from the method of calculation given in section 6 for the determination of λ.

From here on, the coefficients $a_{\nu,r}^{\mu}(\gamma)$ will always represent that solution of the difference equation (4.4) which shows the behavior (4.7) for $r \rightarrow \pm\infty$, belongs to the value $\lambda_\nu^\mu(\gamma)$, and therefore has the boundary values

$$\lim_{\gamma=0} a_r = 0 \qquad (r = \pm 2, \pm 4, \pm 6, \ldots) \tag{4.8}$$

Furthermore, the constant factor which is arbitrary in the coefficients a_r may be determined in a given manner.

4.3 Convergence of the Series Developments of the X-Functions of the First and Second Kind

From the estimates (3.6) and (3.8) as well as from the boundary values (4.7) for $r \rightarrow \pm\infty$ there follows immediately that the series (4.1) and (4.2) in each closed domain, which does not include the points $\xi = \pm 1, \infty$, will converge absolutely and uniformly. One may further conclude that the series (4.1) and (4.2) will converge as well as the exponential series. Since the individual terms of these series are analytic functions in this domain, there follows from the uniform convergence that the sums of the series themselves will again be analytic functions, the singularities of which can lie only at $\xi = \pm 1, \infty$, furthermore, that the series can be differentiated termwise, and therewith the fact that the functions represented by these series are real solutions of the basic equation.

4.4 Further Solutions of the Basic Equation and Their Relation to the X-Functions of the First and Second Kind

Between the general spherical functions $\underline{P}_\nu^\mu$, $\underline{P}_\nu^{-\mu}$, $\underline{P}_{-\nu-1}^\mu$, $\underline{P}_{-\nu-1}^{-\mu}$, $\underline{Q}_\nu^\mu$, $\underline{Q}_\nu^{-\mu}$, $\underline{Q}_{-\nu-1}^\mu$, $\underline{Q}_{-\nu-1}^{-\mu}$ all of which satisfy the same differential equation there exist, in general, six linear relations independent of each other. They can be generalized for the X-functions of the first and second kind. To this end several relations for the coefficients $a_{\nu,r}^\mu(\gamma)$ will be derived.

The system of equations (4.4) and the system of equations originating from it by the substitution $\nu \rightarrow -\nu-1$ and $r \rightarrow -r$ are identical because of

$$\Phi_{-\nu-1,-r}^\mu = \Phi_{\nu,r}^\mu, \qquad q_{-\nu-1,-r}^\mu = p_{\nu,r}^\mu$$

Due to the uniqueness of the solution there follows from it

$$\lambda_{-\nu-1}^\mu(\gamma) = \lambda_\nu^\mu(\gamma) \tag{4.9}$$

Furthermore, the constant factor which is arbitrary in the a_r's can be determined in such a manner that

$$a_{-\nu-1,-r}^\mu(\gamma) = a_{\nu,r}^\mu(\gamma) \tag{4.10}$$

The system of equations (4.4) and the system of equations originating from it by the substitution $\mu \rightarrow -\mu$ become identical if one introduces in the latter instead of the $a_{\nu,r}^{-\mu}$'s the values

$$b_{\nu,r}^\mu(\gamma) = \frac{\Gamma(\nu + r + \mu + 1)}{\Gamma(\nu + r - \mu + 1)} \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} a_{\nu,r}^\mu(\gamma) \tag{4.11}$$

They are determined so that $b_o = a_o$. Then the equations

$$b^{\mu}_{\nu,r}(\gamma) = a^{-\mu}_{\nu,r}(\gamma) \tag{4.12}$$

and

$$\lambda^{-\mu}_{\nu}(\gamma) = \lambda^{\mu}_{\nu}(\gamma) \tag{4.13}$$

are valid.

After these preparations, at first a relation between $X^{-\mu(1)}_{\nu}(\xi;\gamma)$, $X^{\mu(1)}_{\nu}(\xi;\gamma)$, and $X^{\mu(2)}_{\nu}(\gamma)$ is derived, since according to (4.1) and (4.12)

$$X^{-\mu(1)}_{\nu}(\xi;\gamma) = \sum_{r=-\infty}^{\infty} i^r b^{\mu}_{\nu,r}(\gamma) P^{-\mu}_{\nu+r}(\xi) \tag{4.14}$$

is valid.

If one expresses in this equation the spherical function $P^{-\mu}_{\nu+r}(\xi)$ by $P^{\mu}_{\nu+r}(\xi)$ and $Q^{\mu}_{\nu+r}(\xi)$ (under consideration of (4.1), (4.2), and (4.11)) the required relation

$$X^{-\mu(1)}_{\nu}(\xi;\gamma) = \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)}\left[X^{\mu(1)}_{\nu}(\xi;\gamma) - \frac{2}{\pi}e^{-\mu\pi i} \sin \mu\pi \; X^{\mu(2)}_{\nu}(\xi;\gamma)\right] \tag{4.15}$$

will be found at once.

In exactly the same way there result the formulas

$$X^{\mu(1)}_{-\nu-1}(\xi;\gamma) = X^{\mu(1)}_{\nu}(\xi;\gamma) \tag{4.16}$$

$$X_\nu^{-\mu(2)}(\xi;\gamma) = e^{-2\mu\pi i}\,\frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)}\,X_\nu^{\mu(2)}(\xi;\gamma) \qquad (4.17)$$

$$X_\nu^{\mu(2)}(\xi;\gamma)\ \sin\,(\nu + \mu)\pi - X_{-\nu-1}^{\mu(2)}(\xi;\gamma)\ \sin\,(\nu - \mu)\pi = \pi e^{\mu\pi i}\ \cos\,\nu\pi\ X_\nu^{\mu(1)}(\xi;\gamma) \qquad (4.18)$$

$$X_\nu^{\mu(2)}(-\xi;\gamma) = -e^{\pm\nu\pi i}\ X_\nu^{\mu(2)}(\xi;\gamma) \qquad (4.19)$$

$$X_\nu^{\mu(1)}(-\xi;\gamma) = e^{\mp\nu\pi i}X_\nu^{\mu(1)}(\xi;\gamma) - \frac{2}{\pi}\sin\,(\nu + \mu)\pi e^{-\mu\pi i}\ X_\nu^{\mu(2)}(\xi;\gamma) \qquad (4.20)$$

For the last two relations the upper or lower sign is valid depending on whether the imaginary part of ξ is positive or negative.

The relations (4.15) to (4.20) are identical with corresponding relations for the general spherical functions. This was obtained by the determination of the arbitrary constant factor in the a_r's. That determination is not yet unique; it leaves a certain latitude. It is compatible with (4.10), (4.11), and (4.12) if one sets the coefficient of the principal term equal to one; but then it may happen for certain combinations of values ν, μ, and γ that all a_r's with the exception of a_o become infinitely large. Other determinations still possible within this latitude would be the requirements that $\sum_{r=-\infty}^{\infty} i^r a_r \sum_{r=-\infty}^{\infty} i^r b_r = 1$

or $\sum_{r=-\infty}^{\infty} \frac{2\nu + 1}{2\nu + 2r + 1}\,a_r b_r = 1.$

4.5 General Relations between the X-Functions

The following general relations are valid for the general spherical functions in the case $|\xi| > 1$

$$\underline{Q}_\nu^\mu(\xi e^{l\pi i}) = e^{-l(\nu+1)\pi i}\, \underline{Q}_\nu^\mu(\xi) \tag{4.21}$$

$$\underline{P}_\nu^\mu(\xi e^{l\pi i}) = e^{l\nu\pi i}\, P_\nu^\mu(\xi) - \frac{2i}{\pi} \frac{\sin \pi l\left(\nu + \frac{1}{2}\right)}{\sin \pi\left(\nu + \frac{1}{2}\right)} \sin (\nu + \mu)\pi e^{-(\mu+1/2l)\pi i}\, \underline{Q}_\nu^\mu(\xi) \tag{4.22}$$

l is therein an integer, either positive or negative. The coefficients in (4.21) and (4.22) remain unchanged if ν is replaced by $\nu + r$, r being any even number. If one replaces ν in (4.21) and (4.22) by $\nu + r$, multiplies by $i^r a_r$ and forms the sum over all even r from $-\infty$ to ∞, there is formed because of (4.1) and (4.2)

$$X_\nu^{\mu(1)}(\xi e^{l\pi i};\gamma) = e^{l\nu\pi i}\, X_\nu^{\mu(1)}(\xi;\gamma) - \frac{2i}{\pi} \frac{\sin \pi l\left(\nu + \frac{1}{2}\right)}{\sin \pi\left(\nu + \frac{1}{2}\right)} \sin (\nu + \mu)\pi e^{-(\mu+1/2l)\pi i}\, X_\nu^{\mu(2)}(\xi;\gamma) \tag{4.23}$$

$$X_\nu^{\mu(2)}(\xi e^{l\pi i};\gamma) = e^{-l(\nu+1)\pi i}\, X_\nu^{\mu(2)}(\xi;\gamma) \tag{4.24}$$

These two general relations which are valid for $|\xi| > 1$, are in the case $l = 1$ transformed not to exactly (4.19) and (4.20); $\xi e^{\pi i}$, namely, represents in (4.23) and (4.24) an increase of the argument by π, whereby under certain conditions the branch cut may be passed, whereas the argument of $-\xi$ in (4.19) and (4.20) is obtained by choosing such a path from ξ to $-\xi$ that the branch cut extending from $-\infty$ to 1 will not be passed.

5. THE Z-FUNCTIONS OF THE FIRST TO FOURTH KIND

5.1 Definition of the Z-Functions of the First to Fourth Kind

If the two extra-essential singularities of the basic equation (2.4g) are made to join, as indicated in section 2, there originates, aside from an elementary transformation, Bessel's differential equation. It therefore suggests itself to attempt a solution of the basic equation also by series developments in terms of cylindrical functions. The functions defined by the series (which are at first formal)

$$Z_\nu^{\mu(1)}(\zeta;\gamma) = \left(\zeta^2 - \gamma^2\right)^{\mu/2} \zeta^{-\mu} \sum_{r=-\infty}^{\infty} b_{\nu,r}^{\mu}(\gamma)\, \psi_{\nu+r}(\zeta) \tag{5.1}$$

$$Z_\nu^{\mu(2)}(\zeta;\gamma) = \left(\zeta^2 - \gamma^2\right)^{\mu/2} \zeta^{-\mu} \sum_{r=-\infty}^{\infty} b_{\nu,r}^{\mu}(\gamma)\, n_{\nu+r}(\zeta) \tag{5.2}$$

are defined as Z-functions of the first and second kind. In these series $|\arg \zeta| < \pi$; $\arg \dfrac{\zeta^2 - \gamma^2}{\zeta^2} = 0$, if $\arg \zeta^2 = \arg \gamma^2$.

Substitution of these series into the basic equation (2.4g) (it is best to insert it into the transformed form (2.10) of the basic equation), elimination of the first and second derivatives of the indices $\psi_{\nu+r}$ and $n_{\nu+r}$ by means of (3.10) and (3.11), and removal of the denominator ζ^2 by means of (3.12) leads finally, exactly as in the X-functions of the first and second kind, to a three-term recursion system for the $b_{\nu,r}^{\mu}(\gamma)$. It agrees with the recursion system (4.4) for the $a_{\nu,r}^{\mu}(\gamma)$, if the μ there is replaced by $-\mu$. The solution of the recursion system differs from the indices $b^{\mu}_{\nu,r}(\gamma)$ defined in (4.11) only by a constant factor; this factor is selected to equal one. Therefore the relation (4.3) formerly found between the index ν of the generating functions in (5.1) and (5.2) and the separation parameter λ has to be assumed also in this case.

As Z-functions of the third and fourth kind one defines

$$Z^{\mu(3)}_{\nu}(\zeta;\gamma) = Z^{\mu(1)}_{\nu}(\zeta;\gamma) + iZ^{\mu(2)}_{\nu}(\zeta;\gamma) \tag{5.3}$$

$$Z^{\mu(4)}_{\nu}(\zeta;\gamma) = Z^{\mu(1)}_{\nu}(\zeta;\gamma) - iZ^{\mu(2)}_{\nu}(\zeta;\gamma) \tag{5.4}$$

They have the same relation to Hankel's functions as the Z-functions of the first and second kind to Bessel's and Neumann's functions.

5.2 Convergence of the Series Developments of the Z-Functions of the First and Second Kind

It must now be demonstrated that the series (5.1) and (5.2) converge uniformly in a certain domain. One starts from the estimate (3.17) and from the boundary values (4.7) which are also valid for the $b^{\mu}_{\nu,r}$. There results

$$\limsup_{r \to \infty} \sqrt[r]{\left|a_{\pm r} Z_{\nu \pm r}\right|} \leqq \left|\frac{\chi}{\zeta}\right| = \frac{1}{|\xi|} \tag{5.5}$$

The convergence is uniform. Thus the series (5.1) and (5.2) converge uniformly and absolutely in the entire domain $|\xi| > 1$ with the exclusion of the infinitely distant point; they represent therefore analytic functions, can be differentiated any number of times termwise with respect to ζ, and satisfy the basic equation. Only in special cases these series converge also for $|\xi| \leqq 1$.

5.3 General Relations between the Z-Functions

The transition to various function branches over the branch cut from $-\infty$ through to 1 is made possible by the general relations. They can be obtained corresponding to the case of the X-functions from the general relations valid for the separate series terms, thus for the cylindrical functions. (Compare Magnus and Oberhettinger, elsewhere.) They read for $|\zeta| > |\gamma|$ that is, $|\xi| > 1$

$$Z_\nu^{\mu(1)}\left(e^{l\pi i}\zeta;\gamma\right) = e^{l\nu\pi i}\, Z_\nu^{\mu(1)}(\zeta;\gamma) \tag{5.6}$$

$$Z_\nu^{\mu(2)}\left(e^{l\pi i}\zeta;\gamma\right) = e^{-l(\nu+1)\pi i}\, Z_\nu^{\mu(2)}(\zeta;\gamma)$$

$$+ 2ie^{-l\pi i/2} \sin l\pi\left(\nu + \frac{1}{2}\right) \cot\left(\nu + \frac{1}{2}\right)\pi Z_\nu^{\mu(1)}(\zeta;\gamma) \tag{5.7}$$

$$Z_\nu^{\mu(3)}\left(e^{l\pi i}\zeta;\gamma\right) = -e^{-l\pi i/2}\, \frac{\sin\,(l-1)\left(\nu + \frac{1}{2}\right)\pi}{\sin\left(\nu + \frac{1}{2}\right)\pi}\, Z_\nu^{\mu(3)}(\zeta;\gamma)$$

$$- e^{-l\pi i/2}\, e^{-(\nu+1/2)\pi i}\, \frac{\sin\, l\left(\nu + \frac{1}{2}\right)\pi}{\sin\left(\nu + \frac{1}{2}\right)\pi}\, Z_\nu^{\mu(4)}(\zeta;\gamma) \tag{5.8}$$

$$Z_\nu^{\mu(4)}\left(e^{l\pi i}\zeta;\gamma\right) = e^{-l\pi i/2}\, e^{(\nu+1/2)\pi i}\, \frac{\sin\, l\left(\nu + \frac{1}{2}\right)\pi}{\sin\left(\nu + \frac{1}{2}\right)\pi}\, Z_\nu^{\mu(3)}(\zeta;\gamma)$$

$$+ e^{-l\pi i/2}\, \frac{\sin\,(l+1)\left(\nu + \frac{1}{2}\right)\pi}{\sin\left(\nu + \frac{1}{2}\right)\pi}\, Z_\nu^{\mu(4)}(\zeta;\gamma) \tag{5.9}$$

5.4 Asymptotic Developments of the Z-Functions

In order to obtain asymptotic developments of the Z-functions for $|\zeta| \gg 1$, it suggests itself to insert the asymptotic developments of the functions $\psi_{\nu+r}(\zeta)$ and $\eta_{\nu+r}(\zeta)$ into (5.1) and (5.2) and then to interchange the summations. Since

$$\psi_{\nu+r}(\zeta) \pm i\eta_{\nu+r}(\zeta) = \frac{1}{\zeta} e^{\pm i\left(\zeta - \frac{\nu+1}{2}\pi\right)} i^r \left[\sum_{p=0}^{M-1} \frac{\left(\nu + r + \frac{1}{2}, p\right)}{(\mp 2i\zeta)^p} + O(|\zeta|^{-M})\right] \tag{5.10}$$

($-\pi < \arg \zeta < 2\pi$ and $-2\pi < \arg \zeta < \pi$ for the upper and lower sign, respectively) where, for abbreviation

$$\left(\nu + r + \frac{1}{2}, p\right) = \frac{\Gamma(\nu + r + p + 1)}{p!\,\Gamma(\nu + r - p + 1)} \tag{5.11}$$

there results in this manner

$$Z_\nu^{\mu(3,4)}(\zeta;\gamma) = (\zeta^2 - \gamma^2)^{\mu/2}\, \zeta^{-\mu-1}\, e^{\pm i\left(\zeta - \frac{\nu+1}{2}\pi\right)} \left[\sum_{p=0}^{M-1} \frac{C(p)}{(\mp 2i\zeta)^p} + O(|\zeta|^{-M})\right] \tag{5.12}$$

($-\pi < \arg \zeta < 2\pi$ and the upper sign for $Z_\nu^{\mu(3)}(\zeta;\gamma)$, $-2\pi < \arg \zeta < \pi$ and the lower sign for $Z_\nu^{\mu(4)}(\zeta;\gamma)$). The coefficients $C(p)$ are defined by the absolutely convergent series

$$C(p) = \sum_{r=-\infty}^{\infty} \left(\nu + r + \frac{1}{2}, p\right) i^r b_{\nu,r}^{\mu}(\gamma) \tag{5.13}$$

This derivation is not accurate since the asymptotic developments (5.10) are further dependent upon the condition $|\zeta| \gg |\nu + r|$, and this condition is not satisfied for all series terms of equations (5.1) and (5.2), respectively, since the sum has to be formed over all r's from $-\infty$ to ∞. The fact that the developments (5.12) are valid nevertheless is due to the behavior at infinity of the b_r's (compare equation (4.7)) according to which the series terms with sufficiently large values of r do not contribute noticeably to the Z-functions.

Equation (5.12) is proved as follows.

According to general theorems on the asymptotic behavior of the solutions of homogeneous linear differential equations, the coefficients of which are polynomials, one obtains asymptotic series for the solutions by going into the differential equation (2.4g) with a formulation of the form (5.12) and attempts to satisfy it formally. This yields for the present case for the coefficients $C(p)$ the four-term recursion system

$$(p + 1)C(p + 1) + \left[(p + 1)p + \gamma^2 - \lambda\right]C(p) + 4\gamma^2(\mu + p)C(p - 1)$$

$$+ 4\gamma^2(\mu + p)(\mu + p - 1)C(p - 2) = 0$$

$$C(-1) = C(-2) = 0, \quad p = 0, 1, 2, 3, \ldots \tag{5.14}$$

from which they can be calculated recursively. This recursion system, however, is satisfied just then when the series (5.13) are substituted for the coefficients $C(p)$. This substitution leads after slight transformation to

$$\sum_{r=-\infty}^{\infty} i^r b_{\nu,r}^{\mu}(\gamma) \left\{ \frac{\Gamma(\nu + r + p + 1)}{\Gamma(\nu + r - p + 1)} \left[(\nu + r + 1)(\nu + r) + \gamma^2 - \lambda \right] \right.$$

$$\left. + \frac{\Gamma(\nu + r + p - 1)}{\Gamma(\nu + r - p + 3)} 4\gamma^2 p(\mu + p) \left[(\nu + r + 1)(\nu + r) + (\mu + 1)(p - 1) \right] \right\} = 0 \qquad (5.15)$$

for $p = 0, 1, 2, 3, \ldots$ and these relations between the b_r's can be simply derived by multiplying equation (4.4) by $i^r\Gamma(\nu + r + p + 1)/\Gamma(\nu + r - p + 1)$, by forming the sum over r from $-\infty$ to ∞, and finally replacing μ by $-\mu$ and therewith $a_{\nu,r}^{\mu}(\gamma)$ by $b_{\nu,r}^{\mu}(\gamma)$.

Thus the asymptotic series (5.12) with the significance of the coefficients $C(p)$ given in equation (5.13) are actually asymptotic solutions of the basic equation (2.4g); it is now easily understood that they represent asymptotically those solutions $Z_{\nu}^{\mu(3,4)}(\zeta;\gamma)$.

In a special case the developments (5.12) are even convergent: then namely, when ν is a real integer = n. Then the series (5.10) are broken off; the functions $\psi_{n+r}(\zeta)$ and $n_{n+r}(\zeta)$ are elementary functions. Since the series (5.1) and (5.2) converge uniformly, the summations can be exchanged after substituting equation (5.10) in equations (5.1) and (5.2); therefore the series (5.12) are convergent. Their domain of convergence is $|\xi| > 1$, that is, $|\zeta| > |\gamma|$. One can also easily understand with the aid of equations (6.6) and (6.7) that, for $p > n$, the coefficients $C(p)$ are of the order of magnitude γ^{α}, with α being the smaller one of the two even numbers among the four positive numbers $p - n$, $p - n + 1$, $p + n + 1$, and $p + n + 2$. For the case when ν and μ are positive integers and $\nu \geqq \mu \geqq 0$, the series (5.1) of the Z-functions of the first kind converge for all finite ζ.

5.5 Further Solutions of the Basic Equation and Their Relation to the Z-Functions of the First and Second Kind

It is immediately clear that with the Z-functions $Z_\nu^{\mu(1)}(\zeta;\gamma)$ and $Z_\nu^{\mu(2)}(\zeta;\gamma)$ the functions $Z_{-\nu-1}^{\mu(1,2)}(\zeta;\gamma)$ and $Z_\nu^{-\mu(1,2)}(\zeta;\gamma)$ also are solutions of the basic equation. Since there exist only two linearily independent solutions of the basic equation, it must be possible to express all solutions linearily by two of them. Because of the two relations

$$\left.\begin{aligned} \psi_{-\nu-1}(\zeta) &= -\cos\nu\pi n_\nu(\zeta) - \sin\nu\pi\psi_\nu(\zeta) \\ n_{-\nu-1}(\zeta) &= \cos\nu\pi\psi_\nu(\zeta) - \sin\nu\pi n_\nu(\zeta) \end{aligned}\right\} \tag{5.16}$$

there follows with the aid of equations (4.12) and (4.10) from the definitions (5.1) and (5.2)

$$Z_{-\nu-1}^{\mu(1)}(\zeta;\gamma) = -\sin\nu\pi Z_\nu^{\mu(1)}(\zeta;\gamma) - \cos\nu\pi Z_\nu^{\mu(2)}(\zeta;\gamma) \tag{5.17}$$

$$Z_{-\nu-1}^{\mu(2)}(\zeta;\gamma) = \cos\nu\pi Z_\nu^{\mu(1)}(\zeta;\gamma) - \sin\nu\pi Z_\nu^{\mu(2)}(\zeta;\gamma) \tag{5.18}$$

In order to express the functions $Z_\nu^{-\mu(1,2)}(\zeta;\gamma)$ by $Z_\nu^{\mu(1)}(\zeta;\gamma)$ and $Z_\nu^{\mu(2)}(\zeta;\gamma)$, it will be practical to use the asymptotic series; it is sufficient to limit oneself to the first term of the series. Then there becomes

$$Z_\nu^{\mu(3,4)}(\zeta;\gamma) = \frac{1}{\zeta}e^{\pm i\left(\zeta-\frac{\nu+1}{2}\pi\right)} \sum_{r=-\infty}^{\infty} i^r b_{\nu,r}^{\mu}(\gamma)\left[1 + O\left(|\zeta|^{-1}\right)\right] \tag{5.19}$$

The only difference for the asymptotic series for $Z_\nu^{-\mu(3,4)}(\zeta;\gamma)$ is that here $a^\mu_{\nu,r}(\gamma)$ takes the place of $b^\mu_{\nu,r}(\gamma)$. There follows immediately that

$$Z_\nu^{-\mu(i)}(\zeta;\gamma) \sum_{r=-\infty}^{\infty} i^r b^\mu_{\nu,r}(\gamma) = Z_\nu^{\mu(i)}(\zeta;\gamma) \sum_{r=-\infty}^{\infty} i^r a^\mu_{\nu,r}(\gamma)$$

$$(i = 1,\ 2) \tag{5.20}$$

By combination of equations (5.17), (5.18), and (5.20) finally, also the solutions $Z_{-\nu-1}^{-\mu(i)}(\zeta;\gamma)$ of the basic equation can be reduced to the two solutions $Z_\nu^{\mu(1,2)}(\zeta;\gamma)$. As special cases of equations (5.17) and (5.18)

$$\left.\begin{aligned} Z_{-\nu-1}^{\mu(3)}(\zeta;\gamma) &= ie^{\nu\pi i}\, Z_\nu^{\mu(3)}(\zeta;\gamma) \\ Z_{-\nu-1}^{\mu(4)}(\zeta;\gamma) &= -ie^{-\nu\pi i}\, Z_\nu^{\mu(4)}(\zeta;\gamma) \end{aligned}\right\} \tag{5.21}$$

should be noted.

5.6 Laurent-Developments for X- and Z-Functions

The X- and Z-functions were introduced wholly independent of each other. Since they all are, however, solutions of the same differential equation, it must be possible to express, for instance, the Z-functions of the first to fourth kind in general linearily by the X-functions of the first and second kind. It will appear that in general the Z-functions of the first and second kind are not proportional to the X-functions of the first and the second kind, respectively. Thus, it is not possible to define simply functions of the first and second kind for the solutions of the basic equation; it must always be added whether one is dealing with X- or Z-functions.

The problem to express the Z-functions by the X-functions can be solved partly by comparison of the general relations already known for both kinds of functions (4.24), (4.25), (5.6), (5.7), (5.8), and (5.9). A complete solution of this problem, however, is obtained in the following manner. The series (4.1) and (4.2) for the X-functions and equations (5.1) and (5.2) for the Z-functions have the common domain of convergence $1 < |\xi| < \infty$. They even converge uniformly in the domain bounded by the two circles $|\xi| = 1 + \delta$ and $|\xi| = 1 + \vartheta$, where δ is an arbitrarily small and ϑ an arbitrarily large positive number. Since the series terms of these developments are analytic functions regular in this domain, they can be developed in Laurent-series which converge in this entire domain. According to Weierstrass's double series theorem one may exchange the summation over r and the summation of these Laurent-series and thus obtain the X-functions and the Z-functions in a representation by Laurent-series. From the comparison of these Laurent-series then results the representation of the Z-functions as linear combinations of the X-functions of the first and second kind.

The performance of the transformations described just now yields, using equations (3.4) and (3.14),

$$X_\nu^{\mu(2)}(\xi;\gamma) = \left(\xi^2 - 1\right)^{\mu/2} \xi^{-\mu} e^{\mu\pi i} \pi^{1/2} (2\xi)^{-\nu-1}$$

$$\sum_{s=-\infty}^{\infty} (2\xi)^{-s} \Gamma(\nu + s + \mu + 1) \sum_{r=-\infty}^{\infty} \frac{i^r a_{\nu,r}^{\mu}(\gamma)}{\Gamma\left(\nu + \frac{s + r}{2} + \frac{3}{2}\right)\Gamma\left(1 + \frac{s - r}{2}\right)} \qquad (5.22)$$

$$X^{\mu(2)}_{-\nu-1}(\xi;\gamma) = \left(\xi^2 - 1\right)^{\mu/2} \xi^{-\mu} e^{\mu\pi i} \pi^{1/2} (2\xi)^{\nu} \sum_{s=-\infty}^{\infty} (2\xi)^{-s} \Gamma(-\nu + s + \mu) \sum_{r=-\infty}^{\infty} \frac{i^r a^{\mu}_{\nu,r}(\gamma)}{\Gamma\left(-\nu + \frac{s-r}{2} + \frac{1}{2}\right)\Gamma\left(1 + \frac{s+r}{2}\right)} \tag{5.23}$$

$$Z^{\mu(1)}_{\nu}(\zeta;\gamma) = \frac{1}{2}\pi^{1/2}\left(\xi^2 - 1\right)^{\mu/2} \xi^{-\mu}\left(\frac{\gamma}{2}\right)^{2\nu}(2\xi)^{\nu} \sum_{s=-\infty}^{\infty} (2\xi)^{-s}\left(\frac{\gamma}{2}\right)^{-2s} i^s \sum_{r=-\infty}^{\infty} \frac{i^r b^{\mu}_{\nu,r}(\gamma)}{\Gamma\left(\nu + \frac{r-s}{2} + \frac{3}{2}\right)\Gamma\left(1 - \frac{r+s}{2}\right)} \tag{5.24}$$

$$Z^{\mu(1)}_{-\nu-1}(\zeta;\gamma) = \frac{1}{2}\pi^{1/2}\left(\xi^2 - 1\right)^{\mu/2} \xi^{-\mu}\left(\frac{\gamma}{2}\right)^{-2\nu-2}(2\xi)^{-\nu-1} \sum_{s=-\infty}^{\infty} (2\xi)^{-s}\left(\frac{\gamma}{2}\right)^{-2s} i^s \sum_{r=-\infty}^{\infty} \frac{i^r b^{\mu}_{\nu,r}(\gamma)}{\Gamma\left(-\nu - \frac{s+r}{2} + \frac{1}{2}\right)\Gamma\left(1 + \frac{r-s}{2}\right)} \tag{5.25}$$

the summation index s assumes only even values.

5.7 Connection between the X- and $\mathfrak{Z}$-Functions

Due to the equality of the characteristic exponents ν and $-\nu - 1$, respectively, in equations (5.23) and (5.24) or equations (5.22) and (5.25), respectively, $X_{\nu}^{\mu(2)}$ and $\mathfrak{Z}_{-\nu-1}^{\mu(1)}$ on the one hand and $X_{-\nu-1}^{\mu(2)}$ and $\mathfrak{Z}_{\nu}^{\mu(1)}$ on the other only differ every time by a factor independent of ξ. Thus one may equate

$$\mathfrak{Z}_{\nu}^{\mu(1)}(\zeta;\gamma) = \frac{1}{\pi} \sin (\nu - \mu)\pi e^{-(\mu+\nu+1)\pi i} \kappa_{\nu}^{\mu(1)}(\gamma) X_{-\nu-1}^{\mu(2)}(\xi;\gamma) \tag{5.26}$$

$$\mathfrak{Z}_{-\nu-1}^{\mu(1)}(\zeta;\gamma) = e^{(\nu+1)\pi i} \kappa_{\nu}^{\mu(2)}(\gamma) X_{\nu}^{\mu(2)}(\xi;\gamma) \tag{5.27}$$

The various factors, as $\sin (\nu - \mu)\pi$, and so forth, were introduced for convenience. Between $\kappa_{\nu}^{\mu(1)}(\gamma)$ and $\kappa_{\nu}^{\mu(2)}(\gamma)$ there exists the connection

$$\kappa_{-\nu-1}^{\mu(2)}(\gamma) = \frac{1}{\pi} \sin (\nu - \mu)\pi e^{-(\mu+1)\pi i} \kappa_{\nu}^{\mu(1)}(\gamma) \tag{5.28}$$

One further obtains with the aid of equations (4.18) and (5.17)

$$\mathfrak{Z}_{\nu}^{\mu(1)}(\zeta;\gamma) = \kappa_{\nu}^{\mu(1)}(\gamma)\left[e^{-\nu\pi i} \cos \nu\pi\, X_{\nu}^{\mu(1)}(\xi;\gamma) - e^{-(\mu+\nu)\pi i} \frac{\sin (\nu + \mu)\pi}{\pi} X_{\nu}^{\mu(2)}(\xi;\gamma)\right] \tag{5.29}$$

$$\cos \nu\pi\, Z_{\nu}^{\mu(2)}(\xi;\gamma) + \sin \nu\pi\, \mathfrak{Z}_{\nu}^{\mu(1)}(\zeta;\gamma) = e^{\nu\pi i} \kappa_{\nu}^{\mu(2)}(\gamma) X_{\nu}^{\mu(2)}(\xi;\gamma) \tag{5.30}$$

If ν, μ are integers, these relations are essentially simplified; then the X-functions and the $\mathfrak{Z}$-functions of the first and the second kind, respectively, are actually proportional.

Now the calculation of $\kappa_\nu^{\mu(i)}(\gamma)$ is left to be performed. To that end one may select the coefficient of any power of ξ in equations (6.22) and (5.25) and carry out the comparison. One obtains

$$\kappa_\nu^{\mu(1)}(\gamma) = \frac{1}{2}e^{\nu\pi i}\left(\frac{\gamma}{2}\right)^{2\nu-2s} i^s\Gamma(1+\nu-\mu-s)$$

$$\times \frac{\displaystyle\frac{\displaystyle\sum_{r=-\infty}^{\infty} i^r b_{\nu,r}^{\mu}(\gamma)}{\left[\Gamma\left(\nu+\frac{r-s}{2}+\frac{3}{2}\right)\Gamma\left(1-\frac{s+r}{2}\right)\right]}}{\displaystyle\frac{\displaystyle\sum_{r=-\infty}^{\infty} i^r a_{\nu,r}^{\mu}(\gamma)}{\left[\Gamma\left(-\nu+\frac{s-r}{2}+\frac{1}{2}\right)\Gamma\left(1+\frac{s+r}{2}\right)\right]}} \tag{5.31}$$

$$\kappa_\nu^{\mu(2)}(\gamma) = \frac{1}{2}e^{-(\nu+\mu+1)\pi i}\left(\frac{\gamma}{2}\right)^{-2\nu-2s-2} \frac{i^s}{\Gamma(\nu+\mu+s+1)}$$

$$\times \frac{\displaystyle\frac{\displaystyle\sum_{r=-\infty}^{\infty} i^r b_{\nu,r}^{\mu}(\gamma)}{\left[\Gamma\left(-\nu-\frac{r+s}{2}+\frac{1}{2}\right)\Gamma\left(1+\frac{r-s}{2}\right)\right]}}{\displaystyle\frac{\displaystyle\sum_{r=-\infty}^{\infty} i^r a_{\nu,r}^{\mu}(\gamma)}{\left[\Gamma\left(\nu+\frac{s+r}{2}+\frac{3}{2}\right)\Gamma\left(1+\frac{s-r}{2}\right)\right]}} \tag{5.32}$$

Any even number is to be substituted for s in equations (5.31) and (5.32). The value of $\kappa_\nu^{\mu(i)}(\gamma)$ is independent of the selected special value of s. If one replaces in equation (5.32) μ by $-\mu$ and s by $-s$ and then multiplies by equation (5.31),

$$\kappa_{\nu}^{\mu(1)}(\gamma)\ \kappa_{\nu}^{-\mu(2)}(\gamma) = e^{(\mu+1)\pi i}\ \gamma^{-1} \tag{5.33}$$

is originated.

5.8 Wronski's Determinant

Wronski's determinant of the Z-functions of the first and second kind are defined by

$$W_Z \equiv Z_{\nu}^{\mu(1)}(\zeta;\gamma)\ \frac{d}{d\xi} Z_{\nu}^{\mu(2)}(\zeta;\gamma) - Z_{\nu}^{\mu(2)}(\zeta;\gamma)\ \frac{d}{d\xi} Z_{\nu}^{\mu(1)}(\zeta;\gamma) \tag{5.34}$$

From the basic equation (2.4g) there follows in the known way that Wronski's determinant of any two of its solutions is proportional to $(\xi^2 - 1)^{-1}$. The factor of proportionality is determined by substituting their asymptotic series for the Z-functions of the first and second kind; it is sufficient to limit oneself to the first term (5.19). There results

$$W_Z = \frac{1}{\gamma}\ \frac{1}{\xi^2 - 1}\left[\sum_{r=-\infty}^{\infty} i^r b_{\nu,r}^{\mu}(\gamma)\right]^2 \tag{5.35}$$

Wronski's determinant of the X-functions of the first and second kind

$$W_X \equiv X_{\nu}^{\mu(1)}(\xi;\gamma)\ \frac{d}{d\xi} X_{\nu}^{\mu(2)}(\xi;\gamma) - X_{\nu}^{\mu(2)}(\xi;\gamma)\ \frac{d}{d\xi} X_{\nu}^{\mu(1)}(\xi;\gamma) \tag{5.36}$$

results from W_Z by using equation (5.30).

$$W_Z = \kappa_{\nu}^{\mu(1)}(\gamma)\,\kappa_{\nu}^{\mu(2)}(\gamma)W_X \tag{5.37}$$

is originated and therefore

$$W_X = \frac{1}{\gamma}\,\frac{1}{\xi^2 - 1}\,\frac{\left[\sum_{r=-\infty}^{\infty} i^r b_{\nu,r}^{\mu}(\gamma)\right]^2}{\kappa_{\nu}^{\mu(1)}(\gamma)\,\kappa_{\nu}^{\mu(2)}(\gamma)} \tag{5.38}$$

Simplifications result for the important special case $\mu = 0$. First, one agrees upon omitting the index μ when it has the value zero. Now there is valid $b_{\nu,r}(\gamma) = a_{\nu,r}(\gamma)$ and further, according to equation (4.1), because of $\underline{P}_n(1) = 1$,

$$X_{\nu}^{(1)}(1;\gamma) = \sum_{r=-\infty}^{\infty} i^r a_{\nu,r}(\gamma) \tag{5.39}$$

Thus, one can also write for Wronski's two determinants

$$W_X = -\frac{X_{\nu}^{(1)}(1;\gamma)X_{\nu}^{(1)}(1;\gamma)}{\xi^2 - 1} = -\gamma W_Z \tag{5.40}$$

5.9 Other Series Developments of the Solutions of the Basic Equation

Niven (reference 1) investigated series developments of the following form (the functions represented by them are called V- and W-functions):

$$V_{\nu}^{\mu(1)}(w;\gamma) = \left(w^2 + \gamma^2\right)^{1/2} w^{-1} \sum_{r=-\infty}^{\infty} i^r c_{\nu,r}^{\mu}(\gamma)\Psi_{\nu+r}(w) \tag{5.41}$$

$$W_{\nu}^{\mu(1)}(w;\gamma) = \sum_{r=-\infty}^{\infty} i^r d_{\nu,r}^{\mu}(\gamma)\psi_{\nu+r}(w) \tag{5.42}$$

The relation of the variable w to ξ and ζ is:

$$w^2 = \zeta^2 - \gamma^2 = \gamma^2(\xi^2 - 1) \tag{5.43}$$

For the coefficients c_r and d_r there results again a three-term recursion system which can be transformed into equation (4.4). If the coefficient of the principal term is set equal to a_0, the equations

$$c_{\nu,r}^{\mu}(\gamma) = \frac{\Gamma\left(\frac{\nu + r + \mu}{2} + 1\right)\Gamma\left(\frac{\nu - \mu + 1}{2}\right)}{\Gamma\left(\frac{\nu + r - \mu + 1}{2}\right)\Gamma\left(\frac{\nu + \mu}{2} + 1\right)} a_{\nu,r}^{\mu}(\gamma)$$

$$= i^r \frac{d\underline{P}_{\nu+r}^{\mu}(0)/d\xi}{d\underline{P}_{\nu}^{\mu}(0)/d\xi} a_{\nu,r}^{\mu}(\gamma) \tag{5.44}$$

$$d_{\nu,r}^{\mu}(\gamma) = \frac{\Gamma\left(\frac{\nu + r + \mu + 1}{2}\right)\Gamma\left(\frac{\nu - \mu}{2} + 1\right)}{\Gamma\left(\frac{\nu + r - \mu}{2} + 1\right)\Gamma\left(\frac{\nu + \mu + 1}{2}\right)} a_{\nu,r}^{\mu}(\gamma)$$

$$= i^r \frac{\underline{P}_{\nu+r}^{\mu}(0)}{\underline{P}_{\nu}^{\mu}(0)} a_{\nu,r}^{\mu}(\gamma) \tag{5.45}$$

are valid.

The series (5.41) and (5.42) converge uniformly in each closed domain given by $|\gamma| < |w| < \infty$; or, expressed in the ξ-plane: $1 < \left|\sqrt{\xi^2 - 1}\right| < \infty$. The bounding curve $\left|\sqrt{\xi^2 - 1}\right| = 1$ is a lemniscate. Equation (5.42) is, as will be shown later, a limiting case of a general development, which still contains an arbitrary parameter and which yields as a further limiting case the series (5.1) and (5.2) of the $\mathfrak{Z}$-functions.

One can immediately give further series developments of V- and W-functions; to this end one has to replace the functions $\psi_{\nu+r}$ in equations (5.41) and (5.42) by $n_{\nu+r}$, or the indices $\nu + r$ by $-\nu - r - 1$, or μ by $-\mu$, or one has to make two or three of these substitutions simultaneously. One thus obtains a total of eight V-functions and eight W-functions. Their properties will not be investigated here more closely; it should only be mentioned that all of them also can be expressed linearily by the $\mathfrak{Z}$-functions of the first and second kind which is done in the simplest way with the aid of the asymptotic series.

Whereas the asymptotic series of the $\mathfrak{Z}$-functions progress with powers of ζ^{-1}, the asymptotic series of the V- and W-functions one obtains from equation (5.41), and so forth, by substitution of the asymptotic series of the cylindrical functions, contain powers of $\left(\zeta^2 - \gamma^2\right)^{-1/2}$, that is, $\left(\xi^2 - 1\right)^{-1/2}$. According to a suggestion by Wilson (reference 7) one can now also set up asymptotic series which progress with powers of $(\xi \pm 1)^{-1}$. They have compared with the series (5.12) a slight advantage insofar as a three-term recursion system results for their coefficients. Correspondingly, for the solutions of the basic equation also developments in terms of cylindrical functions with the argument $\zeta \pm \gamma = \gamma(\xi \pm 1)$ may be given, of the form

$$\underline{F}_1 = (\xi - 1)^{\mu/2}(\xi + 1)^{-\mu/2} \sum_{t=-\infty}^{\infty} e^{\mu}_{\nu,t}\,\psi_{\nu+t}(\zeta \pm \gamma) \qquad (5.46)$$

where t runs through all integers, the odd as well as the even ones. $\psi_{\nu+t}$ can again be replaced by $n_{\nu+t}$, and so forth.

These developments will, however, not be followed up here.

6. CALCULATION OF THE COEFFICIENTS OF THE SERIES DEVELOPMENTS IN TERMS OF SPHERICAL AND CYLINDRICAL FUNCTIONS

6.1 Continued Fraction Developments

The solution of the recursion system (4.4) which for $r \to \infty$ has the behavior at infinity (4.7), can be represented by the convergent (reference 13) continued fraction

$$\frac{a_r}{a_{r-2}} = \frac{\gamma^2 q_r/\phi_r|}{1} - \frac{\gamma^4 p_r q_{r+2}/\phi_r \phi_{r+2}|}{|1} - \frac{\gamma^4 p_{r+2} q_{r+4}/\phi_{r+2}\phi_{r+4}|}{|1} \ldots \quad (6.1)$$

The solution which has the behavior (4.7) for $r \to -\infty$ can be represented by the convergent continued fraction

$$\frac{a_r}{a_{r+2}} = \frac{\gamma^2 p_r/\phi_r|}{1} - \frac{\gamma^4 q_r p_{r-2}/\phi_r \phi_{r-2}|}{|1} - \frac{\gamma^4 q_{r-2} p_{r-4}/\phi_{r-2}\phi_{r-4}|}{|1} \ldots \quad (6.2)$$

The subnumerators of both continued fractions are in each finite closed domain of γ- and λ-values for sufficiently large values of r in the case (6.1), of $-r$ in the case (6.2) smaller than one-fourth; thus, according to a theorem on uniform convergence of continued fractions, the continued fractions (6.1) and (6.2), respectively, are in each domain of this kind for sufficiently large r's and $-r$'s, respectively, uniformly convergent and are therewith regular analytic functions in γ and λ, since the individual approximation fractions are functions of this kind. For not sufficiently large r's and $-r$'s, respectively, then follows, that these continued fractions are also analytic functions which, however, need not in every case be regular.

A solution of the recursion system (4.4) has now to be found which shows the behavior at infinity (4.7) for $r \to \infty$ as well as for $r \to -\infty$. Then the value of a_r/a_{r-2} calculated from equation (6.1) must equal the value of this expression calculated from equation (6.2). An equation results which for given ν and μ allows calculation of the separation parameter λ as a function of γ. If γ and $\lambda - \nu(\nu + 1)$ both are sufficiently small, the solution $\lambda_\nu^\mu(\gamma)$ is a regular analytic function of γ which assumes for $\gamma = 0$ the value $\nu(\nu + 1)$. Thus the $\lambda_\nu^\mu(\gamma)$ as well as the a_r/a_o can be developed in power series in γ with non-vanishing radius of convergence the magnitude of which will not be investigated here more closely.

6.2 Method for Numerical Calculation of the Separation Parameter and the Development Coefficients

The representation of the coefficient a_r by continued fractions is also for larger values of γ still particularly suitable for the numerical calculation of the separation parameter λ and the a_r's. Mostly ν, μ, and γ are given. Then the values p_r, q_r, and $\phi_r + \lambda$ can be calculated numerically from equation (4.5). One starts from a value for λ which is assumed as close as possible to the actual value and calculates for a selected fixed r the expression a_{r+2}/a_r from equation (6.1) as well as from equation (6.2). Then one repeats this calculation with a slightly altered value of λ and examines whether thereby the agreement of the two values a_{r+2}/a_r is improved. By further variation of λ one can finally obtain an agreement of arbitrary accuracy. Therewith one can find the value $\lambda_\nu^\mu(\gamma)$ with any desired accuracy.

One more investigation has to be made: whether the solution thus found for $\gamma = 0$ goes over continuously into $\nu(\nu + 1)$, that is, into $\lambda_\nu^\mu(0)$ and not perhaps into $\lambda_{\nu+2}^\mu(0)$; for $\lambda_{\nu+2}^\mu(0)$ also is a solution of the present problem as can be recognized from the fact that equations (6.1) and (6.2) contain the values ν and r only in the combination $\nu + r$. This question cannot be decided unless one has already a general picture of the functions $\lambda_\nu^\mu(\gamma)$ as it is given in figure 1 for $\mu = 0$, ν's that are integers, and real γ^2's.

The number of terms of the continued fractions (6.1) and (6.2) to be included in the calculation corresponds to the desired accuracy. For large $|r|$ the partial fractions $\gamma^4 p_r q_{r+2}/\phi_r \phi_{r+2}$ assume the order of magnitude $\gamma^4/(16r^4)$; thus the index r' of the last partial fraction to be included will have to be selected at any rate larger than $|\gamma|/2$.

The calculation of the $a^{\mu}_{\nu,r}(\gamma)$'s is made by taking the value found for $\lambda^{\mu}_{\nu}(\gamma)$ as a base, and calculating a_{r+2}/a_r from equation (6.1) and therefrom $a_{\pm 2}/a_o$, $a_{\pm 4}/a_o$, . . .

6.3 Power Series for Separation Parameter and Development Coefficients

For the numerical calculation of the separation parameter and the development coefficients one can for small values of $|\gamma|$ make good use of the power series developments in terms of γ. If one limits oneself in these to the first terms up to the fifth power of γ^2, inclusive, one obtains, in general, still quite useful approximations up to about $|\gamma^2| = 5$. Therefore, following, the power series for the $\lambda^{\mu}_{\nu}(\gamma)$ shall be calculated explicitly to γ^{10}, inclusive, for the a_r/a_o to γ^8, inclusive. Therewith one more series term is obtained than by Niven (reference 1); compared with Niven's cumbersome treatment, the calculation is essentially simplified.

For the limiting case $\gamma = 0$ there follows from the recursion system (4.4)

$$\left[\lambda^{\mu}_{\nu}(\gamma) + (\nu + r + 1)(\nu + r)\right] a_r = 0 \qquad (6.3)$$

The case where all a_r disappear is not of interest since it leads only to identically disappearing solutions of the basic equation. Thus there becomes $\lambda^{\mu}_{\nu}(0) = \nu(\nu + 1)$, $a_r = 0$ for $r \neq 0$.

For $\gamma = 0$ all ϕ_r with the exception of ϕ_o have nondisappearing limiting values. From the continued fraction (6.1) one can draw the conclusion

$$\frac{a_r}{a_{r-2}} = \frac{\gamma^2 q_r}{\phi_r}\left[1 + O(\gamma^4)\right] \qquad (r = 2, 4, 6, \ldots) \tag{6.4}$$

and from that further

$$\frac{a_r}{a_o} = \gamma^r \frac{q_2 q_4 \cdots q_r}{\phi_2 \phi_4 \cdots \phi_r}\left[1 + O(\gamma^4)\right] \qquad (r = 2, 4, 6, \ldots) \tag{6.5}$$

If one takes the next partial fraction into consideration as well, there results as the next approximation

$$\frac{a_r}{a_o} = \gamma^r \frac{q_2 q_4 \cdots q_r}{\phi_2 \phi_4 \cdots \phi_r}\left[1 + \gamma^4\left(\frac{p_2 q_4}{\phi_2 \phi_4} + \frac{p_4 q_6}{\phi_4 \phi_6} + \cdots + \frac{p_r q_{r+2}}{\phi_r \phi_{r+2}}\right) + O(\gamma^8)\right] \qquad (r = 2, 4, 6, \ldots) \tag{6.6}$$

Accordingly, one obtains

$$\frac{a_r}{a_o} = \gamma^{-r} \frac{p_{-2} p_{-4} \cdots p_r}{\phi_{-2} \phi_{-4} \cdots \phi_r}\left[1 + \gamma^4\left(\frac{q_{-2} p_{-4}}{\phi_{-2} \phi_{-4}} + \frac{q_{-4} p_{-6}}{\phi_{-4} \phi_{-6}} + \cdots + \frac{q_r p_{r-2}}{\phi_r \phi_{r-2}}\right) + O(\gamma^8)\right] \qquad (r = -2, -4, -6, \ldots) \tag{6.7}$$

One now substitutes a_2/a_o from equation (6.6) and a_{-2}/a_o from equation (6.7) into the equation $r = 0$ of the recursion system (4.4) and obtains

$$\frac{\phi_o}{\gamma^4} = \frac{p_o q_2}{\phi_2}\left[1 + \gamma^4 \frac{p_2 q_4}{\phi_2 \phi_4}\right] + \frac{q_o p_{-2}}{\phi_{-2}}\left[1 + \gamma^4 \frac{q_{-2} p_{-4}}{\phi_{-2}\phi_{-4}}\right] + O(\gamma^8) \tag{6.8}$$

This equation permits the calculation of $\lambda_\nu^\mu(\gamma)$ as power series in terms of γ up to the power γ^{10}, inclusive. At first one can see, by having γ approach 0, that $\Phi_o = O(\gamma^4)$. Therewith, however, ϕ_r also is known for any r with the exception of terms of the second and of higher powers in γ^2. If one now inserts Φ_r in this approximation on the right side of equation (6.8), Φ_o becomes already correct up to the third power in γ^2, inclusive. If one repeats this procedure with the new values of the ϕ_r which are correct up to γ^6, inclusive, there results finally ϕ_o and therewith $\lambda_\nu^\mu(\gamma)$ exactly up to γ^{10}, inclusive. The performance of this calculation as well as the calculation of the a_r's is not particularly difficult, therefore the results are given immediately.

In order to make the representation clearer, the following abbreviations are introduced:

$$\Phi_r = D_r\left(1 + \gamma^2 \delta_r\right) + \Phi_o \tag{6.9}$$

where

$$\left.\begin{aligned} D_r &= r(2\nu + r + 1) \\ \delta_r &= \frac{2(4\mu^2 - 1)}{(2\nu + 3)(2\nu - 1)(2\nu + 2r + 3)(2\nu + 2r - 1)} \end{aligned}\right\} \tag{6.10}$$

$$P_2 = \frac{p_0 q_2}{D_2}, \quad P_4 = \frac{p_2 q_4}{D_4}, \quad P_{-2} = \frac{p_{-2} q_0}{D_{-2}}, \quad P_{-4} = \frac{p_{-4} q_{-2}}{D_{-4}} \tag{6.11}$$

$$\left.\begin{aligned} A_i &= P_2\delta_2^{\ i} + P_{-2}\delta_{-2}^{\ i} \\ B_i &= \frac{P_2}{D_2}\delta_2^{\ i} + \frac{P_{-2}}{D_{-2}}\delta_{-2}^{\ i} \\ C &= \frac{P_2P_4}{D_2} + \frac{P_{-2}P_{-4}}{D_{-2}} \\ i &= 0, 1, 2, \ldots \end{aligned}\right\} \tag{6.12}$$

Then there becomes

$$\lambda_\nu^\mu(\gamma) = \nu(\nu + 1) + \gamma^2 \frac{2\nu^2 + 2\nu - 2\mu^2 - 1}{(2\nu + 3)(2\nu - 1)} - A_o\gamma^4 + A_1\gamma^6 + \left(A_oB_o - A_2 - C\right)\gamma^8 + \left[A_3 - 2A_oB_1 - B_oA_1 + \frac{P_2P_4}{D_2}\left(2\delta_2 + \delta_4\right) + \frac{P_{-2}P_{-4}}{D_{-2}}\left(2\delta_{-2} + \delta_{-4}\right)\right]\gamma^{10} + O(\gamma^{12}) \tag{6.13}$$

$$\frac{a_2}{a_o} = \gamma^2 \frac{q_2}{D_2}\left\{1 - \gamma^2\delta_2 + \gamma^4\left(\delta_2^{\ 2} + \frac{P_4 - A_o}{D_2}\right) + \gamma^6\left[\frac{2A_o\delta_2 + A_1}{D_2} - \frac{P_4}{D_2}\left(2\delta_2 + \delta_4\right) - \delta_2^{\ 3}\right]\right\} + O(\gamma^{10}) \tag{6.14}$$

$$\frac{a_4}{a_o} = \gamma^4 \frac{q_2q_4}{D_2D_4}\left[1 - \gamma^2\left(\delta_2 + \delta_4\right) + \gamma^4\left(\delta_2^{\ 2} + \delta_2\delta_4 + \delta_4^{\ 2} + \frac{P_4 - A_o}{D_2} + \frac{P_6 - A_o}{D_4}\right)\right] + O(\gamma^{10}) \tag{6.15}$$

$$\frac{a_6}{a_0} = \frac{q_2 q_4 q_6}{D_2 D_4 D_6} \gamma^6 \left[1 - \gamma^2 \left(\delta_2 + \delta_4 + \delta_6\right)\right] + O(\gamma^{10}) \tag{6.16}$$

$$\frac{a_8}{a_0} = \frac{q_2 q_4 q_6 q_8}{D_2 D_4 D_6 D_8} \gamma^8 + O(\gamma^{10}) \tag{6.17}$$

$$\frac{a_{-2}}{a_0} = \gamma^2 \frac{p_{-2}}{D_{-2}} \left\{1 - \gamma^2 \delta_{-2} + \gamma^4 \left(\delta_{-2}^2 + \frac{P_{-4} - A_0}{D_{-2}}\right)\right.$$
$$\left. + \gamma^6 \left[\frac{2A_0 \delta_{-} + A_1}{D_{-2}} - \frac{P_{-4}}{D_{-2}} \left(2\delta_{-2} + \delta_{-4}\right) - \delta_{-2}^3\right]\right\} + O(\gamma^{10}) \tag{6.18}$$

$$\frac{a_{-4}}{a_0} = \gamma^4 \frac{p_{-2} p_{-4}}{D_{-2} D_{-4}} \left[1 - \gamma^2 \left(\delta_{-2} + \delta_{-4}\right)\right.$$
$$\left. + \gamma^4 \left(\delta_{-2}^2 + \delta_{-2}\delta_{-4} + \delta_{-4}^2 + \frac{P_{-4} - A_0}{D_{-2}} + \frac{P_{-6} - A_0}{D_{-4}}\right)\right] + O(\gamma^{10}) \tag{6.19}$$

$$\frac{a_{-6}}{a_0} = \gamma^6 \frac{p_{-2} p_{-4} p_{-6}}{D_{-2} D_{-4} D_{-6}} \left[1 - \gamma^2 \left(\delta_{-2} + \delta_{-4} + \delta_{-6}\right)\right] + O(\gamma^{10}) \tag{6.20}$$

$$\frac{a_{-8}}{a_0} = \gamma^8 \frac{p_{-2} p_{-4} p_{-6} p_{-8}}{D_{-2} D_{-4} D_{-6} D_{-8}} + O(\gamma^{10}) \tag{6.21}$$

For the case excluded above where ν has fractional values of one-half, the convergence radii of these series equal zero. It seems therefore probable that the convergence radii are functions of ν which can be infinitely large for special cases, but not in general.

6.4 Power Series Developments

Since occasionally power series developments of the solutions of the basic equation (2.4g) also can be useful, they will be briefly discussed below.

One can of course obtain them at once by substituting in the series (4.1) and (4.2) for the X-functions of the first and second kind the known power series developments of the spherical functions in terms of powers of ξ; one thus obtains power series for the solutions of the basic equation which converge in the circle $|\xi| < 1$. The problem of the Laurent-series for $1 < |\xi| < \infty$ need not be discussed further since they are already calculated in equations (5.22) to (5.25). However, one can obtain these developments directly. Therewith a new method for the calculation of these functions and particularly of $\lambda_{\nu}^{\mu}(\gamma)$ is found.

One starts from the differential equation (2.10) which is written in terms of ξ rather than of ζ.

$$(\xi^2 - 1)\frac{d^2v}{d\xi^2} + 2\left(\xi + \frac{\mu}{\xi}\right)\frac{dv}{d\xi} + \left[-\lambda + \gamma^2\xi^2 - \frac{\mu(\mu + 1)}{\xi^2}\right]v = 0 \qquad (6.22)$$

For the integration one tries the statement

$$v = \sum_{s=-\infty}^{\infty} g_{\nu,s}^{\mu}(\gamma) i^s \xi^{\nu+s} \qquad (6.23)$$

Then there results for the g_s (the indices μ and ν as well as the argument γ in general are again omitted) the three-term recursion system

$$(\nu + s - \mu + 2)(\nu + s - \mu + 1)g_{s+2}$$
$$+ \left[(\nu + s + 1)(\nu + s) - \lambda\right]g_s - \gamma^2 g_{s-2} = 0$$
$$(s = 0, \pm 2, \pm 4, \ldots) \tag{6.24}$$

There exists a solution with the behavior at infinity $\frac{g_s}{g_{s-2}} \rightarrow \frac{\gamma^2}{s^2}$ for $s \rightarrow \infty$ and a solution with the behavior at infinity $\frac{g_s}{g_{s-2}} \rightarrow 1$ for $s \rightarrow -\infty$. The quotient of the two solutions is independent of s only then when λ assumes certain distinct values. As one can see by comparing with equation (5.23), these are just the values $\lambda_\nu^\mu(\gamma)$. From the behavior at infinity of the coefficients g_s one can conclude at once that the series (6.23) converges in the domain $1 < |\xi| < \infty$.

If one substitutes in equation (6.24) for the coefficients g_s the coefficients calculated already in equation (5.23), there results after elementary transformations

$$\sum_{r=-\infty}^{\infty} \frac{i^r a_{\nu,r}^\mu(\gamma)}{\Gamma\left(-\nu - \frac{r+s}{2} + \frac{3}{2}\right)\Gamma\left(2 + \frac{r-s}{2}\right)} \left\{(2\nu + r + s - 1)(2 + r - s)\left[(\nu + r + 1)(\nu + r) - \lambda\right] - \gamma^2(\nu + s - \mu - 1)(\nu + s - \mu)\right\} = 0$$
$$(s = 0, \pm 2, \pm 4, \ldots) \tag{6.25}$$

These relations can be used, like equation (5.15), for the control of numerically calculated values of the $a_{\nu,r}^\mu(\gamma)$.

The recursion system (6.24) is, except for the case of ν,μ being integers with $\nu \geq |\mu| \geq 0$, probably less suitable for the numerical calculation of the $\lambda_\nu^\mu(\gamma)$ than the continued fractions (6.1) and (6.2).

Ordinary power series with increasing powers of ξ result for the solutions of the basic equation if one sets equal $g_{-2} = g_{-4} = \ldots = 0$ and requires $g_0 = 0$.

Then there results for ν the determining equation

$$(\nu - \mu)(\nu - \mu - 1) = 0 \tag{6.26}$$

Therefore ν has here a meaning different from the one it had so far. The behavior at infinity of the g_s for $s \rightarrow -\infty$ is simple: all of them disappear. The behavior at infinity for $s \rightarrow \infty$ is given by $\frac{g_s}{g_{s-2}} \rightarrow 1$ or $\rightarrow \frac{\gamma^2}{s^2}$. The first case is the standard case; the power series converges for $|\xi| < 1$. The second case is, for ν and μ being integers with $\nu \geq |\mu| \geq 0$, realized for a solution of the basic equation, the X-function of the first kind; the power series then converges for all finite ξ.

It will be best to make the numerical calculation of the coefficients of these power series which are convergent in the unit circle so that first $\lambda_\nu^\mu(\gamma)$ will be determined according to the method given in section 6.1, or, for smaller values of γ, from the series (6.13); the coefficients g_s can then be calculated from equation (6.24) for each of the two ν-values given by equation (6.26). A special but simple problem will then be left: how the two calculated power series are connected with the X-functions of the first and second kind.

7. EIGENFUNCTIONS OF THE BASIC EQUATION

7.1 Limitation to ν, μ Being Integers; $\nu \geqq |\mu| \geqq 0$

The determining factors for the eigenvalues of the separation parameters λ and μ and, if occasion arises, of the wave coefficient k, are the domain of space which was taken as a basis and the boundary conditions on its boundary. This treatise is limited to the most important type of eigenvalue problems of this kind; for them the domain of space lies either within an ellipsoid of revolution, or between two confocal ellipsoids of revolution, or outside of an ellipsoid of revolution. The first two cases will be called problems of inside space, the last case problem of outside space. The entire domain $-1 \leqq \eta \leqq 1$, $0 \leqq \varphi \leqq 2\pi$ becomes then effective for the two coordinates η and φ. Boundary conditions in η and φ do not appear then; they are replaced by the requirement that the wave function for $\eta = \pm 1$ remains finite and that it is single valued, that is, that it has the same value for $\varphi + 2\pi$ that it has for φ. The latter requirement leads to μ's that are integers, the first one to ν's that are integers $\nu \geqq |\mu| \geqq 0$. That the X-functions of the first kind remain finite at the points $\eta = \pm 1$ follows directly from the series (4.1) by taking the estimate $\left|\underline{P}_\nu^\mu(\xi)\right| \leqq \frac{(\nu - \mu)!}{\nu!}\left|\xi + \sqrt{\xi^2 - 1}\right|^2$, which is valid for this case, as a basis.

Following, n will always be written for ν and m for μ where ν and μ are real integers; for the present, $n \geqq m \geqq 0$ is assumed. The case of negative m's, the absolute amount of which is $\leqq n$, is then obtained at once from equations (4.15), (4.17), and (5.20).

The calculation of these special functions was practically settled amongst other things in the last sections; even though it was assumed there that neither $\nu + \mu$ nor $\nu - \mu$ are integers, almost all results can nevertheless be taken over as simple limiting processes demonstrate. Only a few particularities result, compared with the general case; they will be discussed below.

7.2 Breaking Off of the Series

If $q_{r'} = 0$ for a positive r' or $p_{r'} = 0$ for a negative r', the a_r's break off to the right or to the left, that is,

$$\left.\begin{aligned} &\text{for } q_{r'} = 0; \ r' > 0: \ a_{r'} = a_{r'+2} = a_{r'+4} = \ldots = 0 \\ &\text{for } p_{r'} = 0; \ r' < 0: \ a_{r'} = a_{r'-2} = a_{r'-4} = \ldots = 0 \end{aligned}\right\} \quad (7.1)$$

is valid which follows in the simplest way from the continued fraction developments (6.1) and (6.2). These cases occur when $\mu - \nu$ is a positive integer or when $\mu + \nu$ is not a negative (Sic!) integer. Since it was presumed $0 \leqq m \leqq n$ the first possibility does not occur, but the second one does always occur, that is, for all admissible m,n. Here again two cases must be distinguished which are both originated from $p_{r'} = 0$:

$$n + m + 2 = -r' > 0 \quad \text{or} \quad n + m + 1 = -r' > 0 \qquad (7.2)$$

In the first case $m \pm n$ is an even number, in the second, an odd number; $a_{r'+2}$ is the first nonvanishing a_r. For the b_r's there follows from equation (4.12) that they disappear for all $r \leqq r' + 2m$. Further, all $P^m_{n+r}(\xi)$ disappear for $n + r = -m, \ -m + 1, \ \ldots m - 1$. The developments of the X- and Z-functions of the first kind begin, therefore, for $n - m$ = even with

$$\left.\begin{aligned} X_n^{m(1)}(\xi;\gamma) &= a^m_{n,m-n} i^{m-n} P^m_m(\xi) + a^m_{n,m-n+2} i^{m-n+2} P^m_{m+2}(\xi) + \ldots \\ Z_n^{m(1)}(\xi;\gamma) &= \left(\zeta^2 - \gamma^2\right)^{m/2} \zeta^{-m} \left[b^m_{n,m-n} \psi_m(\zeta) + b^m_{n,m-n+2} \psi_{m+2}(\zeta) + \ldots\right] \end{aligned}\right\} \quad (7.3)$$

and for $n - m =$ odd with

$$\left.\begin{aligned} X_n^{m(1)}(\xi;\gamma) &= a_{n,m-n+1}^{m}\, i^{m-n+1}\, \underline{P}_{m+1}^{m}(\xi) \\ &\quad + a_{n,m-n+3}^{m}\, i^{m-n+3}\, \underline{P}_{m+3}^{m}(\xi) + \ldots \\ Z_n^{m(1)}(\zeta;\gamma) &= \left(\zeta^2 - \gamma^2\right)^{m/2} \zeta^{-m}\Big[b_{n,m-n+1}^{m}\, \psi_{m+1}(\zeta) \\ &\quad + b_{n,m-n+3}^{m}\, \psi_{m+3}(\zeta) + \ldots\Big] \end{aligned}\right\} \tag{7.4}$$

The series for $Z_n^{m(1)}(\zeta;\gamma)$ converges for all finite ζ. The corresponding formulas for the Z-functions of the second kind result if the functions $\psi_{\nu+r}(\zeta)$ are replaced by $n_{\nu+r}(\zeta)$. The developments for the X-functions of the second kind show a special behavior. The spherical functions of the second kind belonging to the vanishing coefficients $a_{r'}$, $a_{r'-2}$, . . . become infinitely large in such a manner that their products have finite limiting values. The coefficients $\alpha_{n,r}^{m}(\gamma)$ are defined by

$$\lim\ a_{\nu,r}^{m}\, \underline{Q}_{\nu+r}^{m}(\xi) = \alpha_{n,r}^{m}\, \underline{P}_{n+r}^{m}(\xi)$$

$$\text{for}\quad m = 0, 1, 2, \ldots \quad\text{and}\quad \nu + r + m \longrightarrow -1, -2, \ldots \tag{7.5}$$

Then there becomes

$$\alpha_{n,r}^{m}(\gamma) = \lim\ (-1)^{m+n+1}\, \Gamma(1 + \nu + m)\Gamma(-\nu-m)\, a_{\nu,r}^{m}(\gamma)$$

$$\text{for}\quad m = 0, 1, 2, \ldots \quad\text{and}\quad \nu + r + m \longrightarrow -1, -2, \ldots \tag{7.6}$$

and the series (4.2) reads

$$X_n^{m(2)}(\xi;\gamma) = \sum_{r=-\infty}^{r'} i^r \alpha_{n,r}^m(\gamma)\underline{P}_{n+r}^m(\xi) + \sum_{r=r'+2}^{\infty} i^r a_{n,r}^m(\gamma)\underline{Q}_{n+r}^m(\xi) \qquad (7.7)$$

The series (5.41) for $V_n^{m(1)}(w;\gamma)$ breaks off only when $n - m$ is an even number; for odd values of $n - m$ the coefficients $c_{n,r}^m(\gamma)$ have the indefinite value $\infty.0$ if $r \leqq r'$ with finite limiting value. The series (5.42) for $W_n^{m(1)}(w;\gamma)$, on the other hand, breaks off only when $n - m$ is an odd number; for even values of $n - m$ the coefficients $d_{n,r}^m(\gamma)$ have the indefinite value $\infty.0$ with finite limiting value if $r \leqq r'$. Similar conditions exist for the other V- and W-functions.

7.3 A Few Special Function Values

From the series (5.1) one obtains when $\arg(\zeta^2 - 1) = \pi$ for $\zeta = 0$

$$Z_n^{m(1)}(0;\gamma) = \begin{cases} \frac{1}{2}\sqrt{\pi}\, i^m(\gamma/2)^m\, b_{n,m-n}^m(\gamma)\Big/\Gamma\left(\frac{3}{2} + m\right) & \text{for } n - m \text{ even} \\ 0 & \text{for } n - m \text{ odd} \end{cases} \qquad (7.8)$$

$$\frac{dZ_n^{m(1)}(0;\gamma)}{d\zeta} = \begin{cases} 0 & \text{for } n - m \text{ even} \\ \frac{1}{4}\sqrt{\pi}\, i^m(\gamma/2)^m\, b_{n,m-n+1}^m(\gamma)\Big/\Gamma\left(\frac{5}{2} + m\right) & \text{for } n - m \text{ odd} \end{cases} \qquad (7.9)$$

The X- and Z-functions of the first kind are for the index values n and m considered here either even or odd functions of ξ or ζ, respectively, according to whether n - m is an even or odd value. Furthermore, because of $\underline{Q}_n^m(\cos\theta + i.0) = (-i)^m Q_n^m(\cos\theta) - \frac{i\pi}{2.}\underline{P}_n^m(\cos\theta + i.0)$ there is

$$X_n^{m(2)}(0;\gamma) = -\frac{1}{2}\pi i\, X_n^{m(1)}(0;\gamma) \qquad \text{for } n - m \text{ even} \qquad (7.10)$$

$$\frac{dX_n^{m(2)}(0;\gamma)}{d\xi} = -\frac{1}{2}\pi i\, \frac{dX_n^{m(1)}(0;\gamma)}{d\xi} \qquad \text{for } n - m \text{ odd} \qquad (7.11)$$

From Wronski's determinant (5.35) follows

$$\gamma Z_n^{m(1)}(0;\gamma)\,\frac{dZ_n^{m(2)}(0;\gamma)}{d\zeta} = -\left[\sum_{r=-\infty}^{\infty} i^r b_{n,r}^m(\gamma)\right]^2 \qquad \text{for } n - m \text{ even} \quad (7.12)$$

$$\gamma Z_n^{m(2)}(0;\gamma)\,\frac{dZ_n^{m(1)}(0;\gamma)}{d\zeta} = \left[\sum_{r=-\infty}^{\infty} i^r b_{n,r}^m(\gamma)\right]^2 \qquad \text{for } n - m \text{ odd} \quad (7.13)$$

Therefrom the Z-function of the second kind and its derivative with respect to ζ for $\zeta = 0$ can be calculated at once.

7.4 Connection between the X- and Z-Functions

If ν,μ are integers, considerable simplifications occur in the relations (4.15) to (4.20), (4.24), (4.25), (5.6) to (5.9), (5.17), and (5.18). They are so obvious that they need not be discussed further. Equations (5.29) and (5.30) now assume the simple form

$$\mathcal{Z}_n^{m(i)}(\zeta;\gamma) = \kappa_n^{m(i)}(\gamma) X_n^{m(i)}(\xi;\gamma) \qquad (i = 1, 2) \tag{7.14}$$

For the $\kappa_n^{m(i)}(\gamma)$ simpler expressions can be obtained if s in equations (5.31) and (5.32) is selected in a suitable manner. The same expressions, however, result in an even simpler way if one substitutes in equation (7.14) and in the derivative of this equation with respect to ξ, respectively, the special value $\xi = 0$. If one expresses $\mathcal{Z}_n^{m(i)}(0;\gamma)$ and $d\mathcal{Z}_n^{m(i)}(0;\gamma)/d\xi$, respectively, according to equations (7.8), (7.9), (7.12), (7.13), using equation (5.13), there originates for $n - m$ = even

$$\left.\begin{aligned} \kappa_n^{m(1)}(\gamma) &= \frac{1}{2}\pi^{1/2}\, i^m \left(\frac{\gamma}{2}\right)^m \frac{b_{n,m-n}^m(\gamma)}{X_n^{m(1)}(0;\gamma)\Gamma\left(\frac{3}{2} + m\right)} \\ \kappa_n^{m(2)}(\gamma) &= -\pi^{-1/2} i^{-m} \left(\frac{\gamma}{2}\right)^{m-1} \frac{X_n^{-m(1)}(0;\gamma)\Gamma\left(\frac{3}{2} - m\right)}{a_{n,-n-m}^m(\gamma)} \end{aligned}\right\} \tag{7.15}$$

and for $n - m$ = odd

$$\left.\begin{aligned} \kappa_n^{m(1)}(\gamma) &= \frac{1}{2}\pi^{1/2}\, i^m \left(\frac{\gamma}{2}\right)^{m+1} \frac{b_{n,m-n+1}^m(\gamma)}{dX_n^{m(1)}(0;\gamma)/d\xi\ \Gamma\left(\frac{5}{2} + m\right)} \\ \kappa_n^{m(2)}(\gamma) &= -\pi^{-1/2}\, i^{-m} \left(\frac{\gamma}{2}\right)^{m-2} \frac{dX_n^{-m(1)}(0;\gamma)/d\xi\ \Gamma\left(\frac{5}{2} - m\right)}{a_{n,-n-m+1}^m(\gamma)} \end{aligned}\right\} \tag{7.16}$$

By $X_n^{m(1)}(0;\gamma)$ and $dX_n^{m(1)}(0;\gamma)/d\xi$ the values of these functions are understood which result when ξ goes towards zero from the positive imaginary half plane.

The distinction between even and odd $n - m$ can be avoided if one sets, for instance, s equal zero in calculating the $\kappa_n^{m(1)}(\gamma)$ from equations (5.31) and (5.32); the formulas (7.15) and (7.16), on the other hand, have the advantage of greater simplicity.

7.5 Normalization and Properties of Orthogonality of the X-Functions of the First Kind

The eigenvalues of the basic equation $\lambda_n^m(\gamma)$ are always real. Proof of it is given in the known manner. Equally simply it can be shown that the functions $X_n^{m(1)}(\xi;\gamma)$ are orthogonal to each other, that is,

$$\int_{-1}^{1} X_n^{m(1)}(\xi;\gamma)X_{n'}^{m(1)}(\xi;\gamma)\,d\xi = 0 \tag{7.17}$$

is valid for $n \neq n'$.

By inserting the series (4.1) into (7.17) one can also express this property of orthogonality for even differences $n - n'$ thus:

$$\sum_{r=-\infty}^{\infty} a_{n,r}^m(\gamma)b_{n',r+n-n'}^m(\gamma)\frac{2}{2n + 2r + 1} = 0 \qquad \text{for } n \neq n' \tag{7.18}$$

For the normalization integral one obtains

$$\int_{-1}^{1} X_n^{m(1)}(\xi;\gamma)X_n^{m(1)}(\xi;\gamma)\,d\xi = \frac{(n+m)!}{(n-m)!}\sum_{r=-\infty}^{\infty}\frac{2}{2n+2r+1}a_{n,r}^m(\gamma)b_{n,r}^m(\gamma) \tag{7.19}$$

7.6 Generalization of F. E. Neumann's Integral Relation

In the case $m = 0$ one obtains a second solution of the basic equation (2.4g) which is independent of $X_n^{(1)}(\xi;\gamma)$ in the form of the integral

$$F(\xi) = \frac{1}{2}\int_{-1}^{1} \frac{e^{i\gamma(t-\xi)}}{\xi - t} X_n^{(1)}(t;\gamma)\, dt \qquad (7.20)$$

The fact that this integral actually represents a solution of equation (2.4g) is confirmed by substitution. The calculation is reproduced in detail in Bouwkamp (reference 10). For large ξ, t and ξ in the denominator of the integrand cancel in first approximation and one can see then at once that $J(\xi)$ is proportional to the Z-function of the fourth kind. The integral over t can then be evaluated according to equations (8.20) and there originates, because of equation (5.12),

$$J(\xi) = -i\gamma\left[Z_n^{(1)}(\gamma\xi;\xi) - iZ_n^{(2)}(\gamma\xi;\xi)\right] Z_n^{(1)}(\gamma;\gamma) \Big/ \sum_{r=-\infty}^{\infty} i^r a_{n,r}^m(\gamma) \qquad (7.21)$$

According to equations (7.8) and (5.33) the Z-functions are now converted to X-functions. Because of equation (5.39) there results finally

$$X_n^{(2)}(\xi;\gamma) - i\gamma\left[\kappa_n^{(1)}(\gamma)\right]^2 X_n^{(1)}(\xi;\gamma) = \frac{1}{2}\int_{-1}^{1} \frac{e^{i\gamma(t-\xi)}}{\xi - t} X_n^{(1)}(t;\gamma)\, dt \qquad (7.22)$$

Therefrom results for $\gamma = 0$ F. E. Neumann's integral relation between spherical functions of the first and second kind.

7.7 Zeros of the Eigenfunctions

For $m > 0$ the zeros of the basic equation are situated at $\xi = \pm 1$, respectively, since they there have the behavior $\left(\xi^2 - 1\right)^{m/2}$. If one divides the eigenfunctions by this expression, the quotient does not have zeros at $\xi = \pm 1$. In order to understand this, one need only enter the basic equation (2.4g) with the expression $(\xi \pm 1)^{m/2}$ multplied by a power series in $(\xi \pm 1)$. The zeros of $\left(\xi^2 - 1\right)^{-m/2} X_n^{m(1)}(\xi;\gamma)$ are all simple; if they were not simple, all higher derivatives of the eigenfunctions would have to disappear there also. Since it is, however, a non-identically vanishing analytic function, this case can never occur. Further properties of the zeros of the eigenfunctions follow from a simple consideration of continuity: namely, that in a nonsingular point of the basic equation a zero cannot be newly originated for a change of . . .* and an already existing one cannot vanish. Therewith the problem of the number of the zeros is essentially reduced to the problem of the number of zeros of Légendre's and their associated polynomials and of Bessel's functions with an index of a fractional value of one-half.

One deals first with the $X_n^{m(1)}(\xi;\gamma)$ with real ξ and γ, that is, with the eigenfunctions of the prolate ellipsoid of revolution. All zeros are real; for this is valid for $\gamma = 0$. If, namely, for a change of γ a complex zero would originate, the conjugate-complex would originate along with it; but it contradicts the simplicity of the zeros, that a real zero splits into two complex zeros. The number of the zeros in the interval $-1 < \xi < +1$ equals $n - m$, that is, the number of zeros of $P_n^m(\xi)$ in this interval. The zeros outside of this interval go over into the zeros of $J_{n+1/2}(\gamma\xi)$ for $\gamma \longrightarrow 0$; the asymptotic distribution of the zeros for large ξ is the same as the distribution of $J_{n+1/2}(\gamma\xi)$ for arbitrary γ.

For the eigenfunctions of the oblate ellipsoid of revolution $X_n^m(\xi;i\gamma)$ with real ξ and γ also $n - m$ zeros are situated in the interval $-1 < \xi < 1$; but now the remaining zeros

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*Translator's note: . . . missing in the original.

lie on the positive and negative imaginary semiaxis of the complex ξ-plane; they agree asymptotically as well as for $\gamma = 0$ with those of $J_{n+1/2}(i\gamma\xi)$.

7.8 Integral Equations for the Eigenfunctions

The most important integral equation for the X-functions of the first kind is derived from the known relation

$$Qi^{n+m} \frac{(n+m)!}{(n-m)!} z^{-m} \psi_n(z) = \int_{-1}^{1} e^{i\eta z} P_n^m(\eta)\left(1 - \eta^2\right)^{m/2} d\eta \tag{7.23}$$

If one writes $\gamma\xi$ for z, replaces n by $n + r$, multiplies by $i^r a_{n,r}^m(\gamma)$ and forms the sum over all even r, there results

$$Qi^{n} \frac{(n+m)!}{(n-m)!}\left(\xi^2 - 1\right)^{-m/2} \gamma^{-m} Z_n^{m(1)}(\gamma\xi;\gamma) = \int_{-1}^{1} e^{i\gamma\xi\eta} X_n^{m(1)}(\eta;\gamma)\left(1 - \eta^2\right)^{m/2} d\eta \tag{7.24}$$

Equation (7.24) is a homogeneous linear integral equation of the Fredholm type for the X- and Z-functions of the first kind, respectively, with the symmetric kernel $e^{i\gamma\xi\eta}(1 - \eta^2)(1 - \xi^2)$. If one replaces in accordance with equation (7.14) the Z-function of the first kind by the X-function of the first kind, one can see that the connecting coefficient $\kappa_n^{m(1)}(\gamma)$ in this integral equation plays essentially the role of the eigenvalue parameter.

The integral equation (7.24) can be generalized. By selecting another path of integration one can, for instance, also express the Z-function of the second kind (that is, also the X-function of the second kind) by an integral over the X-function of the first kind; furthermore, equation (7.24) can be generalized to the case of arbitrary ν, μ, γ. However, the respective results shall not be discussed here. Kotani (reference 8) indicated a general principle for obtaining more general integral equations for the X- and Z-functions, respectively.

Integral representations for the X- and Z-functions have not become known so far. It seems that the integral equations of the type (7.24) or of another kind also can be substituted for them and replace them; thus equation (7.24), for instance, represents a very useful starting point for the investigation of the ξ-asymptotics of the X-functions of the first kind. The integral equation (7.24) can perhaps also be applied when the values of the X-function of the first kind are known only in the interval $-1 < \xi < 1$ and are to be calculated for arbitrary real and complex ξ. (Compare the discussion on the γ-asymptotics of the eigenfunctions in the following section.) As Möglich (reference 4) has shown, the integral equation (7.24) can also be used for obtaining developments of the X-functions of the first kind in terms of powers of γ.

8. ASYMPTOTICS OF THE EIGENVALUES AND EIGENFUNCTIONS

8.1 Asymptotic Behavior of the Eigenvalues and Eigenfunctions for Large ν

The continued fractions (6.1) and (6.2) do not only have the property to yield a development of $\lambda_\nu^\mu(\gamma)$ in terms of powers of γ but in addition one can obtain from them a development in terms of powers of ν^{-1}. It is more favorable to set up a development in terms of powers of $(2\nu + 1)^{-1}$, because then the odd powers of $(2\nu + 1)$ are eliminated because of equation (4.9). The calculation itself is relatively simple so that the result can be given immediately

$$\lambda_\nu^\mu(\gamma) = \nu(\nu + 1) + \frac{1}{2}\gamma^2 + \frac{1}{8(2\nu + 1)^2}\left[(4 - 16\mu^2)\gamma^2 + \gamma^4\right]$$

$$+ \frac{1}{2(2\nu + 1)^4}\left[(4 - 16\mu^2)\gamma^2 + \left(\frac{5}{2} - 6\mu^2\right)\gamma^4\right] + 0\left[(2\nu + 1)^{-6}\right] \quad (8.1)$$

Presumably this series is not convergent but has asymptotic character.

In order to form a judgment on the usefulness of the series (8.1) for numerical purposes, one gives for several cases the numerical value of the remainder term denoted by $0\left[(2\nu + 1)^{-6}\right]$ in comparison with the value of the separation parameter λ itself.

n	2	4	6	8
$\lambda_n^o\left(\sqrt{10}\right)$	11.7904	25.2513	47.10958	77.06246
Remainder term	-0.134	0.0132	0.00095	0.00017

In a similar way one obtains the following expressions for the development coefficients of the eigenfunctions for large values of ν.

$$\frac{a_2}{a_0} = \frac{\epsilon}{8(2\nu + 1)} - \frac{\mu\epsilon}{2(2\nu + 1)^2} + 0\left[(2\nu + 1)^{-3}\right] \quad (8.2)$$

$$\frac{a_{-2}}{a_0} = -\frac{\gamma^2}{8(2\nu + 1)} - \frac{\mu\gamma^2}{2(2\nu + 1)^2} + 0\left[(2\nu + 1)^{-3}\right] \quad (8.3)$$

$$\left.\begin{aligned} \frac{a_4}{a_0} &= \frac{\gamma^4}{32(2\nu + 1)^2} + 0\left[(2\nu + 1)^{-3}\right] \\ \frac{a_{-4}}{a_0} &= \frac{\gamma^4}{32(2\nu + 1)^2} + 0\left[(2\nu + 1)^{-3}\right] \end{aligned}\right\} \quad (8.4)$$

8.2 Asymptotic Behavior of the Eigenvalues for Large Real γ

One limits oneself here to n and m which are integers ($n \geqq m \geqq 0$) and to real large γ which may be assumed to be positive without essential restriction. Thus one obtains the asymptotics of the eigenvalues and eigenfunctions for the wave equation in the coordinates of the prolate ellipsoid of revolution.

An approximate picture of the eigenvalues and eigenfunctions is obtained if one puts the basic equation into the Liouville standard form

$$\left.\begin{aligned} &\frac{d^2q}{d\theta^2} + \left(\lambda - \gamma^2 \cos^2\theta - \frac{\mu^2 - \frac{1}{4}}{\sin^2\theta} + \frac{1}{4}\right)q = 0 \\ &q = F_1(\xi)\left(1 - \xi^2\right)^{1/4} \\ &\xi = \cos\theta \end{aligned}\right\} \tag{8.5}$$

and interprets it as a Schrödinger wave equation in the interval $0 \leqq \theta \leqq \pi$ of the potential energy (in suitable units)

$$\gamma^2 \cos^2\theta + \frac{\mu^2 - \frac{1}{4}}{\sin^2\theta} - \frac{1}{4} \tag{8.6}$$

It has for large γ at $\theta = \frac{\pi}{2}$ a very narrow minimum and can there very well be approximated by a parable. Then, however, there results just Schrödinger's wave equation of the harmonic oscillator, for which eigenvalues and eigenfunctions are known.

In order to obtain also higher approximations it suggests itself to attempt a similar formulation as in equations (4.1) and (5.1). One sets

$$F_1 = \left(\xi^2 - 1\right)^{m/2} \sum_{r=-\infty}^{\infty} \vartheta_r D_{N+r}\left(\xi\sqrt{2\gamma}\right) \tag{8.7}$$

equal, where the D_n are Hermite's orthogonal functions and the functions of the parabolic cylinder, respectively. By substitution of equation (8.7) into equation (2.4g) there results, if one utilizes also the recursion formulas and the differential equation of Hermite's orthogonal functions (see Magnus and Oberhettinger (reference 11)), the five-term recursion system

$$\vartheta_{r-4} - 4\mu\,\vartheta_{r-2} + \Big[-4\lambda + 8\gamma\left(N + r + \tfrac{1}{2}\right) + 4m^2$$

$$- 2(N + r)^2 - 2(N + r) - 3\Big]\vartheta_r + 4\mu(N + r + 2)(N + r + 1)\vartheta_{r+2}$$

$$+ (N + r + 4)(N + r + 3)(N + r + 2)(N + r + 1)\vartheta_{r+4} = 0$$

$$(r = 0, \pm 2, \pm 4, \ldots) \tag{8.8}$$

The series (8.7) is probably not convergent; it rather represents an asymptotic development in the sense that limits $\dfrac{\vartheta_r}{\vartheta_0} = 0$ for all even $r \neq 0$, or, as one concludes from that and from equation (8.8)

$$\left.\begin{aligned} \frac{\vartheta_{\pm(2r+2)}}{\vartheta_0} &= 0(\gamma^{-1-r/2}) \\ \frac{\vartheta_{\pm(2r+4)}}{\vartheta_0} &= 0(\gamma^{-1-r/2}) \\ (r = 0,\ 2,\ 4,\ \ldots) & \end{aligned}\right\} \tag{8.9}$$

By a method of successive approximation the ϑ_r and λ can be represented as power series in γ^{-1}. The calculation is elementary; thus only the result is given. It is

$$\lambda_n^m(\gamma) = (2N + 1)\gamma - \frac{1}{4}(2N^2 + 2N + 3 - 4m^2)$$
$$- \frac{1}{16\gamma}(2N + 1)(N^2 + N + 3 - 8m^2)$$
$$+ \frac{1}{64\gamma^2}\Big[48m^2(2N^2 + 2N + 1)$$
$$- 5(N^4 + 2N^3 + 8N^2 + 7N + 3)\Big] + O(\gamma^{-3}) \qquad (8.10)$$

The connection between N, n, and m is given by counting the zeros. For $\gamma \longrightarrow \infty$ the development (8.8) is reduced to the principal term with the N real zeros of Hermite's N^{th} polynomial, whereas the X-function with the indices n and m to be approximated has exactly n - m real zeros in the interval $-1 < \xi < 1$. Therefrom follows $N = n - m$. For negative m one inserts instead $N = n + m$.

8.3 Asymptotic Behavior of the Eigenfunctions for Large Real γ

The asymptotic representation of the eigenfunctions results by calculation of the coefficients ϑ_r. They read, aside from the terms of the order γ^{-3},

$$\left.\begin{aligned}
\frac{\vartheta_2}{\vartheta_0} &= \frac{m}{4\gamma}\left[1 - \frac{1}{32\gamma}(N^2 - 25N - 36)\right] \\
\frac{\vartheta_{-2}}{\vartheta_0} &= \frac{m}{4\gamma}\left[1 + \frac{1}{32\gamma}(N^2 + 27N - 10)\right]N(N - 1) \\
\frac{\vartheta_4}{\vartheta_0} &= -\frac{1}{32\gamma}\left[1 + \frac{1}{2\gamma}\left(N + \frac{5}{2} - 2m^2\right)\right] \\
\frac{\vartheta_{-4}}{\vartheta_0} &= \frac{1}{32\gamma}\left[1 + \frac{1}{2\gamma}\left(N - \frac{3}{2} + 2m^2\right)\right]N(N - 1)(N - 2)(N - 3) \\
\frac{\vartheta_6}{\vartheta_0} &= -\frac{m}{128\gamma^2}; \qquad \frac{\vartheta_{-6}}{\vartheta_0} = \frac{m}{128\gamma^2}\,\frac{N!}{(N - 6)!} \\
\frac{\vartheta_8}{\vartheta_0} &= \frac{1}{2048\gamma^2}; \qquad \frac{\vartheta_{-8}}{\vartheta_0} = \frac{1}{2048\gamma^2}\,\frac{N!}{(N - 8)!}
\end{aligned}\right\} \tag{8.11}$$

According to the type of derivation, however, the eigenfunctions are approximated by these series only in the interval $-1 < \xi < 1$. In order to obtain an asymptotic series also for other ξ one starts from the integral equation (7.24) and substitutes for the X-functions in the integrand the series (8.8). Therewith the asymptotic development of the eigenfunctions for all ξ is known; in particular, their behavior can be investigated where, besides γ, ξ also is very large. Since now the eigenfunction for all ξ is asymptotically known, one obtains the solutions of the second kind by calculating the integral in equation (7.24) with the asymptotic series of the eigenfunction and by means of another appropriate path of integration.

The zeros of the eigenfunctions located in the interval $-1 < \xi < 1$ crowd for large γ more and more around $\xi = 0$; in order to understand this, one has only to divide the zeros of Hermite's N^{th} polynomial by $\sqrt{2\gamma}$ and therewith to convert to the ξ-scale.

The domain of validity for equation (8.10) and (8.11) extends over the indicated domain; thus originates, for instance, for $m = \pm\frac{1}{2}$ from equation (8.10) the asymptotic representation of the eigenvalues of Mathieu's differential equation found by Ince (reference 14). However, the limits for this domain of validity shall not be submitted to closer investigation here.

8.4 Asymptotic Behavior of the Eigenvalues for Large Purely Imaginary γ

One limits oneself again to n and m that are integers ($n \geqq m \geqq 0$) and to purely imaginary γ of large absolute value. This procedure yields the asymptotics of the eigenvalues and eigenfunctions for the coordinates of the oblate ellipsoid of revolution (reference 10) and for the so-called inner equation for the separation of the wave equation of the ion of the hydrogen molecule (reference 7).

The method applied in equation (8.2) fails here; $\gamma\xi^2$ namely would become purely imaginary and the $D_n\left(\xi\sqrt{2\gamma}\right)$ would, for large ξ, no longer decrease exponentially, but increase exponentially; they would, therefore, be no longer appropriate for the development of the eigenfunctions. The wave mechanical picture of the differential equation (8.5) shows that in the case of purely imaginary γ two domains with low potential energy are present at $\theta = 0$ and $\theta = 2\pi$, which are separated by a high potential peak with the maximum at $\theta = \frac{\pi}{2}$. One may, therefore, expect beforehand that the eigenvalues will degenerate in first approximation; their split-up is exponentially small in $|\gamma|$; it is the larger, the higher the eigenvalue. For each eigenvalue there is an eigenfunction symmetric with respect to $\theta = \frac{\pi}{2}$, that is, $\xi = 0$ and an asymmetric eigenfunction.

The mathematical treatment is as follows. A singularity is made to move to infinity. Then one obtains from equation (2.4g), aside from an elementary transformation, the differential equation of

Laguerre's orthogonal polynomials. This suggests for the solution of equation (2.4g) the formulation of Svartholm (reference 7)

$$F_1(\xi) = \left(1 - \xi^2\right)^{m/2} e^{p\xi} \sum_{t=-\infty}^{\infty} \sigma_t L_{N+t}^{(m)}\left[2p(1 - \xi)\right] \tag{8.12}$$

wherein $i\gamma$ was set equal to p; again it does not mean an essential restriction if $p > 0$ is assumed. By substitution of equation (8.12) into the differential equation 2.4g), application of Laguerre's differential equation, and the recursion formulas for Laguerre's polynomials (compare Magnus and Oberhettinger (reference 11)), there originates in the known way for the σ_t a three-term recursion system. With the abbreviations

$$\lambda = -p^2 + 2\tau p - \frac{1}{2}(\tau^2 + 1 - m^2) + \Lambda \tag{8.13}$$

$$\tau = 2N + m + 1 \tag{8.14}$$

$$4A_t = (\tau + 2t - 1)^2 - m^2; \quad P_t = 2t(\tau - 2p + t) \tag{8.15}$$

the recursion system reads

$$\sigma_{t+1} A_{t+1} + \sigma_{t-1} A_t = (\Lambda + P_t)\sigma_t \quad (t = -N, -N+1, -N+2, \ldots \tag{8.16}$$

Therefrom follows for Λ the transcendent equation

$$\Lambda = \frac{A_1^2|}{\Lambda + P_1} - \frac{A_2^2|}{|\Lambda + P_2} \ldots + \frac{A_0^2|}{\Lambda + P_{-1}} - \frac{A_{-1}^2|}{|\Lambda + P_{-2}} - \ldots \tag{8.17}$$

from which Λ can be obtained as series in terms of powers of p^{-1}. Therefrom then results

$$\lambda_n^m(\gamma) = -p^2 + 2\tau p - \frac{1}{2}(\tau^2 + 1 - m^2) - \frac{\tau}{8p}(\tau^2 + 1 - m^2)$$

$$- \frac{1}{64p^2}\left[5\tau^4 + 10\tau^2 + 1 - 2m^2(3\tau^2 + 1) + m^4\right]$$

$$- \frac{\tau}{512p^3}\left[33\tau^4 + 114\tau^2 + 37 - 2m^2(23\tau^2 + 25) + 13m^4\right] + o\left(|\gamma|^{-4}\right) \quad (8.18)$$

8.5 Asymptotic Behavior of the Eigenfunctions for Large Purely Imaginary γ

For the coefficients of the development (8.12)

$$\left.\begin{aligned}
\frac{\sigma_1}{\sigma_0} &= -\frac{1}{16p}\left[(\tau + 1)^2 - m^2\right]\left(1 + \frac{\tau + 1}{2p}\right) + o\left(|\gamma|^{-3}\right) \\
\frac{\sigma_2}{\sigma_0} &= \frac{1}{512p^2}\left[(\tau + 1)^2 - m^2\right]\left[(\tau + 3)^2 - m^2\right] + o\left(|\gamma|^{-3}\right) \\
\frac{\sigma_{-1}}{\sigma_0} &= \frac{1}{16p}\left[(\tau - 1)^2 - m^2\right]\left(1 + \frac{\tau - 1}{2p}\right) + o\left(|\gamma|^{-3}\right) \\
\frac{\sigma_{-2}}{\sigma_0} &= \frac{1}{512p^2}\left[(\tau - 1)^2 - m^2\right]\left[(\tau - 3)^2 - m^2\right] + o\left(|\gamma|^{-3}\right)
\end{aligned}\right\} \quad (8.19)$$

is valid.

The significance of τ and N results again from counting the zeros. The principal term $L_N^{(m)}\left[-2i\gamma(1 - \xi)\right]$ has N zeros which for large $|\gamma|$ lie all closely to $\xi = 1$. The real eigenfunction has again N zeros in the neighborhood of $\xi = -1$. For odd $n - m$ another zero at $\xi = 0$ is added. The sum total of the zeros $n - m$ equals, therefore, $2N$ for even $n - m$ and $2N + 1$ for odd $n - m$; thus

$$\left.\begin{aligned} \tau &= n + 1 = 2N + m + 1 \qquad \text{for} \quad n - m = \text{even} \\ \tau &= n = 2N + m + 1 \qquad \text{for} \quad n - m = \text{odd} \end{aligned}\right\} \tag{8.20}$$

is valid.

Baber and Hassé (reference 7) calculated the series (8.18) with the exception of the last two terms; only for the special case $N = 0$ they give also the last two terms; Bouwkamp (reference 10) calculated the series (8.18) with the exception of the last term for the special case $m = 0$. The asymptotic series (8.18) can still be used for $m = \pm\frac{1}{2}$; it then goes over, exactly like equation (8.10), into the asymptotic series for the eigenvalues of Mathieu's functions (reference 14).

For large values of $|\gamma|$ the eigenvalues move closer and closer together in pairs so that the asymptotic series (8.18) for the eigenvalues of each pair are the same (see equations (8.20)); that is, the difference of the two eigenvalues has a stronger tendency to vanish with increasing $|\gamma|$ than any power of $1/|\gamma|$. (Compare table 11.)

The series (8.12) for the eigenfunction is useless in the interval $-1 \leqq \xi \leqq 0$. There an approximation must be attempted starting from the point $\xi = -1$. Since the eigenfunctions become exponentially small in the neighborhood of $\xi = 0$, one can build up the eigenfunction in the entire interval $-1 < \xi < 1$ by combination of the two approximations starting from -1 and 1 and one obtains

$$X_n^{m(1)}(\xi;\gamma) = \text{Constant}\left(1 - \xi^2\right)^{m/2} \sum_{t=-\infty}^{\infty} \sigma_t \left\{e^{p\xi} L_{N+t}^{(m)}\left[2p(1 - \xi)\right] \pm e^{-p\xi} L_{N+t}^{(m)}\left[-2p(1 - \xi)\right]\right\} \tag{8.21}$$

For even $n - m$ the positive, for odd $n - m$ the negative sign is to be selected; in the one case the eigenfunction is symmetric, in the other antisymmetric with respect to the point $\xi = 0$.

What was said in section 8.3 is valid for the asymptotic calculation of the eigenfunctions and the functions of the second kind for any complex ξ as well as for the limits of the domain of validity in the variables ν, μ, γ of the asymptotic representations.

In order to show the use of the asymptotic series for numerical purposes one compares for $m = 0$ a few eigenvalues with the values resulting from equations (8.10) and (8.18) by giving the value of the remainder term $O(\gamma^{-3})$ and $O(p^{-4})$, respectively.

n	0	0	0	2
γ^2	10	-25	-100	-100
$\lambda_n(\gamma)$	2.305	-16.07904	-81.02794	-45.48967
Remainder term	-0.025	-0.01616	-0.00008	-0.01528

9. EIGENFUNCTIONS OF THE WAVE EQUATION IN ROTATIONALLY SYMMETRICAL ELLIPTIC COORDINATES

9.1 Lamé's Wave Functions of the Prolate Ellipsoid of Revolution

By separation of the wave equation in the coordinates of the prolate ellipsoid of revolution one obtains the following solutions of the wave equation

$$u = \left[AZ_\nu^{\mu(1)}(\zeta;\gamma) + BZ_\nu^{\mu(2)}(\zeta;\gamma)\right]\left[CX_\nu^{\mu(1)}(\eta;\gamma) + DX_\nu^{\mu(2)}(\eta;\gamma)\right]\left(Ee^{i\mu\varphi} + Fe^{-i\mu\varphi}\right) \tag{9.1}$$

A, B, C, D, E, F are arbitrary constants, ν and μ arbitrary real or complex parameters; the significance of γ is given by equation (2.8), thus γ is real. The coordinates ξ and $\zeta = \gamma\xi$, respectively, η and φ are real as well. Under $X_\nu^{\mu(1,2)}(\eta;\gamma)$ one understands $X_\nu^{\mu(1,2)}(\eta + i \times 0;\gamma)$. According to the kind of the boundary value problem presented, the arbitrariness concerning the constants and parameters is limited; then such solutions of the wave function have to be determined which remain finite for the entire domain of the eigenfunctions.

Following, as before, three-dimensional domains only are dealt with which lie inside or outside of an ellipsoid of revolution or between two confocal ellipsoids of revolution. Then the domain of the coordinates η and φ is given by $-1 \leqq \eta \leqq 1$, $0 \leqq \varphi \leqq 2\pi$. The requirement of single-valuedness and finiteness of the eigenfunctions then leads to $\nu = n$, $\mu = m$, $n \geqq |m| \geqq 0$, and $D = 0$. The eigenfunctions are written in the form

$$u_n^m(\xi,\eta,\varphi;k) = \left[AZ_n^{m(1)}(\zeta;\gamma) + BZ_n^{m(2)}(\zeta;\gamma)\right] X_n^{m(1)}(\eta;\gamma)e^{im\varphi}$$

$$(n = 0, 1, 2, \ldots; \quad m = 0, \pm 1, \pm 2, \ldots, \pm n) \tag{9.2}$$

The domain of variables in ξ is denoted by $\xi_1 \leqq \xi \leqq \xi_2$ and $\zeta_1 \leqq \zeta \leqq \zeta_2$, respectively. For the prolate ellipsoid of revolution there is always $1 \leqq \xi_1$. For inside space problems ξ_2 = finite, for outside space problems infinite. For inside space problems boundary conditions for ξ_1 and ξ_2 are to be prescribed. This results in two linear homogeneous determining equations for A and B; they can be satisfied only for certain distinct values of γ, that is, for certain eigenfrequencies; in that case they fix the ratio A:B. In case $\xi_1 = 1$ a boundary condition can be prescribed only for $\xi_2 > 1$; the boundary condition for $\xi_1 = 1$ is then replaced by the requirement of finiteness of the eigenfunction at the singular point $\xi_1 = 1$; it leads to $B = 0$.

For outside space problems the boundary condition for $\xi_2 = \infty$ is eliminated; the functions (9.2) have for $\xi_2 \longrightarrow \infty$ for arbitrary A and B an oscillating behavior. One can see that immediately from the asymptotic series (5.12). The boundary condition at $\xi = \xi_1$ gives the ratio A:B. For $\xi_1 = 1$ this boundary condition in turn is eliminated and B becomes $B = 0$. A condition for the frequence does not exist; all wave coefficients are admissible, the spectrum is continuous and extends from $k = 0$ to $k = \infty$.

9.2 Lamé's Wave Functions of the Oblate Ellipsoid of Revolution

The solutions of the wave equation originating by separation of the wave equation in the coordinates of the oblate ellipsoid of revolution are obtained from equation (9.1), by replacing γ there by $\pm i\gamma$. Here also only the three-dimensional domains characterized in section 9.1 are dealt with and the eigenfunctions can, therefore, be written in the form

$$u_n^m(\xi,\eta,\varphi;k) = \left[A Z_n^{m(1)}(\zeta;i\gamma) + B Z_n^m(\zeta;i\gamma)\right] X_n^{m(1)}(\eta;i\gamma)e^{im\varphi}$$

$$(n = 0,\ 1,\ 2,\ \ldots;\qquad m = 0,\ \pm 1,\ \pm 2,\ \ldots,\ \pm n) \qquad (9.3)$$

The domain of variables in η and φ is the same as in the coordinates of the prolate ellipsoid of revolution. The domain of variables in ζ is again denoted by $\zeta_1 \leqq \zeta \leqq \zeta_2$. For the oblate ellipsoid of revolution there is $0 \leqq \zeta_1$. What was said in section 9.1 for $\xi_1 > 1$ is valid for inside and outside space problems with $\zeta_1 > 0$. However, whereas there $\xi_1 = 1$ was a singular point of the Z-function of the second kind, here $\zeta = 0$ is a regular point for all Z-functions. Thus, for determination of the eigenvalue problem for $\zeta_1 = 0$ in this case, also a boundary condition must be given. The area $\zeta_1 = 0$ is a circular disc. If such a circular disc actually exists as a physical object, for instance, a circular screen for problems of diffraction or a circular membrane, the boundary condition on the disc results from the physical problem taken as a basis. If, however, this circular disc has geometrical significance only as singular surface of the coordinate system taken as a basis, for instance, for the determination of the acoustic or electrical natural oscillations inside an oblate ellipsoid of revolution, the eigenfunction together with its derivative must be required to be continuous at this circular disc which leads to $B = 0$.

9.3 Normalization of Lamé's Wave Functions for Outside Space Problems

One limits oneself at first to Lamé's wave functions of the prolate ellipsoid of revolution. In the normalization of the eigenfunctions (9.2) one cannot normalize in each coordinate separately although the eigenfunctions contain only factors which every time are dependent on one coordinate only; in the element of volume

$$d\tau = c^3(\xi^2 - \eta^2)\, d\xi\, d\eta\, d\varphi \tag{9.4}$$

namely, the coordinates are not separated. The condition of normalization for outside space problems with continuous spectrum reads

$$\lim_{\xi \to \infty} \int_{k-\Delta k}^{k+\Delta k} d\bar{k} \int_{\xi_1}^{\xi} d\xi \int_{-1}^{1} d\eta \int_{0}^{2\pi} d\varphi\; c^3(\xi^2 - \eta^2) u_n^m(\xi,\eta,\varphi;\bar{k})^* u_n^m(\xi,\eta,\varphi;k) = \frac{1}{N_n^m(\gamma)^2} \tag{9.5}$$

The asterisk (*) signifies the formation of the conjugate-complex expression. $N_n^m(\gamma)$ is called the factor of normalization. If the indices n and m of the two eigenfunctions in the integrand (9.5) would not both agree, the integral would equal zero; the same would be valid if the interval of integration for $\bar{k}$ would not contain the point k. However, a delay by proof of these properties of orthogonality is unnecessary; that proof is elementary.

From the wave equation in elliptic coordinates one obtains the identity

$$(\bar{k}^2 - k^2)c^2\bar{u}^* u = \frac{\partial}{\partial\xi}\left[(\xi^2 - 1)\left(\bar{u}^* \frac{\partial u}{\partial\xi} - u\,\frac{\partial\bar{u}^*}{\partial\xi}\right)\right] + \frac{\partial}{\partial\eta}\left[(1 - \eta^2)\left(\bar{u}^* \frac{\partial u}{\partial\eta} - u\,\frac{\partial\bar{u}^*}{\partial\eta}\right)\right] \tag{9.6}$$

For abbreviation the wave function concerning the parameter values $\bar{k},n,m$ is designated by $\bar{u}$. The φ-integration of equation (9.6) gives 2π. The η-integration of the second term of the sum on the right side of equation (9.6) gives always zero since at the ends of the interval of integration $1 - \eta^2 = 0$. After carrying out the η-integration the ξ-integration of equation (9.6) results finally in

$$c^2(\bar{k}^2 - k^2)\int_{\xi_1}^{\xi} d\xi \int_{-1}^{1} (\xi^2 - \eta^2)\, d\eta \int_0^{2\pi} \bar{u}^* u\, d\varphi = 2\pi \int_{-1}^{1} d\eta \left[(\xi^2 - 1)\left(\bar{u}^* \frac{\partial u}{\partial \xi} - u\frac{\partial \bar{u}^*}{\partial \xi}\right)\right]_{\xi_1}^{\xi} \tag{9.7}$$

The boundary condition is assumed $u = \text{finite}$ for $\xi = 1$ in case $\xi_1 = 1$, or $\alpha u + \beta \frac{\partial u}{\partial n} = 0$ in case $\xi_1 > 1$, where α and β shall both be real and independent of k. Then the content of the brackets on the right side of equation (9.7) at the point ξ_1 vanishes; for large ξ the asymptotic series of the Z-functions may be substituted. Note the fact that, for real boundary conditions, A/B is always real. After division by $\bar{k}^2 - k^2$, performance of the $\bar{k}$-integration and transition to $\xi = \infty$ there results the integral in equation (9.5); thus one obtains as the factor of normalization

$$\frac{1}{N_n^m(\gamma)^2} = \frac{\pi^2}{k^2}(AA^* + BB^*)\left[\sum_{r=-\infty}^{\infty} i^r\, b_{n,r}^m(\gamma)\right]^2 \frac{(n+m)!}{(n-m)!}\sum_{r=-\infty}^{\infty}\frac{2}{2n+2r+1}\, a_{n,r}^m(\gamma)\, b_{n,r}^m(\gamma) \tag{9.8}$$

The normalized wave functions are, therefore, obtained by multiplying $u_n^m(\xi,\eta,\varphi;k)$ by $N_n^m(\gamma)$.

Several changes occur for the coordinates of the oblate ellipsoid of revolution. The eigenfunctions (9.3) have to be substituted in equation (9.5); instead of $\xi^2 - \eta^2$ in equations (9.4), (9.5), and (9.6) one has to write $\xi^2 + \eta^2$; $\xi^2 - 1$ in equations (9.6) and (9.7) is to be replaced by $\xi^2 + 1$. Finally one obtains as the factor of normalization $N_n^m(i\gamma)$ instead of $N_n^m(\gamma)$ in equation (9.8).

9.4 Development of Lamé's Wave Functions in Terms of Spherical and Cylindrical Functions

By the following consideration one obtains a remarkable development which includes a number of the developments of X- and Z-functions given so far as special cases. Any Lamé wave function can be developed in terms of such wave functions as originate by separation, for instance, in polar coordinates. The eigenfunction of the continuous spectrum, in particular, which results from equation (9.2) for $B = 0$ can be developed in terms of the eigenfunctions in polar coordinates $\psi_w(kr)\underline{P}_n^m\left(\frac{z}{r}\right)e^{im\varphi}$. One obtains the development coefficients by making, for instance, x and $y \to 0$, that is, $\frac{z}{r} \to 1$ and comparing the thus originating development with equation (5.1). Thus there results, if one expresses, moreover, x,y,z by ξ,η,φ and equates the coefficients with $e^{im\varphi}$,

$$Z_n^{m(1)}(\zeta;\gamma)X_n^{m(1)}(\eta;\gamma) = \sum_{r=-\infty}^{\infty} i^r b_{n,r}^m(\gamma) \sum_{r=-\infty}^{\infty} a_{n,r}^m(\gamma)\psi_{n+r}(kr)\underline{P}_{n+r}^m\left(\frac{z}{r}\right) \tag{9.9}$$

$$= \sum_{r=-\infty}^{\infty} i^r b_{n,r}^m(\gamma) \sum_{r=-\infty}^{\infty} a_{n,r}^m(\gamma)\psi_{n+r}\left(\gamma\sqrt{\xi^2 + \eta^2 - 1}\right)\underline{P}_{n+r}^m\left(\frac{\xi\eta}{\sqrt{\xi^2 + \eta^2 - 1}}\right) \tag{9.10}$$

where r in the argument is given by $r^2 = x^2 + y^2 + z^2$ and must not be confused with the index r which runs through all even numbers. The development (9.9) is given already by Morse (reference 6). One can interpret equation (9.10) as a development of the Z-functions which contains still an arbitrary parameter η. For $\eta \longrightarrow 1$ there originates, if one divides before by $(1 - \eta^2)^{m/2}$, the series (5.1); for $\eta \longrightarrow 0$ one obtains the series (5.42) for the W-functions of the first kind. If one differentiates equation (9.10) with respect to η and sets then $\eta = 0$, there results the series (5.41) of the V-functions of the first kind. For $\zeta \longrightarrow \infty$ there originates from equation (9.10) the series (4.1) of the X-functions of the first kind.

At this point one can recognize why the formulations (4.1), (4.2), (5.1), (5.2), (5.41), and (5.42), that is, the series developments considered by Niven (reference 1) all had to lead to the same development coefficients a_r.

Whereas equation (9.9) represents a development in terms of eigenfunctions in polar coordinates which have their origin at the point $x = y = z = 0$, Lamé's wave functions can be developed also in terms of eigenfunctions in polar coordinates with the origin $x = y = 0$, $z = c$. This development reads, as shown by a simple calculation,

$$Z_n^{m(1)}(\zeta;\gamma)X_n^{m(1)}(\eta;\gamma) = \sum_{t=-\infty}^{\infty} e_{n,t}^m(\gamma)\psi_{n+t}\left[\gamma(\xi - \eta)\right]P_{n+t}^m\left(\frac{\xi\eta - 1}{\xi - \eta}\right) \quad (9.11)$$

If one multiplies by $\left(1 - \eta^2\right)^{-m/2}$ and then sets $\eta = 1$, equation (9.11) is transformed into the development (5.46). For $\xi \longrightarrow \infty$ one obtains a development in terms of spherical functions multiplied by sin $(\gamma\eta)$ and cos $(\gamma\eta)$, respectively; the special case of this development $m = 0$ is already given by Hanson (reference 5).

If one finally develops Lamé's wave functions in terms of the eigenfunctions originating by separation of the wave equation in cylindric coordinates, there results, with the aid of equation (7.24),

$$Z_n^{m(1)}(\zeta;\gamma)X_n^{m(1)}(\eta;\gamma) = \frac{i^{m-n}}{2}\sum_{r=-\infty}^{\infty} i^r\, b_{n,r}^m(\gamma)\int_{-1}^{1} e^{i\alpha\zeta\eta}\, X_n^{m(1)}(\alpha;\gamma)F_m\left[\sqrt{(\zeta^2-\gamma^2)(1-\eta^2)(1-\alpha^2)}\right]d\alpha \quad (9.12)$$

This integral equation which was derived in another way also by Kotani (reference 8) contains equation (7.24) as special case for $\eta \longrightarrow 1$.

It is obvious that the developments (9.10), (9.11), and (9.12) are capable of generalization; one can consider complex ν,μ,γ and one can replace one or both functions of the first kind on the left and right side of equations (9.10), (9.11), or (9.12) by functions of the second kind; however, reproduction of the thus originating formulas and establishment of their domain of validity will be omitted.

10. THE METHOD OF GREEN'S FUNCTION FOR THE SOLUTION OF BOUNDARY VALUE PROBLEMS, PARTICULARLY OF RADIATION PROBLEMS

10.1 Green's Function of the Wave Equation in Radiation Problems

For development of the plane wave or more generally of the spherical wave in terms of Lamé's wave functions of the ellipsoid of revolution one uses the method of Green's function thought up by Sommerfeld (reference 15). According to this method the spherical wave is a solution of the inhomogeneous wave equation

$$\Delta_P u(P,Q) + k^2 u(P,Q) = \delta(P,Q) \quad (10.1)$$

the right side of which is a (recently so-called) Dirac's δ function. It was introduced first by Sommerfeld (reference 15) and designated by him as prong function. It has a singularity at the source point Q in such a manner that

$$\int_G \delta(P,Q)\, d\tau_P = 1 \tag{10.2}$$

for each domain G which contains the source point Q, whereas the integral has the value zero if the domain G does not contain the source point. One can interpret $\delta(P,Q)$ as limiting case of a function which has for points of influence P in the neighborhood of the source point Q a very steep prong whereas it decreases toward the outside very rapidly to zero.

The solution of equation (10.1) is for outside space problems uniquely determined only when besides the boundary conditions on the bounding areas which are at a finite distance an additional boundary condition at infinity is required, namely, the outgoing radiation condition (or else the incoming radiation condition) introduced by Sommerfeld (reference 15). According to this condition, $u(P,Q)$ for points P at very large distance from the source point Q should behave like an outgoing (or incoming) wave. One designates this solution because of its special properties as Green's function $G(P,Q;k)$ of the wave equation pertaining to the outgoing (or incoming) radiation condition. For physical reasons the case of the incoming radiation condition will not be considered below.

All developments of this section are performed for the coordinates of the prolate ellipsoid of revolution; one obtains the corresponding formulas for the coordinates of the oblate ellipsoid of revolution by replacing γ everywhere by $i\gamma$.

Green's function can be developed in the following way in terms of the eigenfunctions of the continuous spectrum

$$G(P,Q;k) = \int_0^\infty \frac{d\kappa}{k^2 - \kappa^2} \sum_{n,m} u_n^m\left(\xi_P,\eta_P,\varphi_P;\kappa\right) u_n^m\left(\xi_Q,\eta_Q,\varphi_Q;\kappa\right)^* N_n^m(\gamma)^2 \tag{10.3}$$

The integration over κ goes from 0 to ∞, the path of integration deviating at the point $\kappa = k$ in the case of the outgoing radiation condition into the negative-imaginary half plane. The integration over κ can be performed according to Sommerfeld and yields for $\zeta_P > \zeta_Q$

$$G(P,Q;k) = -\frac{\pi i}{2k}\sum_{n,m}\left(A_n^m - iB_n^m\right)Z_n^{m(3)}(\zeta_P;\gamma)\ e^{im(\varphi_P-\varphi_Q)}\ N_n^m(\gamma)^2\left[A_n^{m*}Z_n^{m(1)}(\zeta_Q;\gamma)\right.$$

$$\left.+ B_n^{m*}Z_n^{m(2)}(\zeta_Q;\gamma)\right]\left[X_n^{m(1)}(\eta_P;\gamma)X_n^{m(1)}(\eta_Q;\gamma)\right] \qquad (10.4)$$

In order to characterize the dependence of the constants A and B on n and m, the indices n and m were appended. For $\zeta_P < \zeta_Q$ the arguments ζ_P and ζ_Q in equation (10.4) are to be exchanged since Green's function is symmetric in P and Q. One can recognize that there actually result outgoing waves by substituting for the Z-functions of the third kind in equation (10.4) their asymptotic representation, since an outgoing wave is given by the behavior e^{ikr} for large r if the time dependency as is customary in wave physics is fixed by $e^{-i\omega t}$.

10.2 Development of the Spherical Wave and of the Plane Wave in Terms of Lamé's Wave Functions

If the space does not have any boundaries within a finite region, B = 0 in the eigenfunctions for the coordinates of the prolate as well as of the oblate ellipsoid of revolution. A may then be set equal to 1. On the other hand, it is known that Green's function for the entire space without boundaries within a finite region is given by the spherical wave

$$G(P,Q;k) = \frac{e^{ikr_{PQ}}}{-4\pi r_{PQ}} \tag{10.5}$$

Therefrom one obtains immediately the development of the spherical wave in terms of Lamé's wave functions for $\zeta_Q > \zeta_P$

$$\frac{e^{ikr_{PQ}}}{-4\pi r_{PQ}} = -\frac{\pi i}{2k} \sum_{n,m} Z_n^{m(3)}(\zeta_Q;\gamma) Z_n^{m(1)}(\zeta_P;\gamma) X_n^{m(1)}(\eta_Q;\gamma) X_n^{m(1)}(\eta_P;\gamma) e^{im(\varphi_Q - \varphi_P)} N_n^m(\gamma)^2 \tag{10.6}$$

and correspondingly for $\zeta_Q < \zeta_P$. Morse (reference 6) discovered this development in terms of another method. The development of the plane wave in terms of Lamé's wave functions originates from this if one moves the source point Q to infinity. For $\zeta_Q \gg \zeta_P$, $\zeta_Q \gg 1$

$$G(P,Q;k) \approx \frac{e^{ikr_Q}}{-4\pi r_Q} e^{ik(\underline{e}_Q,\underline{r}_P)} \text{*} \tag{10.7}$$

is valid where r_Q is the distance of the source point from the origin of the coordinates, $\underline{r}_P$ the radius vector of the influence point P and $\underline{e}_Q$ a unit vector in the direction of Q toward the origin of the coordinates, therefore in the direction of the direction of propagation of the plane wave. For equation (10.6) this limiting process is performed

*$\underline{r}$ is used in place of German script $\mathfrak{r}$
$\underline{e}$ is used in place of German script $\mathfrak{e}$

by introducing for the $\mathcal{Z}$-function of the third kind the asymptotic series (5.12) and setting according to equations (2.11g) and (2.11a), respectively, $\zeta_Q \approx kr_Q$. Then there results for the plane wave running in direction $\underline{e}_Q$

$$e^{ik(\underline{e}_Q,\underline{r}_P)} = \frac{2\pi^2}{k^2}\sum_{n,m} i^{-n}\, \mathcal{Z}_n^{m(1)}(\zeta_P;\gamma)\, X_n^{m(1)}(\eta_P;\gamma)\, X_n^{m(1)}(\eta_Q;\gamma)\, e^{im(\varphi_Q-\varphi_P)}\, N_n^m(\gamma)^2 \sum_{r=-\infty}^{\infty} i^r\, b_{n,r}^m(\gamma) \qquad (10.8)$$

The direction in which the source point is situated can be characterized instead of by the vector $-\underline{e}_Q$ by the coordinates η_Q and φ_Q. According to equations (2.1g) and (2.1a)

$$\begin{aligned} -(\underline{e}_Q,\underline{r}_P) &= x_P\sqrt{1-\eta_Q^2}\cos\varphi_Q + y_P\sqrt{1-\eta_Q^2}\sin\varphi_Q + z_P\eta_Q \\ &= c\sqrt{\xi_P^2 \mp 1}\,\sqrt{\left(1-\eta_P^2\right)\left(1-\eta_Q^2\right)}\cos\left(\varphi_Q-\varphi_P\right) + c\xi_P\eta_P\eta_Q \end{aligned} \qquad (10.9)$$

is valid.

10.3 Diffraction of a Scalar Spherical Wave or Plane Wave on the Ellipsoid of Revolution

The method of Green's functions can also be used for treatment of problems of diffraction. In the scope of a scalar diffraction theory a homogeneous boundary condition $\alpha u + \beta \frac{\partial u}{\partial n} = 0$ is prescribed on the diffracting body; at first, the problem of the determination of the eigenfunctions is solved. With these eigenfunctions, Green's functions (10.3) and (10.4), respectively, are formed and the contention made that it solves the problem of diffraction. Actually it represents a wave which comes as spherical wave from the source point Q, satisfies on the diffracting surface the given boundary condition (as does each single term of the sum), and which behaves at infinity like an outgoing spherical wave. If the source point Q, in particular, lies at infinity, Green's functions represent the superposition of a plane wave and of an outgoing spherical wave originating from the diffracting body with an amplitude which, in general, is dependent on direction. Treatment of the diffraction problem for a source point within a finite region is omitted. One starts immediately from equation (10.8) and contends that the solution of the diffraction problem of a plane wave at the ellipsoid of revolution is given by

$$\frac{2\pi^2}{k^2}\sum_{n,m}\left[\mathcal{Z}_n^{m(1)}(\zeta_P;\gamma) - \frac{\mathcal{Z}_n^{m(1)}(\zeta_1;\gamma)}{\mathcal{Z}_n^{m(3)}(\zeta_1;\gamma)}\mathcal{Z}_n^{m(3)}(\zeta_P;\gamma)\right]$$

$$\times X_n^{m(1)}(\eta_P;\gamma)X_n^{m(1)}(\eta_Q;\gamma)e^{im(\varphi_Q-\varphi_P)}\ N_n^m(\gamma)^2\ i^{-n}\sum_{r=-\infty}^{\infty} i^r\ b_{n,r}^m(\gamma) \quad (10.10)$$

in the case of the boundary condition $u = 0$ for $\zeta = \zeta_1$. Under $N_n^m(\gamma)$, one understands therein the factor of normalization (9.8) with $AA^* + BB^* = 1$. In the case of the boundary condition $\frac{\partial u}{\partial \zeta} = 0$ for $\zeta = \zeta_1$ one has to replace the two $\mathcal{Z}$-functions with the argument ζ_1 in equation (10.10) by their derivatives with respect to ζ_P at the point ζ_1. The first term of the sum in the brackets of equation (10.10) yields, when the sum over n,m is formed, exactly the plane wave (10.8); the second term of the sum gives outgoing spherical waves; furthermore, the wave equation and the boundary (surface) condition are satisfied by each separate term of the sum; the contention is therefore proved.

For the diffraction at the infinitely thin wire of finite length, one has to set $\zeta_1 = 1$. $\mathcal{Z}_n^{m(3)}(\zeta_1;\gamma)$ then becomes infinitely large and, in equation (10.10), there remains only the plane wave. Thus an infinitely thin wire does not present an obstacle for a plane wave.

For the diffraction at the infinitely thin circular disk, γ is to be replaced in the formulas by $i\gamma$ and ζ_1 is to be set equal to zero. $g_n^{m(3)}(0;i\gamma)$ has a finite value so that the outgoing spherical waves do not disappear; that is, even an infinitely thin disk represents an essential disturbance for a plane wave striking it.

11. TABLES

11.1 Comments to the Tables

The tables in section 11.2 contain power series developments to γ^{10}, inclusive, for the eigenvalue $\lambda_n^m(\gamma)$ according to equation (6.13) and to γ^6, inclusive, for the coefficients $a_{n,r}^m(\gamma)$ and $b_{n,r}^m(\gamma)$ according to equations (6.14) to (6.21) and equation (4.11). Furthermore, to γ^6, inclusive, the coefficients $\alpha_{n,r}^m/a_{n,o}^m(\gamma)$, according to equation (7.6), are given for all those cases where a_{-2}/a_0, a_{-4}/a_0, and a_{-6}/a_0 disappear. As far as the values of the coefficients a_r/a_o and b_r/b_o are not given in the tables, they disappear; then one must use for the X-functions of the second kind the series (7.7) and the table for the α_r/a_o.

The region of the n- and m-values in the tables extends from $m = 0, 1, 2, \ldots, 9$ and from $n = m, m + 1, \ldots, 9$. For negative m, which are integers, reference is made to the relation (4.12).

The last given digit is, in general, probably certain; only where the following digit after rounding up or off, respectively, is a 5, the last given digit would have to be changed in a few cases by unity. In the cases of the end digits ...5, ..50, .500, and so forth, it is mostly indicated by a line over or under, respectively, the last digit whether the respective decimal fraction had been originated by rounding up or off.

The given broken off series developments in terms of powers of γ^2 are the more useful, the smaller γ^2 and the larger n.

For $n = 0$ the series begin to be useless only at $\gamma^2 = 10$; for larger n they can be used up to far larger values of γ^2. Below, a few of the first eigenvalues for $\gamma^2 = 10$ are given as they follow from the exact numerical calculation and from the power series development to γ^{10}, inclusive.

n	0	2	4
$\lambda_n^o(\sqrt{10})$	2.305040	11.790395	25.251313
$\lambda_n^o(\sqrt{10})$ approximation	2.215	11.880	25.25147

Figure 1 gives a survey on the dependence of the lowest eigenvalues on γ.

The tables in section 11.3 are taken from the thesis of Bouwkamp (reference 10). They contain the eigenvalues $\lambda_n^o(\gamma)$ for a number of pairs of values n, γ^2, and the coefficients $a^m_{n,r}(\gamma)$ of the pertinent X-functions. These latter are fixed so that

$$\sum_{r=-\infty}^{\infty} \frac{2n + 1}{2n + 2r + 1}\left[a^o_{n,r}(\gamma)\right]^2 = 1 \qquad (11.1)$$

The integral of normalization then (compare equation (7.19)) has the value

$$\int_{-1}^{1}\left[X_n^{o(1)}(\xi;\gamma)\right]^2 d\xi = \frac{2}{2n + 1} \qquad (11.2)$$

These tables contain further the values $X_n^{(1)}(1;\gamma)$ and $X_n^{(1)}(0;\gamma)$ for even and $dX_n^{(1)}(0;\gamma)\Big/d\xi$ for odd n. The signs of the a_r are different from those of Bouwkamp since the present series (4.1) and (4.2) contain in the coefficients a factor i^r which is missing in reference 10 by Bouwkamp.

Since the γ^2 assume in these tables only negative values, these functions are appropriate for the treatment of problems concerning the oblate ellipsoid of revolution or for the investigation of the eigenvalues of the ion of the hydrogen molecule, whereas, the tables in section 11.2 where γ^2 can be positive as well as negative, may be used for problems of the oblate as well as of the prolate ellipsoid of revolution.

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11.2 – Eigenvalues $\lambda_n^m(\gamma)$ and Development Coefficients $a_{n,r}^m(\gamma)$, $b_{n,r}^m(\gamma)$;

Represented by Broken-Off Power Series in γ

TABLE 1.– $\left[\lambda_n^m(\gamma) - n(n+1)\right] \times 10^{10}$ AS POWER SERIES IN γ

m = 0, n = 0	$+3333333333\gamma^2$	$-148148148\gamma^4$	$+4703116\gamma^6$	$+135868\gamma^8$	$-24280.96\gamma^{10}$
n = 1	+6000000000	− 68571429	− 609524	+ 25896	+ 872.80
n = 2	+5238095238	+101500918	−4760812	−141089	+24412.17
n = 3	+5111111111	+ 32941763	+ 595989	− 26542	− 887.76
n = 4	+5064935065	+ 17750507	+ 53147	+ 5040	− 132.19
n = 5	+5042735043	+ 11298966	+ 11653	+ 577	+ 14.80
n = 6	+5030303030	+ 7874434	+ 3655	+ 150	+ 0.94
n = 7	+5022624434	+ 5819124	+ 1413	+ 53	+ 0.15
n = 8	+5017543860	+ 4482651	+ 628	+ 22	+ 0.03
n = 9	+5014005602	+ 3562440	+ 310	+ 11	+ 0.01
m = 1, n = 1	$+2000000000\gamma^2$	$-45714286\gamma^4$	$+1219048\gamma^6$	$-21034\gamma^8$	$-205.71\gamma^{10}$
n = 2	+4285714286	− 38872692	+ 144240	+ 5682	− 76.28
n = 3	+4666666667	+ 13647587	−1182504	+ 21357	+ 229.72
n = 4	+4805194805	+ 11902417	− 131503	− 5669	+ 78.30
n = 5	+4871794872	+ 8894604	− 31166	− 337	− 23.65
n = 6	+4909090909	+ 6698782	− 10150	− 26	− 1.93
n = 7	+4932126697	+ 5174730	− 4008	+ 6	− 0.34
n = 8	+4947368421	+ 4099403	− 1807	+ 7	− 0.08
n = 9	+4957983193	+ 3320032	− 899	+ 5	− 0.02
m = 2, n = 2	$+1428571429\gamma^2$	$-19436346\gamma^4$	$+360600\gamma^6$	$-5822\gamma^8$	$+57.71\gamma^{10}$
n = 3	+3333333333	− 22446689	+ 127901	+ 607	− 22.60
n = 4	+4025974026	− 2139874	− 309653	+ 6053	− 56.26
n = 5	+4358974359	+ 2584894	− 104857	− 532	+ 23.27
n = 6	+4545454545	+ 3476672	− 39401	− 205	− 1.22
n = 7	+4660633484	+ 3364211	− 16771	− 67	− 0.58
n = 8	+4736842105	+ 3005596	− 7911	− 22	− 0.30
n = 9	+4789915966	+ 2620834	− 4052	− 8	− 0.07
m = 3, n = 3	$+1111111111\gamma^2$	$-9976306\gamma^4$	$+132638\gamma^6$	$-1698\gamma^8$	$+17.80\gamma^{10}$
n = 4	+2727272727	− 13870427	+ 76421	− 83	− 4.51
n = 5	+3504273504	− 4920040	− 92311	+ 1750	− 18.27
n = 6	+3939393939	− 877352	− 54468	+ 119	+ 4.55
n = 7	+4208144796	+ 755560	− 27781	− 32	+ 0.53
n = 8	+4385964912	+ 1369050	− 14434	− 27	− 0.01
n = 9	+4509803922	+ 1548920	− 7843	− 14	− 0.04
m = 4, n = 4	$+909090909\gamma^2$	$-5779345\gamma^4$	$+57316\gamma^6$	$-578\gamma^8$	$+5.24\gamma^{10}$
n = 5	+2307692308	− 9103323	+ 44360	− 118	− 0.79
n = 6	+3090909091	− 4839057	− 28865	+ 570	− 5.53
n = 7	+3574660633	− 2037903	− 26567	+ 129	+ 0.73
n = 8	+3894736842	− 530540	− 17172	+ 17	+ 0.29
n = 9	+4117647059	+ 244418	− 10467	− 4	+ 0.06
m = 5, n = 5	$+769230769\gamma^2$	$-3641329\gamma^4$	$+27883\gamma^6$	$-244\gamma^8$	$+1.70\gamma^{10}$
n = 6	+2000000000	− 6274510	+ 26419	− 80	− 0.07
n = 7	+2760180995	− 4157529	− 8311	+ 205	− 1.82
n = 8	+3263157895	− 2301599	− 12785	+ 79	+ 0.03
n = 9	+3613445378	− 1096495	− 10175	+ 22	+ 0.10
m = 6, n = 6	$+666666667\gamma^2$	$-2440087\gamma^4$	$+14840\gamma^6$	$-97\gamma^8$	$+0.62\gamma^{10}$
n = 7	+1764705882	− 4499341	+ 16310	− 49	+ 0.04
n = 8	+2491228070	− 3440673	− 1336	+ 80	− 0.65
n = 9	+2997198880	− 2221592	− 6054	+ 44	− 0.07
m = 7, n = 7	$+588235296\gamma^2$	$-1714035\gamma^4$	$+8473\gamma^6$	$-46\gamma^8$	$+0.25\gamma^{10}$
n = 8	+1578947368	− 3332431	+ 10435	− 29	+ 0.05
n = 9	+2268907563	− 2822595	+ 967	+ 32	− 0.25
m = 8, n = 8	$+526315789\gamma^2$	$-1249662\gamma^4$	$+5117\gamma^6$	$-23\gamma^8$	$+0.11\gamma^{10}$
n = 9	+1428571429	− 2535176	+ 6898	− 18	+ 0.03
m = 9, n = 9	$+476190476\gamma^2$	$-938954\gamma^4$	$+3236\gamma^6$	$-13\gamma^8$	$+0.02\gamma^{10}$

TABLE 2.– $\frac{a^m_{n,2}(\gamma)}{a^m_{n,o}(\gamma)} \times 10^{10}$ AS POWER SERIES IN γ

m = 0, n = 0	+1111111111γ^2	−35273369γ^4	−1019008γ^6
n = 1	+ 400000000	+ 3555556	− 151062
n = 2	+ 244897959	+ 302904	+ 208818
n = 3	+ 176366843	+ 66996	+ 46389
n = 4	+ 137741047	+ 21683	+ 18710
n = 5	+ 112963959	+ 8738	+ 9467
n = 6	+ 95726496	+ 4071	+ 5462
n = 7	+ 83044983	+ 2105	+ 3439
n = 8	+ 73325729	+ 1177	+ 2306
n = 9	+ 65640291	+ 700	+ 1622
m = 1, n = 1	+ 133333333γ^2	− 3555556γ^4	+ 61348γ^6
n = 2	+ 122448980	− 454356	− 17898
n = 3	+ 105820106	− 120593	+ 16210
n = 4	+ 91827365	− 43366	+ 9870
n = 5	+ 80688542	− 18723	+ 5930
n = 6	+ 71794872	− 9160	+ 3767
n = 7	+ 64590542	− 4912	+ 2524
n = 8	+ 58660583	− 2826	+ 1768
n = 9	+ 53705693	− 1719	+ 1285
m = 2, n = 2	+ 40816327γ^2	− 757260γ^4	+ 12227γ^6
n = 3	+ 52910053	− 301482	− 1430
n = 4	+ 55096419	− 130098	+ 2386
n = 5	+ 53792361	− 62411	+ 2541
n = 6	+ 51282051	− 32716	+ 2057
n = 7	+ 48442907	− 18420	+ 1579
n = 8	+ 45624898	− 10990	+ 1207
n = 9	+ 42964554	− 6877	+ 931
m = 3, n = 3	+ 17636684γ^2	− 234486γ^4	+ 3003γ^6
n = 4	+ 27548209	− 151781	+ 165
n = 5	+ 32275417	− 87376	+ 621
n = 6	+ 34188034	− 50891	+ 828
n = 7	+ 34602076	− 30700	+ 812
n = 8	+ 34218674	− 19232	+ 718
n = 9	+ 33416876	− 12481	+ 609
m = 4, n = 4	+ 9182736γ^2	− 91068γ^4	+ 918γ^6
n = 5	+ 16137708	− 78638	+ 210
n = 6	+ 20512821	− 54963	+ 242
n = 7	+ 23068051	− 36840	+ 329
n = 8	+ 24441910	− 24727	+ 361
n = 9	+ 25062657	− 16849	+ 353
m = 5, n = 5	+ 5379236γ^2	− 41192γ^4	+ 332γ^6
n = 6	+ 10256410	− 43185	+ 131
n = 7	+ 13840830	− 34735	+ 118
n = 8	+ 16294606	− 25905	+ 152
n = 9	+ 17901898	− 18912	+ 178
m = 6, n = 6	+ 3418803γ^2	− 20793γ^4	+ 136γ^6
n = 7	+ 6920415	− 25086	+ 72
n = 8	+ 9776764	− 22451	+ 64
n = 9	+ 11934598	− 18212	+ 79
m = 7, n = 7	+ 2306805γ^2	− 11403γ^4	+ 62γ^6
n = 8	+ 4888382	− 15307	+ 43
n = 9	+ 7160759	− 14900	+ 37
m = 8, n = 8	+ 1629461γ^2	− 6672γ^4	+ 30γ^6
n = 9	+ 3580380	− 9743	+ 25
m = 9, n = 9	+ 1193460γ^2	− 4114γ^4	+ 16γ^6

TABLE 3.— $\frac{a^m_{n,4}(\gamma)}{a^m_{n,o}(\gamma)} \times 10^{10}$ AND $\frac{a^m_{n,6}(\gamma)}{a^m_{n,o}(\gamma)} \times 10^{10}$ AS POWER SERIES

	$a_4/a_o \times 10^{10}$		$a_6/a_o \times 10^{10}$
m = 0, n = 0	$+19047619\gamma^4$	$-7.696017\gamma^6$	$+137429\gamma^6$
n = 1	+ 4535147	+ 55817	+ 24667
n = 2	+ 2061431	+ 3740	+ 8970
n = 3	+ 1177274	+ 684	+ 4262
n = 4	+ 760709	+ 189	+ 2355
n = 5	+ 531595	+ 67	+ 1437
n = 6	+ 392250	+ 28	+ 940
n = 7	+ 301245	+ 13	+ 649
n = 8	+ 238563	+ 7	+ 466
n = 9	+ 193571	+ 4	+ 346
m = 1, n = 1	$+ 907029\gamma^4$	$- 33490\gamma^6$	$+ 3524\gamma^6$
n = 2	+ 687144	− 3740	+ 2242
n = 3	+ 504546	− 879	+ 1421
n = 4	+ 380350	− 284	+ 942
n = 5	+ 295331	− 111	+ 653
n = 6	+ 235350	− 50	+ 470
n = 7	+ 191701	− 24	+ 349
n = 8	+ 159042	− 13	+ 266
n = 9	+ 134010	− 7	+ 208
m = 2, n = 2	$+ 137429\gamma^4$	$- 3740\gamma^6$	$+ 320\gamma^6$
n = 3	+ 168182	− 1466	+ 355
n = 4	+ 163007	− 608	+ 314
n = 5	+ 147665	− 277	+ 261
n = 6	+ 130750	− 138	+ 214
n = 7	+ 115021	− 73	+ 172
n = 8	+ 101209	− 42	+ 143
n = 9	+ 89340	− 25	+ 119
m = 3, n = 3	$+ 33636\gamma^4$	$- 684\gamma^6$	$+ 51\gamma^6$
n = 4	+ 54336	− 473	+ 79
n = 5	+ 63285	− 277	+ 87
n = 6	+ 65375	− 161	+ 85
n = 7	+ 63900	− 95	+ 79
n = 8	+ 60725	− 58	+ 72
n = 9	+ 56853	− 37	+ 64
m = 4, n = 4	$+ 10867\gamma^4$	$- 170\gamma^6$	$+ 11\gamma^6$
n = 5	+ 21095	− 166	+ 22
n = 6	+ 28018	− 124	+ 28
n = 7	+ 31950	− 86	+ 32
n = 8	+ 33736	− 58	+ 33
n = 9	+ 34112	− 40	+ 32
m = 5, n = 5	$+ 4219\gamma^4$	$- 52\gamma^6$	$+ 3\gamma^6$
n = 6	+ 9339	− 65	+ 7
n = 7	+ 13693	− 58	+ 11
n = 8	+ 16868	− 46	+ 13
n = 9	+ 18951	− 35	+ 15
m = 6, n = 6	$+ 1868\gamma^4$	$- 19\gamma^6$	$+ 1\gamma^6$
n = 7	+ 4564	− 28	+ 3
n = 8	+ 7229	− 28	+ 4
n = 9	+ 9475	− 25	+ 6
m = 7, n = 7	$+ 913\gamma^4$	$- 8\gamma^6$	$+ 0\gamma^6$
n = 8	+ 2410	− 13	+ 1
n = 9	+ 4061	− 15	+ 2
m = 8, n = 8	$+ 482\gamma^4$	$- 3\gamma^6$	$+ 0\gamma^6$
n = 9	+ 1354	− 6	+ 0
m = 9, n = 9	$+ 271\gamma^4$	$- 2\gamma^6$	$+ 0\gamma^6$

TABLE 4.— $\frac{a^m_{n,-2}(\gamma)}{a^m_{n,o}(\gamma)} \times 10^{10}$ AS POWER SERIES IN γ

m = 0, n = 2	$-2222222222\gamma^2$	$+\ 7054674\gamma^4$	$+\ 151971\gamma^6$
n = 3	− 171428571	− 1523810	+ 42927
n = 4	− 136054422	− 168280	− 26150
n = 5	− 112233446	− 42634	− 8241
n = 6	− 95359186	− 15011	− 4452
n = 7	− 82840237	− 6408	− 2790
n = 8	− 73202614	− 3113	− 1889
n = 9	− 65561828	− 1662	− 1346
m = 1, n = 1	$-3333333333\gamma^2$	$-1333333333\gamma^4$	$-540952381\gamma^6$
n = 2	− 666666667	− 63492063	− 6478782
n = 3	− 342857143	+ 9142857	− 197018
n = 4	− 226757370	+ 841400	+ 16156
n = 5	− 168350168	+ 191852	− 7170
n = 6	− 133502861	+ 63047	− 5138
n = 7	− 110453649	+ 25630	− 3381
n = 8	− 94117647	+ 12009	− 2295
n = 9	− 81952286	+ 6232	− 1621
m = 2, n = 2	$-1333333333\gamma^2$	$-\ 634920635\gamma^4$	$-\ 80552890\gamma^6$
n = 3	− 571428571	+ 76190476	+ 8760622
n = 4	− 340136054	+ 6310502	− 122277
n = 5	− 235690236	+ 1342964	− 4268
n = 6	− 178003814	+ 420316	− 4295
n = 7	− 142011834	+ 164766	− 3369
n = 8	− 117647059	+ 75054	− 2449
n = 9	− 100163905	+ 38087	− 1778
m = 3, n = 3	$-\ 857142857\gamma^2$	$+\ 266666667\gamma^4$	$+\ 883800095\gamma^6$
n = 4	− 476190476	+ 20614306	− 2451288
n = 5	− 314253648	+ 4178111	− 64139
n = 6	− 228862047	+ 1260948	− 7860
n = 7	− 177514793	+ 480568	− 3572
n = 8	− 143790850	+ 214043	− 2475
n = 9	− 120196685	+ 106642	− 1829
m = 4, n = 4	$-\ 634920635\gamma^2$	$+\ 494743355\gamma^4$	$-\ 129516487\gamma^6$
n = 5	− 404040404	+ 9669343	− 380104
n = 6	− 286077559	+ 2837133	− 34429
n = 7	− 216962525	+ 1057249	− 6853
n = 8	− 172549020	+ 462332	− 2950
n = 9	− 142050628	+ 226858	− 1908
m = 5, n = 5	$-\ 505050505\gamma^2$	$+\ 189933527\gamma^4$	$-\ 13658317\gamma^6$
n = 6	− 349650350	+ 5449096	− 123724
n = 7	− 260355030	+ 1993669	− 19430
n = 8	− 203921569	+ 858617	− 5180
n = 9	− 165725733	+ 415905	− 2347
m = 6, n = 6	$-\ 419580420\gamma^2$	$+\ 94451007\gamma^4$	$-\ 3470117\gamma^6$
n = 7	− 307692308	+ 3403334	− 52456
n = 8	− 237908497	+ 1446929	− 11529
n = 9	− 191222000	+ 693176	− 3762
m = 7, n = 7	$-\ 358974359\gamma^2$	$+\ 54143957\gamma^4$	$-\ 1238787\gamma^6$
n = 8	− 274509804	+ 2276636	− 25809
n = 9	− 218539428	+ 1080274	− 7154
m = 8, n = 8	$-\ 313725490\gamma^2$	$+\ 34024467\gamma^4$	$-\ 537207\gamma^6$
n = 9	− 247678019	+ 1601021	− 14025
m = 9, n = 9	$-\ 278637771\gamma^2$	$+\ 22814567\gamma^4$	$-\ 265037\gamma^6$

TABLE 5.– $\frac{a^m_{n,-4}(\gamma)}{a^m_{n,o}(\gamma)} \times 10^{10}$ AND $\frac{a^m_{n,-6}(\gamma)}{a^m_{n,o}(\gamma)} \times 10^{10}$ AS POWER SERIES IN γ

	$a_{-4}/a_o \times 10^{10}$		$a_{-6}/a_o \times 10^{10}$
m = 0, n = 4	+ 907029γ^4	− 6731γ^6	
n = 5	+ 687144	+ 2610	
n = 6	+ 504546	+ 371	− 1602γ^6
n = 7	+ 380350	+ 106	− 1207
n = 8	+ 295331	+ 39	− 852
n = 9	+ 235350	+ 17	− 610
m = 1, n = 3	+ 19047619γ^4	+ 338624γ^6	
n = 4	+ 4535147	+ 100968	
n = 5	+ 2061431	− 23492	− 45810γ^6
n = 6	+ 1177274	− 2595	− 11212
n = 7	+ 760700	− 635	− 4830
n = 8	+ 531595	− 213	− 2557
n = 9	+ 392250	− 86	− 1524
m = 2, n = 2	−66666667γ^4	−1269841277γ^6	
n = 3	+ 95238095	+ 8465608	
n = 4	+ 13605442	+ 1514520	+ 1511716γ^6
n = 5	+ 4810005	− 274074	− 320667
n = 6	+ 2354548	− 25945	− 44849
n = 7	+ 1369260	− 5719	− 14490
n = 8	+ 885992	− 1776	− 6392
n = 9	+ 616393	− 677	− 3353
m = 3, n = 3	+285714286γ^4	+ 59259259γ^6	−31746032γ^6
n = 4	+ 31746032	+ 8245723	+10582011
n = 5	+ 9620010	− 1279014	− 1234668
n = 6	+ 4238186	− 108971	− 134546
n = 7	+ 2282100	− 22241	− 36224
n = 8	+ 1392273	− 6514	− 14063
n = 9	+ 924590	− 2370	− 6705
m = 4, n = 4	+ 63492063γ^4	+ 29684601γ^6	+42328042γ^6
n = 5	+ 17316017	− 4144004	− 3848004
n = 6	+ 7063643	− 326912	− 336364
n = 7	+ 3586157	− 62911	− 79692
n = 8	+ 2088409	− 17587	− 28127
n = 9	+ 1335519	− 6162	− 12452
m = 5, n = 5	+ 28860029γ^4	− 10853344γ^6	− 9620010γ^6
n = 6	+ 11100011	− 807274	− 740001
n = 7	+ 5379236	− 148289	− 159385
n = 8	+ 3016591	− 39919	− 52235
n = 9	+ 1869726	− 13555	− 21792
m = 6, n = 6	+ 16650017γ^4	− 1749093γ^6	− 1480001γ^6
n = 7	+ 7770008	− 309394	− 296000
n = 8	+ 4223228	− 80725	− 91412
n = 9	+ 2549627	− 26700	− 36319
m = 7, n = 7	+ 10878011γ^4	− 590661γ^6	− 518001γ^6
n = 8	+ 5758947	− 150108	− 152353
n = 9	+ 3399502	− 48546	− 58111
m = 8, n = 8	+ 7710238γ^4	− 262805γ^6	− 244769γ^6
n = 9	+ 4445503	− 83016	− 89808
m = 9, n = 9	+ 5715647γ^4	− 135197γ^6	− 134712γ^6

TABLE 6.— NUMERICAL VALUES OF THE COEFFICIENTS $\alpha_{n,r}^{m} / a_{n,o}^{m}$

APPEARING IN THE SERIES DEVELOPMENTS (7.7)

$\alpha_{-2}/a_o \times 10^{10}$

m = 0, n = 0	$+5000000000\gamma^2$	$-666666667\gamma^4$	$+66137566\gamma^6$
n = 1	−1666666667	+222222222	−16825397

$\alpha_{-4}/a_o \times 10^{10}$

m = 0, n = 0	$+166666667\gamma^4$	$-24691358\gamma^6$
n = 1	−277777778	+31746032
n = 2	+ 55555556	− 705467
n = 3	+ 4761905	− 28219
m = 1, n = 1	$+277777778\gamma^4$	$+95238095\gamma^6$
n = 2	−166666667	− 6349206

$\alpha_{-6}/a_o \times 10^{10}$

m = 0, n = 0	$+1763668\gamma^6$
n = 1	−5291005
n = 2	+3703704
n = 3	− 529101
n = 4	− 50391
n = 5	− 7635
m = 1, n = 1	$+2659574\gamma^6$
n = 2	−3703704
n = 3	+1058201
n = 4	+ 251953
m = 2, n = 2	$+7407407\gamma^6$
n = 3	−5291005

TABLE 7.- $\dfrac{b_{n,2}^{m}(\gamma)}{b_{n,o}^{m}(\gamma)} \times 10^{10}$ AS POWER SERIES IN γ

[For $m = 0$, there is $b_r = a_r$; compare therefore table 2]

m = 1, n = 1	$+800000000\gamma^2$	$-21333333\gamma^4$	$+368089\gamma^6$
n = 2	+408163265	− 1514520	− 59660
n = 3	+264550265	− 301482	+ 40526
n = 4	+192837466	− 91068	+ 20727
n = 5	+150618612	− 34950	+ 11069
n = 6	+123076923	− 15704	+ 6458
n = 7	+103806228	− 7894	+ 4057
n = 8	+ 89620336	− 4357	+ 2702
n = 9	+ 78768349	− 2522	+ 1885
m = 2, n = 2	$+612244898\gamma^2$	$-11358903\gamma^4$	$+183404\gamma^6$
n = 3	+370370370	− 2110372	− 10012
n = 4	+257116621	− 607123	+ 11136
n = 5	+193652501	− 224681	+ 9149
n = 6	+153846154	− 98147	+ 6171
n = 7	+126874279	− 48243	+ 4135
n = 8	+107544403	− 25905	+ 2844
n = 9	+ 93089868	− 14900	+ 2017
m = 3, n = 3	$+493827160\gamma^2$	$-6565603\gamma^4$	$+84071\gamma^6$
n = 4	+330578512	− 1821369	+ 1978
n = 5	+242065627	− 655319	+ 4654
n = 6	+188034188	− 279902	+ 4553
n = 7	+152249135	− 135080	+ 3574
n = 8	+127097931	− 71435	+ 2666
n = 9	+108604845	− 40562	+ 1980
m = 4, n = 4	$+413223140\gamma^2$	$-4098081\gamma^4$	$+41293\gamma^6$
n = 5	+295857988	− 1441703	+ 3851
n = 6	+225641026	− 604588	+ 2666
n = 7	+179930796	− 287353	+ 2567
n = 8	+148280919	− 150013	+ 2193
n = 9	+125313283	− 84244	+ 1766
m = 5, n = 5	$+355029586\gamma^2$	$-2718639\gamma^4$	$+21880\gamma^6$
n = 6	+266666667	− 1122807	+ 3410
n = 7	+209919262	− 526814	+ 1790
n = 8	+171093368	− 272002	+ 1601
n = 9	+143215181	− 151296	+ 1422
m = 6, n = 6	$+311111111\gamma^2$	$-1892138\gamma^4$	$+12385\gamma^6$
n = 7	+242214533	− 878023	+ 2623
n = 8	+195535278	− 449019	+ 1285
n = 9	+162310538	− 247677	+ 1072
m = 7, n = 7	$+278616609\gamma^2$	$-1368347\gamma^4$	$+7410\gamma^6$
n = 8	+221606648	− 693939	+ 1939
n = 9	+182599356	− 379959	+ 952
m = 8, n = 8	$+249307479\gamma^2$	$-1020891\gamma^4$	$+4643\gamma^6$
n = 9	+204081633	− 555324	+ 1422
m = 9, n = 9	$+226757370\gamma^2$	$-781567\gamma^4$	$+3025\gamma^6$

TABLE 8.– $\frac{b^m_{n,4}(\gamma)}{b^m_{n,o}(\gamma)} \times 10^{10}$ AND $\frac{b^m_{n,6}(\gamma)}{b^m_{n,o}(\gamma)} \times 10^{10}$ AS POWER SERIES IN γ

[For m = 0, there is $b_r = a_r$; compare therefore table 3]

	$b_4/b_o \times 10^{10}$		$b_6/b_o \times 10^{10}$
m = 1, n = 1	$+13605442\gamma^4$	$-502355\gamma^6$	$+98667\gamma^6$
n = 2	+ 4810005	– 26177	+26909
n = 3	+ 2354548	– 4104	+10654
n = 4	+ 1369260	– 1021	+ 5181
n = 5	+ 885992	– 333	+ 2874
n = 6	+ 616393	– 130	+ 1746
n = 7	+ 451867	– 58	+ 1135
n = 8	+ 344592	– 28	+ 777
n = 9	+ 270999	– 15	+ 554
m = 2, n = 2	$+9620010\gamma^4$	$-261769\gamma^6$	$+67273\gamma^6$
n = 3	+ 4238186	– 36934	+23439
n = 4	+ 2282100	– 8508	+10362
n = 5	+ 1392273	– 2615	+ 5337
n = 6	+ 924590	– 972	+ 3056
n = 7	+ 652697	– 417	+ 1892
n = 8	+ 482428	– 198	+ 1243
n = 9	+ 369544	– 102	+ 856
m = 3, n = 3	$+7063643\gamma^4$	$-143633\gamma^6$	$+46878\gamma^6$
n = 4	+ 3586157	– 31198	+19245
n = 5	+ 2088409	– 9154	+ 9339
n = 6	+ 1335519	– 3285	+ 5094
n = 7	+ 913776	– 1362	+ 3027
n = 8	+ 657857	– 630	+ 1921
n = 9	+ 492725	– 317	+ 1284
m = 4, n = 4	$+5379236\gamma^4$	$-84233\gamma^6$	$+33678\gamma^6$
n = 5	+ 3016591	– 23800	+15565
n = 6	+ 1869726	– 8277	+ 8150
n = 7	+ 1246058	– 3343	+ 4678
n = 8	+ 877142	– 1512	+ 2881
n = 9	+ 644333	– 747	+ 1876
m = 5, n = 5	$+4223228\gamma^4$	$-52359\gamma^6$	$+24905\gamma^6$
n = 6	+ 2549627	– 17737	+12595
n = 7	+ 1661411	– 7005	+ 7017
n = 8	+ 1147032	– 3107	+ 4211
n = 9	+ 828428	– 1509	+ 2680
m = 6, n = 6	$+3399502\gamma^4$	$-34159\gamma^6$	$+18892\gamma^6$
n = 7	+ 2172614	– 13231	+10256
n = 8	+ 1474756	– 5770	+ 6016
n = 9	+ 1049342	– 2761	+ 3752
m = 7, n = 7	$+2793361\gamma^4$	$-23197\gamma^6$	$+14651\gamma^6$
n = 8	+ 1868024	– 9966	+ 8423
n = 9	+ 1311678	– 4706	+ 5160
m = 8, n = 8	$+2335030\gamma^4$	$-16290\gamma^6$	$+11581\gamma^6$
n = 9	+ 1620308	– 7602	+ 6981
m = 9, n = 9	$+1980376\gamma^4$	$-11769\gamma^6$	$+9308\gamma^6$

TABLE 9.– $\frac{b^m_{n,-2}(\gamma)}{b^m_{n,o}(\gamma)} \times 10^{10}$ AS POWER SERIES IN γ

[For m = 0, there is $b_r = a_r$; compare therefore table 4]

m = 1, n = 3	$-57142857\gamma^2$	$+1523810\gamma^4$	$-32836\gamma^6$
n = 4	−68027211	+ 252420	+ 4847
n = 5	−67340067	+ 76741	− 2868
n = 6	−63572791	+ 30022	− 2447
n = 7	−59171598	+ 13730	− 1811
n = 8	−54901961	+ 7005	− 1339
n = 9	−50992534	+ 3878	− 1009
m = 2, n = 4	$-22675737\gamma^2$	$+ 420700\gamma^4$	$- 8152\gamma^6$
n = 5	−33670034	+ 191852	− 610
n = 6	−38143674	+ 90068	− 920
n = 7	−39447732	+ 45768	− 936
n = 8	−39215686	+ 25018	− 816
n = 9	−38244400	+ 14542	− 679
m = 3, n = 5	$-11223345\gamma^2$	$+ 149218\gamma^4$	$- 2291\gamma^6$
n = 6	−19071837	+ 105079	− 655
n = 7	−23668639	+ 64076	− 476
n = 8	−26143791	+ 38917	− 450
n = 9	−27317428	+ 24237	− 416
m = 4, n = 6	$- 6357279\gamma^2$	$+ 63047\gamma^4$	$- 765\gamma^6$
n = 7	−11834320	+ 57668	− 374
n = 8	−15686275	+ 42030	− 268
n = 9	−18211619	+ 29084	− 245
m = 5, n = 7	$- 3944773\gamma^2$	$+ 30207\gamma^4$	$- 294\gamma^6$
n = 8	− 7843137	+ 33024	− 199
n = 9	−10926971	+ 27422	− 155
m = 6, n = 8	$- 2614379\gamma^2$	$+ 15900\gamma^4$	$- 127\gamma^6$
n = 9	− 5463486	+ 19805	− 107
m = 7, n = 9	$- 1821162\gamma^2$	$+ 9002\gamma^4$	$- 60\gamma^6$

For n = m and n = m + 1, b_{-2}/b_o disappears.

TABLE 10.– $\frac{b^m_{n,-4}(\gamma)}{b^m_{n,o}(\gamma)} \times 10^{10}$ AND $\frac{b^m_{n,-6}(\gamma)}{b^m_{n,o}(\gamma)} \times 10^{10}$ AS

POWER SERIES IN γ

[For m = 0, there is $b_r = a_r$; compare therefore table 5]

	$b_{-4}/b_o \times 10^{10}$		$b_{-6}/b_o \times 10^{10}$
m = 1, n = 5	$+137429\gamma^4$	$-1566\gamma^6$	
n = 6	+168182	− 371	
n = 7	+163007	− 136	$-172\gamma^6$
n = 8	$+14766\bar{5}$	− 59	−213
n = 9	$+13075\bar{0}$	− 29	−203
m = 2, n = 6	$+ 33636\gamma^4$	$-371\gamma^6$	
n = 7	+ 54336	−227	
n = 8	$+ 6328\bar{5}$	−127	$- 30\gamma^6$
n = 9	$+ 6537\bar{5}$	− 72	− 51
m = 3, n = 7	$+ 10867\gamma^4$	$-106\gamma^6$	
n = 8	$+ 2109\bar{5}$	− 99	
n = 9	+ 28018	− 72	$- 7\gamma^6$
m = 4, n = 8	$+ 4219\gamma^4$	$- 36\gamma^6$	
n = 9	+ 9339	− 43	
m = 5, n = 9	$+ 1868\gamma^4$	$- 14\gamma^6$	

For n = m, n = m + 1, n = m + 2, n = m + 3, b_{-4}/b_o disappears.

For n = m, n = m + 1, n = m + 2, n = m + 3, n = m + 4, n = m + 5, b_{-6}/b_o disappears.

11.3 – Numerical Magnitude of the Eigenvalues and the Development Coefficients for Different n, γ and $m = 0$

TABLE 11.– EIGENVALUES $\lambda_n^o(\gamma)$

$-\gamma^2$	λ_0	λ_1	λ_2	λ_3
3	– 1.144334	+ 0.140119	+ 4.530790	+10.494513
4	– 1.594507	– .505244	+ 4.091201	+10.003864
5	– 2.079939	– 1.162422	+ 3.677958	+ 9.517981
6	– 2.599717	– 1.831051	+ 3.288927	+ 9.036338
7	– 3.151917	– 2.510421	+ 2.923314	+ 8.558395
8	– 3.734090	– 3.200050	+ 2.578205	+ 8.083615
9	– 4.343439	– 3.899400	+ 2.250704	+ 7.611465
10	– 4.976896	– 4.607952	+ 1.938379	+ 7.141428
15	– 8.42084	– 8.27180	+ .49949	+ 4.80616
20	–12.16294	–12.09943	– .91127	+ 2.45867
25	–16.07904	–16.05041	– 2.44860	+ .06093
50	–36.90015	–36.89912	–13.56548	–13.21675
100	–81.02794	–81.02794	–45.48967	–45.48391

$-\gamma^2$	λ_4	λ_5	λ_6	λ_7
4		+28.00092		
5	+17.511596		+39.504499	
10	+15.11342	+25.06949	+37.04822	
15	+12.81726	+22.68771	+34.63123	
20	+10.64634	+20.36028	+32.25386	
25	+ 8.63040	+18.08457	+29.91689	
50	+ .94568	+ 7.25075	+18.92267	
100	–16.06556	–15.32812	+ 2.57368	+11.45564

$-\gamma^2$	λ_8	λ_9	λ_{10}	λ_{12}
25	+59.736180		+97.652659	+143.606898
100	+26.56408	+43.49374	+62.82728	

TABLE 12.– NUMERICAL VALUES OF THE COEFFICIENTS $a^{o}_{o,r}(\gamma)$, AND THE FUNCTION VALUES $X^{(1)}_{o}(1;\gamma)$ AND $X^{(1)}_{o}(0;\gamma)$

r	$-\gamma^2 = 3$	4	5	6
0	+0.987210	+0.976788	+0.963507	+0.947840
2	− .356220	− .478334	− .597277	− .710545
4	+ 18683	+ 33565	+ 52482	+ 74937
6	− 408	− 979	− 1914	− 3279
8	+ 5	+ 16	+ 38	+ 79
10				− 1
$X_o(1)$	+1.362526	+1.489682	+1.615218	+1.736681
$X_o(0)$	+ .815979	+ .749906	+ .683961	+ .619666

r	$-\gamma^2 = 7$	8	9	10
0	+0.930429	+0.911986	+0.892960	+0.874035
2	− .816037	− .912632	− .999698	−1.077418
4	+ .100273	+ .127817	+ .156894	+ .186943
6	− 5114	− 7440	− 10250	− 13535
8	+ 144	+ 238	+ 369	+ 541
10	− 3	− 5	− 8	− 14
$X_o(1)$	+1.852000	+1.960118	+2.060379	+2.152486
$X_o(0)$	+ .558452	+ .501339	+ .448846	+ .401345

r	$-\gamma^2 = 15$	20	25	50	100
0	+0.78915	+0.72576	+0.67909	+0.55601	+0.46036
2	−1.34978	−1.49587	−1.57800	−1.68750	−1.64674
4	+ .33881	+ .47816	+ .60025	+1.01470	+1.38412
6	− 3608	− 6622	− .10110	− .29658	− .63255
8	+ 260	+ 515	+ 965	+ 5161	+ .18483
10	− 10	− 26	− 59	− 595	− 3745
12	+	+ 1	+ 3	+ 48	+ 555
14				− 3	− 63
16					+ 1
$X_o(1)$	+2.51654	+2.77142	+2.96871	+3.61286	+4.35229
$X_o(0)$	+ .23073	+ .13779	+ .08609	+ .01284	+ .00081

TABLE 13.- NUMERICAL VALUES OF THE COEFFICIENTS $a^{o}_{1,r}(\gamma)$, AND THE FUNCTION VALUES $X^{(1)}_{1}(1;\gamma)$ AND $dX^{(1)}_{1}(0;\gamma)/d\xi$

r	$-\gamma^2$ = 3	4	5	6
0	+0.997105	+0.994984	+0.992380	+0.989330
2	− .116098	− .152711	− .188048	− .222236
4	+ 3902	+ 6812	+ 9751	+ 14717
6	− 63	− 150	− 262	− 473
8	+ 1	+ 2	+ 4	+ 9
$X_1(1)$	+1.117169	+1.154659	+1.190445	+1.226765
$X_1(0)'$	+ .830138	+ .778367	+ .728028	+ .682557

r	$-\gamma^2$ = 7	8	9	10
0	+0.985910	+0.982170	+0.978150	+0.973908
2	− .255039	− .286470	− .316613	− .345385
4	+ 19595	+ 25008	+ 30907	+ 37211
6	− 753	− 1066	− 1477	− 1969
8	+ 16	+ 27	+ 42	+ 62
10			− 1	− 1
$X_1(1)$	+1.261293	+1.294741	+1.327190	+1.358536
$X_1(0)'$	+ .638528	+ .597088	+ .558051	+ .521444

r	$-\gamma^2$ = 15	20	25	50	100
0	+0.95067	+0.92654	+0.90339	+0.81333	+0.71269
2	− .47019	− .56735	− .64300	− .84443	− .95736
4	+ 7352	+ .11408	+ .15571	+ .34108	+ .57527
6	− 572	− 1160	− 1935	− 7563	− .20752
8	+ 27	+ 71	+ 146	+ 1060	+ 5010
10	− 1	− 3	− 7	− 102	− 863
12				+ 7	+ 111
14					− 11
16					+ 1
$X_1(1)$	+1.50038	+1.62031	+1.72298	+2.08616	+2.51279
$X_1(0)'$	+ .37136	+ .26571	+ .19192	+ .04430	+ .00421

TABLE 14.– NUMERICAL VALUES OF THE COEFFICIENTS $a^{o}_{2,r}(\gamma)$, AND THE FUNCTION VALUES $X^{(1)}_{2}(1;\gamma)$ AND $X^{(1)}_{2}(0;\gamma)$

r	$-\gamma^2 = 3$	4	5	6
–2	+0.071289	+0.095777	+0.119671	+0.142474
0	+ .985721	+ .974148	+ .959391	+ .941924
2	– 72766	– 96430	– .119654	– .142393
4	+ 1840	+ 3258	+ 55067	+ 7260
6	– 24	– 57	– 111	– 190
8	+ 1	+ 1	+ 1	+ 3
$X_2(1)$	+0.989062	+0.978117	+0.964533	+0.949296
$X_2(0)$	– .537431	– .547702	– .556050	– .562257

r	$-\gamma^2 = 7$	8	9	10
–2	+0.163772	+0.183325	+0.201034	+0.216893
0	+ .922384	+ .901424	+ .879642	+ .857549
2	– .164669	– .186536	– .208085	– .229432
4	+ 9837	+ 12798	+ 16148	+ 19901
6	– 302	– 450	– 640	– 880
8	+ 6	+ 10	+ 16	+ 24
$X_2(1)$	+0.933426	+0.917893	+0.903497	+0.890893
$X_2(0)$	– .566205	– .567935	– .567698	– .565615

r	$-\gamma^2 = 15$	20	25	50	100
–2	+0.27303	+0.30282	+0.31712	+0.28828	+0.22335
0	+ .75076	+ .65357	+ .55985	+ .13410	– .20363
2	– .33636	– .44827	– .56362	– .91568	– .79124
4	+ 4532	+ 8373	+ .13671	+ .50022	+ .89250
6	– 306	– 768	– 1596	– .12202	– .41949
8	+ 13	+ 43	+ 112	+ 1745	+ .11556
10	– 1	– 2	– 5	– 166	– 2129
12				+ 11	+ 240
14				– 1	– 21
16					+ 2
$X_2(1)$	+0.86260	+0.89086	+0.96019	+1.40297	+1.81574
$X_2(0)$	– .53563	– .48567	– .42431	– .13884	– .01312

TABLE 15.– NUMERICAL VALUES OF THE COEFFICIENTS $a^{o}_{3,r}(\gamma)$, AND THE FUNCTION VALUES $X^{(1)}_{3}(1;\gamma)$ AND $dX^{(1)}_{3}(0;\gamma)/d\xi$

r \ $-\gamma^2$	3	4	5	6
−2	+0.049768	+0.065474	+0.080671	+0.095329
0	+ .996217	+ .993384	+ .989910	+ .985786
2	− 52769	− 70252	− 87661	− .104980
4	+ 1057	+ 1877	+ 2929	+ 4212
6	− 12	− 27	− 53	− 92
8			+ 1	+ 1
$X_3(1)$	+1.000287	+1.000066	+0.999883	+0.999742
$X_3(0)'$	−1.447434	−1.427867	+1.407450	−1.386162

r \ $-\gamma^2$	7	8	9	10
−2	+0.109432	+0.122969	+0.135938	+0.148343
0	+ .981091	+ .975879	+ .970201	+ .964106
2	− .122202	− .139316	− .156318	− .173200
4	+ 5725	+ 7466	+ 9433	+ 11626
6	− 145	− 217	− 308	− 422
8	+ 2	+ 4	+ 7	+ 10
$X_3(1)$	+0.999733	+0.999913	+1.000329	+1.001021
$X_3(0)'$	−1.364111	−1.341380	−1.318040	−1.294172

r \ $-\gamma^2$	15	20	25	50	100
−2	+0.20229	+0.24438	+0.27699	+0.35635	+0.37464
0	+ .92878	+ .88803	+ .84416	+ .61460	+ .27353
2	− .25562	− .33423	− .40827	− .68495	− .82807
4	+ 2590	+ 4548	+ 6992	+ .23918	+ .56017
6	− 141	− 332	− 641	− 4395	− .19704
8	+ 5	+ 15	+ 37	+ 508	+ 4388
10			− 2	− 41	− 681
12				+ 2	+ 78
14					− 7
$X_3(1)$	+1.00947	+1.02683	+1.05216	+1.23184	+1.53571
$X_3(0)'$	−1.16949	−1.04147	− .91584	− .42163	− .07382

TABLE 16.- NUMERICAL VALUES OF THE COEFFICIENTS $a^{o}_{4,r}(\gamma)$, AND THE FUNCTION VALUES $X_4^{(1)}(1;\gamma)$ AND $X_4^{(1)}(0;\gamma)$, $X_5^{(1)}(1;\gamma)$ AND $dX_5^{(1)}(0;\gamma)/d\xi$

n = 4

r \ $-\gamma^2$	5	10	15	20	25
−4	+ 2349	+ 9906	+ 2240	+ 4048	+ 6318
−2	+ 67563	+0.137045	+0.19952	+0.26279	+0.32153
0	+0.994218	+ .975942	+ .94636	+ .90051	+ .84069
2	− 68652	− .136070	− .20137	− .26269	− .31828
4	+ 1897	+ 7544	+ 1684	+ 2956	+ 4535
6	− 29	− 234	− 79	− 185	− 357
8	+	+ 5	+ 2	+ 8	+ 18
10					− 1
$X_4(1)$	+0.999582	+0.992656	+0.98827	+0.97338	+0.94971
$X_4(0)$	+ .388021	+ .403891	+ .41853	+ .43549	+ .45131

n = 4

r \ $-\gamma^2$	50	100
−4	+0.17284	+0.19417
−2	+ .45662	+ .25147
0	+ .40843	− .27937
2	− .50209	− .60518
4	+ .16360	+ .59941
6	+ 2753	− .23640
8	+ 291	+ 5458
10	− 21	− 849
12	+ 1	+ 96
14		− 8
16		+ 1
$X_4(1)$	+0.82102	+1.16944
$X_4(0)$	+ .43598	+ .14245

n = 5

r \ $-\gamma^2$	10	15
−4	+ 6543	+ 1424
−2	+0.110949	+0.16441
0	+ .985522	+ .96761
2	− .112180	− .16690
4	+ 5292	+ 1185
6	− 143	− 48
8	+ 3	+ 1
10		
$X_5(1)$	+0.998734	+0.99668
$X_5(0)'$	+1.788369	+1.73792

n = 5

r \ $-\gamma^2$	20	25	50	100
−4	+ 2434	+ 3632	+0.10628	+0.19798
−2	+0.21549	+0.26342	+ .43712	+ .49281
0	+ .94291	+ .91191	+ .68994	+ .18816
2	− .22006	− .27117	− .48639	− .68181
4	+ 2093	+ 3245	+ .12243	+ .39752
6	− 114	− 221	− 1710	− .11798
8	+ 4	+ 10	+ 160	+ 2220
10			− 10	− 293
12				+ 29
14				− 2
$X_5(1)$	+0.99393	+0.99074	+0.98672	+1.11609
$X_5(0)'$	+1.68269	+1.62227	+1.25100	+ .51419

TABLE 17.– NUMERICAL VALUES OF THE COEFFICIENTS $a^{o}_{n,r}(\gamma)$ AS WELL AS OF THE X-FUNCTIONS OF THE FIRST KIND FOR THE ARGUMENTS 1 AND 0, WITH n = 6, 7, 8, 9, 10, 12

	n = 6				
r \ $-\gamma^2$	5	10	15	20	25
−6	+ 20	+ 165	+ 57	+ 136	+ 268
−4	+ 1257	+ 5001	+ 1123	+ 1979	+ 3074
−2	+ 47578	+ 94694	+0.14139	+0.18603	+0.22952
0	+0.997485	+0.989966	+ .98042	+ .96011	+ .93789
2	− 47797	− 95268	− .14252	− .18796	− .23256
4	+ 980	+ 3910	+ 879	+ 1552	+ 2410
6	− 12	− 94	− 32	− 75	− 145
8	+ 1	+ 3	+ 1	+ 2	+ 6
$X_6(1)$	+0.999935	+0.999383	+1.00132	+0.99675	+0.99460
$X_6(0)$	− .317374	− .322430	− .32872	− .33331	− .33924

	n = 6	
r \ $-\gamma^2$	50	100
−6	+ 2977	+0.12226
−4	+0.11872	+ .32925
−2	+ .40974	+ .41829
0	+ .75313	+ .12434
2	− .42313	− .51561
4	+ 9124	+ .28174
6	− 1122	− 7755
8	+ 92	+ 1354
10	− 5	− 166
12		+ 15
14		+ 1
$X_6(1)$	+0.95890	+0.80329
$X_6(0)$	− .38254	− .39599

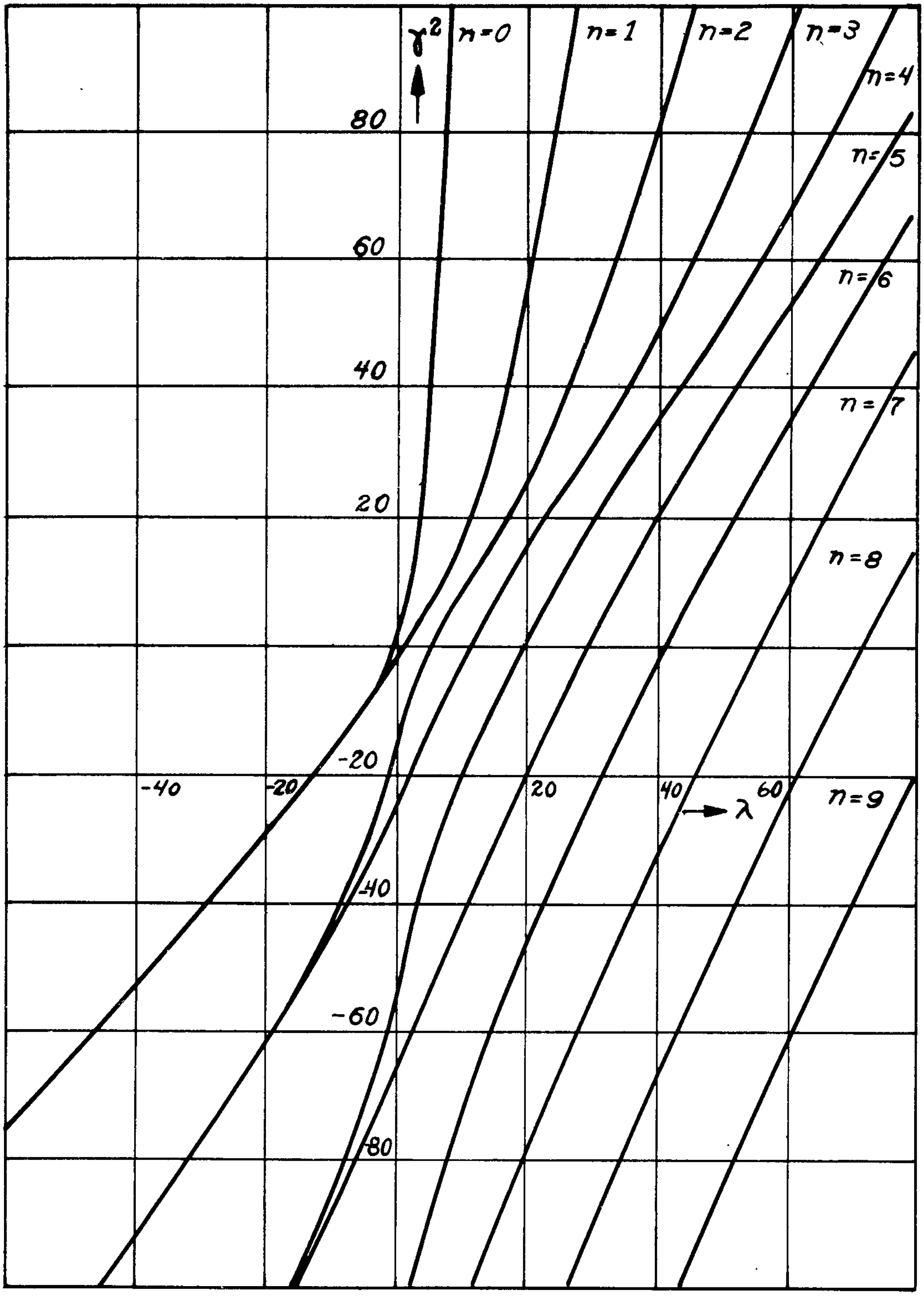
	n = 7	n = 8	
r \ $-\gamma^2$	100	25	100
−8		+ 7	+ 1719
−6	+ 6383	+ 131	+ 7722
−4	+0.25861	+ 1816	+0.23556
−2	+ .52093	+0.17929	+ .51621
0	+ .38995	+ .96481	+ .49454
2	− .58698	− .18035	− .55330
4	+ .24863	+ 1476	+ .20286
6	− .05770	− 72	− 4203
8	+ 880	+ 2	+ 581
10	− 96		− 59
12	+ 8		+ 5
14	− 1		
$X_n(1)$	+0.96695	+0.99830	+0.95851
$X_n(0)$		+ .28617	+ .34251
$X_n(0)'$	−1.36551		

	n = 9
r \ $-\gamma^2$	100
−8	+ 879
−6	+ 5205
−4	+0.19542
−2	+ .50325
0	+ .59410
2	− .52913
4	+ .17076
6	− 3192
8	+ 404
10	− 37
12	+ 3
$X_9(1)$	+0.97926
$X_9(0)'$	+1.99652

	n = 10	
r \ $-\gamma^2$	25	100
−10	+	+ 107
−8	+ 6	+ 769
−6	+ 69	+ 4037
−4	+ 1189	+0.16730
−2	+0.14651	+ .48617
0	+ .97729	+ .67076
2	− .14696	− .50415
4	+ 995	+ .14599
6	− 41	− 2492
8	+ 1	+ 291
10		− 25
12		+ 2
$X_{10}(1)$	+0.99937	+0.99644
$X_{10}(0)$	+ .25343	− .28280

	n = 12
r \ $-\gamma^2$	25
−10	+
−8	+ 2
−6	+ 40
−4	+ 832
−2	+0.12367
0	+ .98408
2	− .12390
4	+ 715
6	− 26
8	+ 1
$X_{12}(1)$	+0.99965
$X_{12}(0)$	+ .23024

11.4.- Course of the Curves $\lambda = \lambda_n(\gamma)$ for Low Values of the Index n



A D D E N D U M

NACA TM 1224

LAMÉ'S WAVE FUNCTIONS OF THE ELLIPSOID OF REVOLUTION

By J. Meixner

April 1949

It has recently been brought to the attention of the NACA by Miss Gertrude Blanch of the Bureau of Standards, Department of Commerce that errors exist in the tabulated values appearing in tables 11 to 17 of TM 1224. Miss Blanch notes that C. J. Bouwkamp, from whom Meixner obtained the values presented, subsequently corrected them in tables appearing in the Journal of Mathematics and Physics, vol. XXVI, no. 2, July 1947, pp. 88-91.

In spite of the difference in symbols and notation in the two papers, reprints of tables I to IX included in the July 1947 issue of the Journal of Mathematics and Physics are attached for the use of those interested in receiving them. The NACA wishes to express its appreciation to the Journal of Mathematics and Physics for permitting these tables to be reproduced for this purpose.

TABLE I

k^2	A_0	A_1	A_2
−10	2.305040	7.285254	11.790394
−9	2.136732	6.820888	11.192939
−8	1.959207	6.342739	10.594773
−7	1.771184	5.850492	9.997253
−6	1.571156	5.343904	9.401958
−5	1.357357	4.822809	8.810735
−4	1.127734	4.287129	8.225713
−3	0.879934	3.736870	7.649318
−2	0.611314	3.172128	7.084258
−1	0.319000	2.593085	6.533473
0	0	2	6
1	−0.348602	1.393206	5.486800
2	−0.729392	0.773098	4.996484
3	−1.144328	0.140119	4.531027
4	−1.594493	−0.505244	4.091509
5	−2.079934	−1.162478	3.677958
6	−2.599668	−1.831051	3.289357
7	−3.151841	−2.510421	2.923796
8	−3.733982	−3.200049	2.578732
9	−4.343293	−3.899400	2.251269
10	−4.976896	−4.607952	1.938420

TABLE II

k^2	A_3	A_4	A_5	A_6
0	12	20	30	42
1	11.492121	19.495277	29.496855	41.497757
2	10.990438	18.994079	28.995964	40.997089
3	10.494513	18.496395	28.497321	40.497988
4	10.003864	18.002228	28.000923	40.000458
5	9.517981	17.511597	27.506763	39.504497
6	9.036338	17.024541	27.014846	39.010106
7	8.558395	16.541109	26.525161	38.517282
8	8.083615	16.061383	26.037710	38.026027
9	7.611465	15.585448	25.552488	37.536339
10	7.141428	15.113424	25.069492	37.048221

TABLE III

Characteristic function $X_0(\xi) = \sum_0^\infty b_{2n} P_{2n}(\xi)$

k^2	b_0	b_2	b_4	b_6	b_8	b_{10}
−10	0.944709	−0.728578	0.110455	−0.007498	0.000288	−0.000007
−9	0.951472	−0.684479	0.094690	−0.005827	0.000203	−0.000005
−8	0.958380	−0.635639	0.079247	−0.004365	0.000136	−0.000003
−7	0.965363	−0.581441	0.064298	−0.003120	0.000085	−0.000001
−6	0.972311	−0.521212	0.050067	−0.002097	0.000049	−0.000001
−5	0.979071	−0.454254	0.036840	−0.001294	0.000025	
−4	0.985428	−0.379882	0.024958	−0.000706	0.000011	
−3	0.991099	−0.297493	0.014835	−0.000316	0.000004	
−2	0.995716	−0.206682	0.006949	−0.000100	0.000001	
−1	0.998846	−0.107374	0.001824	−0.000013		
0	1.000000					
1	0.998691	0.114368	0.001976	0.000014		
2	0.994509	0.233927	0.008138	0.000118	0.000001	
3	0.987210	0.356205	0.018683	0.000408	0.000005	
4	0.976790	0.478301	0.033563	0.000979	0.000016	
5	0.963507	0.597278	0.052483	0.001914	0.000039	
6	0.947848	0.710493	0.074931	0.003279	0.000079	0.000001
7	0.930440	0.815971	0.100266	0.005114	0.000143	0.000003
8	0.911948	0.912502	0.127799	0.007438	0.000238	0.000005
9	0.892980	0.999612	0.156881	0.010250	0.000369	0.000008
10	0.874065	1.077435	0.186946	0.013533	0.000540	0.000014

TABLE IV

Characteristic function $X_1(\xi) = \sum_0^\infty b_{2n+1} P_{2n+1}(\xi)$

k^2	b_1	b_3	b_5	b_7	b_9	b_{11}
−10	0.964429	−0.402104	0.046184	−0.002528	0.000081	−0.000002
−9	0.970923	−0.364436	0.037696	−0.001858	0.000054	−0.000001
−8	0.976877	−0.325710	0.029954	−0.001312	0.000034	−0.000001
−7	0.982232	−0.286082	0.023016	−0.000882	0.000020	
−6	0.986936	−0.245730	0.016934	−0.000556	0.000011	
−5	0.990948	−0.204851	0.011752	−0.000322	0.000005	
−4	0.994236	−0.163656	0.007499	−0.000164	0.000002	
−3	0.996784	−0.122359	0.004197	−0.000069	0.000001	
−2	0.998586	−0.081179	0.001852	−0.000020		
−1	0.999651	−0.040326	0.000459	−0.000002		
0	1.000000					
1	0.999664	0.039616	0.000447	0.000002		
2	0.998683	0.078362	0.001764	0.000019		
3	0.997105	0.116098	0.003902	0.000063	0.000001	
4	0.994984	0.152711	0.006812	0.000147	0.000002	
5	0.992373	0.188112	0.010436	0.000281	0.000004	
6	0.989330	0.222236	0.014716	0.000473	0.000009	
7	0.985910	0.255039	0.019595	0.000733	0.000016	
8	0.982167	0.286500	0.025011	0.001066	0.000027	
9	0.978150	0.316612	0.030908	0.001477	0.000042	0.000001
10	0.973908	0.345386	0.037230	0.001969	0.000062	0.000001

TABLE V

Characteristic function $X_2(\xi) = \sum_0^\infty b_{2n} P_{2n}(\xi)$

k^2	b_0	b_2	b_4	b_6	b_8	b_{10}
0		1.000000				
1	−0.022875	0.998525	0.024445	0.000206	0.000001	
2	−0.046799	0.993846	0.048736	0.000821	0.000007	
3	−0.071286	0.985722	0.072766	0.001840	0.000024	0.000001
4	−0.095772	0.974150	0.096431	0.003258	0.000057	0.000001
5	−0.119671	0.959391	0.119654	0.005067	0.000110	0.000002
6	−0.142464	0.941931	0.142398	0.007260	0.000190	0.000003
7	−0.163759	0.922394	0.164677	0.009837	0.000302	0.000006
8	−0.183310	0.901438	0.186545	0.012799	0.000450	0.000010
9	−0.201017	0.879661	0.208098	0.016150	0.000640	0.000016
10	−0.216892	0.857550	0.229438	0.019902	0.000880	0.000024

TABLE VI

Characteristic function $X_3(\xi) = \sum_0^\infty b_{2n+1} P_{2n+1}(\xi)$

k^2	b_1	b_3	b_5	b_7	b_9	b_{11}
0		1.000000				
1	−0.016979	0.999565	0.017626	0.000118		
2	−0.033587	0.998287	0.035224	0.000470	0.000003	
3	−0.049768	0.996217	0.052770	0.001057	0.000012	
4	−0.065475	0.993406	0.070253	0.001877	0.000027	
5	−0.080671	0.989910	0.087661	0.002929	0.000053	
6	−0.095328	0.985786	0.104979	0.004212	0.000092	0.000001
7	−0.109432	0.981091	0.122202	0.005725	0.000145	0.000002
8	−0.122970	0.975879	0.139316	0.007465	0.000217	0.000004
9	−0.135939	0.970201	0.156318	0.009433	0.000308	0.000006
10	−0.148343	0.964106	0.173200	0.011626	0.000422	0.000010

TABLE VII

Characteristic function $X_4(\xi) = \sum_0^\infty b_{2n} P_{2n}(\xi)$

k^2	b_0	b_2	b_4	b_6	b_8	b_{10}	b_{12}
0			1.000000				
1	0.000091	−0.013588	0.999768	0.013773	0.000076		
2	0.000368	−0.027140	0.999074	0.027528	0.000304	0.000002	
3	0.000834	−0.040653	0.997918	0.041266	0.000684	0.000006	
4	0.001493	−0.054128	0.996300	0.054977	0.001215	0.000015	
5	0.002348	−0.067563	0.994218	0.068651	0.001897	0.000029	
6	0.003404	−0.080957	0.991669	0.082279	0.002729	0.000051	0.000001
7	0.004663	−0.094312	0.988648	0.095855	0.003712	0.000080	0.000001
8	0.006125	−0.107546	0.984408	0.109286	0.004839	0.000120	0.000002
9	0.007806	−0.120900	0.981162	0.122811	0.006122	0.000171	0.000003
10	0.009695	−0.134130	0.976680	0.136173	0.007550	0.000234	0.000005

TABLE VIII

Characteristic function $X_5(\xi) = \sum_{0}^{\infty} b_{2n+1} P_{2n+1}(\xi)$

k^2	b_1	b_3	b_5	b_7	b_9	b_{11}	b_{13}
0			1.000000				
1	0.000068	−0.011218	0.999854	0.011294	0.000046		
2	0.000273	−0.022423	0.999418	0.022584	0.000213	0.000001	
3	0.000611	−0.033609	0.998690	0.033863	0.000478	0.000004	
4	0.001080	−0.044772	0.997673	0.045127	0.000850	0.000009	
5	0.001681	−0.055906	0.996367	0.056373	0.001327	0.000018	
6	0.002408	−0.067006	0.994772	0.067596	0.001910	0.000031	
7	0.003262	−0.078066	0.992888	0.078793	0.002598	0.000050	0.000001
8	0.004236	−0.089080	0.990719	0.089958	0.003391	0.000073	0.000001
9	0.005331	−0.100043	0.988263	0.101088	0.004290	0.000105	0.000002
10	0.006543	−0.110949	0.985522	0.112180	0.005292	0.000143	0.000003

TABLE IX

Characteristic function $X_6(\xi) = \sum_{0}^{\infty} b_{2n} P_{2n}(\xi)$

k^2	b_0	b_2	b_4	b_6	b_8	b_{10}	b_{12}	b_{14}
0				1.000000				
1		0.000050	−0.009535	0.999899	0.009571	0.000039		
2	−0.000001	0.000202	−0.019061	0.999597	0.019140	0.000157	0.000001	
3	−0.000004	0.000454	−0.028580	0.999094	0.028702	0.000353	0.000003	
4	−0.000010	0.000805	−0.038087	0.998390	0.038257	0.000627	0.000006	
5	−0.000020	0.001256	−0.047578	0.997486	0.047802	0.000980	0.000012	
6	−0.000035	0.001807	−0.057050	0.996381	0.057331	0.001410	0.000020	
7	−0.000056	0.002458	−0.066501	0.995076	0.066845	0.001919	0.000032	
8	−0.000084	0.003207	−0.075928	0.993572	0.076341	0.002505	0.000048	0.000001
9	−0.000120	0.004055	−0.085326	0.991867	0.085816	0.003169	0.000068	0.000001
10	−0.000165	0.005001	−0.094694	0.989966	0.095268	0.003910	0.000094	0.000002

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